An Optimal Ordering Rule for A Stochastic Sequencing Model

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In this report an optimal sequencial rule is developed for a class of stochastic sequencial models. Based on pairwise interchange necessary and sufficient conditions for the optimality of the rule is given. The rule is applicable to many situations such as multicharacteristic sequencial testing and job processing on a single machine. The optimal sequencial rule generalizes the deterministic results given in [3,4], for situations when the parameter of the problem are random variables. Several examples are given to demonstrate the utility of the result in the paper.

Keywords: Optimal Sequencial Rule, Random Parameters

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1.0 Introduction

In this report the problem of finding an optimal sequence for a class of problems with random parameters has been studied. An important example from this class is the stochastic multicharacteristic inspection problem, when characteristics need to be ordered for inspection in order to minimize the expected cost of inspection. In such a problem, components with several characteristics are supplied by vendors that have to be inspected. The characteristics have different defective rates and cost of inspection. The cost depends on the type of test and equipment needs in carrying out the quality assurance. The probability of characteristic i is defective is $P_i > 0$, and its cost of inspection is $C_i > 0$. Both P_i and C_i are not known precisely and are assumed to be independent random variables. Inspectors commit two types of errors. Type I error q_i and type II b_i. The inspection of characteristics are independent, and inspection will be carried out until one characteristic is rejected or all characteristics pass the inspection. The component is rejected if one characteristic is defective. The deterministic multicharacteristic inspection problem (where all parameters are assumed to be known constants) is first addressed in [1,2,3] and a rule is given for finding the optimal sequence, however the proofs given in [2,3], only give a sufficient condition for the optimality of the rule. Later the deterministic sequencing problem has been stated in general terms in [4], and necessary and sufficient conditions are given for the optimality of the sequencing rule. Moreover, many other examples from the literature are cited as special cases of the general sequencing model.

The model considered in this paper generalizes the sequencial model given in [4] by allowing some of the input variables to be random. For example, for the multicharacteristic inspection problem usually characteristic's defective rates are not known precisely and the cost of inspection is not exactly a fixed value, the cost varies from component to component. Therefore, the sequencing model with random input represents real life more appropriately and is worthy of study. The objective of the paper is to develop rules for finding optimal sequences that minimizes expected cost of inspection.

The rest of the report is organized as follows: Section 2 provides a brief review of the deterministic sequencial model and the result to be used in the proof of the optimal sequence of the Stochastic Model. Section 3 formulates the Stochastic Sequencial Model and provides the main result of the paper. Section 4 presents examples pertinent to the stochastic case, and Section 5 concludes the report.

2.0 Deterministic Sequencial Model

The general sequencing problem in [4] is defined as follows. Given n objects with no precedence relation among the objects, for each object i there are a positive weight w_i, and a nonzero number P_i and a cost function f_i : R^N Æ R_+, where R^N is all ordered subsets of the set of objects (for example, (1,2,3,4) is different from (1,2,4,3)). The objective is to find the optimal ordering p: $(j_1, j_2, \ldots, j_n)$ that minimizes the total cost function.

$$TC(\pi) = \sum_{i=1}^{n} f_{j_i}(j_1, j_2, \ldots, j_i) \tag{1}$$

It is assumed f_i for each object i is independent of specific ordering of objects $(j_1, j_2, \ldots, j_{i-1})$. Consider a partial sequence $B = (j_1, j_2, \ldots, j_{i-1})$ of the objects and an object l such that l œ B, the following relation is assumed for each f_i, $i = 1,2,\ldots,n$

$$f_{j_i}(j_1, j_2, \ldots, j_a, l, j_{a+1}, \ldots, j_i) - f_{j_i}(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_i) = P_l w_{j_i} H(j_1, j_2, \ldots, j_{i-1}) \tag{2}$$

where H is a positive set function on all subsets of the n objects and H(f) = 1. Note that the set function H does not depend on any specific index ordering. The main theorem given in [4] states:

Theorem 1: For any sequencing problem defined above, a sequence π minimizes the total cost given in (1) if and only if we have

$$h(j_1) \le h(j_2) \le \ldots \le h(j_n) \tag{3}$$

where $h(j_i) = P_{j_i} / w_{j_i}$. The proof is based on pairwise comparisons and is given in [4].

3.0 The Stochastic Sequencing Model

The stochastic sequencing problem can be stated as follows: There are n objects with no precedence relation associated with the objects. Each object i is associated with a positive random weight w_i and a nonzero random variable P_i, both with finite expectations and independent. Also a random cost function $f_i : R^N \to R_+$, where R^N is all ordered subsets of the n objects (for example, (1,2,3,4) is different from (1,2,4,3)). All random variables are assumed to be independent. The objective is to find an ordering $\pi = (j_1, j_2, \ldots, j_n)$ of the n objects to minimize the expected total cost.

$$E[TC(\pi)] = \sum_{i=1}^{n} E[f_{j_i}(j_1, j_2, \ldots, j_i)] \tag{4}$$

It is assumed that f_i for each object i is independent of specific ordering of the objects $(j_1, j_2, \ldots, j_{i-1})$. Consider a partial sequence B of the objects and an object l, such that $l \notin B$, the following relation is assumed for each function f_i, $i = 1,2,\ldots,n$.

$$f_{j_i}(j_1, j_2, \ldots, j_a, l, j_{a+1}, \ldots, j_i) - f_{j_i}(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_i) = P_l w_{j_i} H(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_{i-1}) \tag{5}$$

where H is a random set function defined on all subsets of the n objects with a positive expectation and $H(\phi) = 1$. Note that the set function H does not depend on any specific job ordering and has finite expectation. Taking the expected value operator of equation (5) we get

$$E[f_{j_i}(j_1, j_2, \ldots, j_a, l, j_{a+1}, \ldots, j_i) - f_{j_i}(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_i)] = E[P_l]E[w_{j_i}]E[H(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_{i-1})] \tag{6}$$

Theorem 2: For any stochastic sequencing problem defined above, a sequence π will minimize expected total cost if and only if

$$h(j_1) \le h(j_2) \le \ldots h(j_n) \tag{7}$$

where $h(j_1) = E[P_{j_i}] / E[w_{j_i}]$ for $i = 1,2,\ldots,n$.

Proof:

Let π be a sequence of n objects and let π^1 be a sequence obtained from π by changing the position of ith and $(i + 1)$th objects, then we have

$$\begin{aligned} E[TC(\pi)] - E[TC(\pi^1)] &= E[f_{j_{i+1}}(B, j_i, j_{i+1}) - f_{j_{i+1}}(B, j_{i+1})] \\ &= E[f_{j_i}(B, j_{i+1}, j_i) - f_{j_i}(B, j_i)] \\ &= E[H(B)]E[P_{j_i}]E[w_{j_{i+1}}] - E[H(B)]E[P_{j_{i+1}}]E[w_{j_i}] \\ &= E[H(B)][E[P_{j_i}]E[w_{j_{i+1}}] - E[P_{j_{i+1}}]E[w_{j_i}]] \end{aligned}$$

Since $E[H(B)]$ and $E[w_i]$ are positive then $E[TC(\Pi)] \le E[TC(\Pi^1)]$ if and only if $h(j_i) \le h(j_{i+1})$ and this is independent of $H(B)$. Since the value of h for each specific object i is equal to $h(i) = E[P_i] / E[w_i]$ and positive, then it satisfies the transitivity relation i.e for three objects i,j and k, if $h(i) \le h(j)$ and $h(j) \le h(k)$, then $h(i) \le h(k)$. It follows from transitivity of h that π is an optimal sequence if and only if $h(j_1) \le h(j_2) \le \ldots \le h(j_n)$.

In the following section, generalizations of the deterministic examples in [2,4] are given to demonstrate the utility of theorem 2 in the paper.

4.0 Examples of the Stochastic Model

4.1 The Stochastic Multicharacteristic Inspection Problem

Consider a component with n characteristics such that each characteristic has to be inspected separately. It is assumed that the inspection of characteristics are independent from each other. Each characteristic i has a cost of inspection $C_i > 0$, C_i is not known exactly it is a random variable with finite expectation. Also it has a probability of rejection R_i, R_i is a random variable between zero and one, and is a function of the defective rate P_i and inspectors errors θ_i and β_i. All R_i's and C_i's are independent random variables.

The objective is to find the optimal sequence (order the characteristics for inspections) to minimize the expected total cost.

$$TC(\pi) = C_{j_1} + \sum_{r=2}^{n} C_{j_r} \prod_{t=1}^{r-1} (1 - R_{j_t}) \tag{8}$$

Since all random variables are independent, the expected total cost is given by:

$$E[TC(\pi)] = E[C_{j_1}] + \sum_{r=2}^{n} E[C_{j_r}] \prod_{t=1}^{r-1} (1 - E[R_{j_t}]) \tag{9}$$

where $\pi = (j_1, j_2, \ldots, j_n)$ is an ordering policy (a sequence) and j_r is the rth characteristic to be inspected.

The multicharacteristic inspection problem with random inspection cost and rejection rate is a special case of theorem 2. It can be easily shown if we define:

$w_{j_i} = C_{j_i}$ and $P_{j_i} = -R_{j_i}$

$E[w_{j_i}] = E[C_{j_i}]$ and $\mathrm{E}[P_{j_i}] = -E[R_{j_i}]$

$f_{j_1} = C_{j_1}$ for all objects, and $E(f_{j_1}) = E(C_{j_1})$

$f_{j_i}(j_1, j_2, \ldots, j_i) = C_{j_i}(\prod_{t=1}^{i-1}(1 - R_{j_t}))$ for $i = 2, \ldots, n$

and $E[f_{j_i}(j_1, j_2, \ldots, j_i)] = E[C_{j_i}](\prod_{t=1}^{i-1}(1 - E[R_{j_t}]))$ and $H(\phi) = 1$.

All the conditions of theorem 2 are satisfied and π is the optimal sequence if and only if

$$E[C_{j_1}] / E[R_{j_1}] \le E[C_{j_2}] / E[R_{j_2}] \le \ldots \le E[C_{j_n}] / E[R_{j_n}] \tag{10}$$

The above result in (10) can be obtained from the approach given in [3] for the deterministic case.

4.2 The Stochastic Candidate Selection Problem

This is a generalization of the sequencing selection problem given in [4,5] by making some of its variables random. Assume there are n acceptable candidates to fill a position. Suppose the benefits from candidate i, (i = 1,..., n) for the next M years is a random variable b_j, and the probability of acceptance of the job by candidate i is a random variable $O<R_i<1$ (the probability of rejecting the offer is (1-R_i)). When a candidate receives an offer, there is a fixed period of time, he/she can accept or reject the offers and during this period no offer is made to other candidates. As soon as a candidate accepts the offer the search is terminated. The cost of an offering a job is random variable C. All random variables are assumed to be independent of each other.

The objective is to find an optimal selection ordering $(j_1, j_2,..., j_n)$ among $n!$ permutations to maximize the expected benefit.

The total benefit, TB is given as:

$$TB(\pi)=(R_{j_1}b_{j_1}-C)+\sum_{r=2}^{n}\prod_{i=1}^{r-1}(1-R_{j_i})(R_{j_r}b_{j_r}-C) \tag{11}$$

The expected total benefit is given by

$$E[TB]=(E[R_{j_1}]E[b_{j_1}]-E[C])+\sum_{r=2}^{n}\prod_{i=1}^{r-1}(1-E[R_{j_{i1}}])(E[R_{j_r}]E[b_{j_r}]-E[C]) \tag{12}$$

The candidate selection problem with random parameters can easily be shown to be a special case of theorem 2. Let the random variables w_i, P_i and functions H and f_i for all objects i be defined as follows:

$w_{j_i}=(R_{j_1}b_{j_1}-C)$ and $P_{j_i}=-R_{j_i}$

$f_{j_1}(j_1)=(R_{j_1}b_{j_1}-C)$ for all objects

$f_{j_i}(j_1,j_2,\ldots,j_i)=(R_{j_i}b_{j_i}-C)(\prod_{t=1}^{i-1}(1-\mathrm{R}_{j_t}))$ for i = 2,3,..., n.

$H(j_1,j_2,\ldots,j_i)=(\prod_{t=1}^{i-1}(1-\mathrm{R}_{j_t}))$ and $H(\theta)=1$

The candidate selection problem satisfies the condition of the theorem 2 and hence the sequence π is optimal (maximizes expected total benefit) if and only if

$h(j_1)\le h(j_2)\le\ldots\le h(j_n)$, where $E[R_{j_n}]/(E[R_{j_i}][b_{j_i}]-E[C])$ for i = 1,2,...,n (13)

4.3 Optimization of Jobs with Random Processing Time on a Single Machine

This is a generalization of the problem in [4,6] by making some of its variable random. Consider a set of n independent single operation jobs to be processed by a single machine. Once processing of a job starts, it continues until the processing of all jobs finish. No preemption is assumed. Let t_i, $F_i = t_1 + \ldots + t_i$ and $f_i(t) = \alpha_i \exp(t)$ be processing time, flow time (completion time) and cost function of job i, ($i = 1,2,\ldots,n$), respectively. Assume t_i to be a positive random variable and α_i is a constant. Therefore, F_i and $f_i(t)$ are also random variables, since they are functions of t. The random variables t_i's are independent.

The objective is to find an optimal ordering (optimal sequencing) to minimize the expected total cost. The total cost is given by

$$TC(\pi) = \sum_{i=1}^{n} f_{j_i}(F_{j_i}) \tag{14}$$

The expected total cost is given as:

$$\begin{aligned} E[TC(\pi)] &= \sum_{i=1}^{n} E[f_{j_i}(F_{j_i})] \\ &= \sum_{i=1}^{n} \alpha_i (E[\exp(t_{j_i} + t_{j_2} + \ldots + t_{j_i})] \end{aligned} \tag{15}$$

let w_i, p_i, t_i and H for all jobs as follows:

$w_i = \alpha_i \exp(t_i)$, $\quad P_i = \exp(t_i) - 1$

$f_{j_i}(j_1, j_2, \ldots, j_i) = \alpha_{j_1} \exp(F_{j_i})$, $\quad H(j_1, j_2, \ldots, j_i) = \exp(F_{j_{i-1}})$

for $i = 1,2,\ldots,n$, and $H(\theta) = 1$.

Then the condition of theorem 2 is satisfied and π is optimal if and only if

$$h(j_1) \le h(j_2) \le \ldots \le h(j_n)$$

$$\text{where } h(j_i) = E[\exp(t_{j_i})] - 1 / \alpha_{j_i} E[\exp(t_j)],\ i = 1,2,\ldots,n \tag{16}$$

In the above problem if we assume the cost function is linear instead of exponential, i.e. $f_i(t) = \alpha_j t$. Then the stochastic version of the problem in [7] is obtained. The expected total cost is given as:

$$\begin{aligned} E[TC(\pi)] &= \sum_{i=1}^{n} \alpha_{j_i} E[t_{j_1} + t_{j_2} + \ldots + t_{j_i}] \\ &= \sum_{i=1}^{n} \alpha_{j_i} (E[t_{j_1}] + E[t_{j_2}] + \ldots + E[t_{j_i}]) \end{aligned} \tag{17}$$

If w_i, P_i, f_i and H are defined for all jobs as:

$w_i = \alpha_i$, $P_i = -t_i$ for all jobs

$f_{j_i}(j_1, j_2, \ldots, j_i) = \alpha_{j_1}(F_{j_i}) = \alpha_{j_i}(t_{j_1} + t_{j_2} + \ldots + t_{j_i})$

and $H(j_1, j_2, \ldots, j_i) = 1$ for all jobs.

The conditions of theorem 2 are met and a sequence π is optimal if and only if

$$h(j_1) \leq h(j_2) \leq \ldots \leq h(j_n)$$

$$\text{where } h(j_i) = E[t_{j_i}] / \alpha_{j_i}, \text{ for } i = 1,2,\ldots,n \tag{16}$$

5.0 Conclusion

In this paper a stochastic sequencing problem has been defined. The necessary and sufficient conditions for finding the optimal sequence has been derived. The stochastic sequencing problem generalizes the deterministic sequencing problem given [4] by considering some of the input variables as random. The stochastic sequencing problem represent many practical situations such as sequencing characteristic for inspection, jobs processing and candidate selection. When the random variables are assumed constants in the stochastic sequencing model the results of this paper specialize to results in [3,4].

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