Algebraic Methods for the Calculation of Radiation Exchange in an Enclosure

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Abstract. The algebraic methods of Hottel and Sarofim, Gebhart and one recently published by the authors for the calculation of radiant interchange in an enclosure are examined. Each of these formulations, while written in a different form, is shown to be mathematically equivalent to the others. This equivalence is established by deriving the formulations of Hottel and Sarofim and Gebhart from that of the authors. Because of the analytical complexity of these forms their mathematical equivalence cannot be demonstrated by explicit methods. For this reason proof of equivalence is established by numerical programming using a digital computer. Three different numerical examples are used for this purpose.–In addition, the mathematical structure underlying the absorption factor B_{ij} of Gebhart and the total exchange factor $\overline{S_i S_j}$ and total view factor $\mathscr{F}_{ij}$ of Hottel and Sarofim is developed.

Zusammenfassung. Die algebraischen Methoden zur Berechnung des Strahlungsaustausches in Hohlräumen von Hottel und Sarofim, Gebhart und von den Autoren dieser Arbeit werden untersucht. Trotz verschiedener Schreibweisen sind die einzelnen Formulierungen mathematisch gleichwertig. Das zeigt sich, wenn man die Formulierungen von Hottel und Sarofim sowie von Gebhart aus der unsrigen ableitet. Wegen der komplizierten mathematischen Form kann diese Äquivalenz nicht explizit gezeigt werden. Sie wurde daher numerisch mit Hilfe eines Digitalrechners an drei Beispielen nachgewiesen. Außerdem wurde die mathematische Struktur des Absorptionsfaktors B_{ij} nach Gebhart, des totalen Austauschfaktors $\overline{S_i S_j}$ und des totalen Flächenverhältnisses $\mathscr{F}_{ij}$ nach Hottel und Sarofim abgeleitet.

Nomenclature

A	area, $[ft^2]$
A_{ij}	cofactor of matrix $[a_{ij}]$, Eq. (15), [1]
$[a_{ij}]$	geometric view factor–reflectance matrix, Eq. (8)
B_{ij}	Gebhart absorption factor, Eq. (1), [1]
D	determinant of $[a_{ij}]$ matrix, [1]
DERGEBHOT	name of authors' computer program to calculate Gebhart's B_{ij} and Hottel's $\overline{S_i S_j} = A \mathscr{F}_{ij}$
F_{ij}	geometric view factor, [1]
$[F_{ij}]$	geometric view factor matrix, Eq. (9), [1]
$\mathscr{F}_{ij}$	Hottel's script-F or total view factor, Eq. (32), [1]
H	incident radiation flux, Eq. (3), $[Btu/hr\text{-}ft^2]$
$[I]$	identity matrix, Eq. (10), [1]
J	radiosity flux, Eqs. (14), (16), (17), $[Btu/hr\text{-}ft^2]$
K	$(\epsilon/\rho)\,\sigma T^4$, Eq. (13), $[Btu/hr\text{-}ft^2]$
n	number of surfaces in enclosure, [1]
q_{ij}^*	net radiation heat transfer rate between surfaces A_i and A_j, Eq. (31) $[Btu/hr]$
q_{ij}	energy absorbed at surface A_j originating as emission at surface A_i, $[Btu/hr]$

q_j^*	net radiation heat transfer at surface A_j, $[Btu/hr]$
RADTQO	name of authors' computer program to use with formulation in [12]
T	temperature, $[°R]$
x	coordinate

Greek

α	absorptance or absorptivity, [1]
β	angle, [1]
$[\beta_{ij}]$	matrix, Eq. (25), [1]
γ_{ij}	coefficients of the inverted matrix $[a_{ij}]^{-1}$, Eq. (15), [1]
ϵ	emittance or emissivity, [1]
ρ	reflectance or reflectivity, [1]
σ	Stefan-Boltzmann constant

Subscripts

i	designation of row number in matrix; also ith surface
j	designation of column number in matrix; also jth surface

1. Introduction

The exchange of thermal radiation between the surfaces of an enclosure having arbitrary geometry is an important engineering design problem of current interest. Frequently, the information desired in such calculations is either the temperature of the radiating surfaces for a prescribed heat flux rate or the net rate of heat transfer at the surface for specified surface temperatures. In some instances, an enclosure may have heat flux rates fixed for some of the surfaces and temperatures given on others. The calculation of each of these quantities can, furthermore, be determined by the manner in which the enclosure interacts thermally with its external environment. This interaction can take many forms, including both convective and

radiative exchange with an environment, a fixed exter-
nal heat flux, an enthalpy transfer by the agency of
fluid flow in ducts within the surface or heat genera-
tion in the surface. In terms of enclosure analysis
these interactive effects represent boundary conditions
or constraints imposed on the system of equations
governing the radiation exchange within the enclosure.
There are also other constraints imposed by the laws
of thermodynamics which while not usually explicitly
stated, represent immutable limits on both the physical
system and the set of equations chosen to describe the
system. For a steady-state condition the first law
requires the sum of the net internal radiant heat trans-
fer rates at the surfaces to be zero; the second law
requires the specification of at least one temperature
in the system, either at a surface or in the environ-
ment with which the system interacts.

For many engineering design situations the assump-
tion of gray, diffuse radiating surfaces without the
presence of participating media is satisfactory. The
design calculations in such systems are generally carried
out using an algebraic or numerical formulation with
subsequent reduction of the equations on a digital com-
puter or, in simple systems, by hand calculations.
These kinds of formulations which assume uniform
radiosity flux and uniform properties for each surface
are numerical approximations to the more exact analy-
tical formulations involving the solution of a system of
integral equations. However, the difficulties encoun-
tered in the solution of such equations for an enclosure
of arbitrary geometry are so enormous as to render any
general attempt impractical for design purposes. On
the other hand, the various algebraic methods may be
relatively easily applied to very complex systems provid-
ing results having excellent correspondence with the
more exact formulations. This is particularly true
when a large memory digital computer is available for
programming and solving the governing system of
equations. Furthermore, the condition of variable sur-
face radiosity flux, temperature and surface properties
can be approximated to almost any degree by sub-
dividing a surface into sufficiently small sub-sections.
The penalty for this improvement is an increase in the
magnitude and cost of the calculations.

In recent years several algebraic methods for radia-
tion enclosure analysis and calculation have been
published. Probably the earliest of these is that of
Hottel [1]. Later this work was expanded by Hottel
and Sarofim [2]. The basis for their method is the
concept of the radiosity heat flux and, hence, is some-
times referred to as a radiosity method. Gebhart [3,
4] has approached the problem of enclosure analysis
in a different way. His method is based on the defini-
tion of an absorption factor, the fraction of energy
absorbed at a surface which originates as emission at
another surface. Other work dealing with radiant
exchange between the surfaces of an enclosure include

those of Sparrow [5], Sparrow and Cess [6], Oppen-
heim [7], Eckert and Drake [8], Love [9], Wiebelt [10]
and Siegel and Howell [11].

Recently the authors have published a paper on this
problem [12] which treats an enclosure having gray,
diffuse surfaces that are adiabatic or of known temper-
ature. The formulation of this method is based on the
concept of radiosity heat flux and is similar to that of
Hottel [1, 2] although formulated in somewhat differ-
ent manner. The emphasis is on engineering design
application with the governing equations written for
convenience in computer programing. The analysis is
coupled with the external thermal interactions of the
enclosure with its environment. A number of examples
of the use of the method to enclosures are given and
a comparison is made with the results of Hottel [1, 2]
and Hottel and Sarofim [2] and Gebhart [3].

The various algebraic methods for enclosure ana-
lysis are mathematically identical. Although formu-
lated in different ways their differences represent only
the varying mathematical and physical perspectives of
their authors. For a given set of conditions, each
method produces results identical to that of the others.
Usually certain types of calculations are more conve-
niently made with one method, rather than another,
owing to the manner of its formulation. It is also
possible to derive the form of any of the methods
using another as a starting point.

The purpose of the present paper is to demonstrate
the equivalence of the algebraic methods of Hottel
and Sarofim [1, 2], Gebhart [3] and that given by the
authors [12]. The equivalence will be established by
using the enclosure analysis in [12] to derive the
methods of [1, 2 and 3]. Specific numerical examples
will then be given in which numerical results using
each method are compared. These results will also be
compared with such independent calculations made by
others as are available. All numerical calculations made
by the present authors were performed using the
IBM 360/67 Digital Computer of The University of
Michigan Computing Center.

In addition to showing equivalence of the various
methods for enclosure calculations, a fundamental
derivation of the elements which form the Gebhart
absorption factor, B_{ij}, the Hottel total-view factor,
$\mathscr{F}_{ij}$, and the total-exchange area, $\overline{S_i S_j}$, of Hottel and
Sarofim will be given. All computer calculations have
been performed using the author's program named
DERGEBHOT.

2. Derivation of Gebhart's Absorption Factors

An enclosure of n-surfaces having arbitrary geometry
is shown in Fig. 1. It will be assumed that all surfaces
have gray, diffuse radiation characteristics. The conse-
quences and applicability of this assumption are well
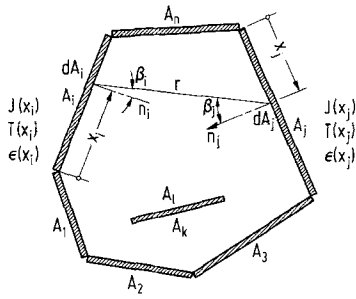documented [2, 6, 12]. Further, the temperature and

Fig. 1. A general n-surface enclosure encompassing 4π-steradians solid angle

radiosity flux are assumed to be uniform over each surface. As a practical matter these assumptions are usually necessary for engineering design calculations and do represent approximations that closely resemble those circumstances often found in practice. The enclosure is filled with a non-participating medium.

Gebhart's absorption factor [3], B_{ij}, is defined as the fraction of radiation emitted by surface A_i that is absorbed by surface A_j considering all paths of radiant exchange between A_i and A_j. Hence,

$$B_{ij} = \frac{\text{Energy Absorbed at } A_j \text{ originating as emission at } A_i}{\epsilon_i A_i \sigma T_i^4}$$

$$= \frac{q_{ij}}{\epsilon_i A_i \sigma T_i^4} . \tag{1}$$

The heat transfer rate incident on surface A_j may be written [12] in terms of the incident flux, H_j, the radiosity fluxes of all n surfaces, $J_1, \ldots J_n$, and the geometric view factors $F_{1j}, \ldots, F_{nj}$, considering emission and reflection from all n-surfaces, as

$$A_j H_j = F_{1j} A_1 J_1 + F_{2j} A_2 J_2 + \ldots + F_{nj} A_n J_n . \tag{2}$$

Since by the reciprocal (geometric) relations $F_{1j} A_1 = F_{j1} A_j$, $F_{2j} A_2 = F_{j2} A_j$, etc., the incident flux on A_j becomes

$$H_j = F_{j1} J_1 + F_{j2} J_2 + \ldots + F_{jn} J_n . \tag{3}$$

$$(j = 1, 2 \ldots n)$$

Now, the energy absorbed at A_j for emission and reflection from all n-surfaces may be written as

$$q_{ij} = \alpha_j A_j H_j . \qquad (j = 1, 2 \ldots n) \tag{4}$$

To relate these results to the absorption factor, B_{ij}, of Eq. (1), it is necessary to identify the energy absorbed at A_j, that is q_{ij}, with that energy which originated only at surface A_i as emission. As written in Eq. (4) q_j is the energy absorbed at A_j resulting from

simultaneous emission and reflection of energy from all the n-surfaces in the enclosure. Owing to the complexity of radiant interchange, as expressed in Eq. (4), it is difficult to isolate that fraction of q_j that originated as emission from A_i. However, it is possible to determine that portion of q_j which originated as emission from A_i if a change is made in the analytical model, Fig. 1. This is to set the temperatures of all n-surfaces to zero, except A_i, which has a finite temperature T_i. In this instance, then, it is only A_i which is a source of energy since all other surfaces, including A_j, being at zero temperature, emit no energy but can only absorb and reflect. Hence, an energy absorbed at any surface, including A_j, under these circumstances originates as emission from A_i as a source. Accordingly, H_j can be computed directly from Eq. (3) in terms of the radiosity fluxes, $J_1, \ldots, J_n$, existing at all the surfaces under the conditions of a finite temperature only at A_i.[1] While this model is hypothetical, the resulting B_{ij} have genuine physical significance.

Thus, the energy absorbed at A_j originating as emission from A_i, including all paths for interchange becomes from Eq. (4)

$$q_{ij} = \alpha_j A_j H_{ij} , \tag{5}$$

$$(i = 1, 2 \ldots n) \quad (j = 1, 2 \ldots n)$$

where, H_{ij} is the partial incident energy flux on A_j originating as emission at A_i and is found using Eq. (3). The radiosity fluxes in Eq. (3) are those existing under the circumstances of zero temperatures on all surfaces except A_i and may be considered as partial radiosity fluxes. To keep the notation as simple as possible, however, these fluxes will be designated with a single subscript only. Thus,

$$q_{ij} = \alpha_j A_j (F_{j1} J_1 + F_{j2} J_2 + \ldots + F_{jn} J_n) , \tag{6}$$

$$(i = 1, 2 \ldots n) \quad (j = 1, 2 \ldots n)$$

The various radiosity fluxes $J_1, J_2, \ldots, J_n$, are determined by the formulation given in [12], as

$$[a_{ij}] [J_j] + [K_i] = 0 , \tag{7}$$

$$(i = 1, 2 \ldots n) \quad (j = 1, 2 \ldots n)$$

where, the following matrixes are defined [12]

$$[a_{ij}] = [F_{ij}] - \frac{1}{\rho_i} [I] \tag{8}$$

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[1] Under these circumstances the fraction of energy absorbed at A_j, originating as emission only at A_i, is designated as q_{ij}, a partial absorbed energy at A_j. Also, the corresponding partial incident energy flux at A_j is written as H_{ij}.

and,

$$[F_{ij}] = \begin{bmatrix} F_{11} & F_{12} & \ldots & F_{1n} \\ F_{21} & F_{22} & \ldots & F_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ F_{n1} & F_{n2} & \ldots & F_{nn} \end{bmatrix} \tag{9}$$

$$[I] = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & 1 \end{bmatrix} \tag{10}$$

$$[J_j] = \begin{bmatrix} J_1 \\ J_2 \\ \cdot \\ \cdot \\ J_n \end{bmatrix} \quad (11), \qquad [K_i] = \begin{bmatrix} K_1 \\ K_2 \\ \cdot \\ \cdot \\ K_n \end{bmatrix} \tag{12}$$

where

$$K_i = \left(\frac{\epsilon}{\rho}\right)_i \sigma T_i^4 . \qquad (i = 1, 2 \ldots n) \tag{13}$$

As will be noted the matrix $[F_{ij}]$ is the geometric view factor matrix, $[I]$ is an identity matrix and both $[J_j]$ and $[K_i]$ are column matrixes. Information required to establish numerical values for each of these matrixes are those physical and geometric data normally available in an engineering problem. For the developments in [12] it was assumed that surface temperatures are specified, the conditions required in this present analysis.

The solution of Eq. (7) for the J_j radiosity fluxes is found from elementary matrix algebra to be [12]

$$[J_j] = - [a_{ij}]^{-1} [K_i] , \tag{14}$$

$$(i = 1, 2 \ldots n) \; (j = 1, 2 \ldots n)$$

where $[a_{ij}]^{-1}$ is the inverse of matrix $[a_{ij}]$ and may be written

$$[a_{ij}]^{-1} = \frac{1}{D} [A_{ji}] = [\gamma_{ij}] , \tag{15}$$

in which D is the determinant of $[a_{ij}]$ and $[A_{ji}]$ is the transposed matrix of cofactors $[A_{ij}]$ of $[a_{ij}]$. γ_{ij} are the elements of the inverted matrix $[a_{ij}]^{-1}$. Thus, the

radiosity fluxes are written as

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ \cdot \\ \cdot \\ J_i \\ \cdot \\ J_n \end{bmatrix} = -\frac{1}{D} \begin{bmatrix} A_{11} & A_{21} & A_{31} & \ldots & A_{ji} & \ldots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \ldots & A_{j2} & \ldots & A_{n2} \\ A_{13} & A_{23} & A_{33} & \ldots & A_{j3} & \ldots & A_{n3} \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ A_{1i} & A_{2i} & A_{3i} & \ldots & A_{ji} & \ldots & A_{ni} \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ A_{1n} & A_{2n} & A_{3n} & \ldots & A_{jn} & \ldots & A_{nn} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ \cdot \\ \cdot \\ K_i \\ \cdot \\ K_n \end{bmatrix} , \tag{16}$$

with K_i given by Eq. (13). In the actual numerical calculations with a digital computer, the elements γ_{ij} to the inverted matrix $[a_{ij}]^{-1}$ are determined rather than the determinant D and the cofactors A_{ji}. However, for purposes of analysis the algebraic representation using cofactors has the advantage of showing the mathematical structure of the result.

Now, with all surface temperatures equal to zero, except T_i, the column matrix $[K_i]$ in Eq. (16) becomes

$$\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ K_i \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Hence, the corresponding radiosity fluxes for zero temperatures on all surfaces except A_i become,

$$J_1 = -\frac{A_{j1}}{D} K_i = -\frac{A_{i1}}{D} K_i$$

$$J_2 = -\frac{A_{j2}}{D} K_i = -\frac{A_{i2}}{D} K_i , \tag{17}$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$J_n = -\frac{A_{jn}}{D} K_i = -\frac{A_{in}}{D} K_i$$

since all products of coefficients in the matrix product $[A_{ji}] [K_i]$ will be zero except those in the column associated with $j = i$ in the $[A_{ji}]$ matrix.

The absorption factor B_{ij} is now formulated from Eq. (1), using the results in Eqs. (6) and (17), as

$$B_{ij} = -\frac{\alpha_j (A_j/A_i)}{D \epsilon_i \sigma T_i^4} \left(\frac{\epsilon}{\rho}\right)_i \sigma T_i^4 \times$$

$$\times (F_{j1} A_{i1} + F_{j2} A_{i2} + \ldots + F_{jn} A_{in}) , \qquad (18)$$

or, by introducing the grayness assumption $\alpha_j = \epsilon_j$ and $\rho_i = 1 - \epsilon_i$ and eliminating the temperature T_i, the absorption factor B_{ij} becomes

$$B_{ij} = -\frac{\epsilon_j (A_j/A_i)}{D(1 - \epsilon_i)} (F_{j1} A_{i1} + F_{j2} A_{i2} + \ldots + F_{jn} A_{in}) , \qquad (19)$$

$$[\beta_{ij}] = -\frac{1}{D} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \ldots & A_{1n} \\ A_{21} & A_{22} & A_{23} & A_{24} & \ldots & A_{2n} \\ A_{31} & A_{32} & A_{33} & A_{34} & \ldots & A_{3n} \\ A_{41} & A_{42} & A_{43} & A_{44} & \ldots & A_{4n} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ A_{n1} & A_{n2} & A_{n3} & A_{n4} & \ldots & A_{nn} \end{bmatrix} \begin{bmatrix} F_{11} & F_{21} & F_{31} & F_{41} & \ldots & F_{n1} \\ F_{12} & F_{22} & F_{32} & F_{42} & \ldots & F_{n2} \\ F_{13} & F_{23} & F_{33} & F_{43} & \ldots & F_{n3} \\ F_{14} & F_{24} & F_{34} & F_{44} & \ldots & F_{n4} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ F_{1n} & F_{2n} & F_{3n} & F_{4n} & \ldots & F_{nn} \end{bmatrix} \qquad (26)$$

in which i, j = 1, 2 . . . n. Eq. (19) may also be written

$$B_{ij} = -\frac{\epsilon_j (A_j/A_i)}{D(1 - \epsilon_i)} \sum_{k=1}^{n} F_{jk} A_{ik} , \qquad (20)$$

$$(i = 1, 2 \ldots n), \quad (j = 1, 2 \ldots n)$$

or, as

$$\left(\frac{A_i}{A_j}\right)\left(\frac{1 - \epsilon_i}{\epsilon_j}\right) B_{ij} = -\frac{1}{D} \sum_{k=1}^{n} F_{jk} A_{ik} . \qquad (21)$$

$$(i = 1, 2 \ldots n), \quad (j = 1, 2 \ldots n)$$

If the quantity β_{ij} is defined as

$$\beta_{ij} = \left(\frac{A_i}{A_j}\right)\left(\frac{1 - \epsilon_i}{\epsilon_j}\right) B_{ij} , \qquad (22)$$

$$(i = 1, 2 \ldots n), \quad (j = 1, 2 \ldots n)$$

it will be noted that each β_{ij}, in Eq. (22) is an element in an n^2-matrix, which may be designated as the $[\beta_{ij}]$-matrix. Thus,

$$[\beta_{ij}] = \left[\frac{A_i}{A_j}\left(\frac{1 - \epsilon_i}{\epsilon_j}\right) B_{ij}\right] . \qquad (23)$$

$$(i = 1, 2 \ldots n), \quad (j = 1, 2 \ldots n)$$

Furthermore, from the summation in Eq. (21) it may be recognized that the column it represents in the $[\beta_{ij}]$-matrix permits the entire matrix to be written

$$[\beta_{ij}] = -\frac{1}{D} \left[[F_{ij}] [A_{ji}]\right]^T , \qquad (24)$$

where, [ ]^T signifies the transpose of the matrix product. By the rules of matrix algebra, the transpose of the product of two matrixes is the product in reverse order of their transposed matrixes. Hence,

$$[\beta_{ij}] = -\frac{1}{D} [A_{ij}] [F_{ji}] \qquad (25)$$

or,

In this result the first matrix is the matrix of cofactors $[A_{ij}]$ for the matrix $[a_{ij}]$ and the second matrix is the transpose of the view factor matrix $[F_{ij}]$. It might be stated again that during machine computation of the $[\beta_{ij}]$-matrix in Eq. (26) the elements γ_{ij} of the inverse $[a_{ij}]$ matrix are used instead of the elements A_{ij}/D. This is done as a matter of computational convenience. For a given column index j Eq. (26) gives an identical result as that found from Eq. (21).

Hence, the absorption factor B_{ij} introduced by Gebhart [3] may be written as

$$\left[\frac{A_i}{A_j}\left(\frac{1 - \epsilon_i}{\epsilon_j}\right) B_{ij}\right] = -\frac{1}{D} [A_{ij}] [F_{ji}] = [\beta_{ij}] . \qquad (27)$$

$$(i = 1, 2 \ldots n) \quad (j = 1, 2 \ldots n)$$

The absorption factor B_{34} would be determined, for example, for i = 3, j = 4, from

$$\beta_{34} = \frac{A_3}{A_4}\left(\frac{1 - \epsilon_3}{\epsilon_4}\right) B_{34} =$$

$$= -\frac{1}{D} (A_{31} F_{41} + A_{32} F_{42} +$$

$$+ A_{33} F_{43} + \ldots + A_{3n} F_{4n}) . \qquad (28)$$

And a typical absorption factor for a surface radiating

to itself is written, for example, for $i = 3$, $j = 3$, as

$$\beta_{33} = \left(\frac{1 - \epsilon_3}{\epsilon_3}\right) B_{33} =$$

$$= \frac{1}{D}(A_{31} F_{31} + A_{32} F_{32} +$$

$$+ A_{33} F_{33} + \ldots + A_{3n} F_{3n}) . \tag{29}$$

This last result shows that while a surface may not radiate directly to itself (i. e., for $F_{33} = 0$) it will radiate to itself indirectly by reflection from the other $n - 1$ surfaces in the enclosure. Further, the mathematical description of the mechanics of this physical event is given by Eq. (29).

The matrixes in Eq. (27) indicate those variables by which the absorption factor is determined. The right hand side of Eq. (27) involves only those quantities found in the (a_{ij}) matrix, Eq. (8), namely, the geometric view factors, F_{ij}, and the reflectances ρ_i of each surface. The terms multiplying B_{ij} in β_{ij} are the surfaces areas A_i and A_j and the reflectance ρ_i and emittance ϵ_j. Thus, from the geometric and physical data used to define the properties of an enclosure the elements B_{ij} of the entire B_{ij}-matrix can be calculated using Eq. (27). This will be done later in this paper for three examples. It is interesting to note that surface temperatures do not influence the calculation of each B_{ij}. The independence of the absorption factors from the surface temperatures is an important characteristic of Gebhart's method, something easily demonstrated by Eq. (27). However, the energy q_{ij}, absorbed at A_j originating as emission at A_i is directly related to the temperature at A_i, by Eq. (1). Thus, it is possible to have finite values of the absorption factor B_{ij} but a zero value for q_{ij} for those cases of T_i equal to zero. An example of this circumstance will be cited later. In general, all absorption factors B_{ij} will be finite for all real, gray systems of a given geometry, as indicated by Eqs. (23)–(29), because these factors are properties of an enclosure and related to the geometric configuration and radiation properties of the surfaces. The rates of heat transfer by absorption and emission, however, are determined, in addition to the B_{ij}, by the surface temperatures. Thus, the absorption factors themselves cannot be used as an indicator of the absolute rates of energy transfer in an enclosure without also including the effect of the surface temperatures.

The net heat transfer rate at A_j resulting from the exchange of radiation with all n-surfaces in the enclosure each having a specified temperature may be written (3) as

$$q_j^* = \epsilon_j A_j \sigma T_j^4 - \sum_{i=1}^{n} \epsilon_i A_i B_{ij} \sigma T_i^4 . \tag{30}$$

$$(j = 1, 2 \ldots n)$$

In the examples that follow the net heat transfer rate at each surface of an enclosure will be computed using Eq. (30), from Gebhart [3], but with the absorption factors determined using the derivation given in this paper, Eq. (27). The temperatures of each surface may be specified arbitrarily in using Eq. (30) and may include some with zero temperature.

3. Derivation of The Hottel and Sarofim Factors $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$

Hottel and Hottel and Sarofim [1, 2] define the two factors $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$ as the total-exchange area and the total view factor, respectively. Their role in radiant interchange is related to the calculation of the net rate of heat transfer q_{ij}^* between A_i and A_j. For the enclosure in Fig. 1 for all surfaces at zero temperature except T_i and T_j, q_{ij}^* is written

$$q_{ij}^* = \overline{S_i S_j} \, \sigma(T_i^4 - T_j^4) \tag{31}$$

$$= A_i \mathscr{F}_{ij} \, \sigma(T_i^4 - T_j^4) . \tag{32}$$

The net rate of heat exchange may also be written in terms of Gebhart's absorption factors for these same conditions, as

$$q_{ij}^* = A_i \epsilon_i \sigma T_i^4 B_{ij} - A_j \epsilon_j \sigma T_j^4 B_{ji} . \tag{33}$$

However, Gebhart has shown [3] that a reciprocal relationship exist between the quantities in Eq. (33) such that

$$A_i \epsilon_i B_{ij} = A_j \epsilon_j B_{ji} . \tag{34}$$

Hence,

$$q_{ij}^* = \epsilon_i A_i B_{ij} \sigma(T_i^4 - T_j^4) . \tag{35}$$

Thus, from the results in Eqs. (31, 32 and 35) the factors $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$ may be related to the absorption factor B_{ij} as

$$\epsilon_i A_i B_{ij} = \overline{S_i S_j} \tag{36}$$

$$= A_i \mathscr{F}_{ij} . \tag{37}$$

Accordingly, from Eq. (27), the factors $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$ can be shown to be

$$\overline{S_i S_j} = A_j \frac{\epsilon_i \epsilon_j}{1 - \epsilon_i} \beta_{ij} , \tag{38}$$

and

$$\mathscr{F}_{ij} = \frac{A_j}{A_i} \cdot \frac{\epsilon_i \epsilon_j}{1 - \epsilon_i} \cdot \beta_{ij} , \tag{39}$$

where β_{ij} is an element in the matrix

$$[\beta_{ij}] = -\frac{1}{D}[A_{ij}][F_{ji}] .\qquad(27)$$

$$(i = 1, 2 \ldots n)\ (j = 1, 2 \ldots n)$$

These formulations for $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$ are different in structure from those of Hottel and Sarofim [1, 2] but are mathematically equivalent to them.

The development leading to Eqs. (38) and (39) indicates that the factors $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$ are composed of the same quantities as is the absorption factor B_{ij} and can, in fact, be derived from B_{ij}, as

$$\overline{S_i S_j} = \epsilon_i A_i B_{ij}\qquad(36)$$

and

$$\mathscr{F}_{ij} = \epsilon_i B_{ij} .\qquad(40)$$

While Eqs. (36) and (40) show the relation of $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$ to B_{ij}, Eqs. (38) and (39) would be used to compute these factors from the corresponding element β_{ij} and the geometric and physical data used to describe an enclosure. Thus, $\overline{S_3 S_4}$ and $\mathscr{F}_{34}$, for example, are written from Eqs. (28), (38) and (39) as,

$$\overline{S_3 S_4} = A_4 \frac{\epsilon_3\,\epsilon_4}{1-\epsilon_3}\,\beta_{34}$$

$$= -\frac{A_4}{D}\cdot\frac{\epsilon_3\,\epsilon_4}{1-\epsilon_3}\times$$

$$\times (A_{31}F_{41} + A_{32}F_{42} + \ldots + A_{3n}F_{4n})$$

$$= A_3\,\mathscr{F}_{34} .\qquad(41)$$

The net heat transfer rate q_j^* at surface A_j considering energy exchange with all n-surfaces in the enclosure can be expressed in terms of the factors $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$ from Eqs. (30), (36) and (37) as

$$q_j^* = \epsilon_j A_j \sigma T_j^4 - \sum_{i=1}^{n} \overline{S_i S_j}\,\sigma T_i^4 ,\qquad(42)$$

$$(j = 1, 2 \ldots n)$$

or

$$q_j^* = \epsilon_j A_j \sigma T_j^4 - \sum_{i=1}^{n} \mathscr{F}_{ij} A_i \sigma T_i^4 ,\qquad(43)$$

$$(j = 1, 2 \ldots n) .$$

For the numerical examples given later, the q_j^* for the method of Hottel and Sarofim is determined from Eqs. (42) and (43) using the values of $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$

calculated from the derivations given in this paper, namely, Eqs. (38) and (39).

Some interesting and related consequences may be derived from this evaluation of $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$. The factor $\mathscr{F}_{ij}$ can be written

$$\mathscr{F}_{ij} = \epsilon_i \frac{q_{ij}}{\epsilon_i A_i \sigma T_i^4}$$

$$= \frac{q_{ij}}{A_i \sigma T_i^4}\qquad(44)$$

or

$$\mathscr{F}_{ij} = \frac{\text{energy absorbed at } A_j \text{ originating as emission at } A_i}{\text{total black radiation from } A_i \text{ at temperature } T_i} .$$

$$(45)$$

Hence, from this result it may be seen that the factor $\mathscr{F}_{ij}$ is really another form of an absorption factor, based on the total black radiation from the originating surface rather than on its gray emission, as is the case with Gebhart's B_{ij}. Further, the factor $\overline{S_i S_j}$ also may be interpreted as an absorption factor since it is simply related to $\mathscr{F}_{ij}$ by the relation

$$\overline{S_i S_j} = A_i\,\mathscr{F}_{ij}$$

$$= \frac{q_{ij}}{\sigma T_i^4}\qquad(46)$$

or

$$\overline{S_i S_j} = \frac{\text{energy absorbed at } A_j \text{ originating as emission at } A_i}{\text{black radiation flux from } A_i \text{ having temperature } T_i}$$

$$(47)$$

From Eq. (37) some additional interpretation concerning the factor $\mathscr{F}_{ij}$ will be included. This factor is related to B_{ij} by

$$\mathscr{F}_{ij} = \epsilon_i B_{ij} .\qquad(48)$$

Now, if the summation is formed over the index j

$$\sum_{j=1}^{n} \mathscr{F}_{ij} = \epsilon_i \sum_{j=1}^{n} B_{ij} ,\quad (i = 1, 2, \ldots n)\qquad(49)$$

and, since by the physical constraints (3) on the radiation

$$\sum_{j=1}^{n} B_{ij} = 1.0 .\qquad (i = 1, 2, \ldots n)\qquad(50)$$

Hence,

$$\sum_{j=1}^{n} \mathscr{F}_{ij} = \epsilon_1 , \qquad (i = 1, 2, \ldots n) \qquad (51)$$

or

$$\mathscr{F}_{11} + \mathscr{F}_{12} + \mathscr{F}_{13} + \ldots + \mathscr{F}_{1n} = \epsilon_1$$
$$\mathscr{F}_{21} + \mathscr{F}_{22} + \mathscr{F}_{23} + \ldots + \mathscr{F}_{2n} = \epsilon_2 \qquad (52)$$
$$\quad \cdot \qquad \cdot \qquad \cdot \qquad \qquad \cdot \qquad \cdot$$
$$\quad \cdot \qquad \cdot \qquad \cdot \qquad \qquad \cdot \qquad \cdot$$
$$\quad \cdot \qquad \cdot \qquad \cdot \qquad \qquad \cdot \qquad \cdot$$
$$\mathscr{F}_{n1} + \mathscr{F}_{n2} + \mathscr{F}_{n3} + \ldots + \mathscr{F}_{nn} = \epsilon_n$$

This means that the sum of the $\mathscr{F}_{ij}$ factors in each row of the $(\mathscr{F}_{ij})$-matrix must equal the emittance of the surface having the same index as that row, i.e., A_i. Should the $[\mathscr{F}_{ij}]$-matrix be employed in the calculation, the constraints in Eq. (51) can be used as a check in the numerical results. This constraint is completely similar as that in each row of the $[F_{ij}]$-matrix, that is

$$\sum_{j=1}^{n} F_{ij} \equiv 1.0 . \qquad (i = 1, 2, \ldots n) \qquad (53)$$

In fact, for black systems Eq. (51) is identical to Eq. (53), since not only will ϵ_i be equal to 1.0, but the factor $\mathscr{F}_{ij}$ is identical to F_{ij} under these circumstances. Furthermore, for black systems the following relations will always hold

$$\mathscr{F}_{ij} = B_{ij}$$
$$\qquad = F_{ij} , \qquad (54)$$

and

$$\overline{S_1 S_j} = A_i F_{ij} . \qquad (55)$$

4. Numerical Results

Three numerical examples are chosen to compare results using the authors' method with computer program RADTQO [12] and those determined from the methods of Gebhart [3] and Hottel and Sarofim [1, 2], as derived in the previous section using Eqs. (27), (30), (38), (39), (42), and (43). The computer program for this later calculation is known as DERGEBHOT. As an additional check the results for the first example will also be compared with an independent calculation prepared by Gebhart [3, 13]. The three examples are shown in Figs. 2, 3, and 4.

The first example, Fig. 2, taken from Gebhart [3], is a 4-surface gray enclosure having 3 specified surface temperatures and 1 adiabatic surface. In Gebhart's original problem surfaces 1 and 3 are black. This condition is approximated here by establishing the emittances of both A_1 and A_3 as 0.99999 and their corresponding reflectances as 0.00001. With the decimal
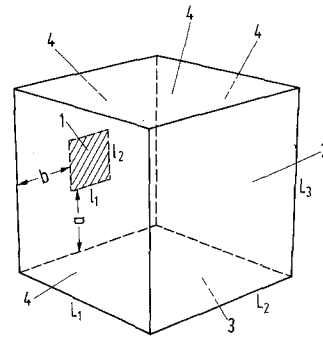

Fig. 2. 4-surface gray enclosure of Gebhart [3]

Input data for RADTQO [12] and DERGEBHOT

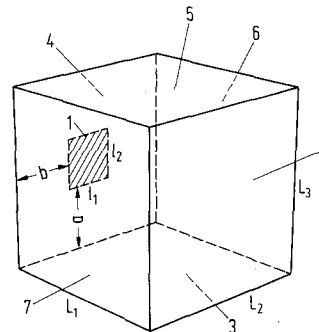|		
$T_1 = 0.0\,R$	$\epsilon_1 = 0.99999$	$A_1 = 1.0\,ft.^2$
$T_2 = 2460.0\,R$	$\epsilon_2 = 0.90000$	$A_2 = 16.0\,ft.^2$
$T_3 = 1860.0\,R$	$\epsilon_3 = 0.99999$	$A_3 = 16.0\,ft.^2$
$T_4 = 2189.22\,R$	$\epsilon_4 = 0.00000,$	$A_4 = 63.0\,ft.^2$
$q_4^* = 0.0\,Btu/hr$	$= 0.50000$	$b = 1.5\,ft.$
$L_1 = 4.0\,ft.$	$l_1 = 1.0\,ft.$	
$L_2 = 4.0\,ft.$	$l_2 = 1.0\,ft.$	
$L_3 = 4.0\,ft.$	$a = 1.5\,ft.$	

Fig. 3. 7-surface gray enclosure (modified 4-surface enclosure of Gebhart [3])

Input data for RADTQO [12] and DERGEBHOT

$T_1 = 0.0\,R$	$\epsilon_1 = 0.99999$	$A_1 = 1.0\,ft.^2$
$T_2 = 2460.0\,R$	$\epsilon_2 = 0.90000$	$A_2 = 16.0\,ft.^2$
$T_3 = 1860.0\,R$	$\epsilon_3 = 0.99999$	$A_3 = 16.0\,ft.^2$
$T_4 = 2192.54\,R$	$\epsilon_4 = 0.00000,$	$A_4 = 15.0\,ft.^2$
$T_5 = T_6 = T_7 =$	$= 0.50000$	$A_5 = 16.0\,ft.^2$
$= 2187.51\,R$	$\epsilon_5 = 0.00000,$	$A_6 = 16.0\,ft.^2$
$q_4^* = 0.0\,Btu/hr$	$= 0.50000$	$A_7 = 16.0\,ft.^2$
	$\epsilon_6 = 0.00000,$	$a = 1.5\,ft.$
$q_5^* = 0.0\,Btu/hr$	$= 0.50000$	$b = 1.5\,ft.$
$q_6^* = 0.0\,Btu/hr$	$\epsilon_7 = 0.00000,$	$L_3 = 4.0\,ft.$
	0.50000	
$q_7^* = 0.0\,Btu/hr$	$l_1 = 1.0\,ft.$	
$L_1 = 4.0\,ft.$	$l_2 = 1.0\,ft.$	
	$L_2 = 4.0\,ft.$	

point capability of computers these values could be extended considerably closer to their theoretical values of one and zero, if desired. An emittance of 1.0 cannot be used as it would produce a reflectance of zero and an infinite value to the $[a_{ij}]$-matrix, Eq. (8). However, it has been established by the present authors
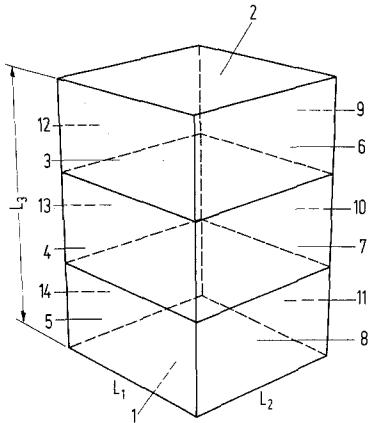
Fig. 4. 14-surface gray enclosure

Initial input data for RADTQO [12] and DERGEBHOT

T_1 = 1500.0 R A_1 = 1.0 ft.^{2}
T_2 = 600.0 R A_2 = 1.0 ft.^{2}
$T_3 - T_{14}$ = 800.0 R $A_3 - A_{14}$ = 0.6667 ft.^{2}
$\epsilon_1 - \epsilon_{14}$ = 0.65000 $\rho_1 - \rho_{14}$ = 0.35000

L_1 = 1.0 ft. L_2 = 1.0 ft. L_3 = 2.0 ft.

that for this example an essentially black condition is achieved by an emittance equal to or greater than 0.99900 and, correspondingly, a reflectance equal to or less than 0.00100. Further, Gebhart considers A_1 to be a non-emitter of radiation. This circumstance is established here by setting the temperature of A_1 at 0.0 R. Numerical results for this problem are summarized in Table 1. For the authors' method the computer program RADTQO [12] was used which not only computes all the radiosity fluxes and net heat transfer rates for the n-surfaces but determines the temperatures of all adiabatic surfaces as well. For calculations using the methods of Gebhart and Hottel and Sarofim,

by Eqs. (27), (30), (38), (42), and (43), the adiabatic surface temperatures determined by RADTQO [12] were used.

As may be seen from Table 1 there is no significant difference in the results using all three methods. The tabulation in the last column represents Gebhart's independent calculations and are in essential agreement with the rest of the results. The calculations made by the present authors were completed for four different values of ϵ_4, the emittance of the adiabatic surface, A_4. These values were 0.0, 0.00001, 0.50000 and 0.99999. To conserve space only the results for ϵ_4 of 0.0 and 0.50000 are given in Table 1. However, the results using the other two values for ϵ_4 were identical to those listed in Table 1. The four different values for ϵ_4, spanning the physically possible range for the emittance, were chosen in order to demonstrate that the computed heat transfer rates for an enclosure do not depend on the values of the emittance of the adiabatic surfaces but that the corresponding values of Gebhart's absorption factors B_{ij} are significantly influenced by these emittances (see Tables 2 and 5). It is, of course, well known that the emittance of an adiabatic surface does not influence its temperature nor its radiosity heat flux. However, it is not as evident, a priori, that with a different set of absorption factors B_{ij}, resulting from different ϵ_4, the computed heat transfer rates would be unaffected. The results in Table 1 demonstrates that this is indeed the case.

Another means for comparing the author's method [12] with that of Gebhart [3] is to compare the elements of the B_{ij} matrix computed by Eq. (27) and those given by Gebhart [3] for the 4-surface enclosure, Fig. 2. The elements B_{ij} for this enclosure determined in this way are given in Table 2 for values of ϵ_4 of 0.0 and 0.50000. The values in parenthesis in the upper

Table 1. Comparative results for the 4-surface enclosure
a. Input data for RADTQO [12] and DERGEBHOT listed on Fig. 2
b. T_4 computed by RADTQO [12] is 2189.22 R

Quantity	Authors' method RADTQO [12]	Eqs. (27, 30, 38, 39, 42, 43)				Results of Gebhart [3]
		Gebhart		Hottel and Sarofim		
		ϵ_4		ϵ_4		
		0.0	0.50000	0.0	0.50000	
q_1^* Btu/hr	− 40,475.2	− 40,475.2	− 40,475.2	− 40,475.2	− 40,475.2	− 40,500.0
q_2^* Btu/hr	399,193.2	399,193.2	399,193.2	399,193.2	399,193.2	399,000.0
q_3^* Btu/hr	−360,138.2	−360,138.2	−360,138.2	−360,000.2	−360,138.2	−358,500.0
q_4^* Btu/hr	0.0	− 0.0*	−0.00054	− 0.0	−0.00054	0.0
Σq_j^* Btu/hr	− 1420.1	− 1420.1	−1420.1	− 1420.1	−1420.1	0
% Σq_j^* of q_2^*	−0.356%	−0.356%	−0.356%	−0.356%	−0.356%	0

* −0.0 signifies a small negative number less than 10^{-6}.

Table 2. Absorption factor B_{ij} matrix for the 4-surface enclosure ($\epsilon_4 = 0.0$)

i/j j 1	2	3	4	$\sum\limits_{j=1}^{4} B_{ij}$ (i = 1, 2, 3, 4)
1	0.0143637	0.468677	0.499916	0.0	0.98296
2	0.0332888 (0.0333)	0.359747 (0.360)	0.608723	0.0	1.00176
3	0.0316420 (0.0308)	0.546941 (0.557)	0.420899	0.0	0.99948
4	0.0240176	0.456820	0.519015	0.0	0.99985

Absorption factor B_{ij} matrix for the 4-surface enclosure ($\epsilon_4 = 0.50000$)

i/j j 1	2	3	4	$\sum\limits_{j=1}^{4} B_{ij}$ (i = 1, 2, 3, 4)
1	0.00428550	0.2769870	0.2821290	0.4196160	0.98302
2	0.01971370	0.1015460	0.3153680	0.5652150	1.00184
3	0.01779540	0.2835760	0.1216770	0.5765180	0.99957
4	0.01355880	0.2578920	0.2930030	0.4354630	0.99992

Table 3. $\mathscr{F}_{ij}$-matrix for the 4-surface enclosure ($\epsilon_4 = 0.0$)

i/j j 1	2	3	4	$\sum\limits_{j=1}^{4} \mathscr{F}_{ij}$ (i = 1, 2, 3, 4)	ϵ_i
1	0.0143636	0.468672	0.499911	0.0	0.98285	0.99999
2	0.0299599	0.323772	0.547851	0.0	0.90158	0.90000
3	0.0316417	0.546936	0.420895	0.0	0.99947	0.99999
4	0.0	0.0	0.0	0.0	0.0	0.0

$\mathscr{F}_{ij}$-matrix for the 4-surface enclosure ($\epsilon_4 = 0.50000$)

i/j j 1	2	3	4	$\sum\limits_{j=1}^{4} \mathscr{F}_{ij}$ (i = 1, 2, 3, 4)	ϵ_i
1	0.00428551	0.276985	0.282126	0.419612	0.98301	0.99999
2	0.0177423	0.0913910	0.283831	0.508693	0.90166	0.90000
3	0.0177953	0.283573	0.121676	0.576512	0.99956	0.99999
4	0.00677942	0.128946	0.146501	0.217732	0.49996	0.50000

table are those reported by Gebhart [3], also for the case of $\epsilon_4 = 0.0$.

The absorption factors B_{21}, B_{22}, B_{31} and B_{32} in Table 2 (for $\epsilon_4 = 0.0$) are in essential agreement with those computed by Gebhart. Although Gebhart's method applied to this problem does not require all the B_{ij}, the complete n x n B_{ij}-matrix with n^2 elements is always determined using Eq. (27) and the computer program DERGEBHOT. By nature of its definition the sum of the B_{ij} elements in each row in Table 2 shown in the last column should be exactly unity. The fact that these elements do not sum to unity (exactly) is probably caused by some uncertainty in the third or fourth decimal place in the magnitude of the geometric view factors, F_{ij}, for the enclosure. Another contributing factor is the numerical round-off effect resulting from the large number of arithmetic operations required to compute each B_{ij}.

Although surface A_1 is at zero temperature and thus cannot emit energy to any of the other surfaces for absorption there, all the absorption factors associated with such an emission-reflection process, i.e., B_{11}, B_{12}, B_{13} and B_{14}, are finite except for B_{14} when ϵ_4 is zero. This is a consequence of the fact that the absorption factors are determined only by the geometric and physical properties of the enclosure and are independent of the surface temperatures, as discussed earlier and demonstrated in Table 2. However, the

Table 4. Comparative results for the 7-surface enclosure
a. Input data for RADTQO [12] and DERGEBHOT listed on Fig. 3
b. Adiabatic surface temperatures computed by RADTQO [12] are: $T_4 = 2192.54\,R$;
   $T_5 = T_6 = T_7 = 2187.51\,R$

Quantity	Author's method RADTQO [12]	Eqs. (27, 30, 38, 39, 42, 43)			
		Gebhart		Hottel und Sarofim	
		$\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7$		$\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7$	
		0.0	0.50000	0.0	0.50000
q_1^* Btu/hr	− 40,412.98	− 40,412.98	− 40,412.98	− 40,412.98	− 40,412.98
q_2^* Btu/hr	399,613.3	399,613.3	399,613.3	399,613.3	399,613.3
q_3^* Btu/hr	−359,669.8	−359,669.8	−359,669.8	−359,669.8	−359,669.8
q_4^* Btu/hr	0.0	−0.0*	−0.00014	−0.0	−0.00014
q_5^* Btu/hr	0.0	−0.0	−0.00005	−0.0	−0.00005
q_6^* Btu/hr	0.0	−0.0	−0.00005	−0.0	−0.00005
q_7^* Btu/hr	0.0	−0.0	−0.00005	−0.0	−0.00005
$\Sigma\, q_j^*$ Btu/hr	−469.45	−469.45	−469.45	−469.45	−469.45
$\%\, \Sigma\, q_j^*$ of q_2^*	−0.117%	−0.117%	−0.117%	−0.117%	−0.117%

* −0.0 signifies a small negative number less than 10^{-6}.

corresponding rates of energy absorption at A_1, A_2, A_3 and A_4, namely q_{11}, q_{12}, q_{13} and q_{14}, originating as emission at A_1, are each exactly zero because T_1 is zero.

Because A_4 is adiabatic the emittance ϵ_4 can be assigned any value between 0.0 and 1.0 without affecting the results. As a matter of convenience in calculation, however, it is recommended that the emittance of all adiabatic surfaces be set to 0.0 when using Gebhart's method or when determining the various B_{ij} from Eq. (27). This will produce exactly 0.0 values to all absorption factors related to absorption at the adiabatic surface. These are the factors B_{i4} in Table 2.

The Hottel and Sarofim factors $\mathscr{F}_{ij}$ also have been computed for the 4-surface example in Fig. 2 using Eqs. (39) and (27) and the computer program DERGEBHOT. Results of this calculation are listed in Table 3.

As may be noted from a comparision of these results with those in Table 2, each element in the $\mathscr{F}_{ij}$-matrix is related to its companion element in the B_{ij}-matrix by Eq. (40). Further, the sum of each row in Table 3, shown in the fifth column, is essentially equal to that of the emittance of the ith surface listed in the last column, as required by Eq. (51). The small numerical differences in these two sets of values is probably due to some decimal point uncertainties in the geometric view factors, F_{ij}, and numerical round-off

resulting from the enormous arithmetic operation required in the computation of the matrices in Eq. (27).

A matrix for the elements $\overline{S_i S_j}$ could also be computed using Eq. (38). However, since each $\overline{S_i S_j}$ element is a simple geometric extension of $\mathscr{F}_{ij}$ on the basis of A_1, Eq. (37), as separate calculation using the matrix reduction in Eq. (38) can be avoided. The corresponding $\overline{S_i S_j}$ matrix can easily be constructed by multiplying each element in the $\mathscr{F}_{ij}$-matrix, Table 3, by its corresponding surface area, A_i.

The second numerical example is shown in Fig. 3. This enclosure is identical to that in Fig. 2 but it poses a new calculation problem in that the adiabatic surface is divided into 4 separate surfaces, A_4, A_5, A_6 and A_7, making this a 7-surface enclosure. A summary of the results for this problem is given in Table 4, for values of the emittance of the adiabatic surfaces or 0.0 and 0.50000. As in the case of the first example, the temperatures of the adiabatic surfaces were determined by RADTQO [12] and used to complete the calculations for the results identified as "Gebhart" and "Hottel and Sarofim" using the authors' program DERGEBHOT. For this example there are no independent calculations available for comparision.

As shown by Table 4 there is an exact correspondence between the results of the three methods. Also, the influence of the emittance of the adiabatic surfaces A_4, A_5, A_6 and A_7 is negligible on the calculation of

Table 5. Absorption factor B_{ij} matrix for the 7-surface enclosure ($\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = 0.0$)

i/j	1	2	3	4	5	6	7	$\sum_{j=1}^{7} B_{ij}$ (i = 1, ..., 7)
1	0.0159081	0.472315	0.503810	0.0	0.0	0.0	0.0	0.99203
2	0.0323387	0.359179	0.608085	0.0	0.0	0.0	0.0	1.00050
3	0.0315906	0.547225	0.421230	0.0	0.0	0.0	0.0	1.00005
4	0.0164970	0.460030	0.524039	0.0	0.0	0.0	0.0	1.00057
5	0.0263256	0.456021	0.517691	0.0	0.0	0.0	0.0	1.00004
6	0.0263256	0.456021	0.517691	0.0	0.0	0.0	0.0	1.00004
7	0.0263256	0.456021	0.517691	0.0	0.0	0.0	0.0	1.00004

Absorption factor B_{ij} matrix for the 7-surface enclosure ($\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = 0.50000$)

i/j	1	2	3	4	5	6	7	$\sum_{j=1}^{7} B_{ij}$ (i = 1, ..., 7)
1	0.00509208	0.278130	0.283295	0.0391638	0.128772	0.128772	0.128772	0.99200
2	0.0197074	0.101360	0.315158	0.134419	0.143255	0.143255	0.143255	1.00041
3	0.0177627	0.283629	0.121739	0.138082	0.146246	0.146246	0.146246	0.99995
4	0.00524832	0.258131	0.294765	0.040265	0.133986	0.133986	0.133986	1.00053
5	0.0161479	0.257845	0.292489	0.125530	0.420419	0.132951	0.132951	0.99996
6	0.0161479	0.257845	0.292489	0.125530	0.132951	0.0420419	0.132951	0.99996
7	0.0161479	0.257845	0.292489	0.125530	0.132951	0.132951	0.0420419	0.99996

Table 6. $\mathscr{F}_{ij}$-matrix for the 7-surface enclosure ($\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = 0.0$)

i/j	j 1	2	3	4	5	6	7	$\sum_{j=1}^{7} \mathscr{F}_{ij}$ (i = 1, ..., 7)	ϵ_i
1	0.0159080	0.472310	0.503805	0.0	0.0	0.0	0.0	0.99202	0.99999
2	0.0299148	0.323262	0.547276	0.0	0.0	0.0	0.0	0.90045	0.90000
3	0.0315903	0.547219	0.421226	0.0	0.0	0.0	0.0	1.00004	0.99999
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

$\mathscr{F}_{ij}$-matrix for the 7-surface enclosure ($\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = 0.50000$)

i/j	j 1	2	3	4	5	6	7	$\sum_{j=1}^{7} \mathscr{F}_{ij}$ (i = 1, ..., 7)	ϵ_i
1	0.00509203	0.278128	0.283292	0.0391634	0.128771	0.128771	0.128771	0.99199	0.99999
2	0.0177366	0.0912243	0.283642	0.120977	0.128930	0.128930	0.128930	0.90037	0.90000
3	0.0177625	0.283626	0.121738	0.138081	0.146244	0.146244	0.146244	0.99994	0.99999
4	0.00262416	0.129066	0.147382	0.0202132	0.0669928	0.0669928	0.0669928	0.50026	0.50000
5	0.00807395	0.128923	0.146244	0.0627648	0.0210209	0.0669928	0.0664755	0.49998	0.50000
6	0.00807395	0.128923	0.146244	0.0627648	0.0664755	0.0210209	0.0664755	0.49998	0.50000
7	0.00807395	0.128923	0.146244	0.0627648	0.0664755	0.0664755	0.0210209	0.49998	0.50000

Table 7. Results for the 14-surface enclosure

a. Input data for RADTQO [12] and DERGEBHOT listed on Fig. 4

| | Authors' method RADTQO [12] | | Eqs. (27, 30, 38, 39, 42, 43) | |
| | | | Gebhart | Hottel and Sarofim |
j	J_j Btu/hr-ft.^{2}	q_j^* Btu/hr	q_j^* Btu/hr	q_j^* Btu/hr
1	6038.40	4900.48	4900.48	4900.48
2	557.98	−623.71	−623.71	−623.71
3	765.62	− 78.70	− 78.70	− 78.70
4	893.70	−237.27	−237.27	−237.27
5	1296.10	−735.49	−735.49	−735.49
6	765.62	− 78.70	− 78.70	− 78.70
7	893.70	−237.27	−237.27	−237.27
8	1296.10	−735.49	−735.49	−735.49
9	765.62	− 78.70	− 78.70	− 78.70
10	893.70	−237.27	−237.27	−237.27
11	1296.10	−735.49	−735.49	−735.49
12	765.62	− 78.70	− 78.70	− 78.70
13	893.70	−237.27	−237.27	−237.27
14	1296.10	−735.49	−735.49	−735.49
q_j^*		70.87	70.87	70.87
$\% \Sigma q_j^*$ of q_1^*		1.44%	1.44%	1.44%

the various net heat transfer rates. As in the case of the 4-surface enclosure of Fig. 2 these calculations were also performed with emittances of the adiabatic surfaces of 0.00001 and 0.99999 with results identical to those given in Table 4.

The [B_{ij}]-matrix computed from Eq. (27) and the [$\mathscr{F}_{ij}$]-matrix from Eq. (39) using the program DER-GEBHOT are given in Tables 5 and 6 for the 7-surface enclosure of Fig. 3 with emittances of the adiabatic surface equal to 0.0 and 0.50000. The [$\overline{S_i S_j}$]-matrix is not tabulated since each of its elements may be obtained by multiplying each of the elements in the $\mathscr{F}_{ij}$-matrix by A_i. Also listed in Tables 5 and 6 are the sums of each row of the matrixes. The row sums in the [B_{ij}]-matrix are very close to unity as required by Eq. (50), the differences from unity being caused by numerical uncertainty in the geometric view factors as well as round-off effects in the arithmetic operations. The sum of the rows in the $\mathscr{F}_{ij}$-matrix is essentially equal to ϵ_i as required by Eq. (51). The values of ϵ_i are included in the last column in Table 6 for comparison with the row sums.

The last example, shown in Fig. 4 with its input data, is a 14-surface enclosure having specified surface temperatures. It is included here to extend the comparison of the three methods and to include a more comprehensive problem. The results for the 14-surface enclosure are given in Table 7.

As in the case of the other examples the results for the 14-surface enclosure using the three methods are in complete agreement.

The absorption factors B_{ij} and the factors $\mathscr{F}_{ij}$ for the 14-surface enclosure in Fig. 4 were also computed using the program DERGEBHOT. However, in the interest of conserving space these data will not be included in this paper. Each set of data would involve 196 separate numerical entries.

5. Conclusions

The equivalence of three algebraic methods for the calculation of radiation heat exchange in an enclosure is demonstrated. The methods compared are those of Hottel and Sarofim [1, 2], Gebhart [3, 4] and one published recently by the authors [12]. Equivalence between the methods is established by deriving the absorption factors, B_{ij}, of Gebhart and the total exchange factors, $\overline{S_i S_j}$, and the total view factors, $\mathscr{F}_{ij}$, of Hottel and Sarofim using the authors' formulation [12] as a starting point. Numerical comparison is also made using three enclosures as examples and computing the radiant exchange rates by the three methods. Essentially exact correspondence is found between the numerical results using each of the three methods to compute the radiant interchange. The computer program used to compute the derived forms of the methods of Gebhart and Hottel and Sarofim is named DERGEBHOT. Numerical results using the authors' method are determined from a computer program named RADTQO [12].

The factors $\overline{S_i S_j}$ and $\mathscr{F}_{ij}$ of Hottel and Sarofim are shown to be other forms of an absorption factor.

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