

A.P.O. Rules are Asymptotically non Deficient for Estimation with Squared Error Loss

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Summary. The problem considered is sequential estimation of the mean θ of a one-parameter exponential family of distributions with squared error loss for estimation error and a cost $c > 0$ for each of an i.i.d. sequence of potential observations X_1, X_2, \dots . A Bayesian approach is adopted, and natural conjugate prior distributions are assumed. For this problem, the asymptotically pointwise optimal (A.P.O.) procedure continues sampling until the posterior variance of θ is less than $c(r_0 + n)$, where n is the sample size and r_0 is the fictitious sample size implicit in the conjugate prior distribution. It is known that the A.P.O. procedure is Bayes risk efficient, under mild integrability conditions. In fact, the Bayes risk of both the optimal and A.P.O. procedures are asymptotic to $2V_0\sqrt{c}$, as $c \rightarrow 0$, where V_0 is the prior expectation of the standard deviation of X_1 , given θ . Here the A.P.O. rule is shown to be asymptotically non-deficient, under stronger regularity conditions: that is, the difference between the Bayes risk of the A.P.O. rule and the Bayes risk of the optimal procedure is of smaller order of magnitude than c , the cost of a single observation, as $c \rightarrow 0$. The result is illustrated in the exponential and Bernoulli cases, and extended to the case of a normal distribution with both the mean and variance unknown.

I. Conjugate Prior Distributions for Exponential Families

Let Ω be an interval and let F_ω , $\omega \in \Omega$, denote a non degenerate exponential family of probability distributions on the Borel sets of $(-\infty, \infty)$: that is, suppose that

$$dF_\omega(x) = \exp\{\omega x - \psi(\omega)\} d\lambda(x) \quad +\infty < x < \infty, \quad \omega \in \Omega.$$

Here Ω denotes the natural parameter space of the family; and Ω is assumed to be open, say $\Omega = (\underline{\omega}, \bar{\omega})$, where $-\infty \leq \underline{\omega} < \bar{\omega} \leq \infty$. The distributions F_ω , $\omega \in \Omega$,

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are mutually absolutely continuous. Their common closed convex support is denoted by X ; and the interior of X is denoted by X^0 . It is well known that if X is a random variable with distribution F_ω , where $\omega \in \Omega$, then the mean and variance of X are $E_\omega(X) = \psi'(\omega)$ and $D_\omega(X) = \psi''(\omega)$. The mean is of special interest here and is denoted by

$$\theta = \psi'(\omega),$$

throughout. It is easily seen that $\theta = \psi'$ is a strictly increasing function from Ω onto X^0 , so the distributions F_ω , $\omega \in \Omega$, could be parameterized by θ as well as ω . See Lehmann (1959, Sect. 2.7) for the elementary properties of exponential families, and Barndorff-Nielson (1978) for a more detailed discussion.

Diaconis and Ylvisaker (1979) have given an interesting characterization of conjugate prior distributions for exponential families. If $r_0 > 0$ and $\mu_0 \in X^0$, then

$$0 < c(r_0, \mu_0) = \int_{\Omega} \exp \{r_0 \mu_0 \omega - r_0 \psi(\omega)\} d\omega < \infty;$$

so,

$$\xi_0(\omega) = (1/c(r_0, \mu_0)) \exp \{r_0 \mu_0 \omega - r_0 \psi(\omega)\}, \quad \omega \in \Omega,$$

defines a density w.r.t. Lebesgue measure on the Borel sets of Ω . The corresponding distribution is denoted by π_0 , so that $d\pi_0(\omega) = \xi_0(\omega) d\omega$, $\omega \in \Omega$. It is then the case that

$$E(\theta) = \int_{\Omega} \psi' d\pi_0 = \mu_0. \tag{1}$$

When applied to the posterior distribution of ω , given an observation $X \sim F_\omega$, Eq. (1) asserts that the posterior expectation $E(\theta|X)$ is linear in X ; and this property is characteristic of prior distributions of the form π_0 , under some additional conditions on X .

The following simple extension of (1) is central to the analysis of asymptotically pointwise optimal rules: if

$$E[\psi''(\omega)] = \int_{\Omega} \psi'' d\pi_0 < \infty,$$

then

$$E[(\theta - \mu_0)^2] = r_0^{-1} E[\psi''(\omega)]. \tag{2}$$

Equation (2) follows from Lemma 1, below, applied to $g(\omega) = (\theta - \mu_0)$, $\omega \in \Omega$.

Lemma 1. *If g is a continuously differentiable function on Ω for which*

$$\int_{\Omega} |g'| d\pi_0 < \infty, \tag{3}$$

then

$$\int_{\Omega} g \xi'_0 d\omega = - \int_{\Omega} g' \xi_0 d\omega.$$

If $\limsup_{\omega \rightarrow \underline{\omega}} g(\omega) < 0 < \liminf_{\omega \rightarrow \bar{\omega}} g(\omega)$, then $|g'|$ may be replaced by $g'_+ = \max \{0, g'\}$ in condition (3).

Proof. It suffices to show that $(g \xi_0)(\omega) \rightarrow 0$ as either $\omega \rightarrow \underline{\omega}$ or $\omega \rightarrow \bar{\omega}$. Diaconis and Ylvisaker (1979) show that $\xi_0(\omega) \rightarrow 0$ as $\omega \rightarrow \underline{\omega}$ or $\omega \rightarrow \bar{\omega}$. This fact is used.

Let $\hat{\omega}_0$ be the prior mode - that is $\psi'(\hat{\omega}_0) = \mu_0$; and observe that $\xi_0(\omega)$ is increasing in $\omega < \hat{\omega}_0$ and decreasing in $\omega > \hat{\omega}_0$. If $\hat{\omega}_0 < \omega_0 < \omega_1 < \bar{\omega}$, then

$$\begin{aligned}
 |g(\omega_1)| \xi(\omega_1) &\leq \left\{ |g(\omega_0)| + \int_{\omega_0}^{\omega_1} |g'(\omega)| d\omega \right\} \xi_0(\omega_1) \\
 &\leq |g(\omega_0)| \xi_0(\omega_1) + \int_{\omega_0}^{\omega_1} |g'(\omega)| \xi_0(\omega) d\omega.
 \end{aligned}
 \tag{4}$$

So,

$$\limsup_{\omega \rightarrow \bar{\omega}} |g(\omega)| \xi_0(\omega) \leq \int_{\omega_0}^{\bar{\omega}} |g'(\omega)| \xi_0(\omega) d\omega,$$

which tends to zero as $\omega_0 \rightarrow \bar{\omega}$. The behavior of $g\xi_0$ at the lower end-point may be analyzed similarly.

If $g(\omega) > 0$ for ω in some neighborhood of $\bar{\omega}$ and only g'_+ is assumed to be integrable w.r.t. π_0 , then (4) holds with $|g'|$ replaced by g'_+ , provided that ω_0 is sufficiently close to $\bar{\omega}$. The remainder of the proof proceeds as above.

II. Sequential Estimation

In this section and the next, π_0 denotes a conjugate prior distribution with $r_0 > 0$ and $\mu_0 \in X^0$; and ω and X_1, X_2, \dots are random variables, defined on some probability space $(\mathcal{S}, \mathcal{A}, P)$, for which $\omega \sim \pi_0$ and X_1, X_2, \dots are conditionally i.i.d. with common distribution F_ω , given ω . Let $\mathcal{A}_0 = \{\phi, \mathcal{A}\}$ be the trivial sigma-algebra and let $\mathcal{A}_n = \sigma\{X_1, \dots, X_n\}$ be the sigma-algebra generated by X_1, \dots, X_n for $n \geq 1$. Then the conditional, or posterior, distribution of ω given \mathcal{A}_n is

$$d\pi_n(\omega) = (1/c(r_n, \mu_n)) \exp\{r_n \mu_n \omega - r_n \psi(\omega)\} d\omega, \quad \omega \in \Omega, \tag{5}$$

where

$$r_n = r_0 + n \quad \text{and} \quad \mu_n = (r_0 \mu_0 + S_n)/r_n$$

with $S_n = X_1 + \dots + X_n$ for $n \geq 1$.

Recall that a stopping time is a random variable t which takes the values $0, 1, 2, \dots$ and ∞ and has the properties $P\{t < \infty\} = 1$ and $\{t = n\} \in \mathcal{A}_n$ for all $n \geq 0$. The sigma-algebra \mathcal{A}_t of events which occur prior to a stopping time t is the class of all $A \in \mathcal{A}$ for which $A \cap \{t = n\} \in \mathcal{A}_n$ for all $n \geq 0$. Then, it is easily seen that the conditional distribution of ω given \mathcal{A}_t is simply π_t on $\{t < \infty\}$.

Now suppose that $E[\psi''(\omega)] < \infty$, so that θ has a finite variance, and consider the problem of sequentially estimating θ with squared error loss for estimation error and a cost $c > 0$ for each observation X_1, X_2, \dots . The Bayesian sequential decision problem is then to find a stopping time t and an \mathcal{A}_t -measurable function $\hat{\theta}_t = \hat{\theta}_t(X_1, \dots, X_t)$ for which

$$\text{Bayes risk}(t, \hat{\theta}_t) = E\{(\hat{\theta}_t - \theta)^2 + ct\}$$

is minimized. It is well known that for any stopping time t , the Bayes risk is minimized by letting $\hat{\theta}_t = \mu_t = E(\theta | \mathcal{A}_t)$. See, for example, Ferguson (1967, pp. 314-15). Moreover, it follows from (2) that, for any stopping time t ,

$$E\{(\theta - \mu_t)^2\} = E\{E[(\theta - \mu_t)^2 | \mathcal{A}_t]\} = E\{r_t^{-1} U_t\},$$

where

$$U_n = E[\psi''(\omega) | \mathcal{A}_n] = \int_{\Omega} \psi'' d\pi_n, \quad n \geq 0.$$

It is relevant that $U_n, n \geq 0$, is a uniformly integrable martingale for which $U_n \rightarrow \psi''(\omega)$ w.p.1 as $n \rightarrow \infty$. Next let

$$L_n(c) = r_n^{-1} U_n + cr_n, \quad n \geq 0, \quad c > 0.$$

Then

$$\text{Bayes risk}(t, \mu_t) = E\{L_t(c)\} - cr_0, \quad c > 0, \tag{6}$$

for any stopping time t ; so, minimizing the Bayes risk is equivalent to minimizing $E\{L_t(c)\}$ by choice of t .

It follows from Theorems 4.4 and 4.5 of Chow, Robbins and Siegmund (1970) that an optimal stopping time, one which minimizes $E\{L_t(c)\}$, exists for each $c > 0$. In principle, the optimal stopping time may be computed by backward induction and a limiting operation; but the exact determination of the optimal stopping time appears to be a formidable task, in practice. Bickel and Yahav (1967, 1968, 1969a) have developed an interesting class of asymptotic solutions to the minimization problem, as $c \rightarrow 0$. They describe methods for finding stopping times $t = t_c$ which are asymptotically pointwise optimal (A.P.O.) in the sense that

$$L_t(c) \sim \inf_n L_n(c) \sim 2\sqrt{[c\psi''(\omega)]}$$

w.p.1 as $c \rightarrow 0$. Replacing n by r_n in Bickel and Yahav's (1967) A.P.O. rule suggests the stopping times

$$t = t_c = \inf\{n \geq 0: U_n < cr_n^2\}, \quad c > 0.$$

In fact, a careful reading of their derivation yields the following: if $s = s_c$ are stopping times, then, as $c \rightarrow 0$,

$$L_s(c) \sim \inf_n L_n(c) \quad \text{iff} \quad s_c \sim t_c, \tag{7}$$

except for a set of probability zero.

In the next section, the A.P.O. rule $t = t_c$ of (6) is shown to be asymptotically optimal in the following strong sense: let

$$\rho_0(c) = \inf_s E[L_s(c)], \quad c > 0, \tag{8}$$

where the infimum extends over all stopping times s ; then $E\{L_t(c)\} = \rho_0(c) + o(c)$, as $c \rightarrow 0$, under a mild integrability condition. That is, the A.P.O. rule t is asymptotically non-deficient in the sense that the additional cost which results from using the sub-optimal procedure t is of smaller order of magnitude than the cost of a single observation. Bickel and Yahav (1968) have shown that $E\{L_t(c)\}/\rho_0(c) \rightarrow 1$ in a context which is more general than the present one; and Alvo (1977) has shown that the deficiency of a closely related procedure is of

order c , under conditions which are related to the present ones but are not strictly comparable. The representation (6) for the Bayes risk is implicit in his examples. Shapiro and Wardrop (1980a) have used a representation which is similar to (6) in their study of the myopic procedure. They consider only the term of order \sqrt{c} in the asymptotic expansion, however. Finally, Vardi (1979) has recently given an interesting discussion of Bayes procedures for diffuse priors.

3. Asymptotic Optimality

The proof of asymptotic optimality depends on the simple identity,

$$L_n(c) = r_n^{-1} U_n + c r_n = 2\sqrt{cU_n} + r_n^{-1} [\sqrt{U_n} - r_n \sqrt{c}]^2 \tag{9}$$

for $n \geq 0$ and $c > 0$. It is shown below that the last term on the right side of (9) contributes only $o(c)$ to the expected loss for both the optimal and A.P.O. rules. To understand the first, let

$$V_n = E[\sqrt{\psi''(\omega)} | \mathcal{A}_n] \quad \text{and} \quad W_n = U_n - V_n^2, \quad n \geq 0.$$

Then $V_n, n \geq 0$, is a uniformly integrable martingale; so,

$$E[V_s] = V_0 = \int_{\Omega} \sqrt{\psi''} d\pi_0$$

for every stopping time s , by the optional stopping theorem. (See, for example, Theorem 2.3 of Chow, Robbins, and Siegmund (1970).) Observe that W_n is the posterior variance of $\sqrt{\psi''(\omega)}$ given \mathcal{A}_n and that $0 < W_n \rightarrow \psi''(\omega) - \psi''(\omega) = 0$ w.p.l as $n \rightarrow \infty$. Now

$$\sqrt{U_n} = V_n + \frac{W_n}{\sqrt{U_n} + V_n}, \quad n \geq 0,$$

so

$$E[\sqrt{U_s}] = V_0 + E\left[\frac{W_s}{\sqrt{U_s} + V_s}\right] \tag{10}$$

for any stopping time s . Clearly, an expansion for $E[W_s/(\sqrt{U_s} + V_s)]$ is needed.

In the next lemma, the following result is used: let $\hat{\omega}_n$ denote the posterior mode, so that $\hat{\omega}_n$ solves the equation $\psi'(\hat{\omega}_n) = \mu_n$; as $n \rightarrow \infty$, the posterior distribution of $\sqrt{[r_n \psi''(\hat{\omega}_n)]} \cdot (\omega - \hat{\omega}_n)$ converges to the standard normal distribution w.p.l. See Bickel and Yahav (1969b) for this result in a more general context; or see Johnson (1970) for a more detailed result for exponential families.

Lemma 2. Let $\gamma(\theta) = \sqrt{\psi''(\omega)}$, where $\theta = \psi'(\omega)$; and suppose that

$$\int_{\Omega} \gamma'(\theta)^2 \psi'' d\pi_0 < \infty.$$

Then

$$r_n W_n \leq E[\gamma'(\theta)^2 \psi''(\omega) | \mathcal{A}_n], \quad \text{w.p.1, } n \geq 1,$$

and

$$r_n W_n \rightarrow \gamma'(\theta)^2 \psi''(\omega) \quad \text{w.p.1, } n \rightarrow \infty.$$

Proof. For fixed X_1, \dots, X_n , define a function g on Ω by $g(\hat{\omega}_n) = 0$ and $g(\omega) = [\gamma(\theta) - \gamma(\mu_n)]^2 / (\theta - \mu_n)$ for $\omega \neq \hat{\omega}_n$. Then

$$r_n W_n \leq r_n \int_{\Omega} [\gamma(\theta) - \gamma(\mu_n)]^2 d\pi_n = - \int_{\Omega} g \xi'_n d\omega,$$

where ξ_n denotes the posterior density, $\xi_n(\omega) = (1/c(r_n, \mu_n)) \exp\{r_n \mu_n \omega - r_n \psi(\omega)\}$, $\omega \in \Omega$. Now

$$\begin{aligned} \frac{d}{d\theta} g(\omega) &= 2\gamma'(\theta) \left[\frac{\gamma(\theta) - \gamma(\mu_n)}{\theta - \mu_n} \right] - \left[\frac{\gamma(\theta) - \gamma(\mu_n)}{\theta - \mu_n} \right]^2 \\ &= \gamma'(\theta)^2 - \left\{ \gamma'(\theta) - \left[\frac{\gamma(\theta) - \gamma(\mu_n)}{\theta - \mu_n} \right] \right\}^2 \leq \gamma'(\theta)^2 \end{aligned}$$

and

$$g'(\omega) = \frac{d}{d\theta} g(\omega) \cdot \psi''(\omega) \leq \gamma'(\theta)^2 \psi''(\omega), \quad \omega \neq \hat{\omega}_n.$$

Thus, $g'(\omega)$ is bounded above by a function which is integrable w.r.t. π_0 . Since $g(\omega) < 0$ for $\omega < \hat{\omega}_n$ and $g(\omega) > 0$ for $\omega > \hat{\omega}_n$, Lemma 1 yields

$$\begin{aligned} - \int_{\Omega} g \xi'_n d\omega &= \int_{\Omega} g' \xi_n d\omega \\ &\leq \int_{\Omega} \gamma'(\theta)^2 \psi'' d\pi_n = E[\gamma'(\theta)^2 \psi''(\omega) | \mathcal{A}_n], \quad n \geq 1. \end{aligned}$$

This establishes the first assertion of the lemma.

To establish the second assertion, first observe that $E[\gamma'(\theta) \psi''(\omega) | \mathcal{A}_n] \rightarrow \gamma'(\theta)^2 \psi''(\omega)$ w.p.1 as $n \rightarrow \infty$ by the martingale convergence theorem; so, $\limsup r_n W_n \leq \gamma'(\theta)^2 \psi''(\omega)$ w.p.1 as $n \rightarrow \infty$. The reverse inequality follows from the fact that the posterior distribution of ω is asymptotically normal with mean $\hat{\omega}_n$ and variance $1/r_n \psi''(\hat{\omega}_n)$ and Fatou's Lemma.

Lemma 3. Suppose that

$$\int_{\Omega} \gamma'(\theta)^{4\alpha} \psi'' d\pi_0 < \infty > \int_{\Omega} [\psi'' + (1/\psi'')] d\pi_0 \tag{11}$$

for some $\alpha > 1$. If $s = s_c$ are any stopping times for which $\sqrt{c} \cdot s_c \rightarrow \sqrt{\psi''(\omega)}$ w.p.1 as $c \rightarrow 0$, then

$$\liminf_{c \rightarrow 0} c^{-\frac{1}{2}} E \left[\frac{W_s}{\sqrt{U_s + V_s}} \right] \geq \frac{1}{2} \int_{\Omega} \gamma'(\theta)^2 d\pi_0; \tag{12}$$

and if $s_c = t_c$ is the A.P.O. rule, then the limit exists and there is equality in (12).

Proof. If $s = s_c$ are stopping times for which $\sqrt{c} \cdot s_c \rightarrow \sqrt{\psi''(\omega)}$ w.p.1, then

$$c^{-\frac{1}{2}} \left(\frac{W_s}{\sqrt{U_s + V_s}} \right) = \left(\frac{1}{\sqrt{c} \cdot r_s} \right) \left(\frac{r_s W_s}{\sqrt{U_s + V_s}} \right) \rightarrow \frac{\gamma'(\theta)^2 \psi''(\omega)}{2\psi''(\omega)} = \frac{1}{2} \gamma'(\theta)^2$$

w.p.1 as $c \rightarrow 0$; so (12) follows from Fatou's Lemma. Moreover, if $s = t$ is the A.P.O. rule, then $\sqrt{c} \cdot r_t \geq \sqrt{U_t}$, so that

$$\left(\frac{1}{\sqrt{c} \cdot r_t} \right) \left(\frac{r_t W_t}{\sqrt{U_t + V_t}} \right) \leq \frac{r_t W_t}{U_t} \leq \frac{E[\gamma'(\theta)^2 \psi''(\omega) | \mathcal{A}_t]}{E[\psi''(\omega) | \mathcal{A}_t]}. \tag{13}$$

Thus, it suffices to show that the right side of (13) is uniformly integrable.

Uniform integrability is demonstrated by showing that the expected α^{th} power remains bounded. In the proof let

$$Y = \psi''(\omega) / E[\psi''(\omega)] \quad \text{and} \quad Z = \gamma'(\theta)^2;$$

and let P^* be the measure defined by $dP^* = Y dP$. Then the quantity of interest may be written $E(YZ | \mathcal{A}_t) / E(Y | \mathcal{A}_t) = E^*(Z | \mathcal{A}_t)$; and

$$\begin{aligned} \int E^*(Z | \mathcal{A}_t)^\alpha dP &= \int E^*(Z | \mathcal{A}_t)^\alpha E^*(1/Y | \mathcal{A}_t) dP^* \\ &\leq \sqrt{[E^*(Z^{2\alpha}) E^*(1/Y^2)]} = \sqrt{[E(YZ^{2\alpha}) E(1/Y)]}, \end{aligned}$$

which is finite and independent of $c > 0$. Uniform integrability follows.

The asymptotic behavior of $r_t^{-1} [r_t \sqrt{c} - \sqrt{U_t}]^2$ is considered next.

Lemma 4. *Let*

$$Y_n = r_n \left[\sqrt{\left(\frac{U_{n-1}}{U_n} \right)} - 1 \right]_+^2, \quad n \geq 0.$$

If

$$E[\sup_{n \geq 1} Y_n] < \infty, \tag{14}$$

then

$$E[r_t^{-1} (r_t \sqrt{c} - \sqrt{U_t})^2] = o(c), \quad \text{as } c \rightarrow 0.$$

Proof. Clearly, $t \geq 1$ for sufficiently small $c > 0$; and for such c , one has

$$r_t^{-1} [r_t \sqrt{c} - \sqrt{U_t}]^2 \leq 2r_t^{-1} c + 2r_t^{-1} [\sqrt{U_{t-1}} - \sqrt{U_t}]^2$$

and

$$r_t^{-1} [\sqrt{U_{t-1}} - \sqrt{U_t}]^2 = r_t^{-2} U_t \cdot Y_t \leq c Y_t,$$

by definition of the A.P.O. rule t . Thus, since $t \rightarrow \infty$ w.p.1 as $c \rightarrow 0$, it suffices to show that $Y_n \rightarrow 0$ w.p.1 as $n \rightarrow \infty$. The results of Johnson (1970) imply that $U_n = \psi''(\hat{\omega}_n) + 0(1/n)$ w.p.1; and $\psi''(\hat{\omega}_n) - \psi''(\hat{\omega}_{n-1}) = \gamma^2(X_n) - \gamma^2(X_{n-1}) = o(1/\sqrt{n})$ w.p.1, since γ is continuously differentiable. The lemma follows.

Theorem 1. *Let t denote the A.P.O. rule (6). If (11) and (14) hold, then*

$$\rho_0(c) = 2V_0\sqrt{c} + c \int_{\Omega} \gamma'(\theta)^2 d\pi_0 + o(c) \tag{15}$$

and

$$E[L_t(c)] = \rho_0(c) + o(c), \quad \text{as } c \rightarrow 0.$$

Proof. It follows directly from (9) and Lemmas 3 and 4 that

$$\begin{aligned} E[L_t(c)] &= 2\sqrt{c} E[\sqrt{U_t}] + E[r_t^{-1}(r_t\sqrt{c} - \sqrt{U_t})^2] \\ &= 2\sqrt{c} \{V_0 + \frac{1}{2}\sqrt{c} \int_{\Omega} \gamma'(\theta)^2 d\pi_0 + o(\sqrt{c})\} + o(c), \end{aligned}$$

so that $E[L_t(c)]$ is given by the right side of (15).

Since $\rho_0(c) \leq E[L_t(c)]$ for all $c > 0$, it now suffices to show that $\rho_0(c)$ is at most as large as the right side of (15). Let $\tau = \tau_c$ denote the optimal stopping time - that is, $\rho_0(c) = E[L_{\tau}(c)]$, $c > 0$. Then it follows from (7) that $\sqrt{c}\tau_c \rightarrow \sqrt{\psi''(\omega)}$ w.p.1 as $c \rightarrow 0$; for otherwise one would have $\liminf c^{-\frac{1}{2}}L_{\tau}(c) \geq 2\sqrt{\psi''(\omega)}$ with strict inequality on a set of positive probability, in which case Fatou's Lemma would require that $\liminf c^{-\frac{1}{2}}E[L_{\tau}(\psi)] > 2V_0$, contradicting $\rho_0(c) \leq 2V_0\sqrt{c} + O(c)$. Thus, Lemma 3 is applicable to $s = \tau$; and, when combined with (9), Lemma 3 asserts that

$$E[L_{\tau}(c)] \geq 2\sqrt{c} E[\sqrt{U_t}] \geq 2\sqrt{c} \{V_0 + \frac{1}{2}\sqrt{c} \int_{\Omega} \gamma'(\theta)^2 d\pi_0 + o(\sqrt{c})\}$$

to complete the proof of the theorem.

The limitations imposed by the theorem's conditions are illustrated by two examples.

Example 1. Suppose that F_{ω} is the exponential distribution with unknown failure rate $|\omega|$, where $-\infty < \omega < 0$: that is, suppose that F_{ω} has density $f_{\omega}(x) = |\omega| \exp(\omega x)$ for $x > 0$ and $-\infty < \omega < 0$, w.r.t. Lebesgue measure. Then F_{ω} , $-\infty < \omega < 0$, form an exponential family with $\psi(\omega) = \log(1/|\omega|)$; and the mean and variance of F_{ω} are $\theta = \psi'(\omega) = 1/|\omega|$ and $\psi''(\omega) = 1/\omega^2 = \theta^2$ for $-\infty < \omega < 0$. In particular, $\gamma(\theta) = \sqrt{\psi''(\omega)} = \theta$, $0 < \theta < \infty$. The conjugate prior distributions are gamma distributions for $|\omega|$,

$$\xi_0(\omega) = \Gamma(a_0)^{-1} b_0^{a_0} |\omega|^{a_0-1} e^{-b_0\omega}, \quad -\infty < \omega < 0,$$

with shape parameter $a_0 = r_0 + 1 > 1$ and scale parameter $b_0 = r_0 \mu_0 > 0$.

Clearly, condition (11) is satisfied iff both θ^2 and $1/\theta^2$ are integrable w.r.t. π_0 ; and this is the case iff $a_0 > 2$. Condition (14) is also satisfied if $a_0 > 2$. Indeed, letting $S_n = X_1 + \dots + X_n$, one finds easily that

$$U_n = E(\theta^2 | \mathcal{A}_n) = (b_0 + S_n)^2 / (a_0 + n - 1)(a_0 + n - 2)$$

and

$$U_{n-1}/U_n \leq (a_0 + n - 1) / (a_0 + n - 3) = 1 + 2 / (a_0 + n - 3), \quad n \geq 1.$$

Thus, Theorem 1 is applicable under the sole assumption that $a_0 > 2$, which is necessary for θ to have a finite variance. The terms in the asymptotic expansion for $\rho_0(c)$ may be easily computed as

$$V_0 = \frac{b_0}{a_0 - 1} \quad \text{and} \quad \int_{\Omega} \gamma'(\theta)^2 d\pi_0 = 1.$$

Analoguees of this problem in continuous time have received recent attention. Shapiro and Wardrop (1980b) consider the problem when $S_n, n \geq 1$, are the arrival times of a Poisson Process which is observed continuously. They show that the infinitesimal look ahead rule is optimal in this case. Rasmussen (1980) considers the problem in which the partial sums $S_n, n \geq 1$, are replaced by a continuously observed gamma process with mean θ per unit time. She develops an optimal procedure which is closely related to the A.P.O. rule (6).

Example 2. If $X=0$ and 1 with probabilities $1-\theta$ and θ , where $0 < \theta < 1$ is unknown, then the distributions of X form an exponential family with $\omega = \log \theta - \log(1-\theta)$ and $\psi(\omega) = \log(1/(1-\theta))$. The mean and variance of X are θ and $\psi''(\omega) = \theta(1-\theta)$; so $\gamma(\theta) = \sqrt{[\theta(1-\theta)]}$, $0 < \theta < 1$. The conjugate prior distributions are beta distributions for θ ,

$$\xi^*(\theta) = \frac{\Gamma(a_0 + b_0)}{\Gamma(a_0)\Gamma(b_0)} \theta^{a_0-1} (1-\theta)^{b_0-1}, \quad 0 < \theta < 1,$$

where $r_0 = a_0 + b_0$ and $\mu_0 = a_0/(a_0 + b_0)$. In this case, the conditions of Theorem 1 are satisfied if $a_0 > 1 < b_0$; and the coefficients in the asymptotic expansion are

$$V_0 = \Gamma(a_0 + \frac{1}{2}) \Gamma(b_0 + \frac{1}{2}) / \Gamma(a_0) \Gamma(b_0) (a_0 + b_0)$$

and

$$\int_{\Omega} \gamma'(\theta)^2 d\pi_0 = \frac{1}{4} \left[\frac{b_0}{a_0 - 1} + \frac{a_0}{b_0 - 1} - 2 \right].$$

It is disappointing that Theorem 1 is not applicable to the case of a uniform prior distribution, $a_0 = 1 = b_0$. In fact, the coefficient of c in the asymptotic expansion for $\rho_0(c)$ diverges to ∞ as $a_0 \rightarrow 1$ or $b_0 \rightarrow 1$ from above. Alvo and Cabilio (1979) have shown that the A.P.O. rule has deficiency of order c when the prior distribution is uniform, but they do not determine the coefficient of c .

Theorem 1 may be specialized to the Poisson and geometric cases too.

4. The Normal Case

In this section X_1, X_2, \dots denote random variables which are i.i.d. and normally distributed, given the common mean θ and variance σ^2 ; and (θ, σ^2) is assumed to have a conjugate prior, normal-gamma distribution. That is, the conditional distribution of θ , given σ^2 , is normal with mean m_0 and variance $r_0^{-1} \sigma^2$, where $-\infty < m_0 < \infty$ and $r_0 > 0$; and the marginal distribution of σ^{-2} is

gamma with shape parameter $\frac{1}{2}a_0$ and scale parameter $\frac{1}{2}b_0$, where $a_0 > 0 < b_0$. The posterior distribution of (θ, σ^2) , given X_1, \dots, X_n , is then again a normal-gamma distribution with updated parameters

$$a_n = a_0 + n, \quad r_n = r_0 + n, \quad m_n = (r_0 m_0 + S_n)/r_n,$$

and

$$b_n = b_0 + \sum_{i=1}^n (X_i - \bar{X}_n)^2 + \frac{r_0 n}{r_n} (\bar{X}_n - m_0)^2$$

where $\bar{X}_n = S_n/n$ with $S_n = X_1 + \dots + X_n$, $n \geq 1$. The problem considered is again sequential estimation of the mean θ with a cost $c > 0$ for each observation, but with loss $|\hat{\theta} - \theta|^{2q}$ for estimation error, where $0 < q < \frac{1}{2}a_0$.

As above, a decision procedure is a pair $(t, \hat{\theta}_t)$, where t is a stopping time w.r.t. $\mathcal{A}_n = \sigma\{X_1, \dots, X_n\}$, $n \geq 1$, and $\hat{\theta}_t$ is a real valued \mathcal{A}_t -measurable function; and the Bayes risk of $(t, \hat{\theta}_t)$ is the expected value of $|\theta - \hat{\theta}_t|^{2q} + ct$. For any stopping time t , the expected loss due to estimation error is minimized by letting $\hat{\theta}_t = m_t = E(\theta | \mathcal{A}_t)$, and the conditional expected loss may be written in the form

$$E[|\theta - m_t|^{2q} | \mathcal{A}_t] = r_t^{-q} K_q U_t$$

where

$$K_q = 2^q \Gamma(\frac{1}{2} + q) / \Gamma(\frac{1}{2}),$$

$$U_n = E[\sigma^{2q} | \mathcal{A}_n] = C_q(a_n) (a_n^{-1} b_n)^q, \quad n \geq 0,$$

and

$$C_q(a) = (\frac{1}{2}a)^q \Gamma(\frac{1}{2}a - q) / \Gamma(\frac{1}{2}a), \quad a > 2q.$$

Let

$$L_n(c) = r_n^{-q} K_q U_n + cr_n \quad n \geq 0, \quad c > 0.$$

Then the Bayes risk of a sequential procedure (t, m_t) , where t is a stopping time, is $E[L_t(c)] - cr_0$; so, the problem of finding a decision procedure which minimizes the Bayes risk is equivalent to finding a stopping time which minimizes $E[L_t(c)]$. Let

$$\rho_0(c) = \inf_s E[L_s(c)], \quad c > 0,$$

where the infimum extends over all stopping times s ; and let

$$t = t_c = \inf\{n \geq 0: U_n < c(1/qK_q)r_n^{q+1}\}, \quad c > 0,$$

denote the A.P.O. rule. See Bickel and Yahav (1968).

Theorem 2. Suppose $0 < q < \frac{1}{2}a_0$. If t denotes the A.P.O. rule, then

$$\rho_0(c) = \left(\frac{q+1}{q}\right) (qK_q)^{\frac{1}{q+1}} V_0 c^{\frac{q}{q+1}} + \left(\frac{q^2}{q+1}\right) c + o(c)$$

and

$$E[L_t(c)] = \rho_0(c) + o(c), \quad \text{as } c \rightarrow 0,$$

where

$$V_0 = E[\sigma^{2p}] = C_p(a_0) (a_0^{-1} b_0)^p$$

with

$$p = q/(1 + q).$$

Proof. For fixed n , a Taylor series expansion of $L_n(c)$ about

$$r(c, U_n) = c^{q+1} (qK_q U_n)^{\frac{1}{q+1}}$$

yields

$$L_n(c) = L_{n,1}(c) + L_{n,2}(c),$$

with

$$L_{n,1}(c) = c^{\frac{q}{q+1}} \left(\frac{q+1}{q} \right) (qK_q U_n)^{\frac{1}{q+1}}$$

and

$$L_{n,2}(c) = \frac{1}{2} q(q+1) K_q U_n \left(\frac{1}{r_n^*} \right)^{q+2} [r_n - r(c, U_n)]^2,$$

where r_n^* denotes an intermediate point between r_n and $r(c, U_n)$. The proof of Theorem 2 now proceeds along the same lines as that of Theorem 1. The details are omitted.

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