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Upper and lower bounds for the pressure error in the rectilinear flow along a slot with a pressure gradient

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With 6 figures

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1. Introduction

Viscometric flows, locally equivalent to simple shearing, are very important in the theory of incompressible viscoelastic fluids. It is well known, e.g. see the review article by *Pipkin* and *Tanner* (1), that in these flows, the two normal stress differences N_1 and N_2 and the shear stress function τ play crucial roles. With respect to a local Cartesian system, if $\boldsymbol{v} = \kappa y \boldsymbol{i}$ is the velocity field, then N_1, N_2 and τ are defined through:

$$N_1 = T\langle xx\rangle - T\langle yy\rangle, \qquad [1.1]$$

$$N_2 = T\langle yy\rangle - T\langle zz\rangle, \qquad [1.2]$$

$$\tau = T\langle xy\rangle, \qquad [1.3]$$

$$N_i = N_i(\kappa), \quad i = 1,2; \quad \tau = \tau(\kappa), \qquad [1.4]$$

$$N_i(\kappa) = N_i(-\kappa), \quad i = 1,2; \quad \tau(-\kappa) = -\tau(\kappa). \qquad [1.5]$$

In [1.1] – [1.3], $T\langle ij\rangle$ are the physical components of the stress tensor $\boldsymbol{T}$ and in [1.4] – [1.5], κ is called the rate of shear. Much work, theoretical and experimental, has been done in determining the various configurations wherein N_1, N_2 and τ may be measured. For example, apart from *Pipkin* and *Tanner* (1), one may consult *Lodge* (2), or *Huilgol* (3), or *Walters* (4) for an exhaustive study of these aspects. This would reveal that in a number of situations, measurement of the stress distribution on a flat plate (e.g. in cone and plate flow, torsional flow) or at a point on a curved surface (e.g. in Couette flow) are required. One approach to the measurement is to drill a hole or cut a slot for the insertion of a pressure sensitive device. The question this raises is: does the presence of a hole induce errors in the values of the stresses so recorded?

Twenty-five years ago, *Greensmith* and *Rivlin* (5) did a number of tests to determine whether the holes would induce "pressure-hole errors". They concluded, by varying the diameters of holes, that these errors did not exist. The matter rested here until *Adams* and *Lodge* (6) found that the normal stress measurements obtained from the cone-and-plate apparatus did not agree with those obtained from the parallel-plate apparatus.

Following upon this discovery, *Broadbent* et al. (7) published an article in which they pointed out that if a systematic error, p_e, independent of the hole size but dependent on the shear stress τ, was assumed to exist, then these inconsistencies could be resolved. These authors found an equation connecting p_e and τ, which was somewhat empirical.

The next advance was made by *Tanner* and *Pipkin* (8), who considered the creeping flow of an incompressible second order fluid (*Coleman* and *Noll* (9)) across an infinitely deep slot. They found that

$$p_e = -N_1(\kappa)/4, \qquad [1.6]$$

where for the second-order fluid, described by:

$$\boldsymbol{T} + p\boldsymbol{1} = \eta_0 \boldsymbol{A}_1 + \beta \boldsymbol{A}_1^2 + \gamma \boldsymbol{A}_2, \qquad [1.7]$$

with η_0, β and γ constants, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ being the first two Rivlin-Ericksen (10) tensors,

$$N_1(\kappa) = -2\gamma\kappa^2. \qquad [1.8]$$

Using this very constitutive relation [1.7], *Kearsley* (11) found that in the creeping, rectilinear flow along a deep slot,

$$p_e = N_2(\kappa)/2, \qquad [1.9]$$

$$N_2(\kappa) = (\beta + 2\gamma)\kappa^2. \qquad [1.10]$$

Later, *Kearsley* (12) discussed the rectilinear motion of the second-order fluid between two infinitely parallel plates D apart, with the top plate moving with a constant speed V and the bottom plate at rest. The bottom plate had a slot, of finite depth d and width W, in the direction of the motion. *Kearsley* obtained an exact solution to this problem and found explicitly that

$$p_e = \frac{N_2(\kappa)}{2} \cdot G(k,\alpha), \qquad [1.11]$$

where $N_2(\kappa)$ is evaluated at $\kappa = V/D$, and

$$G(k,\alpha) = \frac{1 - k^2}{1 - k^2 \operatorname{sn}^2 \alpha} \qquad [1.12]$$

with k and $\operatorname{sn} \alpha$ being the *modulus* and *modular sine* respectively.

In this paper, we study the above problem of the rectilinear flow when along with the moving top plate, a constant pressure gradient c exists in flow direction. We have not been able to solve the problem completely; however, we have obtained upper and lower bounds to the error p_e. By putting $c = 0$, we can recover bounds for the case considered by *Kearsley* (12). A comparison with his solution (see fig. 4) has been made to indicate the position of the bounds relative to the exact values. If we put $V = 0$, $c \neq 0$, then our results should be applicable to the "Omega flow" considered more recently by *Lobo* and *Osmers* (13) to measure N_1.

In this paper, we assume that $N_2(\kappa) \leq 0$.

2. Formulation of the problem

We are interested in the determination of the velocity field, in a Cartesian frame:

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = w(x,y). \qquad [2.1]$$

This steady flow takes place between a fixed bottom plate with a longitudinal slot which is along the direction of the flow, and a moving upper plate. Both plates are assumed to be infinitely long and wide, and the upper plate moves in the z-direction with a constant speed V. There is also a uniform pressure gradient $\partial p/\partial z = \eta_0 c$ as well, where $c \leq 0$ is a constant. The continuity equation is trivially satisfied and the equations of motion imply that w satisfies

$$\Delta w = c, \qquad [2.2]$$

when body forces are ignored and the constitutive eq. [1.7] is used. In eq. [2.2] Δ is the two-dimensional Laplacian:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad [2.3]$$

The boundary conditions (see fig. 1) are:

$$w = \begin{cases} V \text{ on } y = d + D; \\ 0, |x| \leq (W/2);\ y = 0; \\ 0, |x| = (W/2);\ 0 \leq y \leq d; \\ 0, |x| \geq (W/2);\ y = d; \\ w \to w(y) \text{ as } |y| \to \infty. \end{cases} \qquad [2.4]$$

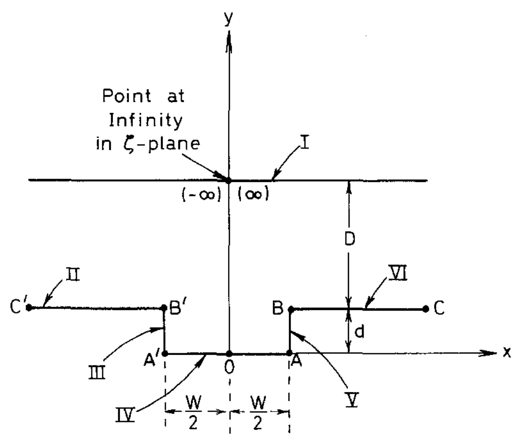

Fig. 1. Flow domain in the z-plane and the boundary regions I, II, III, IV, V and VI

We shall digress a little and note that the corresponding problem for a Newtonian fluid, i.e., the fluid defined by [1.7] with $\beta = \gamma = 0$, is also given by eqs. [2.1]–[2.4]. Thus the Newtonian and second-order fluid velocity fields are identical, but the pressure fields are not. In the Newtonian fluid, $p^0 = p^0(z)$, while in the second-order fluid, as shown by *Kearsley* (11), the pressure p is given in the steady flow [2.1] by:

$$p = p^0 - (\beta + 2\gamma)\frac{1}{\eta_0}\frac{dp^0}{dz}w, \qquad [2.5]$$

i.e., by

$$p = p^0 - c(\beta + 2\gamma)w. \qquad [2.6]$$

Using eqs. [1.7], [2.1] and [2.6], one can calculate the normal stress T_{yy} to be:

$$T_{yy} = -p^0 + \tfrac{1}{2}(\beta + 2\gamma)\left[(w_{,x}^2 - w_{,y}^2) - 2cw\right], \qquad [2.7]$$

where the comma denotes the partial derivative and $w_{,x}^2 = (\partial w/\partial x)^2$, etc.

We assume that in this flow, the normal stress is measured at $x = 0$, $y = 0$, while the undisturbed normal stress, i.e., the true normal stress, is the one at $x = \pm\infty$, $y = d$. The *pressure error* p_e is then defined through:

$$p_e = T_{yy}(\pm\infty, d) - T_{yy}(0,0). \quad [2.8]$$

Now, along the fixed plate, $w = 0$ and $w_{,x} = 0$ as well. Therefore, eqs. [2.7]–[2.8] lead to (12):

$$p_e = \frac{\beta + 2\gamma}{2} \cdot [w_{,y}^2(\pm\infty, d) - w_{,y}^2(0,0)]. \quad [2.9]$$

If we calculate the second normal-stress difference for this fluid based on the shear rate at $x = \pm\infty$, $y = d$, and denote this by $\bar{N}_2$, [2.9] becomes:

$$p_e = \tfrac{1}{2}\bar{N}_2 G, \quad [2.10]$$

where

$$G = \left[1 - \frac{w_{,y}^2|_0}{w_{,y}^2|_C}\right]. \quad [2.11]$$

Here 0 denotes the origin and C denotes the point at (∞, d) as shown in figure 1.

To sum up, the determination of p_e reduces to that of finding the function G, which depends on the dimensions of the flow: d, D, W as well as V and c.

3. Bounds for the error

If the solution of the Poisson's eq. [2.2], subject to the boundary conditions [2.4] is obtained, then G will be known. This is a formidable task because the conformal mapping which transforms the region in figure 1 – hereafter called the z-plane which should not be confused with the Cartesian co-ordinate – onto the upper half-plane in the ζ-plane is given by (*Carter* (14)) a cumbersome result:

$$\frac{dz}{d\zeta} = M\left(\frac{1 - k^2\zeta^2}{1 - \zeta^2}\right)^{1/2} \frac{1}{1 - k^2\zeta^2 \operatorname{sn}^2\alpha}. \quad [3.1]$$

We have already introduced the parameter k and $\operatorname{sn}\alpha$ in eq. [1.12] above, and M is the Schwarz-Christoffel constant to be determined. (The Appendix treats these matters in a more detailed fashion to which the reader is referred). Having accomplished this mapping, the usual procedure is to map this upper half-plane onto a unit circle in the γ-plane (fig. 3b), say, and determine the mapping:

$$z = \sum_{n=0}^{\infty} a_n \gamma^n. \quad [3.2]$$

If one finds the coefficients a_n, there still remains the task of solving the Poisson's equation in the γ-plane, which will now be $\Delta w = f(\gamma_1, \gamma_2)$ rather than $\Delta w = c$. These remarks indicate why we have sought to obtain bounds for the solution to w and hence derive bounds for G itself.

To achieve these bounds, let us decompose w:

$$w = w_0 + w_1, \quad [3.3]$$

where w_0 obeys:

$$\Delta w_0 = c, \quad [3.4]$$

and

$$w_0 = \begin{cases} V, & y = d + D \\ 0, & y = d. \end{cases} \quad [3.5]$$

The solution is trivially shown to be:

$$w_0 = \frac{c}{2} y^2 + by + e, \quad [3.6]$$

where b and e are the two constants:

$$b = \frac{V}{D} - \frac{c}{2}(D + 2d), \quad [3.7]$$

$$e = \frac{cd}{2}(d + D) - \frac{Vd}{D}. \quad [3.8]$$

Breaking up figure 1 into regions I, II, III, IV, V und VI, we can show that w_1 satisfies:

$$\left.\begin{aligned} &\Delta w_1 = 0, \quad [3.9] \\ &w_1 = \begin{cases} 0, \text{ on I, II and VI}, \\ -\frac{c}{2}y^2 - by - e \text{ on III and V}, \\ -e \text{ on IV}. \end{cases} \\ &w_1 \to 0 \text{ as } |x| \to \infty, \end{aligned}\right\} \quad [3.10]$$

so that $w \to w_0$ as $|x| \to \infty$. Next, the constants c and V are such that $c \leq 0$ and $V \geq 0$. From this, one obtains that $b \geq 0$ and $e \leq 0$ and using these it can be shown that the boundary data for w_1 is non-negative. We shall now introduce two other harmonic functions w_1^L and and w_1^U. These obey:

$$\Delta w_1^L = 0, \quad \Delta w_1^U = 0. \quad [3.11]$$

The boundary data for w_1^L and w_1^U are:

$$w_1^L = \begin{cases} -e \text{ on IV} \\ 0, \text{ elsewhere}, \end{cases}$$

$$w_1^L \to 0 \text{ as } |x| \to \infty,$$

$$w_1^U = \begin{cases} -e \text{ on III, IV and V} \\ 0, \text{ elsewhere}, \end{cases}$$

$$w_1^U \to 0 \text{ as } |x| \to \infty. \qquad [3.12]$$

We claim that $w_1^L < w_1 < w_1^U$. To make this transparent, consider the mapping of the upper half of the ζ-plane (fig. 2) onto the unit circle in the γ-plane, (see fig. 3), together with

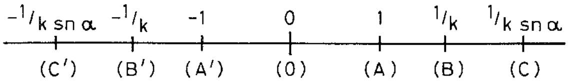

Fig. 2. Points in the ζ-plane corresponding to the flow domain

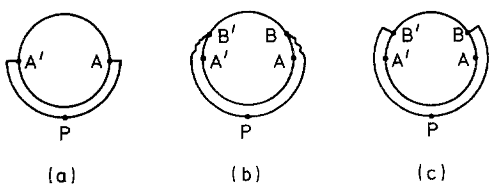

Fig. 3. Boundary data for w_1^L, w_1 and w_1^U in the γ-plane

the relevant boundary data for w_1^L, w_1 and w_1^U. Clearly, the functions $(w_1 - w_1^L)$ and $(w_1^U - w_1)$ are harmonic and the boundary data are non-negative. Hence, by the maximum principle (*Protter* and *Weinberger* (15)):

$$w_1^L < w_1 < w_1^U \qquad [3.13]$$

as asserted.

Moreover, at the point P in the γ-plane which corresponds to 0 in the z-plane,

$$\frac{\partial}{\partial \gamma_2}(w_1^U - w_1) \geq 0,$$

$$\frac{\partial}{\partial \gamma_2}(w_1 - w_1^L) \geq 0. \qquad [3.14]$$

A glance at eq. [3.1] shows that these mappings are all conformal at $z = 0$ and hence

$$\frac{\partial w_1^U}{\partial y} \geq \frac{\partial w_1}{\partial y} \geq \frac{\partial w_1^L}{\partial y} \quad \text{at} \quad 0. \qquad [3.15]$$

Now, by the nature of the boundary conditions on w_1^L and w_1^U, $w_{1,x}^L \to 0$, $w_{1,x}^U \to 0$ at $|x| \to \infty$. Moreover, as we shall see, $w_{1,y}^L \to 0$, $w_{1,y}^U \to 0$ as $|x| \to \infty$. Bearing in mind the fact that $w_{1,x} \to 0$ as $|x| \to \infty$, as well as the ellipticity of the problem with boundary data prescribed over a small region near the origin in the z-plane, one has that $w_{1,y} \to 0$ as $|x| \to \infty$ as well.

We shall now proceed to calculate $w_{1,y}^L$ and $w_{1,y}^U$. This part is straightforward because w_1^L and w_1^U are harmonic with constant boundary data over $(-1,1)$ and $(-1/k, 1/k)$, respectively, in the ζ-plane. Thus (for full details, see *Rajagopal* (16)):

$$\frac{\partial w_1^L}{\partial y} = -\frac{e}{M\Pi}\left[\left(\frac{1-\zeta^2}{1-k^2\zeta^2}\right)^{1/2} \cdot \left(\frac{1-k^2\zeta^2 \operatorname{sn}^2\alpha}{\zeta^2-1}\right) + \left(\frac{1-\bar{\zeta}^2}{1-k^2\bar{\zeta}^2}\right)^{1/2} \cdot \left(\frac{1-k^2\bar{\zeta}^2 \operatorname{sn}^2\alpha}{\bar{\zeta}^2-1}\right)\right], \qquad [3.16]$$

$$\frac{\partial w_1^U}{\partial y} = -\frac{ek}{M\Pi}\left[\left(\frac{1-\zeta^2}{1-k^2\zeta^2}\right)^{1/2} \cdot \left(\frac{1-k^2\zeta^2 \operatorname{sn}^2\alpha}{\zeta^2 k^2-1}\right) + \left(\frac{1-\bar{\zeta}^2}{1-k^2\bar{\zeta}^2}\right)^{1/2} \cdot \left(\frac{1-k^2\bar{\zeta}^2 \operatorname{sn}^2\alpha}{\bar{\zeta}^2 k^2-1}\right)\right], \qquad [3.17]$$

where $\bar{\zeta}$ is the conjugate of ζ. Note that when $\zeta = \pm 1/k \operatorname{sn}\alpha$, which corresponds to the point $|x| \to \infty$ in the z-plane, these derivatives vanish.

Now, let us obtain the upper and lower bounds to w itself as:

$$w^U = w_0 + w_1^U, \qquad w^L = w_0 + w_1^L. \qquad [3.18]$$

Then, from the foregoing

$$\left.\begin{aligned} &w_{,y}^U|_0 = \frac{2ek}{M\Pi} + b, \quad w_{,y}^L|_0 = \frac{2e}{M\Pi} + b, \\ &w_{,y}^U|_C = w_{,y}^L|_C = cd + b. \end{aligned}\right\} \qquad [3.19]$$

We now define two functions G_1 and G_2 as:

$$G_1 = 1 - (w_{,y}^U|_0)^2/(cd+b)^2, \qquad [3.20]$$

$$G_2 = 1 - (w_{,y}^L|_0)^2/(cd+b)^2. \qquad [3.21]$$

Though, as we have shown, $w_{,y}^L|_0 \leq w_{,y}^U|_0$, it need not be true that $G_1 \leq G_2$ always, because the squares of these derivatives are included in [3.20]–[3.21]. However, it will always be true that one of the following inequalities holds for the function G defined in [2.10]:

$$G_1 \leq G \leq G_2, \quad \text{or} \quad G_2 \leq G \leq G_1. \qquad [3.22]$$

Bearing in mind that $\bar{N}_2 \leq 0$, one can obtain bounds for p_e, as defined in [2.11], by a correct choice of one of the inequalities in [3.22].

4. Results and discussion

For values of $d/W \leq 0.7$, we have found by direct calculation that $w^L_{,y}|_0 > 0$ when $D/W = 0.5$ and the parameter T (see [4.3] below) lies between 0 and 0.5. Because $w^L_{,y}|_0 > 0$, we have the result $(w^U_{,y}|_0)^2 > (w^L_{,y}|_0)$, and therefore G satisfies:

$$G_1 \leq G \leq G_2\,, \qquad [4.1]$$

and because $\bar{N}_2 \leq 0$,

$$\frac{\bar{N}_2 G_2}{2} \leq p_e \leq \frac{\bar{N}_2 G_1}{2}\,. \qquad [4.2]$$

To facilitate the presentation of the results, we introduce a parameter T through

$$T = cD/\kappa_0\,, \quad \kappa_0 = V/D\,, \qquad [4.3]$$

where κ_0 is a measure of the shear rate. Thus G_1 and G_2 can be written as:

$$G_1 = 1 - \frac{\left[\frac{\Pi M b}{W\kappa_0} + \frac{kTd}{W}\left(1+\frac{d}{D}\right) - 2\frac{d}{W}\right]^2}{\Pi^2\left(\frac{M}{W}\right)^2\left(\frac{cd+b}{\kappa_0}\right)^2} \qquad [4.4]$$

$$G_2 = 1 - \frac{\left[\frac{\Pi M b}{W\kappa_0} + \frac{Td}{W}\left(1+\frac{d}{D}\right) - 2\frac{d}{W}\right]^2}{\Pi^2\left(\frac{M}{W}\right)^2\left(\frac{cd+b}{\kappa_0}\right)^2}\,. \qquad [4.5]$$

The Appendix lists the method chosen to compute the numbers k and M if d, D and W are given. Using these in [4.4]–[4.5], we have plotted three sets of curves in figures 4–6. In figure 4, $T = 0$ and we have been able to compare our bounds with the exact volume of G in eq. [1.12] above due to *Kearsley* (12). Figures 5 and 6 treat the cases for $T = 0.25$ and $T = 0.5$ respectively. The results seem to indicate that if $D/W = 0.5$, the bounds come together as T increases.

Finally, we can derive the limiting values of G as $V \to 0$, i.e., as the flow occurs due to the pressure gradient alone. If $V \to 0$, then from [3.7]:

$$\left.\begin{aligned} cd + b &\to CD/2,\\ T\kappa_0 &\to cD\,. \end{aligned}\right\} \qquad [4.6]$$

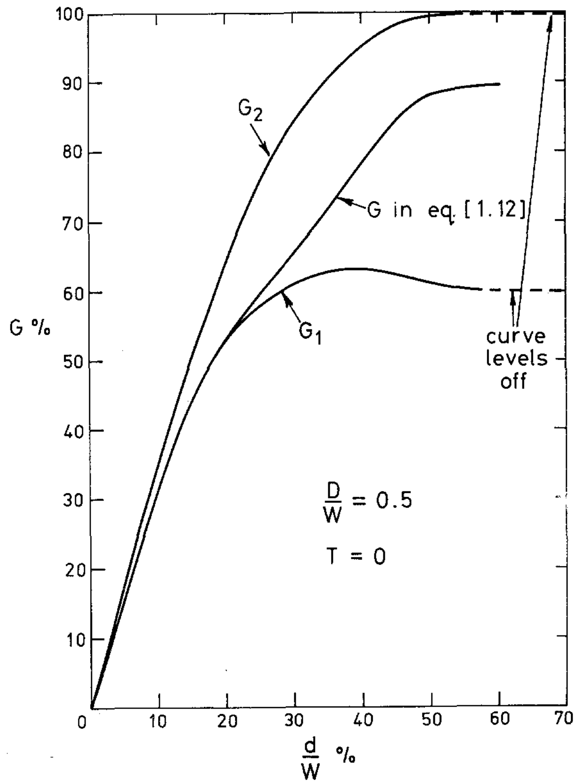

Fig. 4. Upper and lower bounds compared with the exact value of G for $c = 0$, $V > 0$, $T = 0$

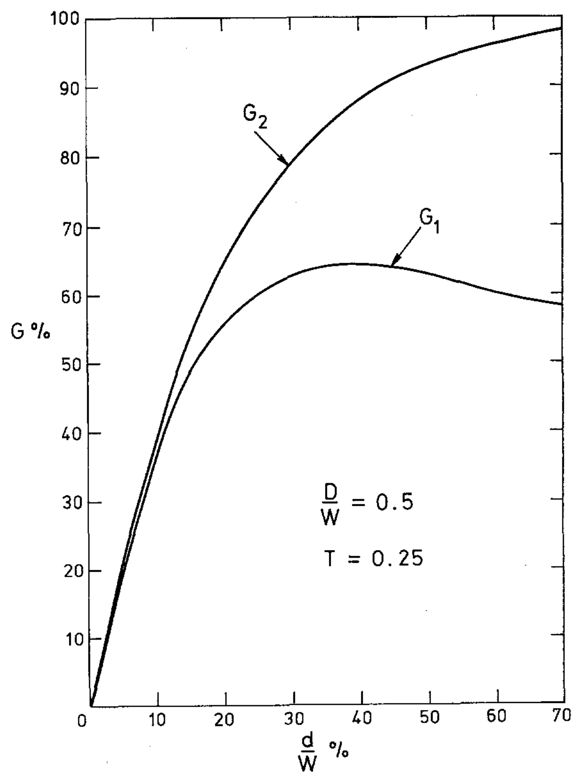

Fig. 5. Upper and lower bounds for G, $c \neq 0$, $V > 0$, $T = 0.25$

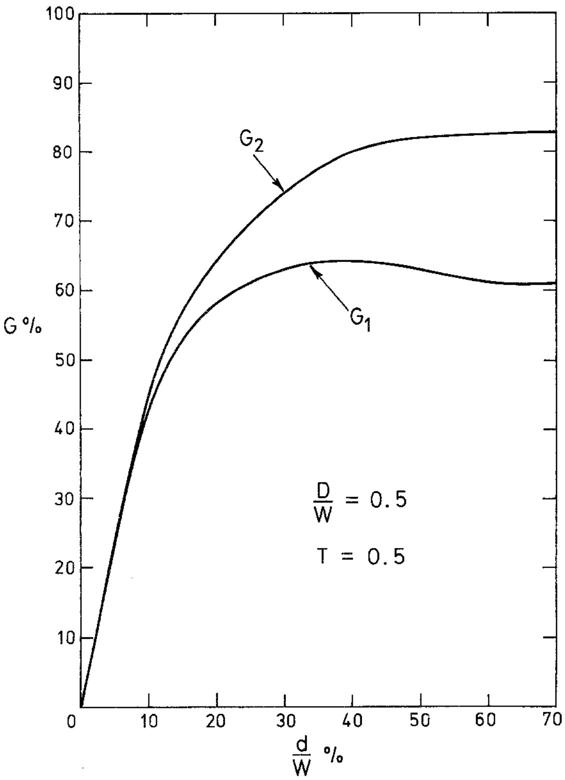

Fig. 6. Upper and lower bounds for G, $c \neq 0$, $V > 0$, $T = 0.5$

Thus, as $V \to 0$,

$$G_1 \to 1 - \left(\frac{2kd(1 + d/D)}{\Pi M}\right)^2, \qquad [4.7]$$

$$G_2 \to 1 - \left(\frac{2d(1 + d/D)}{\Pi M}\right)^2. \qquad [4.8]$$

These show that G_1 and G_2 become independent of the magnitude of the pressure gradient c. Of course p_e will depend on c through the shear rate κ_0 at the wall far away from the slot.

Appendix

Given d, D and W, one obtains (*Carter* (14)) that:

$$d/W = (2K' - \alpha(D/W))/K \qquad [A\,1]$$

where $K = K(k)$ is the complete elliptic integral of the first kind, $K' = K'(k')$, $k' = (1 - k^2)^{1/2}$. Eq. [A 1] yields a relationship between α and k. Next,

$$\frac{D}{W} = \frac{\pi \operatorname{cn} \alpha}{4\left[K \operatorname{cn} \alpha \operatorname{dn} \alpha - K \operatorname{cn} \alpha\, E(\alpha,k) + \alpha \operatorname{cn} \alpha\, E(k)\right]}. \qquad [A\,2]$$

Here $E(k)$ is the complete elliptic integral of the second kind and $E(\alpha, k)$ the elliptic integral of the second kind. For all of the above formulae and their meanings, see *Hancock* (17).

Here, we list the method we have followed to calculate the parameters k and M needed in § 4. This procedure is based to a certain extent on the Appendix 3 of *Carter* (14).

(i) Given d/W and D/W, calculate

$$\sigma = \frac{2}{\pi}\left\{\arctan\left(\frac{2D}{W}\right) - \frac{W}{4D}\ln\left(1 + \left(\frac{2D}{W}\right)^2\right)\right\}. \qquad [A\,3]$$

(ii) Use the elliptic integral of the first kind, i.e., F to calculate $k \operatorname{sn} \alpha$ as follows:

$$\frac{F((\pi/2) - \Theta, \pi/2)}{F(\Theta, \pi/2)} = \frac{2(d + D - \sigma D)}{W}, \qquad [A\,4]$$

where

$$k \operatorname{sn} \alpha = \sin\Theta. \qquad [A\,5]$$

(iii) Let

$$\operatorname{sn} \alpha = \sin \varphi \qquad [A\,6]$$

and calculate, as a first approximation, $\operatorname{sn} \alpha$ from:

$$\operatorname{sn} \alpha \doteq \left[1 + \left(\frac{2D}{W}\right)^2\right]^{-1/2}. \qquad [A\,7]$$

(iv) Since [A 4] has yielded the *exact* value of $k \operatorname{sn} \alpha$ and $\operatorname{sn} \alpha$ is *approximately* known from [A 7], we can find a first guess to k. Use this to find $K(k)$ and thus α from

$$\tan\left(\frac{\pi\alpha}{2K}\right) = \frac{W}{2D}. \qquad [A\,8]$$

The correct relationship between k, φ and α is:

$$\alpha = F(k, \varphi). \qquad [A\,9]$$

(v) Check if the α, k and $\operatorname{sn} \alpha$ so determined satisfy [A 1]–[A 2]. If not, assign a new value to $\operatorname{sn} \alpha$, recalculate k and employ [A 9]. Repeat until agreement with [A 1]–[A 2] is obtained to the desired accuracy, say to 3 decimal places.

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Summary

The rectilinear flow of a second-order fluid is considered between two infinitely wide and long parallel plates. The bottom plate is at rest and has a longitudinal slot in the direction of the flow, while the top plate moves in the flow direction with a constant speed. Upper and lower bounds for the pressure error are ob-

tained by the use of the maximum principle applied to harmonic functions.

Zusammenfassung

Es wird die geradlinige Strömung einer Flüssigkeit zweiter Ordnung zwischen zwei unendlich ausgedehnten parallelen Platten untersucht. Die mit einer rechteckigen, in Strömungsrichtung orientierten Nute versehene Grundplatte befindet sich in Ruhe, wohingegen die glatte Deckplatte sich mit konstanter Geschwindigkeit in Strömungsrichtung bewegt. Untere und obere Schranken für die Abweichung des Druckes infolge der Nute ("pressure error") werden durch Anwendung des Maximumprinzips auf harmonische Funktionen berechnet.

References

1) *Pipkin, A. C., R. I. Tanner*, in: *S. Nemat-Nasser* (ed.), Mechanics Today, Vol. 1 (1972). Pergamon (Oxford 1974).

2) *Lodge, A. S.*, Elastic Liquids. Academic Press (New York 1964).

3) *Huilgol, R. R.*, Continuum Mechanics of Viscoelastic Liquids. Hindustan Pub. Corp., Delhi and Halsted Press (New York 1975).

4) *Walters, K.*, Rheometry. Chapman and Hall (London 1975).

5) *Greensmith, H. W., R. S. Rivlin*, Phil. Trans. Roy. Soc. **A 245**, 399 (1953).

6) *Adams, N., A. S. Lodge*, Phil. Trans. Roy. Soc. **A 256**, 149 (1964).

7) *Broadbent, J. M., A. Kaye, A. S. Lodge, D. G. Vale*, Nature **217**, 55 (1968).

8) *Tanner, R. I., A. C. Pipkin*, Trans. Soc. Rheol. **13**, 471 (1969).

9) *Coleman, B. D., W. Noll*, Arch. Ratl. Mech. Anal. **6**, 355 (1960).

10) *Rivlin, R. S., J. L. Ericksen*, J. Ratl. Mech. Anal. **4**, 323 (1955).

11) *Kearsley, E. A.*, Trans. Soc. Rheol. **14**, 419 (1970).

12) *Kearsley, E. A.*, Lecture at the 44th Annual Meeting of Soc. of Rheology (Montreal 1973).

13) *Lobo, P. F., H. R. Osmers*, Rheol. Acta **13**, 457 (1974). See also, Trans. Soc. Rheol. **20**, 239 (1976).

14) *Carter, F. W.*, J. Inst. Elec. Engrs. **64**, 1115 (London 1926).

15) *Protter, M. H., H. F. Weinberger*, Maximum Principles in Differential Equations. Prentice-Hall (Englewood Cliffs 1967).

16) *Rajagopal, K. R.*, M. Sc. Thesis. Ill. Inst. Technology (Chicago 1974).

17) *Hancock, H.*, Theory of Elliptic Functions. Dover (New York 1958).

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