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THE "WORLD'S SIMPLEST AXIOM OF CHOICE" FAILS

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We use topos-theoretic methods to show that intuitionistic set theory with countable or dependent choice does not imply that every family, all of whose elements are doubletons and which has at most one element, has a choice function.

1. Introduction

The axiom in question (WSAC), originally formulated by F. Richman, states, $\forall F$

$$\begin{array}{l} \forall x,y \in F \cdot x = y \\ \wedge \\ \forall x \in F \cdot \exists v,w \cdot (v \neq w \quad x = \{v,w\}) \end{array} \to \exists f : F \to \cup F \quad .$$

Of course, for such a family, any such f is a choice function, $\forall x \in F \cdot f(x) \in x$. Also, F is completely determined by $\cup F$

$$F = \{\cup F \mid \exists v \cdot v \in \cup F\} \ .$$

Furthermore, $\cup F$ may be any set which if it is inhabited is a doubleton. If A is such that

$$\exists v \cdot v \in A \to \exists v,w \cdot (v \neq w \quad A = \{v,w\}),$$

let $F = \{A \mid \exists x \cdot x \in A\}$, then $A = \cup F$.

We use the technique (originating with Joyal) of considering the universal example of such a set A. It is easy to check that the interpretation of higher-order logic in the classifying topos [12] [16] satisfies countable and dependent choice, CC & DC but not WSAC. This is perhaps the world's simplest example of this technique. We show that such examples can arise in the well-founded part of a topos [7] by embedding this universal example

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universally in 2^N (as in [3] §5).

For details on classifying topoi we refer to Tierney [16] or Makkai & Reyes [11]. The interpretation of intuitionistic type-theory and set-theory in a topos is described by Fourman [2] and [3] and more concretely, by Osius [13] and Scott [15].

2. The Basic Model

We introduce universally a set A which if it is inhabited is a doubleton. The classifying topos is the functor category $S^{\mathbb{C}}$ where $\mathbb{C}$ is the category whose non-identities are

$$E \xrightarrow{\alpha} D \circlearrowleft \beta$$

with $\beta \circ \alpha = \alpha$ and $\beta^2 = \text{id}$. This is (equivalent to) the category of finitely presented such sets (E is the empty set and D a doubleton) and monomorphisms (monos because the equality on such a set is decidable and this would be reflected in any geometric axiomatisation by the addition of predicates or operations forcing homomorphisms to be monos).

The models we consider in this section are thus functors from $\mathbb{C}$ into the category of sets. These are like Kripke models. The category $\mathbb{C}$ replaces the usual partial order and instead of restrictions we need transition maps corresponding to the morphisms of our category. In particular, in our case, a model X will be a pair of domains X(E) and X(D) together with an automorphism $X(\beta) : X(D) \to X(D)$ of order two and a restriction map $X(\alpha) : X(E) \to X(D)$ whose image is fixed by $X(\beta)$. The interpretation of logic in such a presheaf topos is given by a straightforward generalisation of Kripke's definitions [9]. Function spaces are modelled by the categorical exponents which are easily calculated using the Yoneda lemma [11] [10].

The universal A we want is given by the forgetful functor $\mathbb{C} \to \text{Sets}$. The family $F = \{A \mid \exists x, x \in A\}$ is then represented by the functor $F(E) = \emptyset$, $F(D) = \{A\}$. Since $A = \bigcup F$ we now calculate the function space A^F.

$$A^F(E) \cong [\underline{E}, A^F] \cong [F \times \underline{E}, A]$$

$$A^F(D) \cong [F \times \underline{D}, A] \quad \text{similarly.}$$

In words, $A^F(E)$ is represented by the set of natural transformations from $F \times \underline{E}$ to A (where $\underline{E}$ is the representable functor). Since $(F \times \underline{E})(D)$ is a singleton and $A(D)$ is a doubleton with the non-trivial automorphism, there is no map $(F \times \underline{E})(D) \to A(D)$ which respects the automorphism and so no natural transformation $F \times \underline{E} \to A$. Thus $A^F(E) = \emptyset$ and $E \not\Vdash \exists f : F \to \bigcup F$. So $\not\models \exists f : F \to \bigcup F$. However, since $\models$ "if A is inhabited A is a doubleton" (by construction), the hypotheses of WSAC are valid. Thus $\not\models$ WASC.

3. Generalities for Presheaf Models

We firstly show that in any presheaf topos $S^{\mathbb{C}}$ countable choice, CC and dependent choice DC hold (assuming the corresponding principles in our metatheory). The natural numbers in any presheaf topos are given by the constant functor $\mathbb{N}(D) = N$. Now suppose $D \in |\mathbb{C}|$ and that

$$D \Vdash \forall n \in \mathbb{N}\ \exists y \in Y\ \varphi(n,y)$$

where Y is some functor in $S^{\mathbb{C}}$. Then by CC for each $n \in N$ we have $y_n \in Y(D)$ such that

$$D \Vdash \varphi(n,y_n).$$

We define a natural transformation

$$f : \underline{D} \times \mathbb{N} \to Y$$

by $f(\alpha,n) = (Y(\alpha))(y_n)$ for $(\alpha,n) \in (\underline{D} \times \mathbb{N})(B)$

(i.e. for $\alpha : D \to B$ and $n \in N \cong \mathbb{N}(B)$).

By Yoneda (as before) f represents an element of

$Y^{\mathbb{N}}(D)$ and by the persistence property of forcing $D \Vdash \forall n \in \mathbb{N}. \varphi(n, f(n))$. The argument for DC is similar.

We now consider induction over trees of finite sequences. If A is a set, the principle of induction over the A-splitting tree, I_A states that if a collection of finite sequences $K \subset A^{<N}$ is persistent

$$\forall e \in K \cdot \forall a \in A \cdot e^\frown\langle a\rangle \in K, \qquad 1)$$

a cover (or bar)

$$\forall \alpha \in A^N \cdot \exists e \in K \cdot e \subseteq \alpha \qquad 2)$$

and inductive

$$\forall e \, [\forall a \in A \cdot e^\frown\langle a\rangle \in K \to e \in K] \qquad 3)$$

then $\langle\ \rangle \in K$. We call 1), 2) and 3) (conjoined) the hypotheses of I_A for K.

The fan theorem, FT and bar induction, BI are special cases ($A = 2$ and $A = N$). Such principles may fail in sheaf models [5], [14].

In presheaf topoi, finite and countable sequences are computed pointwise:

$$A^{\mathbb{N}}(D) \cong A(D)^N$$

and

$$A^{<\mathbb{N}}(D) \cong A(D)^{<\mathbb{N}} .$$

If K is a subpresheaf of $A^{<N}$ and

$$D \Vdash \text{"hypotheses of } I_A \text{ for } K\text{"}$$

then $K(D) \subseteq A(D)^{<N}$ is a persistent cover. In general, there is no reason why it should be inductive. However, if each restriction map from $A(D)$ is onto and every extension of e in $A^{<N}(D)$ belongs to $K(D)$ then from

$$D \Vdash e^\frown\langle a\rangle \in K \quad \text{for each} \quad a \in A(D)$$

we obtain

$$D \Vdash \forall a \cdot e\frown\langle a\rangle \in K .$$

Thus $D \Vdash e \in K$ and $K(D)$ is inductive. By appeal to $I_{K(D)}$

$$D \Vdash \langle\ \rangle \in K .$$

Since 2 and $\mathbb{N}$ are given by constant presheaves, this shows that FT and BI pass to presheaf models. There are probably easier ways to see this but we want to take the analysis a little further. We show that our model of §2 satisfies I_A for any presheaf A. Since every restriction from A(D) is an automorphism (hence onto), by the remarks above we have

$$D \Vdash I_A .$$

Now suppose that

$$E \Vdash \text{"hypotheses of } I_A \text{ for } K\text{"} .$$

Then restricting to D we see that $D \Vdash \langle\ \rangle \in K$ whence $D \Vdash e \in K$ for every $e \in A^{<N}(D)$.

This suffices to show that K(E) is inductive. By appeal to $I_{K(E)}$ we obtain $E \Vdash \langle\ \rangle \in K$. Thus

$$\models I_A .$$

A more general treatment of such induction principles in presheaf topoi requires a discussion of induction over categories. We shall not pursue this here.

4. A Well-Founded Example

Here we work inside the model constructed in §2 and show that the set A may be embedded in $2^{\mathbb{N}}$ (in a suitable extension) without destroying CC or DC or adding a choice function for the corresponding family F. Again we consider the universal solution or classifying topos. To add an A-indexed family of elements of $2^{\mathbb{N}}$ (Cantor space) we take sheaves on the space $(2^{\mathbb{N}})^A$. This type of model (sheaves on a space) is discussed in detail in Fourman & Scott [6] and Fourman & Hyland [5]. In any such model the internal Cantor space is represented as the sheaf of continuous $2^{\mathbb{N}}$-valued functions. In our example, the required elements of Cantor space are given by the various projections from $(2^{\mathbb{N}})^A$ to $2^{\mathbb{N}}$.

The projections from any inhabited open of $(2^{\mathbb{N}})^A$ from a set isomorphic to A. They generate a subsheaf $\mathbb{A}$ of the internal Cantor space which satifies internally the condition that if it is inhabited it is a doubleton. The corresponding family $\mathbb{F}$ is represented as the trivial sheaf generated by a family of global sections isomorphic to F.

We now show that if there is any morphism $\mathbb{F} \to \mathbb{A}$ defined over some inhabited open of $(2^{\mathbb{N}})^A$ then there is a function $F \to A$ (contrary to the results of 2). Suppose that $\alpha \in [[f : \mathbb{F} \to \mathbb{A}]]$ then for each $x \in F$ there is a unique projection $\pi_a : (2^{\mathbb{N}})^A \to 2^{\mathbb{N}}$ such that $\alpha \in [[f(x) = \pi_2]]$. As π_a determines a this would yield a function $F \to A$. This shows that our new model cannot satisfy WSAC.

To see that CC and DC hold in this model we note that the space $(2^{\mathbb{N}})^A \cong (2^A)^{\mathbb{N}}$ is zero-dimensional in the sense that every open cover has a refinement by mutually disjoint clopen sets because in the model of §2 we have $I_{(2^A)}$ and 2^A is decidable. Thus any existence statement can be realized by a global section and we can use the corresponding principles in the model of §2 to choose sequences of global sections which define the required internal functions.

$$ZF + DC \nvdash WSAC.$$

5. Remarks

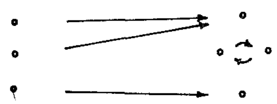
The choice principle WSAC arose from the consideration of separable closures in constructive mathematics: if $X^2 + 1$ has a root it has two and they are distinct.

In the field,

$$R \to \mathbb{C}^2,$$

which lives in the model of §2, with the transition maps given by the inclusion and complex conjugation, the roots of $X^2 + 1$ are just our set A. There can be no field extension in which the set of roots becomes inhabited since this would yield a choice function.

However, adding a root universally gives us an open covering. This corresponds to taking sheaves in $S^{\mathbb{C}}$ on the internal locale



with the transition maps indicated. This locale is generated by the elements $[\![f(*) = a_1]\!]$ and $[\![f(*) = a_2]\!]$ which are permuted by the action of β.

The use of $I_{(2^A)}$ in the model of §2 to see that CC or DC pass to our final model may be circumvented if we use the formal locale $(2^A)^{\mathbb{N}}$ in place of the space $(2^A)^{\mathbb{N}}$. These coincide if $I_{(2^A)}$ is valid [4].

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References

1 DUMMETT, M.A.E.: Elements of Intuitionism . Oxford University Press, Oxford (1977)

2 FOURMAN, M.P.: The logic of topoi. In: Handbook of Mathematical Logic (ed. J. Barwise), North Holland, Amsterdam (1977), 1053-1090

3 FOURMAN, M.P.: Sheaf models for set theory. J. Pure and Applied Algebra 19 (1980), 91-101

4 FOURMAN, M.P. and GRAYSON, R.J.: Formal spaces induction principles and completeness theorems. In preparation

5 FOURMAN, M.P. and HYLAND, J.M.E.: Sheaf models for analysis. In: Applications of Sheaves (Proc. L.M.S. Durham Symposium, 1977), Springer Lecture Notes in Math. 753 (1979)

6 FOURMAN, M.P. and SCOTT, D.S.: Sheaves and logic. In: Applications of Sheaves (Proc. L.M.S. Durham Symposium, 1977), Springer Lecture Notes in Math. 753 (1979)

7 FREYD, P.: The axiom of choice. J. Pure and Applied Algebra 19 (1981), 103-125

8 JOYAL, A. and REYES, G.: Generic models in categorical logic. J. Pure and Applied Algebra (to appear)

9 KRIPKE, S.: Semantical analysis of intuitionistic logic I. Z. math. logik u Grundl. Math. 9 (1963), 67-96

10 LAWVERE, F.W.: Introduction. In: Toposes, Algebraic Geometry and Logic. Springer Lecture Notes in Math. 274 (1972), 1-12

11 MACLANE, S.: Categories for the Working Mathematician. Springer-Verlag, Heidelberg

12 MAKKAI, M. and REYES, G.: First-order categorical logic. Springer Lecture Notes in Math. 611 (1977)

13 OSIUS, G.: Logical and Set-Theoretical Tools in Elementary Topoi. In: Model Theory and Topoi. Springer Lecture Notes in Math. 445 (1975), 297-346

14 ŠČEDROV, A.: Consistency and Independence Proofs in intuitionistic set theory. Proceedings of the New Mexico Conference on Constructive Mathematics (F. Richman, ed.), Springer Lecture Notes (to appear)

15 SCOTT, D.S.: Sheaf models for set theory. To appear

16 TIERNEY, M.: Forcing Topologies and Classifying Topoi, in Algebra, Topology and Category Theory; a collection of papers in honor of Samuel Eilenberg (A. Heller, ed.), Academic Press, New York (1976), 211-219

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