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Construction of Markov processes and associated multiplicative functionals from given harmonic measures

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Summary. Let E be a noncompact locally compact second countable Hausdorff space. We consider the question when, given a family of finite nonzero measures on E that behave like harmonic measures associated with all relatively compact open sets in E (i.e. that satisfy a certain consistency condition), one can construct a Markov process on E and a multiplicative functional with values in $[0, \infty)$ such that the hitting distributions of the process "inflated" by the multiplicative functional yield the given harmonic measures. We achieve this construction under weak continuity and local transience conditions on these measures that are natural in the theory of Markov processes, and a mild growth restriction on them. In particular, if the space E equipped with the measures satisfies the conditions of a harmonic space, such a Markov process and associated multiplicative functional exist. The result extends in a new direction the work of many authors, in probability and in axiomatic potential theory, on constructing Markov processes from given hitting distributions (i.e. from harmonic measures that have total mass no more than 1).

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0 Introduction

Let E be a locally compact second countable Hausdorff space. Let $\{E_n\}$ be a (countable) open covering of E where each E_n is relatively compact and has a nonempty complement. Denote

$$\mathscr{U}=\{U : U \text{ is open}, U\subset E_n \text{ for some } n\}, \qquad \mathscr{D}=\{D : D=E-U \text{ for some } U\in\mathscr{U}\}.$$

Consider a family of nonzero finite measures $Q_D(x,\cdot)$ on E, where $D\in\mathscr{D}$, $x\in E$, such that each $Q_D(x,\cdot)$ is concentrated on D, if $x\in D$ it is the point mass at x, and $Q_D(\cdot, B)$ is Borel measurable (or more generally nearly-Borel measurable – see Sect. 1) if B is Borel. Assume the family satisfies the consistency condition that, if $F\subset D$, $Q_F(x,\cdot)=\int Q_D(x, dy)\, Q_F(y,\cdot)$. The $Q_D(x,\cdot)$ are then called harmonic measures, (and one may call the space E equipped with such a family $Q_D(x,\cdot)$ a *general* harmonic or balayage space). It is natural to ask that under what (additional) conditions on the $Q_D(x,\cdot)$ one can construct a Markov process (more precisely, a right process) $Y=(Y_t, P^x, \zeta)$ on E, ζ the lifetime, and a multiplicative functional (M_t) in Y with values in $[0,\infty)$ such that for all x, $D\in\mathscr{D}$ and bounded Borel f on E

$$\int Q_D(x,dy)f(y)=P^x[f(Y(T_D))\,M(T_D);\,T_D<\zeta] := \int_{[T_D<\zeta]} f(Y(T_D))\,M(T_D)\,dP^x \tag{0.1}$$

where $T_D=\inf\{t\geqq 0\colon Y_t\in D\}$, the (first) hitting time of D.

When the $Q_D(x,\cdot)$ are subprobability measures, the multiplicative functional is not needed, and the problem is that of constructing a Markov process Y with the $Q_D(x,\cdot)$ as its hitting distributions $P^x[Y(T_D)\in\cdot, T_D<\zeta]$. Since 1963 this latter problem has been studied in a number of papers, by P.A. Meyer, F. Knight and S. Orey, D.A. Dawson, N. Boboc and C. Constantinescu and A. Cornea, W. Hansen, C.T. Shih, J.C. Taylor, J. Bliedtner and W. Hansen, J.B. Gravereux and J. Jacod; see references in [9]. Different constructions under different sets of conditions were introduced in these papers, some of which use mostly potential theoretical methods. All of these papers assume stronger continuity conditions than [7], which constructs all transient Hunt processes (up to a time change) and whose result is needed in this article. (We note that, using the theorem in [8], the construction in [7] easily extends to the recurrent Hunt processes as well, and the result is stated as Theorem 3 in [9]; [9] also corrected a gap in [7] in the proof of convergence of the time scale when holding points exist.)

In the present article the measures $Q_D(x,\cdot)$ may have total mass greater than 1. We assume the following natural conditions on the $Q_D(x,\cdot)$: local transience, intrinsic right continuity, and quasi-left-continuity, (see Sect. 1 for their meaning). Furthermore assume the following growth condition: if $F\subset D^\circ$ (interior of D) then for any x

$$\int Q_D(x,dy)\,q_F(y)\log q_F(y)<\infty \tag{0.2}$$

where $q_F(y)=Q_F(y,E)$. (Note that $\int Q_D(x,dy)\,q_F(y)=q_F(x)<\infty$ by the consistency condition; so (0.2) is not a severe restriction.) The following result is proved. Fix for each n an open set U_n with closure $\bar{U}_n\subset E_n$ and denote $\mathscr{D}'=\{D\in\mathscr{D}\colon E-D\subset U_n$ for some $n\}$; note that if there is a subsequence of E_n increasing to E, the U_n can be chosen so that $\mathscr{D}'=\mathscr{D}$. Then there exists a right process Y on E, which is actually a standard process because of the quasi-left-continuity, and a multiplicative functional (M_t) such that (0.1) holds for all x, $D\in\mathscr{D}'$ and bounded Borel f on E.

To our knowledge this result is new even in the context of the theory of harmonic spaces (see e.g. Constantinescu and Cornea [3]) and the more recent theory of balayage spaces (see Bliedtner and Hansen [1]), where the continuity conditions assumed are substantially stronger than the one assumed in this article, which consists of the intrinsic right continuity (a necessary condition) and the quasi-left-continuity. In this connection, however, it should be noted that our method is entirely probabilistic.

Incidentally, the continuity condition assumed in the theory of harmonic spaces (or in that of balayage spaces) implies that for all bounded Borel f the function $y\to\int Q_F(y,dz)f(z)$ is continuous on $E-F$; therefore $q_F(y)$ is continuous on $E-F$ and so locally bounded there, and consequently, since in a harmonic space each $Q_D(x,\cdot)$ is concentrated on the boundary of D if $x\notin D$, the growth conditions (0.2) is always satisfied in such a space.

The results are stated in Sect. 1. In Sect. 2 "local processes" are constructed based on the result in [7]. The heart of this work, however, is in Sects. 3 and 4, where "transformation (multiplicative) functionals" are constructed to connect the local processes, from which a *consistent* family of local processes and associated local multiplicative functionals are obtained, and finally a global process and associated multiplicative functional are defined by piecing together the local ones (with time changes). The reader not familiar with [7], but with one of the constructions (of a Markov process from hitting distributions) referred to above, may replace (some of) the hypotheses in Sect. 1 by his/her familiar ones, omit reading Sect. 2 except the definitions of local processes and their associated local multiplicative functionals (see Theorem 2.1), and proceed to read Sects. 3 and 4.

1 Statement of results

As in the introduction E is a locally compact second countable Hausdorff space. Let $E_\Delta=E\cup\{\Delta\}$ be its one-point compactification. Let $\mathscr{E}_\Delta$ (resp. $\mathscr{E}_\Delta^*$) denote the σ-algebra of Borel (resp. universally measurable) sets in E_Δ. $f\in b\mathscr{E}_\Delta$ (resp. $f\in\mathscr{E}_\Delta^+$), e.g., means f is a bounded (resp. positive) real $\mathscr{E}_\Delta$-measurable function. All real functions f on E_Δ will satisfy $f(\Delta)=0$ as a convention.

Let $\{E_n, n\geqq 1\}$ be an open covering of E where each E_n has (compact) closure $\bar{E}_n\subset E$, and $E-E_n$ is nonempty. (Again) let

$$\mathscr{U}=\{U: U \text{ is open}, U\subset E_n \text{ for some } n\}$$
$$\mathscr{D}=\{D: D=U^c=E_\Delta-U \text{ for some } U\in\mathscr{U}\}.$$

Consider a family $\{Q_D(x,\cdot): D\in\mathscr{D}, x\in D\}$ of (positive) measures on E_Δ that satisfies the following hypotheses (Q1) through (Q7).

(Q1) $0<q_D(x):=Q_D(x,E_\Delta)<\infty$; $Q_D(x,\cdot)$ is concentrated on D; $Q_D(x,\cdot)$ is the point mass ε_x if $x\in D$; $Q_D(x,\{\Delta\})=0$ if $x\neq\Delta$; $Q_D(\cdot,B)\in\mathscr{E}_\Delta^*$ if $B\in\mathscr{E}_\Delta$.

(Q2) (*Consistency*) If $F\subset D$, $Q_F(x,\cdot)=\int Q_D(x,dy)\,Q_F(y,\cdot)$, i.e. $Q_D(x,Q_F f)=Q_F(x,f)$ for all $f\in b\mathscr{E}_\Delta^*$ where $Q_D f(y)=Q_D(y,f):=\int Q_D(y,dz)f(z)$.

To state the next hypothesis, we need to define the nearly Borel sets (relative to the family $Q_D(x,\cdot)$). A set $A\subset E_\Delta$ is nearly Borel if for every n and finite measure μ on E_Δ there exist Borel sets B_1, B_2 with $B_1\subset A\cap E_n\subset B_2\subset E_n$ such that for all compact $C\subset B_2-B_1$ we have

$$\int\mu(dx)\,Q_{E_n^c\cup C}(x,C)=0.$$

It will be seen that the family $\mathscr{E}_\Delta^n$ of nearly Borel sets is a σ-algebra. It is easy to see that $\mathscr{E}_\Delta\subset\mathscr{E}_\Delta^n\subset\mathscr{E}_\Delta^*$.

(Q3) (*Nearly-Borel measurability*) $Q_D(\cdot,B)\in\mathscr{E}_\Delta^n$ if $B\in\mathscr{E}_\Delta$.

(Q4) (*Local transience*) If $F\subset D$ and $x\notin D$, then for some compact neighborhood C of x with $C\subset F^c$

$$\int Q_D(x,dy)\,Q_{F\cup C}(x,F)>0.$$

(Q5) (*Intrinsic right continuity*) For any x, F and increasing sequence D_n with $F \subset D_n$, if $Q_{D_n}(x, dy)\, q_F(y)$ converges weakly to a finite measure $\nu(dy)$, then for any C with $F \subset C$ and $f \in b\mathscr{E}_\Delta$

$$Q_{D_n}(x, dy)\, q_F(y)\, 1_{[Q_C(y, f q_F) \in da]} \to \nu(dy)\, 1_{[Q_C(y, f q_F) \in da]}$$

vaguely when restricted as measures on the locally compact $(E_\Delta - C) \times \mathbf{R}$.

(Of course, weak convergence of measures on E_Δ means convergence of integrals of all (real) continuous functions, and vague convergence of measures on a locally compact space means convergence of integrals of all continuous functions vanishing at infinity.)

(Q6) (*Quasi-left-continuity*) For any $x, D_n \downarrow D$ and $F \subset D$, $Q_{D_n}(x, dy)\, q_F(y)$ converges weakly to $Q_D(x, dy)\, q_F(y)$.

(Q7) For any x and D, F with $F \subset D^\circ$, $\int Q_D(x, dy)\, q_F(y) \log q_F(y) < \infty$.

Theorem 1 *Let $\{Q_D(x, \cdot): D \in \mathscr{D}, x \in E_\Delta\}$ be a family of measures on E_Δ satisfying hypotheses* (Q1) *through* (Q7). *Assume $E_n \uparrow E$. Then there exists a right process $Y = (Y_t, P^x)$ on E_Δ with Δ as the death point, (which is actually a standard process), and a multiplicative functional $M = (M_t)$ with values in $[0, \infty)$ that is* a.s. *strictly positive on $[0, T_\Delta)$, $T_\Delta = \inf\{t \geqq 0: Y_t = \Delta\}$ the lifetime, such that for all $x \in E, D \in \mathscr{D}$, $f \in b\mathscr{E}_\Delta$*

$$Q_D f(x) = P^x[f(Y(T_D))\, M(T_D);\, T_D < T_\Delta] \tag{1.1}$$

where $T_D = \inf\{t \geqq 0: Y_t \in D\}$.

If $\{E_n\}$ is arbitrary, i.e. does not necessarily contain a subsequence increasing to E, then, as in the introduction, we fix for each n an open U_n with $\bar{U}_n \subset E_n$ and denote

$$\mathscr{D}' = \{D \in \mathscr{D}: E_\Delta - D \subset U_n \text{ for some } n\}.$$

Theorem 2 *Let $\{Q_D(x, \cdot): D \in \mathscr{D}, x \in E_\Delta\}$ be a family of measures on E_Δ satisfying hypotheses* (Q1) *through* (Q7). *Then there exists a right process $Y = (Y_t, P^x)$ on E_Δ with Δ as the death point, (which is actually a standard process), and a multiplicative functional $M = (M_t)$ with values in $[0, \infty)$ that is* a.s. *strictly positive on $[0, T_\Delta)$, such that* (1.1) *is satisfied for all $x \in E$, $f \in b\mathscr{E}_\Delta$ and $D \in \mathscr{D}'$.*

Of course Theorem 1 is just a special case of Theorem 2. But it is separately stated because it is the important case; also, in Sect. 4 we prove this case first because the proof is easier.

In this article a multiplicative functional (MF) (M_t) in a right process (X_t, P^x) on E_Δ is permitted to take values in $[0, \infty)$, not just in $[0, 1]$; otherwise it satisfies the usual conditions: (M_t) is adapted to $(\mathscr{F}_t)$, the usual (right continuous and suitable completed) filtration generated by the paths of (X_t); $t \to M_t$ is right continuous a.s. (i.e. a.s. P^x for all $x \in E$); $M_0 = 1$ a.s.; and for any stopping times T, S w.r.t. $(\mathscr{F}_t)$, $M(T + S \circ \theta_T) = M(T) \cdot M(S) \circ \theta_T$ a.s. on $\{T + S \circ \theta_T < \infty\}$ (θ the shift operator). See [2], [6] or [4] for information about standard and right processes.

A MF (Ψ_t) is a *transformation* (*multiplicative*) *functional* (TMF) if $P^x[\Psi_t] \leqq 1$ for all $x \in E$ and t. It is well-known that if (Ψ_t) is a TMF in a right process

(X_t, P^x) on E_Δ, then there exists a right process $(\tilde{X}_t, \tilde{P}^x)$ on E_Δ (with Δ as the death point) such that for all $x, t, f \in b\mathscr{E}_\Delta$ (note the convention $f(\Delta)=0$

$$\tilde{P}^x[f(\tilde{X}_t)] = P^x[f(X_t)\,\Psi_t].$$

See e.g. [5]. Such a process $(\tilde{X}_t)$ will be denoted $(X_t \cdot \Psi_t)$.

Some remarks need to be made about the hypotheses.

Remarks about (Q2): The universal measurability is a preliminary measurability condition; without it (Q2) and (Q3) cannot be stated. The point Δ will only serve as the *adjoined* death point for (Y_t) and the transformed processes $(X_t \cdot \Psi_t)$ constructed from the local processes (X_t); thus it carries no mass in the measures $Q_D(x, \cdot)$ for $x \in E$. If it is desirable to add mass to the measures $Q_D(x, \cdot)$ at an adjoined point, it is preferable to add an isolated point ∂ to E_Δ for this purpose, (∂ is then included in all $D \in \mathscr{D}$).

No remarks need to be made about (Q2).

Remarks about (Q3): First, if the conclusion of the theorem (Theorem 2) holds, it is easy to see that $\mathscr{E}_\Delta^n$ is exactly the σ-algebra of nearly Borel sets for (Y_t); for $M_t > 0$ on $[0, T_\Delta)$ a.s. Of course Borel measurability of $\{Q_D(x, \cdot)\}$ implies nearly-Borel measurability, and the latter is a natural measurability assumption. In Sect. 2 we will see that it is equivalent to: $Q_D(\cdot, B) \in \mathscr{E}_\Delta^n$ if $B \in \mathscr{E}_\Delta^n$.

Remark about (Q4): Under the conclusion of the theorem (Q4) must hold. For if

$$\int Q_D(x, dy)\, Q_{F \cup C}(y, F) = P^x[M(T_D) \cdot M(T_{F \cup C}) \circ \theta(T_D);\; Y(T_{F \cup C}) \circ \theta(T_D) \in F - \{\Delta\}]$$

(for convenience of writing we use the conventions $Y_t \circ \theta_\infty = \Delta$, $Y_\infty = \Delta$ and $M_\infty = 0$ here and below) is 0 for all (sufficiently small) compact neighborhoods C of x, then

$$P^x[Y(T_{F \cup C}) \circ \theta(T_D) \in F - \{\Delta\}] = 0$$

for all such C, and by (Q6) we have

$$P^x[Y(T_{F \cup \{x\}}) \circ \theta(T_D) \in F - \{\Delta\}] = 0.$$

It is then easy to see that $P^x[Y(T_F) \in F - \{\Delta\}] = 0$, which implies $Q_F(x, E_\Delta) = 0$.

Remarks about (Q5): Under the conclusion of the theorem, this condition follows from the intrinsic right continuity of the right process (Y_t), i.e. the right continuity of the filtration $(\mathscr{F}_t)$, relative to which the strong Markov property holds. To see this, first, with $g \in b\mathscr{E}_\Delta$ continuous and $T_n = T_{D_n}$ and T denoting the decreasing limit of T_n (of course $T = T_A$ where $A = \bigcup_n D_n$), we have

$$\begin{aligned}\int Q_{D_n}(x, dy)\, q_F(y)\, g(y) &= P^x[g(Y(T_n))\, M(T_n) \cdot M(T_F) \circ \theta(T_n)] \\ &= P^x[g(Y(T_n))\, M(T_F)] \to P^x[g(Y(T))\, M(T_F)].\end{aligned}$$

Thus $Q_{D_n}(x, dy)\, q_F(y)$ converges weakly to $v(dy) = P^x[M(T_F); Y(T) \in dy]$ (note these measures all have total mass $q_F(x)$). Next, for $f \in b\mathscr{E}_\Delta^+$, $y \to Q_C(y, f q_F) = P^y[f(Y(T_C))\, M(T_F)]$ is easily checked to be excessive on $E_\Delta - C$ and so a.s. $t \to Q_C(Y_t, f q_F)$ is right continuous on t-intervals when $Y_t \in E_\Delta - C$. It follows that $Q_C(Y(T_n)), f q_F) \to Q_C(Y(T), f q_F)$ a.s. P^x on $\{Y(T) \in E_\Delta - C\}$ for $f \in b\mathscr{E}_\Delta$. (Actually the latter fact is easy to prove directly by martingale convergence.) Thus we have the following vague convergence on $(E_\Delta - C)\times \mathbf{R}$

$$\begin{aligned} Q_{D_n}(x,dy)\,q_F(y)\,1_{[Q_C(y,fq_F)\in da]} &= P^x[M(T_F);\ Y(T_n)\in dy,\ Q_C(Y(T_n),fq_F)\in da] \\ &\to P^x[M(T_F);\ Y(T)\in dy,\ Q_C(Y(T),fq_F)\in da] \\ &= \nu(dy)\,1_{[Q_C(y,fq_F)\in da]}. \end{aligned}$$

We remark that $Q_{D_n}(x,dy)\,q_F(y)$ actually always converges weakly. This follows from the other hypotheses, essentially (Q4) and (Q6); see a remark in Sect. 2 about (H5).

Remarks about (Q6): This condition is satisfied if the underlying right process (Y_t) is a standard process. For if $g\in b\mathscr{E}_\Delta$ is continuous, $D_n\downarrow D$, $F\subset D$, then by the quasi-left-continuity of (Y_t), $T_{D_n}\uparrow T_D$ and $Y(T_{D_n})\to Y(T_D)$ on $\{T_F<T_\Delta\}$ a.s. P^x, and so

$$\begin{aligned} &\int Q_{D_n}(x,dy)\,q_F(y)g(y) \\ &= P^x[g(Y(T_{D_n}))\,M(T_F);\ T_F<T_\Delta] \to P^x[g(Y(T_D))\,M(T_F);\ T_F<T_\Delta] \\ &= \int Q_D(x,dy)\,q_F(y)\,g(y). \end{aligned}$$

Note that we did not write (Q6) in the following form:

if $D_n\downarrow D$ then $Q_{D_n}(x,dy)$ converges weakly to $Q_D(x,dy)$,

which would reflect the additional requirement that the MF (M_t) is also "quasi-left-continuous".

2 Local processes

In this section we fix an (open) set $G\in\mathscr{U}$ and denote $F=G^c=E_\Delta-G$. Let $\mathscr{D}_0=\{D\in\mathscr{D}: F\subset D\}$. We will prove the following theorem.

Theorem 2.1 *There exists a right process (X_t, P^x) on E_Δ such that for all $D\in\mathscr{D}_0$, $x\in E_\Delta$ the hitting distribution $P^x[X(T_D)\in\cdot, T_D<\infty]$ (where $T_D=\inf\{t\geqq 0: X_t\in D\}$) is*

(2.1) $$H_D(x,dy) := Q_D(x,dy)\,q_F(x)^{-1}\,q_F(y)$$

and such that each $x\in F$ is an absorbing point. Furthermore,

(2.2) $$J_t := q_F(X_0)\,q_F(X_t)^{-1}$$

defins a MF *(with values in $[0,\infty)$) such that for all x, $D\in\mathscr{D}_0, f\in b\mathscr{E}_\Delta$*

(2.3) $$Q_D(x,f) = P^x[f(X(T_D))\,J(T_D);\ T_D<\infty].$$

The P^x in (X_t, P^x) is not the same as that in (Y_t, P^x) in Theorems 1, 2. Note the measures $H_D(x,\cdot)$ are probability measures (since $\int Q_D(x,dy)\,q_F(y) = q_F(x)$ by (Q2)) and $H_D(x,\{\Delta\})=0$ if $x\neq\Delta$; therefore $T_F<\infty$ a.s. and the lifetime $T_\Delta=\infty$ a.s. P^x for $x\in E$. (T_F will be called the proper lifetime of (X_t).) For (J_t) to be a MF, one needs only to prove that J_t is right continuous a.s. (2.3) is immediate from (2.1) and (2.2).

For the existence of (X_t) we will need Theorem 1 of [7]. Denote $K=G\cup\{\Delta^*\}$ the one-point compactification of G. K is (obtained from) E_Δ by identifying all points in F as the single point Δ^*. $\mathscr{D}_0$ is also regarded as the family of compact sets D in K that contain Δ^*, i.e. $\{(G-U)\cup\Delta^*: U\in\mathscr{U}, U\subset G\}$ (note that we will often write $\{\Delta^*\}$ as Δ^*). Thus $H_D(x,\cdot)$ in (2.1) is also taken to be a measure on K (when $D\subset K$), with of course $H_D(x,\Delta^*)$ equal to the original $H_D(x,F)$.

With $\{H_D(x,\cdot): D\in\mathscr{D}_0,\ x\in K\}$ regarded as a family of measures on K, we will prove that it satisfies hypothesis (H1) through (H6) in [7], (in doing so D is often regarded as a set in E_Δ, x as in E_Δ, and $H_D(x,\cdot)$ as a measure on E_Δ, and no harm will be done). Denote by $\mathscr{B}$ the σ-algebra of Borel sets in K and $\mathscr{B}^*$ that of universally measurable sets in K.

(H1) Each $H_D(x,\cdot)$ is a probability measure concentrated on D; $H_D(x,\cdot)=\varepsilon_x$ if $x\in D$; $H_D(\cdot,B)\in\mathscr{B}$ if $B\in\mathscr{B}^*$.

Proof. It has been observed that $H_D(x,\cdot)$ is a probability measure; the rest is immediate from (Q1). □

(H2) (*Markov property, or consistency*) If $D_2\subset D_1$, $H_{D_2}(x,\cdot)=\int H_{D_1}(x,dy)\,H_{D_2}(y,\cdot)$.

Proof. By (Q2)

$$\begin{aligned}H_{D_2}(x,dz)&=Q_{D_2}(x,dz)\,q_F(x)^{-1}\,q_F(z)\\&=\int Q_{D_1}(x,dy)\,q_F(x)^{-1}\,q_F(y)\,Q_{D_2}(y,dz)\,q_F(y)^{-1}\,q_F(z)\\&=\int H_{D_1}(x,dy)\,H_{D_2}(y,dz).\quad\square\end{aligned}$$

The family $\mathscr{B}^n$ of nearly Borel sets of K relative to the family $H_D(x,\cdot)$ is defined as follows. A subset B of K is in $\mathscr{B}^n$ if for any finite measure μ on G there exist Borel B_1, B_2, with $B_1\subset B\cap G\subset B_2\subset G$, such that for all compact $C\subset B_2-B_1$,

$$H_{C\cup\Delta^*}(\mu,C):=\int\mu(dx)\,H_{C\cup\Delta^*}(x,C)=0. \tag{2.4}$$

Obviously $\mathscr{B}\subset\mathscr{B}^n\subset\mathscr{B}^*$. From the definitions of $\mathscr{E}_\Delta^n$ and $\mathscr{B}^n$, it is easy to see that $A\in\mathscr{E}_\Delta^n$ iff $A\cap G\in\mathscr{B}^n\cap G$ for all $G\in\mathscr{U}$ (note $\mathscr{B}^n$ depends on G). It is proved in [7, Sect. 2], that $\mathscr{B}^n$ is a σ-algebra; from this the following proposition is immediate.

Proposition 2.2 *$\mathscr{E}_\Delta^n$ is a σ-algebra.*

(H3) (*Nearly-Borel measurability*). $H_D(\cdot,B)\in\mathscr{B}^n$ if $D\in\mathscr{D}_0$, $B\in\mathscr{B}$.

Proof. Let $f\in b\mathscr{B}^+$ and $h=H_D(\cdot,f)$; we show $h\in\mathscr{B}^n$. By a standard argument it suffices to show that, given a finite measure μ on G, there exist h_1, h_2 in $\mathscr{B}^n$ with $0\leqq h_1\leqq h\leqq h_2$ such that, for all compact $C\subset\{h_2>h_1\}$, (2.4) holds. Below we regard $D\subset E_\Delta$, $f\in\mathscr{E}_\Delta^+$ with f constant on F, and write $C\cup\Delta^*$ as $F\cup C$. Now

$$h(x)=H_D(x,f)=q_F(x)^{-1}\,Q_D(x,f\,q_F).$$

Note $q_F=Q_F(\cdot,E_\Delta)\in\mathscr{E}_\Delta^n$ by (Q3). Define $\nu(\cdot)=\int\mu(dx)\,q_F(x)^{-1}\,Q_D(x,\cdot)$. Then there exist functions k_1, k_2 in $\mathscr{E}_\Delta$ with $0\leqq k_1\leqq q_F\leqq k_2$ and $k_1=q_F=k_2=1$ on F such that for all compact $C_1 \subset \{k_2 > k_1\}$, $Q_{F \cup C_1}(\nu, C_1) = 0$. We may change k_1, k_2 on D^c to $k_1 = k_2 = 0$ there. Now let

$$h_i(x) = q_F(x)^{-1} Q_D(x, f k_i).$$

Then $h_i \in \mathscr{E}^n_\Delta$ and $0 \leqq h_1 \leqq h \leqq h_2$. Let us show that for all compact $C \subset \{h_2 > h_1\}$, $H_{C \cup \Delta^*}(\mu, C) = 0$, equivalently $Q_{F \cup C}(\mu, C) = 0$ (although $Q_{F \cup C}(\mu, \cdot)$ may not be a finite measure). Suppose $Q_{F \cup C}(\mu, C) > 0$ for some such C; then we claim

$$(2.5) \qquad Q_{F \cup C_1}(\mu, C_1) > 0 \quad \text{for some compact } C_1 \subset \{k_2 > k_1\}.$$

But this leads to a contradiction since $C_1 \subset D$ and so

$$Q_{F \cup C_1}(\mu, C_1) = \int Q_D(\mu, dy)\, Q_{F \cup C_1}(y, C_1) = 0,$$

which follows from $Q_{F \cup C_1}(\nu, C_1) = 0$, a consequence of the choice of k_i. To show (2.5), let $C' = C \cap D$; then since obviously $\{h_2 > h_1\} \cap D \subset \{k_2 > k_1\}$, $C' \subset \{k_2 > k_1\}$. By (Q2) we have

$$Q_{F \cup C'}(\mu, C') = \int Q_{F \cup C}(\mu, dy)\, Q_{F \cup C'}(y, C').$$

If $Q_{F \cup C}(\mu, C') > 0$ then $Q_{F \cup C'}(\mu, C') > 0$. Otherwise $Q_{F \cup C}(\mu, C - C') > 0$. But if $y \in C - C'$, then since $h_2(y) > h_1(y)$, $Q_D(y, \{k_2 > k_1\}) > 0$, and so

$$Q_{F \cup C'}(y, C') = \int Q_D(y, dz)\, Q_{F \cup C'}(z, C') > 0$$

if C' is sufficiently large (i.e. C is sufficiently large). Thus (2.5) follows, where C_1 is C' when C is sufficiently large. □

It is proved in [7, Sect. 2], that $H_D(\cdot, B) \in \mathscr{B}^n$ for $D \in \mathscr{D}_0$, $B \in \mathscr{B}^n$, using an argument resembling the above. In the present situation, we can prove the fact (not needed in the sequel) that $Q_D(\cdot, B) \in \mathscr{E}^n_\Delta$ for $D \in \mathscr{D}$, $B \in \mathscr{E}^n_\Delta$, using a similar but slightly more careful argument. Of course, once we obtain a process (Y_t), this also follows from a standard fact in a right process.

(H4) (*Quasi-left-continuity*) If $D_n \downarrow D$, $H_{D_n}(x, \cdot)$ converges weakly to $H_D(x, \cdot)$.

Proof. This is immediate from (Q6) and the definition of $H_D(x, \cdot)$. □

(H5) (*Intrinsic right continuity*) For any x in K, increasing sequence D_n in $\mathscr{D}_0$, and letting (W_n) be the nonhomogeneous reversed Markov chain on K (under a single probability measure P) with $P(W_n \in \cdot) = H_{D_n}(x, \cdot)$ and $P(W_{n+1} \in \cdot \mid W_m, m \leqq n) = H_{D_{n+1}}(W_n, \cdot)$, we have: if $W_\infty = \lim_n W_n$ exists a.s., then, for any $C \in \mathscr{D}_0$ and $f \in b\mathscr{B}$, $H_C f(W_n) \to H_C f(W_\infty)$ a.s. on $\{W_\infty \in K - C\}$.

Proof. We first show that (H5) follows from condition (H5′) below and then prove (H5′) from (Q5).

(H5′). For x, D_n as in (H5), if $H_{D_n}(x, \cdot)$ converges weakly (as measures on K) to a measure μ, then for any C, f as in (H5) the measures

$$H_{D_n}(x, dy)\, 1_{[H_C(y,f) \in da]} \to \mu(dy)\, 1_{[H_C(y,f) \in da]}$$

vaguely when restricted to $(K - C) \times \mathbf{R}$.

If $W_n \to W_\infty$ a.s. in (H5), then $H_{D_n}(x,\cdot)$ converges weakly to $\mu(\cdot)=P(W_\infty\in\cdot)$. Now by an easy application of martingale convergence $H_C f(W_n)$ converges a.s. on $\{W_\infty \in K-C\}$. The conclusion in (H5′) says the measures

$$P(W_n\in dy, H_C(W_n,f)\in da) \to P(W_\infty\in dy, H_C(W_\infty,f)\in da)$$

vaguely when restricted to $\{W_\infty\in K-C\}$. From this it is clear that the conclusion of (H5) follows. To prove (H5′), assume $H_{D_n}(x,\cdot)$ converges weakly to μ as measures on K; then we claim that $Q_{D_n}(x,dy)\,q_F(y)=q_F(x)\,H_{D_n}(x,\cdot)$ converges weakly as measures on E_Δ (which has a coarser topology at the boundary ∂G). For if not, it is not difficult to construct a sequence $D'_n\downarrow D'$ where $F\subset D'$ such that $Q_{D'_n}(x,dy)\,q_F(y)$ does not converge weakly, and so (Q6) is contradicted. (We do not give the detail of this part of the proof since we feel that it would be quite reasonable to assume in (Q5) that $Q_{D_n}(x,dy)\,q_F(y)$ converges weakly. Incidentally, it was proved in [7] that the (W_n) in (H5) always converges weakly, essentially by (H6) below; therefore also does $H_{D_n}(x,\cdot)$ in (H5′). In this article, it can also be proved that $Q_{D_n}(x,dy)\,q_F(y)$ in (Q5) always converges weakly, using (Q4) and (Q6).) Let ν be the weak limit of $Q_{D_n}(x,dy)\,q_F(y)$; then $\mu(dy)=q_F(x)^{-1}\nu(dy)$ with the understanding $\mu\{\Delta^*\}=q_F(x)^{-1}\nu(F)$. Applying (Q5) with $C=F$ and $f=1$ we have

$$Q_{D_n}(x,dy)\,q_F(y)\,1_{[q_F(y)\in da]} \to \nu(dy)\,1_{[q_F(y)\in da]}$$

vaguely on $(E_\Delta-F)\times\mathbf{R}$. Combining this with the vague convergence of

$$Q_{D_n}(x,dy)\,q_F(y)\,1_{[Q_C(y,fq_F)\in da]} \to \nu(dy)\,1_{[Q_C(y,fq_F)\in da]}$$

on $(E_\Delta-C)\times\mathbf{R}$ for the given f, and noting that $q_F(y)>0$, we obtain

$$Q_{D_n}(x,dy)\,q_F(y)\,1_{[q_F(y)^{-1}Q_C(y,fq_F)\in da]} \to \nu(dy)\,1_{[q_F(y)^{-1}Q_C(y,fq_F)\in da]}$$

vaguely on $(E_\Delta-C)\times\mathbf{R}$. Dividing the above by $q_F(x)$ we have the conclusion of (H5′). □

(H6) (*Transience*) For any $D\in\mathscr{D}_0$ and $x\notin D$, there exists a compact neighborhood C of x such that $\int H_D(x,dy)\,H_{C\cup\Delta^*}(y,C)<1$.

Proof. This is immediate from (Q4) and the fact that the $H_D(y,\cdot)$ are probability measures. □

Now that (H1) though (H6) are established, it follows from Theorem 1 of [7] that there exists a right process (actually a Hunt process) $(\bar{X}_t, P^x)$ on K with Δ^* as the death point and with finite lifetime T_{Δ^*} such that the hitting distributions are the given $H_D(x,\cdot)$. The process (X_t,P^x) in Theorem 2.1 can be obtained as follows. One shows that for $f\in b\mathscr{E}_\Delta^+$, $\lim_{t\uparrow T_{\Delta^*}} H_F f(\bar{X}_t)$ exists a.s. P^x for each $x\in G$ and has P^x-expectation $H_F f(x)$ (because $H_F f$ is excessive and in fact harmonic on G). Thus we can define $(X_t, t<T_F)$ as $(\bar{X}_t, t<T_{\Delta^*})$, and then define the distributions $P^x[X(T_F)\in dy\,|\,X_t, t<T_F]$ by requiring $P^x[f(X(T_F))\,|\,X_t, t<T_F]=\lim_{t\uparrow T_F} H_F f(X_t)$ for all continuous f. Then since each $y\in F$ is to be an absorbing point, the existence of (X_t) as desired follows.

However, there is an easier way to define (X_t). Let X_t^1 be the process obtained from (X_t) by changing each $x \in F - \Delta$ to a holding point, from which a jump is made to Δ with probability 1. We know what the hitting distributions $H_D^1(x,\cdot)$ should be. Let $\mathscr{D}_\Delta = \{D: D$ is closed in $E_\Delta, \Delta \in D\}$. Define $\{H_D^1(x,\cdot): D \in \mathscr{D}_\Delta, x \in E_\Delta\}$ as follows:

$$H_D^1(x,\cdot) = \varepsilon_x(\cdot) \quad \text{if } x \in D; \qquad H_D^1(x,\cdot) = \varepsilon_\Delta(\cdot) \quad \text{if } x \in F - D;$$

and otherwise

$$\begin{aligned} H_D^1(x, B) &= \int H_{F \cup D}(x, dy)\, H_D^1(y, B) \\ &= H_{F \cup D}(x, D \cap B) + 1_B(\Delta)\, H_{F \cup D}(x, F - D). \end{aligned}$$

(In particular, if $F \subset D$, $H_D^1(x,\cdot) = H_D(x,\cdot)$.) It is easy to see that $\{H_D^1(x,\cdot)\}$ satisfies (H 1) through (H 6). Thus there exists a right process (X_t^1) on E_Δ with the $H_D^1(x,\cdot)$ as its hitting distributions. Now change each holding, point $x \in F - \Delta$ to an absorbing point to obtain (X_t) from (X_t^1).

To complete the proof of Theorem 2.1, it remains to show (J_t) defined is (2.2) is a MF. Note $J_0 = 1$ and $J_{t+s} = J_t \cdot J_s(\theta_t)$ for all t, s. So one needs only to show J_t is right continuous a.s. This of course will follow from

Proposition 2.3 *$q_F(X_t)$ is right continuous* a.s.

Proof. By a standard theorem it suffices to show q_F is nearly Borel and finely continuous w.r.t. (X_t). We have already that q_F is nearly Borel. Since each $x \in F$ is absorbing, to prove that q_F is finely continuous we need only to show the fine continuity at each $x \in G$. Define $A = \{y \in G: |q_F(y) - q_F(x)| \geqq \delta\}$ where $\delta > 0$. We show that $P^x(T_A > 0) = 1$. Suppose not; then $P^x(T_A = 0) = 1$. Let C_n be increasing compact subsets of A such that $T_{C_n} \downarrow T_A$ a.s. P^x. Let $D_n = C_n \cup F$. Then $Q_{D_n}(x, dy)\, q_F(y) = q_F(x)\, P^x[X(T_{D_n}) \in dy]$ converges weakly to $v(dy)$ $= q_F(x)\, P^x[X(T_A) \in dy] = q_F(x)\, \varepsilon_x(dy)$. By (Q 5) with $C = F$ and $f = 1$ we thus have that $q_F(X(T_{D_n}))$ under P^x converges in distribution to $Q_F(x, q_F) = Q_F(x, 1) = q_F(x)$, which is a contradiction. □

3 The multiplicative functional connecting two local processes

From now on a right process (X_t, P^x) is simply written as X_t and a MF (M_t) as M_t. We will occasionally write the probability measure P_x^i in (X_t^i, P_x^i) as P^x. All (right) processes X_t will have as sample space the space Ω of right continuous paths $\omega: [0, \infty) \to E_\Delta$ such that $\omega_t = \Delta$ implies $\omega_s = \Delta$ for $s > t$; furthermore we can require all ω to have left limits on $(0, T_\Delta)$ where $T_\Delta = \inf\{t: \omega_t = \Delta\}$. Of course $X_t(\omega) = \omega_t$, and the same notation $T_D = \inf\{t \geqq 0: X_t \in D\}$ is therefore used for all X_t. A MF M_t (in a process X_t) is said to be "trajectory-dependent" (more appropriately trajectory-dependent-only) if for any finite measure μ on E_Δ there exists Λ such that $P^\mu(\Lambda^c) = 0$ and for all ω_1, ω_2 in Λ and $t_1, t_2 > 0$, and increasing homeomorphism $\sigma: [0, t_1] \to [0, t_2]$ with $X_t(\omega_1) = X_{\sigma(t)}(\omega_2)$ on $[0, t_1]$, we have $M_t(\omega_1) = M_{\sigma(t)}(\omega_2)$ on $[0, t_1]$. All MFs will be trajectory-dependent, but we will still usually mention the fact explicitly for emphasis. If U is an open set in E, a MF M_t is said to be "constant off U" if a.s. the following holds: for any $t_1 < t_2$, if $X_t(\omega) \in U^c$ for all $t \in [t_1, t_2]$ then $M_t(\omega)$ is constant on $[t_1, t_2]$.

If two processes X_t^1 and X_t^2 are time changes of each other (which is the case iff they have the same hitting distributions), we will write $X_t^1 \sim X_t^2$. If U is an open set in E, the process Z_t obtained by *stopping* X_t at the exit from U, (i.e. $Z_t = X_{t\wedge T}$ where $T = T_{U^c}$), will be written as $X_t|_U$. In this section we prove the following theorem that is of central importance.

Theorem 3.1 *Let G_1, G_2 and U be in $\mathscr{U}$ with $\bar{U} \subset G_1 \subset G_2$. Let X_t^i be the right process X_t defined in Sect. 2 when $G = G_i$, $i = 1, 2$. Then there exists a* TMF *Ψ_t in X_t^2 such that $X_t^2 \cdot \Psi_t|_U \sim X_t^1|_U$. Furthermore, Ψ_t satisfies the following properties:*

(i) *it is trajectory-dependent;*

(ii) *it is constant off U;*

(iii) a.s. *Ψ_t is strictly positive on $[0, T_\Delta)$.*

Let $F_i = G_i^c$, $q_i = q_{F_i}$, and denote $\zeta_i = T_{F_i}$ (again for both X_t^1 and X_t^2), the proper lifetime of X_t^i. The hitting distributions of X_t^i for $D \in \mathscr{D}$ containing F_i are

$$H_D^i(x, dy) = Q_D(x, dy)\, q_i(x)^{-1} q_i(y) \tag{3.1}$$

and with the MF J_t^i in X_t^i defined by

$$J_t^i = q_i(X_0^i)\, q_i(X_t^i)^{-1} \tag{3.2}$$

we have, for $x \in E$, $D \in \mathscr{D}$ containing F_i, $f \in b\mathscr{E}_\Delta$

$$Q_D f(x) = P^x[f(X^i(T_D))\, J^i(T_D)].$$

We will regard J_t^1 as also defined in X_t^2 by $J_t^1 = q_1(X_0^2)\, q_1(X_t^2)^{-1}$. Denote $Z_t = X_t^2|_{G_1}$. The following defines a MF in Z_t:

$$\phi_t = J_{t\wedge\zeta_1}^2 / J_{t\wedge\zeta_1}^1. \tag{3.3}$$

Denote by $\mathscr{F}_t^2$ the filtration generated by the paths of X_t^2 that is, as always, right continuous and completed in the usual way. Of course ϕ_t is adapted to $\mathscr{F}_t^2$.

Lemma 3.2 (i) *Let T, S be $\mathscr{F}_T^2$-stopping times with $T \leq S \leq \zeta_1$ and $\rho \in (\mathscr{F}_T^2)^+$;* then

$$P_x^2[\rho\, \phi(S)/\phi(T)] = P_x^2[\rho]$$

for all x.

(ii) *If S is as above, $P_x^2[\phi(S)] = 1$ for all x; consequently ϕ_t is a* TMF *in Z_t.*

(iii) $Z_t \cdot \phi_t \sim X_t^1$.

Proof. (i) First, by (3.1), (3.2) and (3.3)

$$P_y^2[\phi(\zeta_1)] = \int Q_{F_1}(y, dz)\, q_2(y)^{-1} q_2(z) [q_2(y)\, q_2(z)^{-1}] / [q_1(y)\, q_1(z)^{-1}] = 1,$$

since $q_{F_1}(y) = Q_{F_1}(y, q_{F_1})$. Thus

$$\begin{aligned} P_x^2[\rho\, \phi(S)/\phi(T)] &= P_x^2\left[\rho \cdot \frac{\phi(S)}{\phi(T)} \cdot P_{X(S)}^2(\phi(\zeta_1))\right] = P_x^2\left[\rho \cdot \frac{\phi(\zeta_1)}{\phi(T)}\right] \\ &= P_x^2[\rho\, \phi(\zeta_1) \circ \theta_T] = P_x^2[\rho\, P_{X(T)}^2(\phi(\zeta_1))] = P_x^2[\rho]. \end{aligned}$$

(ii) follows from (i) with $\rho=1$, $T=0$.
(iii) will follow if $Z_t \cdot \phi_t$ and X_t^1 have the same hitting distributions on sets $D \in \mathscr{D}$ containing F_1, noting that each $x \in F_1$ is an absorbing point for both processes. But by a computation similar to the above, the hitting distributions of $Z_t \cdot \phi_t$ for such D are

$$P_x^2[f(X^2(T_D))\,\phi(T_D)] = \int Q_D(x, dy)\, f(y)\, q_1(x)^{-1} q_1(y) = H_D^1(x, f). \quad \square$$

We will construct the part of the TMF Ψ_t in Theorem 3.1 up to time ζ_1 (i.e. its part in Z_t); its part after time ζ_1 will be determined by the requirement that Ψ_t be a MF and be constant off U, noting $\bar{U} \subset G_1$ and $\zeta_1 = T_{F_1}$, (the part of Ψ_t up to ζ_1 has this latter property). To this end choose compact sets C_n, $n \geqq 0$, with $C_n \uparrow U$ and $C_0 = \emptyset$ (the empty set), denote $V = U^c$, and define, for $n \geqq 1$, $\mathscr{F}_t^2$-stopping times T_{nj}, R_{nj}, $j \geqq 0$, and S_{nj}, $j \geqq 1$, as follows:

$$\begin{aligned}
&T_{n0} = 0, \qquad R_{n0} = T_V (= \inf\{t \geqq 0 : X_t^2 \in V\}), \quad \text{and for } j \geqq 1\\
&T_{nj} = R_{n,j-1} + T_{C_n \cup F_1} \circ \theta(R_{n,j-1}),\\
&S_{nj} = T_{nj} + T_{C_{n-1} \cup V} \circ \theta(T_{nj}),\\
&R_{nj} = S_{nj} + T_V \circ \theta(S_{nj}) = T_{nj} + T_V \circ \theta(T_{nj}).
\end{aligned}$$

Since paths (of X_t^2) have left limits on $(0, T_\Delta)$, we have for a fixed n, $T_{nj} = S_{nj} = R_{nj} = \zeta_1$ for all sufficiently large j. Note the intervals $[T_{nj}, S_{nj})$, $n \geqq 1$, $j \geqq 1$, (most of them being $\emptyset$), are disjoint, and their union is $(T_V, \zeta_1) \cap \{t : X_t^2 \in U\}$. Note also $[T_{nj}, R_{nj})$, $j \geqq 1$, is $[T_{nj}, S_{nj})$ or its union with some $[S_{n-1,i}, R_{n-1,i})$. Define

$$\Phi_n(t) = \prod_{j=0}^{\infty} \frac{\phi(R_{nj} \wedge t)}{\phi(T_{nj} \wedge t)} = \prod_{j=0}^{\infty} \phi(T_V \wedge (t - T_{nj})^+) \circ \theta(T_{nj})$$

(where in the last expression both T_{nj} are evaluated at ω). Note the products are finite, and $\Phi_n(t) = \Phi_n(t \wedge \zeta_1)$. $\Phi_n(t)$ is not a MF (in Z_t), but satisfies the following properties: $\Phi_n(0) = 1$; $\Phi_n(t)$ is right continuous; $\Phi_n(t+s) = \Phi_n(t) \cdot \Phi_n(s) \circ \theta_t$ for all t, s with $t < \zeta_1$ and $X_t^2 \notin U - C_n$; it is trajectory-dependent; it is constant off U and $\Phi_n(t) > 0$ for all t. Observe that

$$\frac{\Phi_n(t)}{\Phi_{n-1}(t)} = \prod_{j=1}^{\infty} \frac{\phi(S_{nj} \wedge t)}{\phi(T_{nj} \wedge t)} = \prod_{j=1}^{\infty} \phi(T_{C_{n-1} \cup V} \wedge (t - T_{nj})^+) \circ \theta(T_{nj}).$$

Lemma 3.3 $P_x^2[\Phi_n(S)] \leqq 1$ *for all* x *and* $\mathscr{F}_t^2$*-stopping time* $S \leqq \zeta_1$.

Proof. $\Phi_n(S)$ is the limit, as $j' \to \infty$, of $\prod_{j=0}^{j'} \phi(R_{nj} \wedge S)/\phi(T_{nj} \wedge S)$, whose P_x^2-expectation is $P_x^2[1] = 1$ by repeatedly using (i) of Lemma 3.2. $\square$

Lemma 3.4 $P_x^2[\log \phi(\zeta_1)] > -\infty$ *for all* x.

Proof.

$$\begin{aligned}
P_x^2[\log \phi(\zeta_1)] &= \int Q_{F_1}(x, dy)\, q_2(x)^{-1} q_2(y) \log[q_2(x)\, q_2(y)^{-1} q_1(x)^{-1} q_1(y)]\\
&= \int Q_{F_1}(x, dy)\, q_2(x)^{-1} q_2(y) [\log(q_2(x)\, q_1(x)^{-1}) - \log q_2(y)]\\
&= \log(q_2(x)\, q_1(x)^{-1}) - q_2(x)^{-1} \int Q_{F_1}(x, dy)\, q_2(y) \log q_2(y) > -\infty
\end{aligned}$$

by applying (Q7) with $D=F_1$, $F=F_2$; in the second equality we used $q_1=1$ on F_1. □

Theorem 3.5 *For all x and* $\delta>0$, $\sup_{n_2>n_1} P_x^2\left[\sup_t \left|\frac{\Phi_{n_2}(t)}{\Phi_{n_1}(t)}-1\right|>\delta\right]\to 0$ as $n_1\to\infty$.

Proof. Suppose not. Then for some x and $\delta>0$ there exist $1\leqq m_1<m_1'\leqq \ldots \leqq m_l <m_l'\leqq \ldots$ such that for all l

$$P_x^2\left[\sup_t \left|\frac{\Phi_{m_l'}(t)}{\Phi_{m_l}(t)}-1\right|>\delta\right]>\delta. \tag{3.4}$$

Fix l and denote $n_1=m_l$, $n_2=m_l'$. In the above t can be restricted to belonging to

$$\bigcup_{n_1<n\leqq n_2} \bigcup_{1\leqq j\leqq j_0} [T_{nj}, S_{nj})$$

for some j_0. Let $\gamma=\inf\left\{t: \left|\frac{\Phi_{n_2}(t)}{\Phi_{n_1}(t)}-1\right|\geqq\delta\right\}$; note that if $\gamma<\infty$, the inf is a.s. a minimum by the a.s. right continuity of $\phi(t)$ and therefore of $\Phi_n(t)$. Arrange the disjoint intervals $[T_{nj}\wedge\gamma, S_{nj}\wedge\gamma)$, $n_1<\leqq n_2$, $1\leqq j\leqq j_0$, (some of which may be empty), as $[\tau_1,\sigma_1), \ldots, [\tau_{k'},\sigma_{k'})$, where $k'=j_0(n_2-n_1)$, with $\tau_1\leqq\sigma_1\leqq\ldots\leqq\tau_{k'} \leqq\sigma_{k'}\leqq\zeta_1$ being $\mathscr{F}_t^2$-stopping times. By (3.4)

$$P_x^2\left[\left|\frac{\Phi_{n_2}(\sigma_{k'})}{\Phi_{n_1}(\sigma_{k'})}-1\right|\geqq\delta\right]>\delta.$$

Now $\Phi_{n_2}(\sigma_{k'})/\Phi_{n_1}(\sigma_{k'})=\prod_{k\leqq k'}\phi(\sigma_k)/\phi(\tau_k)$. From Lemma 3.2(i) (see Lemma 3.3)

$$P_x^2\left[\prod_{k\leqq k'}\phi(\sigma_k)/\phi(\tau_k)\right]=1.$$

Combining the above we have

$$P_x^2\left[\log\prod_{k\leqq k'}\phi(\sigma_k)/\phi(\tau_k)\right]<\varepsilon \tag{3.5}$$

where $\varepsilon>0$ depends only on δ. Denote the above k' by k_l, and τ_k, σ_k by τ_{lk}, σ_{lk}, (recall $n_1=m_l$, $n_2=m_l'$). The intervals $[\tau_{lk},\sigma_{lk})$, $1\leqq k\leqq k_l$, $1\leqq l\leqq l_1$, (l_1 arbitrary), are disjoint. Let the complement in $[0,\zeta_1)$ of their union be the union of disjoint intervals $[\tau_{0k},\sigma_{0k})$, $1\leqq k\leqq k_0$, (again some may be empty), where $\tau_{01}\leqq\sigma_{01} \leqq\ldots\leqq\tau_{0k_0}\leqq\sigma_{0k_0}$ are $\mathscr{F}_t^2$-stopping times. Of course $P_x^2\left[\prod_{k\leqq k_0}\phi(\sigma_{0k})/\phi(\tau_{0k})\right]=1$. Now

$$\phi(\zeta_1)=\phi(\zeta_1)/\phi(0)=\prod_{l=0}^{l_1}\prod_{k=1}^{k_l}\phi(\sigma_{lk})/\phi(\tau_{lk}).$$

So by (3.5) and Lemma 3.4

$$-\infty<P_x^2[\log\phi(\zeta_1)]=\sum_{l=0}^{l_1}P_x^2\left[\log\prod_{k=0}^{k_l}\phi(\sigma_{kl}/\sigma(\tau_{kl})\right]<-l_1\varepsilon.$$

This is a contradiction since l_1 is arbitrary. □

Corollary 3.6 *For any x and* $\delta>0$, $\sup_{n_2>n_1} P_x^2[\sup_t |\Phi_{n_2}(t)-\Phi_{n_1}(t)|>\delta]\to 0$ *as* $n_1\to\infty$.

Proof. Fix n and $b>0$ and let $S=\zeta_1\wedge\inf\{t:\Phi_n(t)>b\}$. Then $\Phi_n(S)\geqq b$ if $\sup_t \Phi_n(t)>b$. Since $P_x^2[\Phi_n(S)]\leqq 1$ by Lemma 3.3, $P_x^2[\sup_t \Phi_n(t)>b]\leqq 1/b$ for all n. Thus

$$P_x^2[\sup_t |\Phi_{n_2}(t)-\Phi_{n_1}(t)|>\delta]\leqq P_x^2\left[\sup_t\left|\frac{\Phi_{n_2}(t)}{\Phi_{n_1}(t)}-1\right|>\delta/b\right]+1/b$$

and the corollary follows from Theorem 3.5. □

Definition. *For each x let integers* $1=n_1(x)<\ldots<n_k(x)<\ldots$ *be defined inductively by*

$$n_{k+1}(x)=\inf\{n>n_k(x): \sup_{n'>n} P_x^2[\sup_t |\Phi_{n'}(t)-\Phi_n(t)|>2^{-k}]<2^{-k}\}$$

(the $n_k(x)$ *exist by Corollary* 3.6*); then set*

$$\Phi_\infty(t)=\varliminf_k \Phi_{n_k(X_0^2)}(t).$$

Theorem 3.7 (i) A.s. *(*P_μ^2 *for any finite measure* μ*)* $\Phi_{n_k(X_0^2)}(t)\to\Phi_\infty(t)$ *uniformly in t.*

(ii) A.s. $\Phi_\infty(t)$ *is right continuous;* $\Phi_\infty(0)=1$; $\Phi_\infty(t)=\Phi_\infty(t\wedge\zeta_1)$; $\Phi_\infty(t)$ *is adapted to* $\mathscr{F}_t^2$; $\Phi_\infty(t)$ *is trajectory-dependent, and is constant off U.*

(iii) $P_x^2[\Phi_\infty(S)]\leqq 1$ *for any* $\mathscr{F}_t^2$*-stopping time* $S\leqq\zeta_1$.

(iv) *For* $\mathscr{F}_t^2$*-stopping time T, S with* $T+S\circ\theta_T\leqq\zeta_1$, $\Phi_\infty(T+S\circ\theta_T)=\Phi_\infty(T)\cdot\Phi_\infty(S)\circ\theta_T$ a.s.

(v) $Z_t\cdot\Phi_\infty(t)|_U\sim X_t^1|_U$, *(from the above* $\Phi_\infty(t)$ *is a* TMF *in* Z_t*).*

(vi) A.s. $\Phi_\infty(t)$ *is strictly positive on* $[0,\zeta_1]$ *(or on* $[0,\infty)$*).*

Proof. (i) follows form the definition of $n_k(x)$. (ii) follows from (i), the corresponding properties of $\Phi_n(t)$ and the fact that $\{x: n_k(x)=n\}\in\mathscr{E}_\Delta^*$. (iii) follows from (i) and Lemma 3.3. To show (iv) holding a.s. P_x^2, choose a subsequence $n_{k(i)}(x)$ of $n_k(x)$ such that a.s. P_x^2, $n_{k(i)}(x)>n_i(X_T^2)$ for all sufficiently large i; then using the fact that, if $n(i)>n_i(y)$ for all large i, $\Phi_{n(i)}(t)$ and $\Phi_{n_i(y)}(t)$ converge uniformly in t to the same limit a.s. P_y^2, and using the fact $\Phi_n(T+S)\circ\theta_T)=\Phi_n(T)\cdot\Phi_n(S)\circ\theta_T$ except possibly on $\{X_T^2\in U-C_n\}$, which set $\downarrow\emptyset$ since $C_n\uparrow U$, one obtains the desired equality. (v) follows from Lemma 3.2(iii) since $\Phi_\infty(t)=\Phi_n(t)=\phi(t)$ for $t\leqq T_V=T_{E_\Delta-U}$. It remains to show (vi). Let

$$\hat{\Phi}_n(t)=\phi(t)/\Phi_n(t).$$

Then we can write

$$\hat{\Phi}_n(t)=\prod_{j=1}^{\infty}\frac{\phi(\tau_{nj}\wedge t)}{\phi(\gamma_{nj}\wedge t)}$$

where $\gamma_{n1} \leqq \tau_{n1} \leqq \dots \leqq \gamma_{nj} \leqq \tau_{nj} \leqq \dots$ are $\mathscr{F}_t^2$-stopping times with $\gamma_{nj} = \tau_{nj} = \zeta_1$ for all sufficiently large j; in fact the union of $[\gamma_{nj}, \tau_{nj})$ is $[0, \zeta_1) - \bigcup_{j=0}^{\infty} [T_{nj}, R_{nj})$. As is with $\Phi_n(t)$ we have $P_x^2[\hat{\Phi}_n(S)] \leqq 1$ for $\mathscr{F}_t^2$-stopping times $S \leqq \zeta_1$. Since $\hat{\Phi}_{n_2}(t)/\hat{\Phi}_{n_1}(t) = \Phi_{n_1}(t)/\Phi_{n_2}(t)$, Theorem 3.5 implies that for all x and $\delta > 0$

$$\sup_{n_2 > n_1} P_x^2[\sup_t |\hat{\Phi}_{n_2}(t)/\hat{\Phi}_{n_1}(t) - 1| > \delta] \to 0 \quad \text{as } n_1 \to \infty.$$

It follows that Corollary 3.6 also holds for $\hat{\Phi}_n(t)$. Let $\hat{n}_k(x)$ be defined in a similar way to $n_k(x)$ but with the additional requirement of being a subsequence of $n_k(x)$: $\hat{n}_1(x) = 1$ and

$$\hat{n}_{k+1}(x) = \inf\{n > \hat{n}_k(x) : n = n_i(x) \text{ for some } i, \\ \sup_{n' > n} P_x^2[\sup_t \hat{\Phi}_{n'}(t) - \hat{\Phi}_n(t)| > 2^{-k}] < 2^{-k}\}.$$

Define $\hat{\Phi}_\infty(t) = \underline{\lim}_k \hat{\Phi}_{\hat{n}_k(X_0^2)}(t)$. Then a.s. $\hat{\Phi}_{\hat{n}_k(X_0^2)}(t) \to \hat{\Phi}_\infty(t)$ uniformly in t; so a.s.

$$\phi(t) = \lim_k \Phi_{\hat{n}_k(X_0^2)}(t)\, \hat{\Phi}_{\hat{n}_k(X_0^2)}(t) = \Phi_\infty(t)\, \hat{\Phi}_\infty(t) \quad \text{on } [0, \zeta_1].$$

Since $\phi(t)$ is strictly positive on $[0, \zeta_1]$, (vi) follows. □

We now define Ψ_t on $[0, \zeta_1]$ by

$$\Psi_t = \Phi_\infty(t), \qquad t \leqq \zeta_1$$

and extend to $t \in [0, \infty)$ by the requirement that Ψ_t be a MF (in X_t^2) and be constant off U (using the fact that $\Psi_t, t \leqq \zeta_1$, is constant off U). To make this definition precise, let T_n be defined by $T_0 = 0$ and

$$T_{2k+1} = T_{2k} + T_{\bar{U}} \circ \theta(T_{2k}), \qquad k \geqq 0;$$
$$T_{2k} = T_{2k-1} + T_{F_1} \circ \theta(T_{2k-1}), \qquad k \geqq 1.$$

Of course $T_n = \infty$ for sufficiently large n. Define

$$\Psi_t = \prod_{k=0}^{\infty} \Phi_\infty(\zeta_1 \wedge (t - T_{2k+1})^+) \circ \theta(T_{2k+1})$$
$$= \begin{cases} \prod_{k=0}^{m-1} \Phi_\infty(\zeta_1) \circ \theta(T_{2k-1}) & \text{if } T_{2m} \leqq t \leqq T_{2m+1} \\ \Psi(T_{2m-1}) \cdot \Phi_\infty(t - T_{2m+1}) \circ \theta(T_{2m+1}) & \text{if } T_{2m+1} < t < T_{2m+2} \end{cases}$$

(an empty product stands for 1).

Proof of Theorem 3.1 From Theorem 3.7 and the definition of Ψ_t we have: a.s. Ψ_t is right continuous, $\Psi_0 = 1$, $\Psi_t > 0$ on $[0, \infty)$, and Ψ_t is constant off U (note Ψ_t is constant on $[T_{2m}, T_{2m+1}]$). It is routine to show that Ψ_t is adapted to $\mathscr{F}_t^2$, and that for two $\mathscr{F}_t^2$-stopping times T, S we have $\Psi(T + S \circ \theta_T)$

$=\Psi(T)\cdot\Psi(S)\circ\theta_T$ a.s. We now show $P_x^2[\Psi_t]\leqq 1$ for all x, t. For this it suffices to show $P_x^2[\Psi(t\wedge T_n)]\leqq 1$. Now

$$P_x^2[\Psi(t\wedge T_n)]=P_x^2[\Psi(t); t<T_{n-1}]+P_x^2[\Psi(t\wedge T_n); t\geqq T_{n-1}].$$

The second term on the right hand side is $P_x^2[\Psi(T_{n-1}); t\geqq T_{n-1}]$ if $n-1$ is even, (since Ψ_t is constant on $[T_{2m}, T_{2m+1}]$), and if $n-1$ is odd it is

$$\begin{aligned} &P_x^2[\Psi(T_{n-1})\cdot\Phi_\infty(\zeta_1\wedge(t-T_{n-1}))\circ\theta(T_{n-1}); t\geqq T_{n-1}]\\ &\quad=P_x^2[\Psi(T_{n-1})\,P^2_{X^2(T_{n-1})}\{\Phi_\infty(\zeta_1\wedge(t-T_{n-1}))\}; t\geqq T_{n-1}]\\ &\quad\leqq P_x^2[\Psi(T_{n-1}); t\geqq T_{n-1}]\end{aligned}$$

by Theorem 3.7(iii). Therefore,

$$P_x^2[\Psi(t\wedge T_n)]\leqq P_x^2[\Psi(t\wedge T_{n-1})]\leqq\ldots\leqq P_x^2[\Psi(t\wedge T_1)]\leqq 1.$$

Thus Ψ_t is a TMF in X_t^2. Finally, the assertion $X_t^2\cdot\Psi_t|_U\sim X_t^1|_U$ is just Theorem 3.7(v). □

4 The global process and associated multiplicative functional

We prove Theorem 1 using two methods. The first method is simpler but not quite rigorous (but perhaps could satisfy some readers); the second method is involved but rigorous. The two methods are not essentially different in the case of Theorem 1. The proof of Theorem 2 uses an extension of the second method.

4.1 First proof of Theorem 1

Here $E_m\uparrow E$. We may assume $E_m\subset\bar{E}_m\subset E_{m+1}$. Choose $U_m\in\mathscr{U}$ such that $E_{m-1}\subset\bar{U}_m\subset E_m$. Denote $F_m=E_m^c$, $q_m=q_{F_m}$ and $\mathscr{D}_m=\{D\in\mathscr{D}: F_m\subset D\}$. Let X_t^m be the right process X_t constructed in Sect. 2 with $G=E_m$, and J_t^m be the MF in all X_t^N, $N\geqq m$, defined by

$$J_t^m=q_m(X_0^N)\,q_m(X_t^N)^{-1}.$$

Then

$$Q_D(x,f)=P_x^m[f(X^m(T_D))\,J^m(T_D)],\qquad x\in E_\Delta,\qquad D\in\mathscr{D}_m. \tag{4.1}$$

Let $\zeta_m=T_{F_m}$. Let Ψ_t^m be the TMF (in X_t^{m+1}) constructed in Sect. 3 with $G_1=E_m$, $G_2=E_{m+1}$, $U=U_m$ and X_t^1, X_t^2 there being X_t^m, X_t^{m+1}. Thus

$$\Psi_t^m=J_t^{m+1}/J_t^m=\frac{q_{m+1}}{q_m}(X_0^{m+1})\,\frac{q_m}{q_{m+1}}(X_t^{m+1}),\qquad\text{for } t\leqq T_{U_m^c} \tag{4.2}$$

and

$$X_t^{m+1}\cdot\Psi_t^m|_{U_m}\sim X_t^m|_{U_m}. \tag{4.3}$$

Now the TMF Ψ_t^{m-1} in X_t^m depends only on the trajectories of $X_t^m|_{E_{m-1}}$ and is constant off U_{m-1}. Since $E_{m-1}\subset U_m$, (4.3) implies that the trajectory-dependent Ψ_t^{m-1} can be regarded as a TMF in $X_t^{m+1}\cdot\Psi_t^m$, first defined up to time ζ_{m-1}, then to all t by the requirement that it be a MF and be constant off U_{m-1}, by the procedure at the end of Sect. 3, (note a slight abuse of notation is involved because this Ψ_t^{m-1} should be written as $\Psi_t^{m+1,m-1}$). Thus we can form the transformed process $(X_t^{m+1}\cdot\Psi_t^m)\cdot\Psi_t^{m-1}$, which will be written as $X_t^{m+1}\cdot\Psi_t^m\,\Psi_t^{m-1}$. (It is not difficult to argue that Ψ_t^{m-1} exists as a MF in X_t^{m+1} and $\Psi_t^m\,\Psi_t^{m-1}$ is a TMF. However we will use a different method in Subsect. 4.2 to construct directly (in X_t^{m+1}) TMF's $\Psi_t^m\,\Psi_t^{m-1}$ and $\Psi_t^m\,\Psi_t^{m-1}\dots\Psi_t^k$ below.) It satisfies

$$X_t^{m+1}\cdot\Psi_t^m\,\Psi_t^{m-1}|_{U_{m-1}}\sim X_t^{m-1}|_{U_{m-1}},\qquad X_t^{m+1}\cdot\Psi_t^m\,\Psi_t^{m-1}|_{U_m}\sim X_t^m\cdot\Psi_t^{m-1}|_{U_m}.$$

Proceeding in this manner we construct processes

$$(\dots(X_t^{m+1}\cdot\Psi_t^m)\dots)\cdot\Psi_t^k=X_t^{m+1}\cdot\Psi_t^m\dots\Psi_t^k$$

(again Ψ_t^j in the above should be written as $\Psi_t^{m+1,j}$), which satisfy

$$X_t^{m+1}\cdot\Psi_t^m\dots\Psi_t^k|_{U_k}\sim X_t^k|_{U_k}$$

and more importantly

$$X_t^{m+1}\cdot\Psi_t^m\dots\Psi_t^k|_{U_m}\sim X_t^m\cdot\Psi_t^{m-1}\dots\Psi_t^k|_{U_m}. \tag{4.4}$$

Define right processes Y_t^m on E_Δ by

$$Y_t^1=X_t^1|_{U_1};\qquad Y_t^m=X_t^m\cdot\Psi_t^{m-1}\dots\Psi_t^1|_{U_m},\qquad m\geqq 2.$$

Since each X_t^m is a Hunt process, each Y_t^m is a standard process; we again call U_m the proper state space of Y_t^m (each $x\in V_m=U_m^c$ is absorbing), and its proper lifetime $\zeta_m'=T_{U_m^c}$ is finite. By (4.4)

$$Y_t^{m+1}|_{U_m}\sim Y_t^m,\qquad m\geqq 1. \tag{4.5}$$

From the theorem in [8], (4.5) implies that there exists a right process Y_t on E_Δ (which is a standard process with lifetime $T_\Delta=\lim_m\zeta_m'$) such that

$$Y_t|_{U_m}\sim Y_t^m,\qquad m\geqq 1. \tag{4.6}$$

Let us define the desired MF M_t in Y_t. First, if X_t is a right process on E_Δ with death point Δ and if Ψ_t is a TMF in X_t that is strictly positive on $[0, T_\Delta)$, then $\Psi_t^{-1}\cdot 1_{[t<T_\Delta]}$, which we simply write as Ψ_t^{-1}, is a MF of the transformed process $\tilde{X}_t=X_t\cdot\Psi_t$, (noting that both X_t and $\tilde{X}_t$ have sample space Ω — see the beginning paragraph of Sect. 3, and we regard Ψ_t as defined in $\tilde{X}_t$ as well with the understanding (or requirement) that if $\omega_1(s)=\omega_2(s)\in E$ for all $s\in[0,t]$ then $\Psi_t(\omega_1)=\Psi_t(\omega_2)$), and

$$\tilde{P}^x[f(X_t\cdot\Psi_t)\,\Psi_t^{-1}]=P^x[f(X_t)\,\Psi_t\,\Psi_t^{-1}]=P^x[f(X_t)],\qquad f\in b\mathscr{E}_\Delta \text{ (with } f(\Delta)=0).$$

Now in $X_t^m \cdot \Psi_t^{m-1} \dots \Psi_t^1$ we have the MF $(\Psi_t^{m-1} \dots \Psi_t^1)^{-1}$, (again the TMF $\Psi_t^{m-1} \dots \Psi_t^1$ is to be rigorously defined in Subsect. 4.2), which satisfies

$$(4.7) \qquad \tilde{P}_x^m[f(X_t^n \cdot \Psi_t^{m-1} \dots \Psi_t^1)(\Psi_t^{m-1} \dots \Psi_t^1)^{-1}] \\ = P_x^m[f(X_t^m)(\Psi_t^{m-1} \dots \Psi_t^1)(\Psi_t^{m-1} \dots \Psi_t^1)^{-1}] = P_x^m[f(X_t)].$$

Define M_t^m by

$$(4.8) \qquad M_t^1 = J_{t \wedge \zeta_1'}^1; \qquad M_t^m = J_{t \wedge \zeta_m'}^m (\Psi_{t \wedge \zeta_m'}^{m-1} \dots \Psi_{t \wedge \zeta_m'}^1)^{-1}, \qquad m \geqq 2.$$

M_t^m is a MF in Y_t^m. We have the consistency

$$(4.9) \qquad M_t^{m+1} = M_t^m, \qquad t \leqq \zeta_m'$$

since by (4.2)

$$J_t^{m+1}(\Psi_t^m)^{-1} = J_t^{m+1}(J_t^{m+1}/J_t^m)^{-1} = J_t^m, \qquad t \leqq \zeta_m'.$$

Since all MFs are trajectory-dependent, (4.6) implies that M_t^m can be regarded as defined in Y_t. The consistency (4.9) permits the definition

$$(4.10) \qquad \begin{aligned} M_t &= M_t^m \quad \text{on } [0, \zeta_m'], \quad m \geqq 1 \\ &= 0 \quad \text{on } [T_\Delta = \lim_m \zeta_m', \infty]. \end{aligned}$$

M_t is obviously a MF in Y_t. It remains to show equality (1.1). For $x \in E$, $D \in \mathscr{D}$ (so $U_m^c \subset D$ for some m), $f \in b\mathscr{E}_\Delta$, and with P^x, P_x^m, $\hat{P}_x^m$ denoting the probability measures for Y_t, X_t^m, Y_t^m and writing $T = T_D$, (also noting $M_t = 0$ on $[T_\Delta, \infty]$)

$$\begin{aligned} P^x[f(Y_T) M_T] &= \hat{P}_x^m[f(Y_T^m) M_T^m] = \hat{P}_x^m[f(Y_T^m) J_T^m (\Psi_T^{m-1} \dots \Psi_T^1)^{-1}] \\ &= P_x^m[f(X_T^m)(\Psi_T^{m-1} \dots \Psi_T^1) J_T^m (\Psi_T^{m-1} \dots \Psi_T^1)^{-1}] \\ &= P_x^m[f(X_T^m) J_T^m] = Q_D f(x) \end{aligned}$$

by (4.10), (4.5), the definition of Y_t^m, the computation (4.7), and (4.1).

4.2 Second proof of Theorem 1

Fix $N \geqq 2$. We will construct directly in X_t^N MFs Ψ_t^m, $1 \leqq m < N$, that appeared in 4.1, and show that $\Psi_t^{N-1} \dots \Psi_t^m$ are TMFs. Let C_{mn} be compact sets with $C_{m0} = \emptyset$, $C_{mn} \uparrow U_m$ as $n \to \infty$ and C_{mn} increasing in m for each n. With X_t^N, X_t^m (respectively X_t^{m+1}), E_N, E_m (respectively E_{m+1}), and U_m playing the roles of X_t^2, X_t^1, G_2, G_1 and U in Sect. 3, and with $C_{mn} = C_n$, we obtain a TMF $\bar{\Psi}_t^{Nm}$ (respectively $\tilde{\Psi}_t^{Nm}$) that is the TMF Ψ_t in Sect. 3. (Note $\tilde{\Psi}_t^{N,N-1} \equiv 1$.) Thus

$$X_t^N \cdot \bar{\Psi}_t^{Nm}|_{U_m} \sim X_t^m|_{U_m}, \qquad X_t^N \cdot \tilde{\Psi}_t^{Nm}|_{U_m} \sim X_t^{m+1}|_{U_m}.$$

Recall the approximating functionals $\Phi_n(t)$ for $\Phi_\infty(t) = \Psi_{t \wedge \zeta_1}$ in Sect. 3; if we extend the definition of $\Phi_n(t)$ to $t \leqq \zeta_2$, by replacing $\zeta_1 = T_{F_1}$ by $\zeta_2 = T_{F_2}$ in the definition of $\Phi_n(t)$, then obviously we have a.s. $\Phi_{n_k(X_0^2)}(t) \to \Psi(t)$ uniformly on $[0, \zeta_2]$ (or on $[0, \infty)$), where $n_k(x)$ is similarly defined (see below). Let $\bar{\Phi}_n^{Nm}(t)$ (respectively $\tilde{\Phi}_n^{Nm}(t)$) denote the approximating functional $\Phi_n(t)$ when $\Psi(t)$ is the above $\bar{\Psi}^{Nm}(t)$ (respectively $\tilde{\Psi}^{Nm}(t)$), (they will be explicitly defined in the proof of Theorem 4.1 below). Thus a.s. in X_t^N

$$\bar{\Phi}^{Nm}_{n_k(X_0^N)}(t) \to \bar{\Psi}^{Nm}(t), \qquad \tilde{\Phi}^{Nm}_{n_k(X_0^N)} \to \tilde{\Psi}^{Nm}(t)$$

uniformly in t on $[0, \zeta_N]$, where $n_1(x)=1$ and

$$n_{k+1}(x)=\inf\{n>n_k(x): \sup_{n'>n} P_x^N[\sup_t |\bar{\Phi}^{Nm}_{n'}(t)-\bar{\Phi}^{Nm}_n(t)|>2^{-k}]<2^{-k}, \text{and}$$
$$\sup_{n'>n} P_x^N[\sup_t |\tilde{\Phi}^{Nm}_{n'}(t)-\tilde{\Phi}^{Nm}_n(t)|>2^{-k}]<2^{-k}, \text{for } 1\leqq m<N\}.$$

Since a.s. $\tilde{\Psi}^{Nm}(t)>0$ for all t, we can define

$$\Psi^m(t)=\Psi^{Nm}(t)=\bar{\Psi}^{Nm}(t)/\tilde{\Psi}^{Nm}(t).$$

$\Psi^m(t)$ is a MF (but not a TMF) in X_t^N. We have a.s. $\bar{\Phi}^{Nm}_n(t)/\tilde{\Phi}^{Nm}_n(t)\to\Psi^m(t)$ uniformly on $[0,\zeta_N]$, as $n=n_k(X_0^N)\to\infty$.

Theorem 4.1 *$\Psi_t^{N-1}\,\Psi_t^{N-2}\ldots\Psi_t^m$ is a* TMF *in X_t^N for $1\leqq m<N$.*

Proof. We need only to prove the case $m=1$. Let

$$L_n(t)=\prod_{m=1}^{N-1}\bar{\Psi}^{Nm}_n(t)/\tilde{\Phi}^{Nm}_n(t).$$

It suffices to prove that for any x and $\mathscr{F}_t^N$-stopping time $S\leqq\zeta_N$, we have $P_x^N[L_n(S)]\leqq 1$. Define for $1\leqq m_1<m_2\leqq N$

(4.11) $$\phi_t^{m_2m_1}=J_t^{m_2}/J_t^{m_1}=(q_{m_2}\,q_{m_1}^{-1})(X_0^N)(q_{m_2}^{-1}\,q_{m_1})(X_t^N)$$

and to simplify expressions write for $r\leqq s$ (and only when $X_t^N\in U_{m_1}$ for $r\leqq t<s$)

$$\phi_t^{m_2m_1}[r,s)=\phi^{m_2m_1}(s)/\phi^{m_2m_2}(r).$$

Define $\mathscr{F}_t^N$-stopping times T_{nj}^m, R_{nj}^m, $1\leqq m<N$, $n\geqq 1$, $j\geqq 0$ as follows (for a fixed m, these are the stopping times T_{nj}, R_{nj} in Sect. 3 when $C_{mn}=C_n$ and $U_m=U$, but with $\zeta_1=T_{F_1}$ there replaced by $\zeta_2=T_{F_2}$; here ζ_2 is $\zeta_N=T_{F_N}$), where $V_m=U_m^c$:

(4.12) $$T_{n0}^m=0;\qquad R_{nj}^m=T_{nj}^m+T_{V_m}\circ\theta(T_{nj}^m);$$
$$T_{n,j+1}^m=R_{nj}^m+T_{F_N\cup C_{mn}}\circ\theta(R_{nj}^m).$$

Again, for sufficiently large j, $T_{nj}^m=R_{nj}^m=\zeta_N$. Note $[T_{nj}^m, R_{nj}^m)\subset\{t: X_t^N\in U_m\}$. Now for $m\leqq m_1\leqq m_2\leqq N$ define

(4.13) $$\Phi_n^{m_2m_1m}(x)=\prod_{j=0}^{\infty}\phi^{m_2m_1}[T_{nj}^m\wedge t, R_{nj}^m\wedge t),$$
$$\Phi_n^{m_2m}(t)=\Phi^{m_2mm}(t).$$

(Note $\bar{\Phi}^{Nm}_n(t)=\Phi^{Nmm}_n(t)$, $\tilde{\Phi}^{Nm}_n(t)=\Phi_n^{N,m+1,m}(t)$, and so since $\phi_t^{m+1,m}=\phi_t^{Nm}/\phi_t^{N,m+1}$ we have a.s.

$$\Phi_n^{m+1,m}(t)=\Phi_n^{Nmm}(t)/\Phi_n^{N,m+1,m}(t)\to\Psi^m(t), \text{as } n=n_k(X_0^N)\to\infty.)$$

Now define $\mathscr{F}_t^N$-stopping times τ_{nj}^m, γ_{nj}^m with $\tau_{n0}^m \leqq \gamma_{n0}^m \leqq \ldots \leqq \tau_{nj}^m \leqq \gamma_{nj}^m \leqq \ldots$ such that $[\tau_{nj}^1, \gamma_{nj}^1) = [T_{nj}^1, R_{nj}^1)$, and for $m \geqq 2$, $\bigcup_j [\tau_{nj}^m, \gamma_{nj}^m) = \bigcup_j [T_{nj}^m, R_{nj}^m) - \bigcup_j [T_{nj}^{m-1}, R_{nj}^{m-1})$, (note $\bigcup_j [T_{nj}^m, R_{nj}^m)$ is increasing in m). We have

$$\begin{aligned} L_n(t) &= \prod_{m=1}^{N-1} \Phi_n^{m+1,m}(t) = \prod_{m=1}^{N-1} \prod_{j=1}^{\infty} \phi^{m+1,m}[T_{nj}^m \wedge t, R_{nj}^m \wedge t) \\ &= \prod_{m=1}^{N-1} \prod_{j=0}^{\infty} \prod_{k=m}^{N-1} \phi^{k+1,k}[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t) \\ &= \prod_{m=1}^{N-1} \prod_{j=1}^{\infty} \phi^{Nm}[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t). \end{aligned}$$

As alway this is a finite product. As in Lemma 3.3, to show $P_x^N[L_n(S) \leqq 1]$ is suffices to prove

$$P_x^N\left[\prod_{m=1}^{N-1} \prod_{j=1}^{j_1} \phi^{Nm}[\tau_{nj}^m \wedge S, \gamma_{nj}^m \wedge S)\right] \leqq 1. \tag{4.14}$$

As in Lemma 3.2, for $\mathscr{F}_t^N$-stopping times $T_1 \leqq T_2 \leqq \zeta_N$ with $[T_1, T_2) \subset \{t: X_t^N \in U_m\}$, and $\rho \in \mathscr{F}^N(T_1)^+$

$$P_x^N[\rho\, \phi^{Nm}(T_2)/\phi^{Nm}(T_1)] = P_x^N[\rho] \tag{4.15}$$

whose proof relies only on the fact that $P_y^N[\phi^{Nm}(\zeta_m)] = 1$ for $y \in E_m$. By rearranging the stochastic intervals $[\tau_{nj}^m \wedge S, \gamma_{nj}^m \wedge S)$ in all possible increasing orders and applying (4.15) repeatedly to the probability in (4.14), we obtain (4.14). □

We now have the transformed processes $X_t^N \cdot \Psi_t^{N-1} \ldots \Psi_t^1$. They satisfy

$$X_t^{N+1} \cdot \Psi_t^N \Psi_t^{N-1} \ldots \Psi_t^1|_{U_m} \sim X_t^N \cdot \Psi_t^{N-1} \ldots \Psi_t^1|_{U_m}.$$

For $X_t^{N+1} \cdot \Psi_t^N|_{U_m} \sim X_t^N|_{U_m}$ and in these processes, $\Psi_t^{N-1} \ldots \Psi_t^1$ are the "same" trajectory-dependent functional; alternatively, one can argue directly that for $U_m^c \subset D$, $f \in b\mathscr{E}_\Delta$

$$P_x^{N+1}[f(X_T^{N+1})\, \Psi_T^N \Psi_T^{N-1} \ldots \Psi_T^1] = P_x^N[f(X_T^N)\, \Psi_T^{N-1} \ldots \Psi_T^1]$$

where $T = T_D$. The rest of the proof of Theorem 1 (involving the definition of Y_t^N, Y_t, M_t^N and M_t) is identical to that in Subsect. 4.1 (after (4.4)).

4.3 *Proof of Theorem 2*

Assume as we may that the E_m are all distinct, $\{E_m\}$ is closed w.r.t. finite non-empty intersection, and $E_n \not\subset E_m$ if $m < n$. Choose open U_m with $\bar{U}_m \subset E_m$ such that $\{U_m\}$ is a covering of E, $U_m \subset U_n$ if $E_m \subset E_n$, and $U_m = U_{m_1} \cap \ldots \cap U_{m_k}$ if $E_m = E_{m_1} \cap \ldots \cap E_{m_k}$. Let compact sets $C_{mn} \uparrow U_m$ with $C_{m0} = \emptyset$ and satisfy $C_{mn} = C_{m_1 n} \cap \ldots \cap C_{m_k n}$ if $E_m = E_{m_1} \cap \ldots \cap E_{m_k}$.

We write $m_1 \dot{<} m_2$ if $E_{m_1} \subset E_{m_2}$ and $m_1 < m_2$ (so that $E_{m_1} \neq E_{m_2}$). If $m_1 \dot{<} m_2$, we write $m_2 \backslash m_1 = 1$ if there exists no m with $m_1 \dot{<} m \dot{<} m_2$ and $m_2 \backslash m_1 > 1$ otherwise. If $m_1 \dot{<} m_2$, a "route" connecting m_1 and m_2 is a sequence $(n_1, \ldots, n_k)$ with $m_1 = n_1 \dot{<} n_2 \dot{<} \ldots \dot{<} n_k = m_2$ such that $n_{i+1} \backslash n_i = 1$. Two routes $(n_1, \ldots, n_k)$, $(n'_1, \ldots, n'_l)$ connecting m_1 and m_2 are said to be distinct if $n_i \neq n'_j$ except $n_1 = n'_1, n_k = n'_l$. If $m_1 \dot{<} m_2$, the number of distinct routes connecting m_1, m_2 is denoted $\nu(m_1, m_2)$.

For each m we have a right process X_t^m on E_Δ with proper state space E_m and finite proper lifetime $\zeta_m = T_{F_m}$, and an associated MF J_t^m, as in Subsect. 4.1.

Fix N for which there exist $m \dot{<} N$. For $m_1 \dot{<} m_2 \dot{\leqq} N$ ($m_2 \dot{<} N$ or $m_2 = N$), let $\Psi_t^{m_2 m_1}$ be the MF in X_t^N defined in a similar way to the MF Ψ_t^m at the beginning of Subsect. 4.2, with the roles of X_t^m, X_t^{m+1} and U_m replaced by $X_t^{m_1}$, $X_t^{m_2}$ and U_{m_1}. That is, with $\bar{\Psi}_t^{Nm}$ (resp. $\tilde{\Psi}_t^{Nm_2m_1}$) denoting the TMF Ψ_t in Sect. 3 when X_t^N, $X_t^{m_1}$ (respectively $X_t^{m_2}$), E_N, E_{m_1} (respectively E_{m_2}) and U_{m_1} play the roles of X_t^2, X_t^1, G_2, G_1 and U in Sect. 3, we have

$$\Psi_t^{m_2 m_1} = \bar{\Psi}_t^{N m_1} / \tilde{\Psi}_t^{N m_2 m_1}.$$

(A precise definition of $\Psi_t^{m_2 m_1}$ is in the proof of Theorem 4.2 below.)

Definition.

$$\hat{\Psi}_t^N = \prod_{\substack{m_1 \dot{<} m_2 \dot{\leqq} N \\ m_2 \backslash m_1 = 1}} \Psi_t^{m_2 m_1} \prod_{\substack{m_1 \dot{<} m_2 \dot{\leqq} N \\ m_2 \backslash m_1 > 1}} (\Psi_t^{m_2 m_1})^{-\nu(m_1, m_2)+1}.$$

Note that in the situation of Theorem 1, i.e. that in Subsect. 4.2, $\hat{\Psi}_t^N$ reduces to $\Psi_t^{N-1} \ldots \Psi_t^1$.

Theorem 4.2 (i) *$\hat{\Psi}_t^N$ is a* TMF *in X_t^N.*
(ii) *If $N \dot{<} N'$, $X_t^{N'} \cdot \hat{\Psi}_t^{N'}|_{U_N} \sim X_t^N \cdot \hat{\Psi}_t^N|_{U_N}$.*

Proof. The proof of (i) is similar to that of Theorem 4.1. Define, for $m_1 \dot{<} m_2 \dot{\leqq} N$, $\phi_t^{m_2 m_1}$ as in (4.11). Define, for $m \dot{<} N$, $\mathscr{F}_t^N$-stopping times T_{nj}^m, R_{nj}^m as in (4.12). Then define for $m \dot{\leqq} m_1 \dot{<} m_2 \dot{\leqq} N$ the approximating functionals $\Phi_n^{m_2 m_1 m}(t)$ and $\Phi_n^{m_2 m}(t)$ as in (4.13). We have a.s.

$$\Phi_n^{m_2 m_1}(t) \to \Psi^{m_2 m_1}(t) \text{ uniformly in } t, \quad \text{as } n = n_k(X_0^N) \to \infty$$

where $n_k(x)$ is defined in the same way as in Subsect. 4.2 to guarantee a sufficient number of a.s. uniform convergences. ($\Psi_t^{m_2 m_1}$ is defined to be $\varliminf_k \Phi_{n_k(X_0^N)}^{m_2 m_1}(t)$. Since all $\Psi_t^{m_2 m_1}$ are strictly positive MFs $\hat{\Psi}_t^N$ is a well-defined MF. To show it is a TMF, it suffices to show $P_x^N[L_n(S)] \leqq 1$ for an $\mathscr{F}_t^N$-stopping time $S \leqq \zeta_N$, where

$$L_n(t) = \prod_{\substack{m_1 \dot{<} m_2 \dot{\leqq} N \\ m_2 \backslash m_1 = 1}} \Phi_n^{m_2 m_1}(t) \cdot \prod_{\substack{m_1 \dot{<} m_2 \dot{\leqq} N \\ m_2 \backslash m_1 > 1}} (\Phi_n^{m_2 m_1}(t))^{-\nu(m_1, m_2)+1}.$$

Define for $m \dot{<} N$ disjoint (increasing) stochastic intervals $[\tau_{nj}^m, \gamma_{nj}^m)$ by the requirement

$$\bigcup_j [\tau_{nj}^m, \gamma_{nj}^m) = \bigcup_j [T_{nj}^m, R_{nj}^m) - \bigcup_{m' \dot{<} m} \bigcup_j [T_{nj}^{m'}, R_{nj}^{m'})$$

(the set difference being proper), so

$$\bigcup_j [T_{nj}^{m_1}, R_{nj}^{m_1}) = \bigcup_{m \leqq m_1} \bigcup_j [\tau_{nj}^m, \gamma_{nj}^m).$$

Now

$$\begin{aligned} L_n(t) = & \prod_{\substack{m_1 \dot{<} m_2 \dot{\leqq} N \\ m_2 \backslash m_1 = 1}} \prod_j \phi^{m_2 m_1}[T_{nj}^{m_1} \wedge t, R_{nj}^{m_1} \wedge t) \\ & \cdot \prod_{\substack{m_1 \dot{<} m_2 \dot{\leqq} N \\ m_2 \backslash m_1 > 1}} \prod_j (\phi^{m_2 m_1}[T_{nj}^{m_1} \wedge t, R_{nj}^{m_1} \wedge t))^{-\nu(m_1, m_2)+1} \\ = & \prod_{m \dot{<} N} \prod_j \Big[\prod_{\substack{m \dot{\leqq} m_1 \dot{<} m_2 \dot{\leqq} N \\ m_2 \backslash m_1 = 1}} \phi^{m_2 m_1}[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t) \\ & \cdot \prod_{\substack{m \dot{\leqq} m_1 \dot{<} m_2 \dot{\leqq} N \\ m_2 \backslash m_1 > 1}} (\phi^{m_2 m_1}[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t))^{-\nu(m_1, m_2)+1} \Big]. \end{aligned}$$

The product inside the brackets simplifies to $\phi^{Nm}[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t)$. The reason is roughly that if $m_1 \dot{<} m_2$ and $\nu(m_1, m_2) = \nu$, then there exist ν distinct routes connecting m_1, m_2, and corresponding to each such route $(m_1 = n_1, n_2, \ldots, n_k = m_2)$ the first product inside the above brackets contains the subproduct $(\phi^{n_2 n_1} \phi^{n_3 n_2} \ldots \phi^{n_k n_{k-1}})[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t) = \phi^{m_2 m_1}[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t)$, and now its ν^{th} power multiplied by the corresponding factor in the second product inside the brackets is $\phi^{m_2 m_1}[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t)$. Successive "cancellations" like this will finally yield the simplified expression. Therefore,

$$L_n(t) = \prod_{m \dot{<} N} \prod_j \phi^{Nm}[\tau_{nj}^m \wedge t, \gamma_{nj}^m \wedge t).$$

Now the rest of the proof of (i) is the same as the last part (the simplification of $L_n(t)$ there) of the proof of Theorem 4.1. To prove (ii), we claim

$$\hat{\Phi}_t^{N'} = \phi_t^{N'N} \hat{\Phi}_t^N \quad \text{on } [0, T_{U_N^c}]. \tag{4.16}$$

For on $[0, T_{U_N^c}]$, if $N \dot{\leqq} m_1 \dot{<} m_2 \dot{\leqq} N'$ then $\Psi_t^{m_2 m_1} = \phi_t^{m_2 m_1}$; so

$$\prod_{\substack{N \dot{\leqq} m_1 \dot{<} m_2 \dot{\leqq} N' \\ m_2 \backslash m_1 = 1}} \Psi_t^{m_2 m_1} \prod_{\substack{N \dot{\leqq} m_1 \dot{<} m_2 \dot{\leqq} N' \\ m_2 \backslash m_1 > 1}} (\Psi_t^{m_2 m_1})^{-\nu(m_1, m_2)+1} = \phi_t^{N'N}$$

by the same argument as the one used to simplify $L_n(t)$, and (4.16) follows from this and the definitions of Φ_t^N and $\Phi_t^{N'}$. Since $X_t^{N'} \cdot \phi_t^{N'N}|_{U_N} \sim X_t^N|_{U_N}$, (ii) is established. □

Define for each N

$$\begin{aligned} Y_t^N &= X_t^N|_{U_N} && \text{if there exist no } m \dot{<} N, \\ &= X_t^N \cdot \hat{\Psi}_t^N|_{U_N} && \text{if there exist } m \dot{<} N. \end{aligned}$$

Y_t^N is a right process (actually a standard process) on E_Δ with proper state space U_N and finite proper lifetime $\zeta_N' = T_{U_N^c}$. From (ii) of Theorem 4.2 the family $\{Y_N\}$ is consistent, in the sense that for any N, N' we have $Y_N|_{U_N \cap U_{N'}} \sim Y_{N'}|_{U_N \cap U_{N'}}$.

Again by the theorem in [8] there exists a right process (actually a standard process) Y_t on E_Δ such that

$$Y_t|_{U_N} \sim Y_t^N, \qquad N \geqq 1.$$

Define a MF M_t^N in Y_t^N by

$$\begin{aligned} M_t^N &= J_{t\wedge\zeta_N}^N && \text{if there exist no } m \dot{<} N \\ &= J_{t\wedge\zeta_N}^N (\hat{\Psi}_{t\wedge\zeta_N}^N)^{-1} && \text{if there exist } m \dot{<} N. \end{aligned}$$

(Compare with (4.8)). The trajectory-dependent M_t^N are consistent: $M_t^N = M_t^{N'}$ on $[0, \zeta'_N \cap \zeta'_{N'}]$; see (4.9). Therefore we can define a MF M_t in Y_t is an obvious way so that $M_t = M_t^N$ on $[0, \zeta'_N]$ for all N. That (Y_t, M_t) is as desired is proved as at the end of Subsect. 4.1.

References

1. Bliedtner, J., Hansen, W.: Potential theory. Berlin Heidelberg New York: Springer 1986
2. Blumenthal, R.M., Getoor, R.K.: Markov processes and potential theory. New York: Academic Press 1968
3. Constantinescu, C., Cornea, A.: Potential theory in harmonic spaces. Berlin Heidelberg New York: Springer 1972
4. Getoor, R.K.: Markov processes: Ray processes and right processes. (Lect. Notes Math., vol. 440) Berlin Heidelberg New York: Springer 1975
5. Ito, K., Watanabe, S.: Transformation of Markov processes by multiplicative functionals. Ann. Inst. Fourier **15**, 13–30 (1965)
6. Sharpe, M.J.: General theory of Markov processes. San Diego: Academic Press 1988
7. Shih, C.T.: Construction of right processes from hitting distributions. In: Çinlar, E. et al. (eds.) Seminar on stochastic processes, pp. 189–236. Boston Basel Stuttgart: Birkhäuser 1984
8. Shih, C.T.: On piecing together locally defined processes. In: Çinlar, E. et al. (eds.) Seminar on stochastic processes, pp. 321–333. Boston Basel Stuttgart: Birkhäuser 1991
9. Shih, C.T.: Construction of Markov processes from hitting distributions without quasi-left-continuity. In: Çinlar, E. et al. (eds.) Seminar on stochastic processes, pp. 187–231. Boston Basel Stuttgart: Birkhäuser 1992