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Technical Note

THE STAR-HEIGHT OF REGULAR EXPRESSIONS

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1. INTRODUCTION*

Kleene[3] was the first to introduce the concept of regularity for expressions and sets of words. Others, including Copi, Elgot and Wright,[1] Myhill,[5] and McNaughton and Yamada,[4] have discussed this topic and its relation to finite automata. We refer the reader especially to Ref. 1 for a presentation similar to ours.

Let $\mathcal{A}$ be a finite set of objects, say $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$. Let $\mathfrak{A}$ be the free semigroup with identity generated by $\mathcal{A}$, where we write the operation multiplicatively and denote it by juxtaposition. We denote by A^m the element $\underbrace{AAA \ldots AA}_{m \text{ times}}$. Let Θ denote the identity element, so $\Theta A_i = A_i\Theta = A_i$ for all $i = 1, \ldots n$. We will also call $\mathfrak{A}$ the set of words on the alphabet $\mathcal{A}$, and hence an element of $\mathfrak{A}$ a word (on the alphabet $\mathcal{A}$).

Let $\mathcal{S} = \mathcal{A} \cup \{\Lambda, \Theta\}$, where Λ and Θ are distinct objects not in $\mathcal{A}$. Let "V" and "·" be associative binary operations and "*" a unary operation and define $\mathfrak{S} = (\mathcal{S}, V, \cdot, *)$ to be the free algebra generated by $\mathcal{S}$ with operations V, · and *. Thus $\mathcal{S} \subseteq \mathfrak{S}$, and $\sigma, \omega \in \mathfrak{S}$ implies $\sigma V \omega$, $\sigma \cdot \omega$ and $\sigma^* \in \mathfrak{S}$. We call any element of $\mathfrak{S}$ a <u>regular expression</u> (on $\mathcal{S}$). As usual, we shall write $\sigma\omega$ for $\sigma \cdot \omega$.

We next define the concept of regular set (or regular event). Define the mapping $|\,|$ from $\mathfrak{S}$ into $2^{\mathfrak{A}}$ inductively as follows: (we read $|\sigma|$ as "the set denoted by σ")

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(i) $|A_i| = \{A_i\}$, $i = 1, \ldots, n$, $|\Lambda| = \phi$, $|\Theta| = \{\Theta\}$.

(ii) For $\sigma, \omega \in \mathfrak{S}$,

$$|\sigma V \omega| = |\sigma| \cup |\omega|,$$

$$|\sigma\omega| = \{xy;\ x \in |\sigma|,\ y \in |\omega|\},$$

$$|\sigma^*| = |\Theta| \cup |\sigma| \cup |\sigma\sigma| \cup |\sigma\sigma\sigma| \cup \ldots .$$

Thus $\mathfrak{A} = |\mathcal{A}^*| = |(A_1 V A_2 V \ldots V A_n)^*|$, so we generally write $\mathcal{A}^*$ for the semi-group with identity generated by the set $\mathcal{A}$.

We define any subset of $\mathfrak{A}$ to be an <u>event</u>. We say that an event Σ is <u>regular</u> in case there exists a regular expression $\sigma \in \mathfrak{S}$ such that $|\sigma| = \Sigma$. Thus the class of <u>regular events</u> (or <u>regular sets</u>) is the range of the function $||$ just defined. It is well known ([1, 3, 4, 5]) that the class of regular events is a Boolean Algebra of sets.

Finally we define the concept of *-height (read: star-height). We define the function $h: \mathfrak{S} \to Z$, (where Z denotes the integers) inductively as follows:

(1) For $s \in \mathcal{S}$, $h(s) = 0$

(2) For $\sigma, \omega \in \mathfrak{S}$,

$$h(\sigma\omega) = h(\sigma V \omega) = \max\{h(\sigma), h(\omega)\}$$

$$h(\sigma^*) = h(\sigma)+1.$$

For example:

$$h(A_1^*(A_2 V A_3^*)^*) = 2,$$

$$h(A_1^* V (A_1^* A_2 V A_3)^*) = 2.$$

We call $h(\sigma)$ the <u>*-height</u> of σ. If Σ is a regular event, we define the *-height of Σ by

$$h(\Sigma) = \min\{h(\sigma):\ |\sigma| = \Sigma\}.$$

It is clear that for each integer n there is a regular expression of *-height n. Our main result is that for each integer n, there is a regular event of *-height n. We, in fact, show somewhat more; namely, that if a regular event Σ contains words of a certain type, then $h(\Sigma) \geq n$.

It should be mentioned that there is some hope of relating the *-height of the behavior of a finite automata or nerve net to the number and complexity of its feedback cycles (cf., Ref. 1 for terminology, and also cf., Ref. 2).

2. PRELIMINARIES

In this section we introduce some notation and conventions and prove some preliminary results.

Let $A_1, A_2, \ldots, A_n$, be an infinite sequence of distinct letters. Let $\mathcal{S}_k = \{A_1, A_2, \ldots, A_j\}$ where $j = 2^k-1$, and let $\mathcal{T}_k = \{A_{j+1}, \ldots, A_{2j}\}$ so $\mathcal{S}_k \cup \mathcal{T}_k \cup \{A_{2^{k+1}-1}\} = \mathcal{S}_{k+1}$. $\mathcal{S}$ will denote a finite alphabet, but we shall presume that $\mathcal{S}_k \subseteq \mathcal{S}$ whenever we speak of $\mathcal{S}_k$. Thus although $\mathcal{S}$ may become arbitrarily large, at any point in our discussion it will be finite. All sets and expressions considered will be presumed to be regular subsets of $\mathcal{S}^*$ or regular expressions on $\mathcal{S}$. (Recall $\mathcal{S}^*$ denotes the semigroup with identity θ generated by $\mathcal{S}$.) Small Greek letters will denote regular expressions, capital Greek letters the corresponding regular set, lower case Roman letters will denote words and upper case Roman letters will denote letters of the alphabet.

If $\Sigma \subseteq \mathcal{S}^*$, let

$$\bar{\Sigma} = \left\{ a \in \mathcal{S}^*;\ \exists x, y \ni xay \in \Sigma \right\},$$

i.e., $\bar{\Sigma}$ is the set of all subwords of words of Σ. Let $h(\sigma)$ and $h\,\Sigma$ be the *-heights of the expression σ and the set Σ, respectively.

We define the properties Φ_k, $k = 1, 2, \ldots$, on the class of regular sets as follows: Let Σ be a regular set. Σ has property Φ_1 (relative to the letter A_1) in case

$$(\forall n)(\exists m \geqslant n)\left(A_1^m \in \bar{\Sigma}\right).$$

Thus Σ has property Φ_1 on the letter A_1 if Σ contains arbitrarily long strings of the letter A_1 as subwords. If a word $\underline{a}$ contains A_1^m for some $m \geqslant n$, then we say that $\underline{a}$ is a <u>1-word for exponent</u> n (on the letter A_1).

We proceed by induction. Suppose that u is a (k-1) word for exponent n on $\mathcal{S}_{k-1}$ and v is a (k-1) word for exponent n on $\mathcal{T}_{k-1}$ and let $j = 2^k-1$. Then any word of the form $(uvA_j)^m$ for $m \geqslant n$ is a <u>k-word for exponent</u> n (on $\mathcal{S}_k$). We say that Σ has property Φ_k relative to $\mathcal{S}_k$ in case (i) $\bar{\Sigma}$ contains k-words for arbitrarily large exponent (on $\mathcal{S}_k$), (ii) Σ has property Φ_{k-1} relative to each of the sets $\mathcal{S}_{k-1}$ and $\mathcal{T}_{k-1}$ and (iii) for $A_p \in \mathcal{T}_{k-1}$, $A_q \in \mathcal{S}_{k-1}$, $A_pA_q \notin \bar{\Sigma}$.

Thus, for example, Σ has property Φ_2 on $\mathcal{S}_2$ in case $(\forall n)(\exists m_1, m_2, m_3 \geqslant n)$ $\left((A_1^{m_1} A_2^{m_2} A_3)^{m_3} \in \bar{\Sigma} \text{ and } A_2A_1 \notin \bar{\Sigma}\right)$. We remark here that it is clear that if Σ contains no k-subword for exponent m, then $\Sigma^2 = \{\sigma_1\sigma_2; \sigma_1, \sigma_2 \in \Sigma\}$ contains no k-subword for exponent 2m.

We next state two lemmas, the first of which follows from the fact that concatenation distributes over union.

<u>Lemma 1</u>. <u>If</u> $h\left(\Sigma\right) = n$, <u>then there is a regular expression</u> σ <u>which denotes</u> Σ <u>such that</u>

$$\sigma = \gamma_1 V \gamma_2 V \ldots V \gamma_m,$$

where each γ_i is of the form

$$x_1\ \alpha_1^*\ x_2\ \alpha_2^*\ \ldots\ x_s\ \alpha_s^*\ x_{s+1}, \qquad (1)$$

where $x_i \in \mathcal{S}^*$ and $0 \leqslant h(\alpha_i) \leqslant n-1$, for each i.

Lemma 2. If $|\gamma_1 V \gamma_2 V \ldots V \gamma_m|$ has property Φ_k, then for some j, $|\gamma_j|$ has property Φ_k. If $|x_1\ \alpha_1^*\ \ldots\ x_s\ \alpha_s^*\ x_{s+1}|$ has property Φ_k, then so does some $|\alpha_j^*|$.

Proof. The first statement is clear. Let $\gamma = x_1\alpha_1^* \ldots x_s\alpha_s^* x_{s+1}$. If we suppose that $|\gamma|$ has Φ_k, then clearly each $|\alpha_j^*|$ satisfies (ii) and (iii) for the definition of Φ_k. Now each of the x_j's is a word and hence of finite length. Thus, since $|\gamma|$ contains arbitrarily long subwords of a certain type, i.e., $|\gamma|$ contains k-subwords for arbitrarily large n, at least one of the $|\alpha_j^*|$ must also have this property.

Definition. Let P_k be the following proposition: If Ω is regular and if Ω satisfies Φ_k, then $h(\Omega) \geqslant k$.

We shall show by induction in the next section that P_k is true for all k. To simplify this proof, we now prove a lemma.

Lemma 3. Let $k \geqslant 1$ be an integer and suppose that proposition P_k is true. If $h(\Sigma) \leqslant k-1$ and if Σ^* has Φ_k relative to $\mathcal{S}_k$, then Σ contains a word on the letters of $\mathcal{S}_k$.

Proof. Since P_k is true, there clearly exists an N such that Σ contains no k-subword for exponent n. (Note that all k-words are relative to $\mathcal{S}_k$.) Recall that by a previous remark Σ^2 contains no k-subword for exponent 2N. Suppose now that Σ contains no word on $\mathcal{S}_k$. Then any word of Σ, and hence Σ^2, which contains the k-subword t is of the form xty where at least one of x and y contains a letter not in $\mathcal{S}_k$. If further $xty \in \Sigma^2$ and t is for exponent N, then both x and y contain a letter not in $\mathcal{S}_k$. This follows since xty must be of the form $x't_1t_2y'$ where $x't_1$, $t_2y' \in \Sigma$ and x' and y' contain letters not in $\mathcal{S}_k$. Thus Σ^3 also contains no k-subword for exponent 2N. By induction we obtain that, for all n, Σ^n contains no k-subword for exponent 2N so A* also has this property. But this contradicts the fact that A* has Φ_k.

3. THE MAIN RESULT

In this section we prove the main result and give some examples.

Theorem. P_k is true for all positive integers k.

Proof. The proof is by induction. If Ω satisfies Φ_1, then clearly Ω is infinite so $h(\Omega) \geqslant 1$ and this establishes P_1.

Suppose then that P_k is true. Let Ω have property Φ_{k+1} relative to $\mathcal{S}_{k+1}$, but suppose $h(\Omega) \leqslant k$. Since Ω also has Φ_k, we have that $h(\Omega) = k$. Let ω denote the set Ω. By Lemma 1 we may assume $\omega = \gamma_1 V \ldots V\gamma_m$ where each γ_i has the form (1). By Lemma 2, some γ_i, say γ_1, has Φ_{k+1}. Let $\gamma_1 = x_1\alpha_1^* \ldots x_s\alpha_s^*x_{s+1}$ where each $x_i \in \mathcal{S}^*$ and $h(\alpha_i) \leqslant k\text{-}1$. By Lemma 2, some α_i^*, say d_r^*, has property Φ_{k+1}. Hence α_r^* has Φ_k relative to $\mathcal{S}_k$ and also Φ_k relative to $\mathcal{T}_k$. Thus by Lemma 3, since $h(|\alpha_r|) \leqslant h(\alpha_r) \leqslant k\text{-}1$, we have that $|\alpha_r|$ contains a word, say a, on the letters of $\mathcal{S}_k$ and $|\alpha_r|$ also contains a word, say b, on the letters of $\mathcal{T}_k$. Then $ba \in |\alpha^*|$ so there exists $A_p \in \mathcal{T}_k$ and $A_q \in \mathcal{S}_k$ such that $A_pA_q \in |\alpha_r^*| \subseteq \overline{\Omega}$. But this contradicts condition (iii) of property Φ_{k+1}. Therefore $h(\Omega) \geqslant k+1$. This completes the proof of the theorem.

Corollary 1. For any positive integer k, there exists a regular set of *-height k.

Proof. Let $\beta_1 = A_1$, let $\beta_2 = (A_1^* A_2^* A_3)$, and let $\gamma_1 = A_2$, $\gamma_2 = (A_4^* A_5^* A_6)$. Then define β_k and γ_k inductively by

$$\beta_k = (\beta_{k-1}^* \gamma_{k-1}^* A_{2^k-1}),$$

and

γ_k is obtained from β_k by adding 2^k-1 to each subscript. Thus γ_k is on the letters of T_k. The sets $|\beta_k^*|$ clearly satisfy Φ_k.

We would next like to note one obvious generalization of property Φ_k for which P_k is also true.* We may replace the occurrence of the A_j in the definition of a k-word by a word, say b_k, on the letters of $\mathcal{S}$, provided of course condition (iii) is not violated. Thus, for example, we would have the following 3-word for exponent 5 $\left(\left(A_1^5 A_2^{10}(A_2A_3A_1A_3)\right)^6\left(A_4^{11}A_5^9A_6A_4A_6A_5A_6\right)^{100}\left(A_4A_6A_7A_1A_2\right)\right)^{10}$, and a typical expression would be $\left((A_1^* A_2^* b_2)^*(A_4^* A_5^* c_2)^* b_3\right)^*$. Note that (iii) requires the words b_k and c_k to contain the letters A_j and A_{2j}, respectively, where $j = 2^k-1$.

Before proving the final corollary, we prove a lemma which may be of some independent interest. Recall that regular events form a Boolean algebra. We use $\sim$ and $\cap$ to denote relative complement and intersection, respectively.

Lemma 4. Let α and β be regular expressions. Then

$$|(\alpha \vee \beta)^*| \sim |\alpha\alpha^*| = |(\alpha^*\beta\alpha^*)^*| \qquad (2)$$

if and only if

$$|(\alpha^*\beta\alpha^*)^*| \cap |\alpha\alpha^*| = \phi. \qquad (3)$$

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*It is also clear that by an appropriate coding one can obtain a regular set of arbitrarily high *-height on the two letter alphabet {0,1}.

Proof. (2) $\Longrightarrow$ (3) is trivial. Suppose that (3) holds. Then clearly $|(\alpha^*\beta\alpha^*)^*| \subseteq |(\alpha V\beta)^*| \sim |\alpha\alpha^*|$. Let the word a be in $|(aV\beta)^*| \sim |\alpha\alpha^*|$. Then a must be of the form a_1ba_2 where a_1, $a_2 \in |(\alpha V\beta)^*|$ and $b \in |\beta|$. But clearly such a word $a_1ba_2 \in |(\alpha^*\beta\alpha^*)^*|$.

Corollary. If α and β are regular expressions, if $\theta \notin |\alpha|$ and if no element of $|\beta|$ is a subword of a word of $|\alpha|$, i.e., if $|\beta| \cap \overline{|\alpha|} = \phi$, then $|(\alpha V\beta)^*| \sim |\alpha\alpha^*| = |(\alpha^*\beta\alpha^*)^*|$.

We shall call an application of a star to a regular expression σ (or regular event Σ) non-trivial in case for all positive integers n, $\Sigma^n \neq \Sigma^*$, where $\Sigma = |\sigma|$. We now know by Corollary 1 that there are expressions for which the application of a star is non-trivial. However, as the next corollary shows, an application of the star may be non-trivial and yet not raise the *-height of the regular set.

Corollary 2. For every positive integer k, there is a regular event Σ_k, of star height k for which the application of the star is non-trivial, but such that Σ_k^* also has height k.

Proof. For k = 1, let $\sigma_1 = A_1^*A_2$ in which case $|\sigma_1^*| = |(A_1VA_2)^*A_2V\theta|$. For $k \geq 1$, let β_k and γ_k be as in Corollary 1, and let $j = 2^{k+1}-1$. Define

$$\Sigma_{k+1} = |\beta_{k+1}^*| \sim |A_j^2A_j^*|$$

and let

$$\pi_k = \beta_k\beta_k^*A_jV\gamma_k\gamma_k^*A_jV\beta_k\beta_k^*\gamma_k\gamma_k^*A_j$$

so that $h(\Sigma_{k+1}) \geq k+1$, by the theorem, and $h(\pi_k) = k$. Now $|\beta_{k+1}| = |\beta_k^*\gamma_k^*A_j| = |A_jV\beta_k\beta_k^*A_jV\gamma_k\gamma_k^*A_jV\beta_k\beta_k^*\gamma_k\gamma_k^*A_j| = |A_jV\pi_k|$ (simply replace β_k^* and γ_k^* by $(\theta V\beta_k\beta_k^*)$ and $(\theta V\gamma_k\gamma_k^*)$ and multiply out). Thus by applying the Corollary to Lemma 4, we obtain

$$|\beta^*_{k+1}| \sim |A_j A^*_j| = |(A^*_j \pi_k A^*_j)^*|.$$

Hence

$$\Sigma_{k+1} = |\beta^*_{k+1}| \sim |A^2_j A^*_j| = |(A^*_j \pi_k A^*_j)^* V A_j|$$

and $h\left((A^*_j \pi_k A^*_j)^* VA\right) = k+1$ so $h(\Sigma_{k+1}) = k+1$.

Note that the example Σ_1 given in this Corollary has the properties:

(1) $\Sigma^j_1 \cap \Sigma^i_1 = \phi$ for $i \neq j$

and

(2) For any regular event Ω, $h(\Omega) < h(\Sigma^*_1) \Rightarrow \Omega^* \neq \Sigma^*_1$.

The first follows simply because every word in Σ^j_1 contains exactly j occurrences of the letter A_2. The second follows because Σ^*_1 satisfies Φ_1 for the letters A_1 and A_2 so that if $h(\Omega) = 0$ and $\Omega^* = \Sigma^*_1$ then Ω must contain a word on the letter A_1, by Lemma 3, which is a contradiction.

Thus we have an example of an event Σ^* of height 1 which satisfies (2). I do not know of any examples of sets of greater height which also have this property.

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