Cutpoints in the conjunction of two graphs

By

Paul Hafner and Frank Harary

1. Definitions and examples. Let $G = (V, E)$ be a graph with vertex set V and edge set E; similarly let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The *conjunction* $G = G_1 \wedge G_2$ is defined by $V = V_1 \times V_2$ and $\{u, v\} = \{(u_1, u_2), (v_1, v_2)\} \in E$ if and only if

$$\{u_1, v_1\} \in E_1 \quad \textit{and} \quad \{u_2, v_2\} \in E_2 .$$

This binary operation on graphs was introduced by Weichsel [6] and later termed conjunction by Harary and Wilcox [4].

Weichsel proved that $G_1 \wedge G_2$ is connected if and only if both G_1 and G_2 are connected and one of them contains an odd cycle. If both G_1 and G_2 are connected and bipartite, their conjunction G consists of two connected components constructed as follows. Color both V_1 and V_2 red and green. Then one component of G contains all vertices (u_1, u_2) where u_1, u_2 have the same color and the other component contains the pairs of vertices of opposite color.

We now develop some definitions and examples leading to a criterion for $G = G_1 \wedge G_2$ to be 2-connected, in other words, nonseparable or a block. Following the notation of [2], we write C_n for the cycle of length n, and P_n for the path with n vertices which we now label $1, 2, 3, \ldots, n$ starting at one endpoint and moving to the other. If u is a point of graph H we will denote the points of $H \wedge P_n$ by $u1$, $u2$, etc. Terminology not defined here may be found in [2].

Weichsel remarked that the conjunction $K_{1,r} \wedge K_{1,s}$ of two stars has the two connected components $K_{1,rs}$ and $K_{r,s}$. One acquires familiarity with the operation

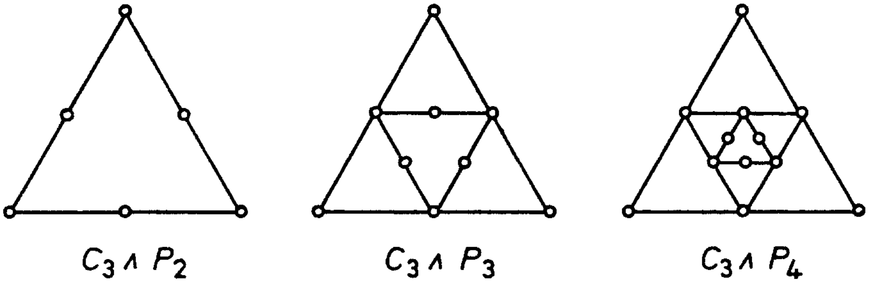

Fig. 1. Three conjunctions of a triangle and a path

by verifying that $C_n \wedge P_2 = C_{2n}$ if n is odd and $H \wedge P_2 = 2H$ if H is bipartite; it is also instructive to construct $C_n \wedge K_{1,m}$. As another illustration Figure 1 shows $C_3 \wedge P_m$ for $m = 2, 3, 4$. One sees at once that the conjunction of a cycle and a path has no cutpoints (although it may be disconnected as $C_{2n} \wedge P_3$).

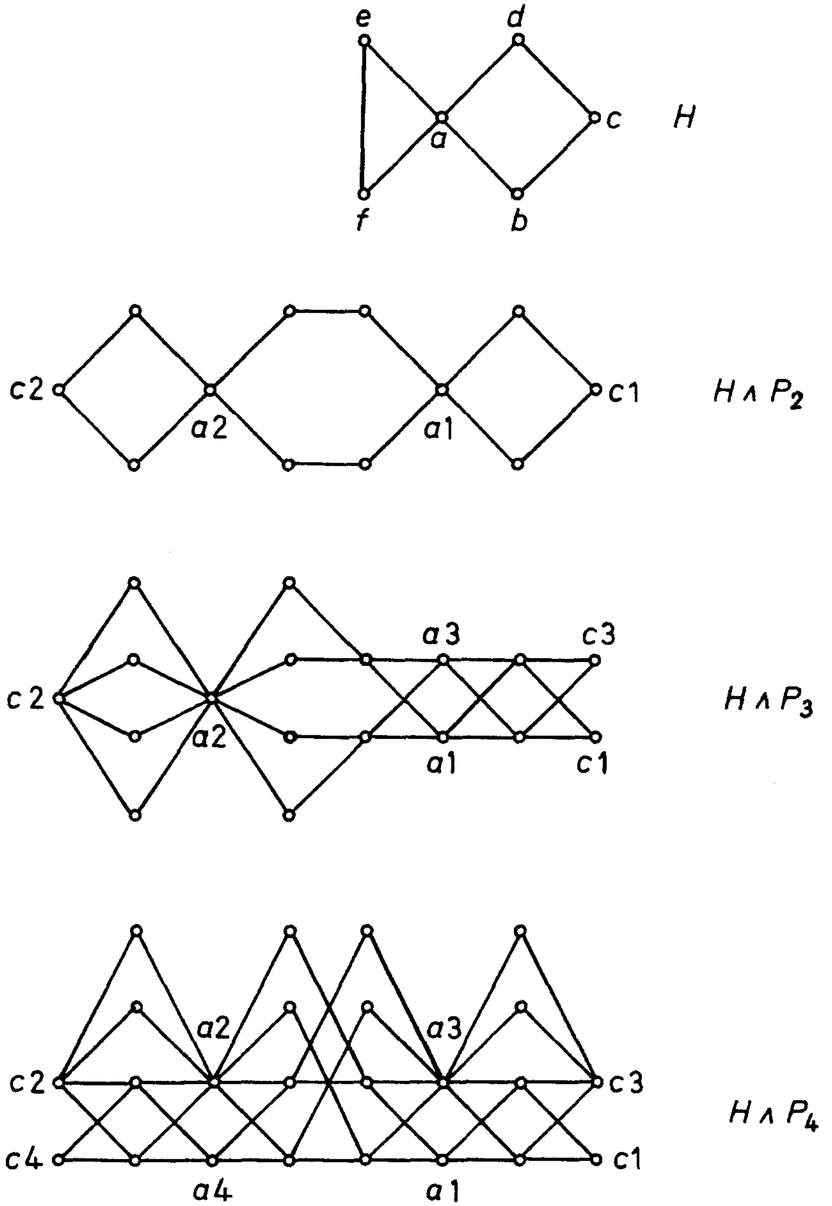

Fig. 2. Three conjunctions of a graph with a path

We will find that the typical behaviour regarding the occurrence of cutpoints in a conjunction is seen in the diagrams of Figure 2 which show $H \wedge P_2$, $H \wedge P_3$, $H \wedge P_4$ when $H = C_3 \cdot C_4$ is the graph obtained by identifying a point of C_3 with a point of C_4.

If u is a cutpoint of the connected graph H then the removal of u disconnects H, leaving a number of connected graphs $H_1, \ldots, H_r$. The subgraphs $F_1, F_2, \ldots, F_r$ of H induced by u and H_1, u and $H_2, \ldots$ are called the branches of the cutpoint u. Note that u is not a cutpoint of any F_i.

2. A necessary condition.

Proposition. *Let G_1 and G_2 be connected graphs and assume $G = G_1 \wedge G_2$ is connected. If (x, y) is a cutpoint of G then x is a cutpoint of G_1 and y is a cutpoint of G_2.*

Proof. Denote by F one of the branches of (x, y) in G and by H the union of the remaining branches of (x, y). Let $(a_1, b_1), \ldots, (a_r, b_r)$ be the vertices of F adjacent to (x, y) in G and $(c_1, d_1), \ldots, (c_s, d_s)$ the remaining vertices adjacent to (x, y) in G. We shall prove that in G_1 any path from a_i to c_j goes through x provided a_i is not terminal, and similarly in G_2. (If a_i is terminal then obviously x is a cutpoint of G_i.) So we assume that there exists a point $a_i' \in V(G_1)$ adjacent to a_i with $a_i' \neq x$. Then (x, y), (a_i, b_i), (a_i', y), (a_i, d_j) is a path in G which avoids (x, y) and hence $(a_i, d_j) \in V(F)$; this proves that $a_i \neq c_j$ for all j. Now let $a_i, p_1, p_2, \ldots, p_t, c_j$ be a path in G_1 which avoids x. If t is odd we get a path (a_i, b_i), (p_1, y), $(p_2, d_j), \ldots, (p_t, y)$, (c_j, d_j) in G which contradicts the fact that in G the point (c_j, d_j) can only be reached from (a_i, b_i) via (x, y). If t is even then G_1 contains an odd C: $x, a_i, p_1, \ldots, p_t, c_j$. We now consider the path b_i, y, b_j or b_i, y in G_2 depending on whether $b_i \neq b_j$ or $b_i = b_j$ and denote it by P_3 or P_2 respectively. Clearly $C \wedge P_3$ and $C \wedge P_2$ have no cutpoints as seen earlier in Figure 1. Hence there is a path in G connecting (a_i, b_i) and (c_j, b_j) which avoids (x, y). As this is a contradiction, it follows that every path in G_1 joining a_i and b_j meets x, i.e., x is a cutpoint of G_1. If b_i is not terminal (in which case y is trivially a cutpoint) one finds that y is a cutpoint of G_2 in the same fashion.

Now we know that if we want to find a cutpoint of G it is sufficient to consider the pairs of cutpoints of G_1 and G_2. The following section shows, however, that this produces cutpoints only in rare cases.

3. The scarcity of cutpoints in G.

Lemma. *Let G_1 be connected and not bipartite and assume x is a cutpoint of G_1.*

(a) *$x1$ is a cutpoint of $G_1 \wedge P_2$ if and only if x has a branch which is bipartite.*

(b) *$x1$ and $x3$ are not cutpoints of $G_1 \wedge P_3$. Also $x2$ is a cutpoint of $G_1 \wedge P_3$ if and only if x has a branch which is bipartite.*

(c) *$x3$ is a cutpoint of $G_1 \wedge P_4$ if and only if x is adjacent to a terminal vertex.*

Proof. Clearly (a) is implied by (b); also (a) follows from the construction given in Figures 1 and 2.

We now prove (b). As 1 and 3 are not cutpoints of P_3, the Proposition implies that $x1$ and $x3$ cannot be cutpoints of $G_1 \wedge P_3$. To study $x2$, let $X_1, X_2, \ldots, X_n$ be the branches of x in G_1 and recall that x is not a cutpoint of any of them. If X_i is bipartite then $X_i \wedge P_3$ has 2 connected components, one containing $x1$ and $x3$ and the other containing $x2$. The removal of $x2$ will disconnect the latter component from the rest of $G_1 \wedge P_3$. If every branch X_i contains an odd cycle then all $X_i \wedge P_3$ are connected and $x2$ is not a cutpoint of any of them (since x is not a cutpoint of any X_i). Now $G_1 \wedge P_3$ can be obtained by identifying the vertices $x1$ in $X_1 \wedge P_3$, $X_2 \wedge P_3$ etc., and the same with $x2$ and $x3$. When the vertex $x2$ is removed the remainder of the $X_i \wedge P_3$ still hold together thanks to $x1$ and $x3$.

To prove (c), if x is adjacent to the terminal vertex u then $u4$ is terminal and adjacent to $x3$ in $G_1 \wedge P_4$; hence $x3$ is a cutpoint of $G_1 \wedge P_4$. If on the other hand all vertices adjacent to x are nonterminal we show that $x3$ is not a cutpoint. Assume u and w are adjacent to x, and $v \neq x$ is adjacent to u, $z \neq x$ is adjacent to w. Typical paths across $x3$ are

(i) $u2,\ x3,\ u4,$
(ii) $u2,\ x3,\ w2,$
(iii) $u2,\ x3,\ w4.$

But one can always bypass $x3$ as follows:

(i) $u2,\ v3,\ u4,$
(ii) $u2,\ x1,\ w2,$
(iii) $u2,\ x1,\ w2,\ z3,\ w4.$

We may summarize these observations by saying that the connected conjunction $G = G_1 \wedge G_2$ has no cutpoints provided G_1 has an odd cycle, every vertex of G_2 is contained in a path P_4, and not both G_1 and G_2 contain terminal vertices. The latter condition is a trivial one and is therefore incorporated into the hypothesis of the following criterion.

Theorem. *Let G_1 and G_2 be both connected, not both bipartite, and not both containing terminal vertices. Then $G = G_1 \wedge G_2$ has a cutpoint if and only if one of G_1 and G_2 is a star and the other has a bipartite block.*

Proof. Assume G has a cutpoint and G_1 contains an odd cycle. The remark just before the theorem excludes any possibility apart from G_2 being a star $K_{1,n}$, $n \geqq 1$ (since G_2 must have a cutpoint and no P_4). The Lemma then forces the condition that G_1 has a bipartite block, as G_1 must contain a cutpoint having a bipartite branch.

If, on the other hand, G_2 is a star and G_1 contains a vertex with a bipartite branch, and (since G is assumed connected) also contains an odd cycle, then the proof of part (b) of the Lemma implies the existence of a cutpoint in G.

4. Unsolved problems.

A. Connectivity. Weichsel [6] found the conditions for $G = G_1 \wedge G_2$ to be connected. In the Theorem above, we derived a criterion for G to be 2-connected. When is G n-connected?

B. Digraphs. The conjunction of two digraphs was defined by McAndrew [5]; its connectedness categories were determined in Harary and Trauth [3]. What is the generalization of our Theorem to digraphs?

C. Factorization under conjunction. Some binary operations on graphs enjoy a unique factorization, but the conjunction does not. This is seen at once by the example $G = K_{2,2} \wedge P_3 \cong K_{2,4} \wedge P_2$. There exist also examples of connected graphs with nonunique factorization. Dörfler [1] has shown how to obtain all "prime" factorizations from a given one. Which graphs have a unique conjunction-factorization?

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Anschrift der Autoren:

Paul Hafner
Department of Mathematics
University of Auckland
Auckland, New Zealand

Frank Harary
Department of Mathematics
University of Michigan
Ann Arbor, Michigan, U.S.A.