THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

VISCOUS FLOW THROUGH SMALL CLEARANCES WITH APPLICATION TO THE PROBLEM OF LEAKAGE IN RECIPROCATING PUMPS

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LIST OF SYMBOLS

A = exponent in the viscosity-temperature relationship.

B = exponent in the viscosity-temperature relationship.

C = exponent in the viscosity-temperature relationship.

C_p = specific heat per pound at constant pressure.

C_v = specific heat per pound at constant volume.

D = a constant.

d = width of clearance.

E = exponent in the viscosity-temperature relationship.

e = eccentricity of the piston with respect to the cylinder.

F = frictional force.

f = a constant.

G = a constant.

g = gravitational acceleration.

H = enthalpy.

h = enthalpy per pound.

i = a constant.

j = exponent in the viscosity-temperature relationship.

L = length of piston.

ℓ = length of connecting rod.

m = a dummy variable.

N = dimensionless number.

n = attitude.

P = pressure of the oil.

P_1 = initial pressure of the oil (pressure difference).

p = dimensionless pressure.

LIST OF SYMBOLS (CONT'D)

Q = rate of flow or leakage.

q = rate of heat transfer.

R = radius of the piston or cylinder bore.

r = radius of crank.

T = temperature.

t = time, dimensionless temperature.

U = overall coefficient of heat transfer.

u = velocity at any point inside the clearance.

V = maximum velocity of the piston.

v = velocity of the piston.

W = rate of work done by the piston.

Y = dimensionless ratio.

Z = dimensionless ratio.

α = ratio of flow number to velocity number.

α_c = thermal coefficient of expansion of the cylinder material.

α_p = thermal coefficient of expansion of the piston material.

β = dimensionless ratio.

Γ = gamma function.

γ = specific weight.

Δ = average clearance.

ϵ = an angle.

η = dimensionless length.

θ = velocity number.

Λ = dimensionless ratio

λ = ratio of adiabatic flow to isothermal flow.

LIST OF SYMBOLS (CONT'D)

μ = absolute viscosity.

ν = kinematic viscosity.

ξ = an angle.

ρ = density.

σ = dimensionless kinematic viscosity.

τ = shear stress.

φ = flow or leakage number.

χ = an angle.

ψ = pressure number.

ω = angular velocity of crank.

I. INTRODUCTION

The problem of leakage of oils in reciprocating pumps is essentially a problem of flow between two coaxial cylinders due to a pressure difference with one cylinder moving relative to the other. Since the clearance between the two cylinders is usually much smaller than the radii of the cylinders, the cylinders can be treated as two parallel plates with no end effects.

The flow between two parallel plates and between two coaxial cylinders has been under investigation for some time. In most of these investigations the viscosity of the flowing fluid as well as its temperature, has been assumed constant. This approach produced satisfactory results only in those cases in which the isothermal assumption (constant viscosity and temperature) was appropriate.

Becker[2] investigated the flow of water, air and steam through thin annular slits between coaxial cylinders using small clearances. His results, on the flow of water, show that the laws of laminar flow hold satisfactorily up to a Reynolds number of 1000, with the clearance taken as the characteristic length.

Davies and White[6] investigated the flow of water between stationary parallel plates using clearances as small as 0.0154 cm. The Reynolds number varied from 60 to 4600, the clearance being taken as the characteristic length. They concluded that:

1. The transition from laminar to turbulent flow occurred at a Reynolds number of about 1000. This number becomes 2000 when the criteria of the hydraulic radius is used.

2. Below this number, the distance along the plates in which a disturbance can survive, after it has been created, is a function of the Reynolds number only.
3. There exists a lower limit of 140 for the Reynolds number below which this distance is zero. This means that below this number (highly laminar flow), a disturbance cannot travel at all and must die out as soon as it is created.
4. Surface roughness up to 2 percent of the clearance (distance between plates) has no measurable effect on laminar flow.

In the problem of leakage in reciprocating pumps, the oil leaks from the pressure chamber through the annular clearance between the piston and the cylinder. The face of the piston acts as a source of disturbances. According to the conclusions of reference 6, these disturbances cannot travel in the annulus and must die out immediately, if the flow in the annulus is highly laminar, i.e., the Reynolds number is less than 140. This makes the shape of the face of the piston completely unimportant in such cases.

The surface roughness is another source of disturbances. In laminar flow through pipes, ordinary surface roughness produces disturbances which ultimately die out and the overall problem is not affected.[20] However, if the height of the roughness compared to the diameter of the pipe is so large that it obstructs the flow, then the flow problem is affected.[7] For laminar flow between parallel plates, the size of the surface roughness below which the flow problem is not affected is, according to reference 6, 2 percent of the clearance. While larger roughness may decrease the flow, and hence the leakage in reciprocating pumps, it may also reduce the lubricating quality of the surfaces necessary for high speed operation.

Piercy and Winny,[18] and Cornish[5] showed mathematically that the flow due to a pressure difference between two coaxal cylinders having maximum eccentricity is two and one-half times that with no eccentricity at all (perfect concentricity).

The above investigators assumed isothermal conditions (constant viscosity and temperature) and neglected the energy dissipated by friction.

Teichmann[19] suggested the calculation of the rise in the temperature of the fluid assuming that the flow takes place adiabatically. Then from this and the initial temperature of the fluid, the average viscosity may be determined and used in Poisseuille's equation for the flow between two parallel plates to yield a refined result over that using the viscosity at the initial temperature.

Wilson and Mitchell[21] mathematically investigated the flow between parallel plates due to a pressure difference with one plate moving at constant velocity relative to the other plate, taking into consideration the energy dissipated by friction and the subsequent change in viscosity. Adiabatic flow was assumed. No experimental work was reported.

II. VISCOUS FLOW THROUGH SMALL CLEARANCES WITH RELATIVE MOTION OF THE BOUNDARY

A. Idealization of the Problem

The following mathematical analysis is carried out in conformity with the technique used in the experimental work. A constant pressure difference exists across a piston throughout each cycle of reciprocation while the piston reverses its motion in each cycle. The following assumptions are introduced in order to simplify the mathematical work:

1. The problem of the flow through small annuli is mathematically equivalent to that of the flow between two parallel plates with no end effects. Hence only the latter will be analyzed.
2. The motion of the piston is caused by the rotation of a crank. In Appendix A, an expression is derived for the velocity of the piston. It consists of several harmonics. Since the ratio of the length of the crank to the length of the connecting rod in the experimental set-up was made 1/13, and this ratio is fairly small, it will be assumed that the motion of the piston is simple harmonic.
3. It will be assumed that the axes of the piston and the cylinder remain parallel during operation.

Furthermore, in the operating range of temperature and pressure, it will be assumed that

4. The fluid is incompressible and its density does not change with temperature. Hyde[12] found that at a pressure of 2000 psi the densities of some lubricating oils changed only by 1 percent. The specific gravity of the oils used in this work change very

little with temperature, as can be seen from the properties of these oils given in Appendix D.

5. The viscosity of the fluid depends on the temperature only. It does not change with pressure. Hersey and Snyder[11] report that at 3600 psi, the viscosity of castor oil increases by about 45 percent. Hyde,[12] and Hersey and Shore[10] found that at 2000 psi the viscosities of some lubricating oils changed from about 30 to 40 percent.

The eccentricity of the piston with respect to the cylinder must be accounted for in the equivalent problem of parallel plates. This means that one plate must be flat while the other should have a cylindrical form with its elements parallel to the flat plate and in the direction of the flow. Hence the clearance, or the distance between the plates, is constant in the direction of the flow but variable in the other direction, which is perpendicular to the first.

The idealized problem then consists of an incompressible viscous vluid flowing between two plates under the influence of a pressure difference. One plate is flat and stationary and the other is cylindrical in form and has a simple harmonic motion.

B. Mathematical Analysis

Figure 1 shows the coordinate system chosen. u_1 is the velocity distribution at any instance in the fluid due to the motion of the cylindrical plate alone, while u_2 is that due to the pressure difference alone. From Lamb,[13] the momentum equation in the x direction gives

$$u_1 = \pm \frac{v}{d} y, \tag{2-1}$$

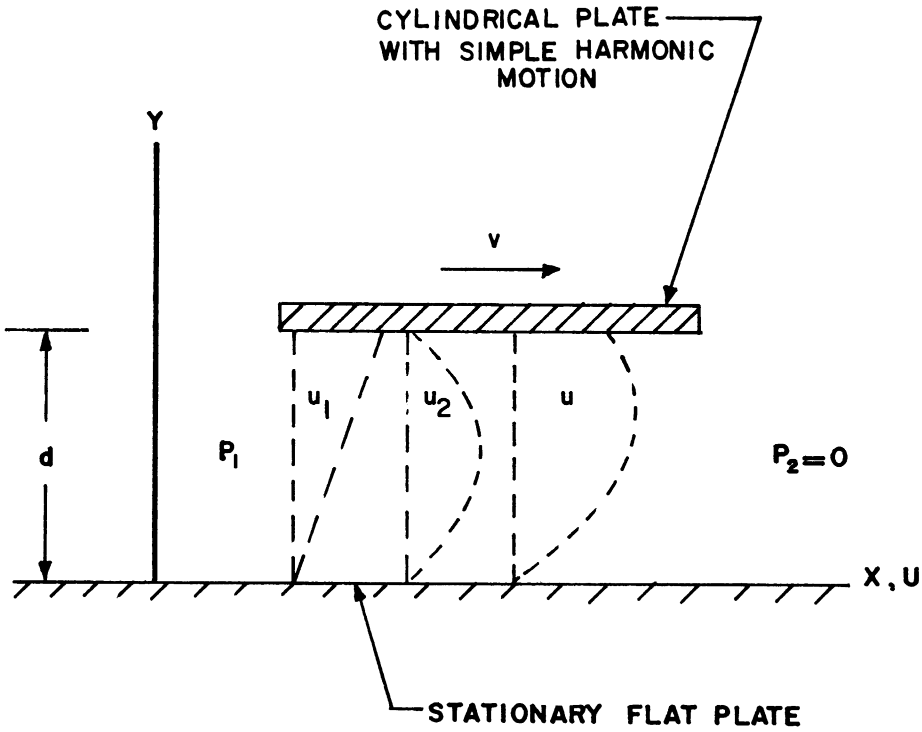

Figure 1. The Coordinate System Used in the Mathematical Analysis

and

$$u_2 = -\frac{1}{2\mu}\frac{dP}{dx}(yd - y^2) \quad . \tag{2-2}$$

The combined velocity distribution u is the sum of u_1 and u_2. Hence,

$$u = \pm \frac{v}{d} y - \frac{1}{2\mu}\frac{dP}{dx}(yd - y^2) \quad . \tag{2-3}$$

The average flow rate Q during each cycle is due to the pressure difference only since the velocity distribution u_1 increases the flow rate during one-half of the cycle and decreases it by the same amount during the other half of the cycle. Hence,

$$Q = -\frac{R}{2\mu}\frac{dP}{dx}\int_0^{2\pi}\int_0^{d}(yd - y^2)\,dy\,d\xi$$

$$= -\frac{R}{12\mu}\frac{dP}{dx}\int_0^{2\pi} d^3 d\xi \quad . \tag{2-4}$$

The relationship between d and ξ is developed in Appendix B.

$$d = \Delta(1 - n\cos\xi) \quad . \tag{2-5}$$

Substituting in Equation (2-4) and integrating, gives

$$Q = -\frac{\pi R \Delta^3}{6\mu}\frac{dP}{dx}\left(1 + \frac{3}{2}n^2\right) \quad . \tag{2-6}$$

If the viscosity and temperature are assumed constant, then it is customary to let the pressure be a linear function of the length x, so that

$$\frac{dP}{dx} = -\frac{P_1}{L} \quad .$$

Also,

$$\mu = \mu_1 \quad .$$

Hence, the flow rate based on the assumption of constant viscosity will be designated as Q_o and is given as

$$Q_o = \frac{\pi R \Delta^3 P_1 \ (1 + 3/2 \ n^2)}{6\mu_1 \ L} \quad , \qquad (2\text{-}7)$$

and will be referred to as the isothermal flow rate.

If the temperature is not assumed constant but allowed to vary, then the viscosity would vary too. The temperature change results from the dissipation of energy into heat by the action of the viscous forces. A portion of the energy required to move the cylindrical plate would be dissipated in this manner. In other words, the cylindrical plate does some work on the fluid in the annulus. The result is to raise the temperature of the fluid. This rise in temperature would in turn initiate some heat transfer from the fluid to the plates, provided that these plates are heat conductors and are at a lower temperature. There are cases in which the fluid in the annulus does work on the cylindrical plate and consequently, the temperature of the fluid decreases. However, such cases fall in limited ranges of velocities and pressures which are not practically significant, and hence will not be taken up here.

In Figure 2, the element ABCD of thickness dx is taken as a free body. The element is enlarged in Figure 3 with all the quantities that contribute to a heat balance shown on it. In steady state operation, the temperature of the cylindrical plate (piston) reaches a steady state value and the heat transfer from the fluid in the annulus to this plate becomes very small. For this reason, only the heat transfer to the flat plate (cylinder wall) will be considered.

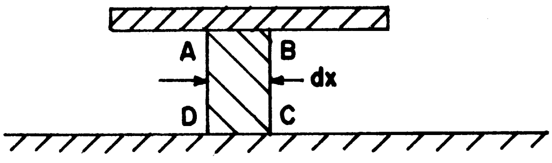

Figure 2. The Free Body

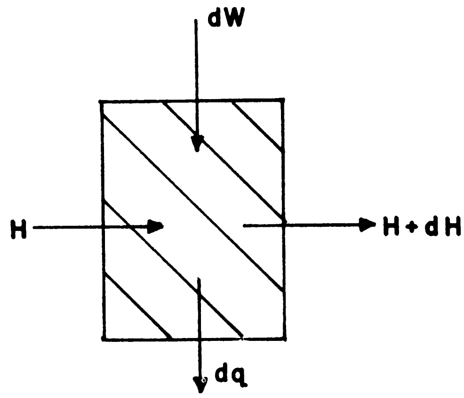

Figure 3. The Free Body with the Energy Terms

Applying the steady state energy equation to the element in Figure 3, gives

$$dH + dq = dW \quad .$$

On the bases of one pound of flowing fluid,

$$dh + \frac{dq}{\gamma Q} = \frac{dW}{\gamma Q} \quad . \qquad (2\text{-}8)$$

The rate of heat transfer dq is given as

$$dq = 2\pi RU\,(T - T_1)\,dx \quad , \qquad (2\text{-}9)$$

where T_1 is the temperature of the surrounding outside the flat plate. It will be assumed that this temperature is the same as the initial temperature of the fluid before entering the annulus.

The enthalpy of an incompressible fluid is

$$dh = C_v dT + \frac{1}{\gamma}\,dP \quad . \qquad (2\text{-}10)$$

Substituting Equations (2-9) and (2-10) in Equation (2-8), gives

$$C_v dT + \frac{2\pi RU}{\gamma Q}\,(T - T_1)\,dx = \frac{dW}{\gamma Q} - \frac{1}{\gamma}\,dP \quad . \qquad (2\text{-}11)$$

Dividing through by C_v and dx,

$$\frac{dT}{dx} + \frac{2\pi RU}{\gamma C_v Q}\,(T - T_1) = \frac{1}{\gamma C_v Q}\,\frac{dW}{dx} - \frac{1}{\gamma C_v}\,\frac{dP}{dx} \quad . \qquad (2\text{-}12)$$

Now

$$\frac{dW}{dx} = \pm\, v\,\frac{dF}{dx} \qquad (2\text{-}13)$$

where F is the shear force between the cylindrical plate and the fluid in the element ABCD. The plus or minus signs are inserted to account for both directions of motion of the cylindrical plate. Also,

$$\frac{dF}{dx} = R \int_{0}^{2\pi} \tau \, d\,\xi \qquad (2\text{-}14)$$

and

$$\tau = \mu \left(\frac{du}{dy}\right)_{y=d}$$

Using Equation (2-3),

$$\tau = \mu \left(\pm \frac{v}{d} + \frac{d}{2\mu}\frac{dP}{dx}\right) \quad . \qquad (2\text{-}15)$$

Substituting in Equation (2-14), gives

$$\frac{dF}{dx} = \mu \, R \int_{0}^{2\pi} \left(\pm \frac{v}{d} + \frac{d}{2\mu}\frac{dP}{dx}\right) d\xi \quad . \qquad (2\text{-}16)$$

Substituting Equation (2-5) in Equation (2-16) and integrating, gives

$$\frac{dF}{dx} = 2\,\pi\mu R \left(\pm \frac{v}{\Delta\sqrt{1-n^2}} + \frac{\Delta}{2\mu}\frac{dP}{dx}\right) \quad . \qquad (2\text{-}17)$$

Substituting in Equation (2-13),

$$\frac{dW}{dx} = 2\,\pi\mu R v \left(\frac{v}{\Delta\sqrt{1-n^2}} \pm \frac{\Delta}{2\mu}\frac{dP}{dx}\right) \quad . \qquad (2\text{-}18)$$

The velocity of the cylindrical plate varies with time during each cycle. Hence dW/dx varies with time. In order to eliminate the time dependency in Equation (2-12), the average value of dW/dx over the whole cycle will be determined and used in Equation (2-12).

Now

$$\left(\frac{dW}{dx}\right)_{ave.} = \frac{\int_{-\pi}^{\pi} \left(\frac{dW}{dx}\right) d(wt)}{2\pi} \quad . \qquad (2\text{-}19)$$

From Equation (2-18),

$$\frac{dW}{dx} = \frac{2\pi\mu R}{\Delta\sqrt{1-n^2}} v^2 \pm \pi R v \Delta \frac{dP}{dx} \quad .$$

From Appendix A,

$$v = V \sin(\omega t) \quad ,$$

where

$$V = \omega r \quad .$$

So that

$$\frac{dW}{dx} = \frac{2\pi\mu R V^2}{\Delta\sqrt{1-n^2}} \sin^2(\omega t) \pm \pi R V \Delta \frac{dP}{dx} \sin(\omega t) . \qquad (2\text{-}20)$$

Substituting in Equation (2-19) and integrating, gives

$$\left(\frac{dW}{dx}\right)_{ave.} = \frac{\pi\mu R V^2}{\Delta\sqrt{1-n^2}} \quad . \qquad (2\text{-}21)$$

Substituting Equation (2-21) in Equation (2-12), gives

$$\frac{dT}{dx} + \frac{2\pi R U}{\gamma C_v Q}(T - T_1) = \frac{\pi R \mu V^2}{\gamma C_v Q \Delta\sqrt{1-n^2}} - \frac{1}{\gamma C_v}\frac{dP}{dx} \quad . \qquad (2\text{-}22)$$

From Equation (2-6),

$$\frac{dP}{dx} = - \frac{6\, Q\, \mu}{\pi R \Delta^3 (1 + 3/2\, n^2)} \quad . \qquad (2\text{-}23)$$

Equation (2-22) is the energy equation, while Equation (2-23) is the momentum equation. Both of these equations must hold in the present problem.

The kinematic viscosity ν is given as

$$\nu = \frac{\mu}{\rho} \quad , \qquad (2\text{-}24)$$

and

$$\gamma = \rho\ g$$

Substituting Equation (2-23) in Equation (2-22), gives

$$\frac{dT}{dx} + \frac{2\pi RU}{\gamma C_v Q}(T - T_1) = \left(\frac{\pi\ R\ V^2}{gC_v Q\Delta\sqrt{1-n^2}} + \frac{6\ \rho\ Q}{\gamma C_v \pi R\Delta^3\ (1 + 3/2\ n^2)}\right)\nu \qquad (2\text{-}25)$$

and

$$\frac{dP}{dx} = -\frac{6\ \rho\ Q}{\pi R\Delta^3\ (1 + 3/2\ n^2)}\ \nu\ . \qquad (2\text{-}26)$$

At this point, it is convenient to introduce dimensionless variables.

Let

$$P = p\ P_1 \qquad (2\text{-}27)$$

$$T = t\ T_1$$

$$x = \eta\ L$$

$$\nu = \sigma\ \nu_1$$

Also let

$$Y = \frac{6\ \rho Q\nu_1 L}{P_1 \pi R\Delta^3\ (1 + 3/2\ n^2)}$$

$$(2\text{-}28)$$

$$Z = \frac{\pi RV^2 \nu_1 L}{T_1 g C_v Q\Delta\sqrt{1-n^2}} + \frac{6\ \rho Q\nu_1 L}{T_1 \gamma C_v \pi R\Delta^3\ (1 + 3/2\ n^2)}\ ,$$

and

$$\Lambda = \frac{2\pi RUL}{\gamma C_v Q_o}$$

$$(2\text{-}29)$$

$$\lambda = \frac{Q}{Q_o}$$

Substituting in Equations (2-25) and (2-26), gives

$$\frac{dt}{d\eta} + \frac{\Lambda}{\lambda}(t - 1) = Z\,\sigma \ , \tag{2-30}$$

and

$$\frac{dp}{d\eta} = -\,Y\,\sigma \ . \tag{2-31}$$

It is significant to point out that the term

$$\frac{\Lambda}{\lambda}(t - 1)$$

in Equation (2-30) represents the contribution of the heat conductivity of the flat plate (cylinder wall) If Λ/λ is very large, then the problem approaches essentially an isothermal case with the temperature and viscosity of the fluid being constant. If Λ/λ is very small, then the problem approaches the adiabatic case with little or no heat conducted through the flat plate.

Rearranging Equation (2-30) gives

$$\frac{dt}{Z\sigma - \frac{\Lambda}{\lambda}(t - 1)} = d\eta \ . \tag{2-32}$$

If a suitable viscosity-temperature relationship is introduced, and σ is eliminated in terms of t, then equation (2-32) becomes ready for integration. After trying a few practical relationships and consulting various tables of integrals, this writer was unsuccessful in proceeding with the integration. The reason for this difficulty is that Equation (2-30) is a first order, nonlinear differential equation, and the resulting integral of Equation (2-32) is not an elementary integral. Hence, the work will be carried on for the adiabatic case only. This means that $\Lambda/\lambda = 0$. Equation (2-30) becomes

$$\frac{dt}{d\eta} = Z\,\sigma \quad . \tag{2-33}$$

Eliminating σ from Equations (2-31) and (2-33), yields

$$\frac{dt}{d\eta} = -\frac{Z}{Y}\frac{dp}{d\eta} \quad . \tag{2-34}$$

Integrating

$$\int_1^t dt = -\frac{Z}{Y}\int_1^p dp \quad .$$

Hence

$$t - 1 = -\frac{Z}{Y}(p - 1) \quad . \tag{2-35}$$

Applying the boundary condition that when

$$\eta = 1, \quad t_2 = \frac{T_2}{T_1} \ , \text{ and } p_2 = 0 \quad ,$$

Equation (2-35) becomes

$$T_2 - T_1 = \frac{Z}{Y}T_1 \quad . \tag{2-36}$$

Equation (2-36) is a general equation which holds true regardless of the viscosity-temperature relationship. However, this equation alone is not sufficient to predict the adiabatic rate of flow.

Equation (2-33) can be written as

$$\frac{1}{\sigma}\frac{dt}{d\eta} = Z \quad . \tag{2-37}$$

This equation can be integrated if a viscosity-temperature relationship is introduced.

There are several types of viscosity-temperature relationships for various hydraulic fluids and oils. The following are three such relationships:

1. $\nu = D\, e^{-AT}$

2. $\nu = G\, T^{-E}$

3. $\nu = e^{iT^{-j}}$

If the operating range of temperature is small, then it is possible to use any one of the above relationships without introducing large errors. The analysis will be carried out using each one of the above relationships.

FIRST CASE:

The relationship

$$\nu = D\, e^{-AT} \tag{2-38}$$

gives a straight line when plotted on semi-log paper. The temperature T is in degrees absolute. Using Equations (2-27), Equation (2-38) becomes

$$\sigma = e^{-AT_1(t-1)} \quad . \tag{2-39}$$

Substituting in Equation (2-37) and integrating, gives

$$\int_1^t e^{AT_1(t-1)}\, dt = Z \int_0^\eta d\eta$$

or

$$\frac{1}{AT_1} \left[e^{AT_1(t-1)} - 1 \right] = Z\, \eta \ . \tag{2-40}$$

Applying the boundary condition that when

$$\eta = 1,\ t = \frac{T_2}{T_1}\ , \tag{2-41}$$

Equation (2-40) becomes

$$e^{A(T_2 - T_1)} - 1 = AT_1 Z$$

or

$$T_2 - T_1 = \frac{1}{A} \ln (1 + AT_1 Z) \quad . \qquad (2\text{-}42)$$

Let

$$\alpha = \frac{Q}{\pi RV\Delta}$$

$$\theta = \frac{2\nu_1 VLA}{gC_v\Delta^2} \quad , \theta' = \frac{\theta}{1 + 3/2\ n^2}$$

$$\psi = \frac{AP_1}{\gamma C_v} \qquad (2\text{-}43)$$

$$\varphi = \alpha\theta \qquad , \varphi' = \alpha\theta'$$

$$N = \frac{1 + 3/2\ n^2}{\sqrt{1 - n^2}}$$

Combining Equations (2-28) and (2-43) gives

$$Y = \frac{3\ \varphi'}{\psi} \qquad (2\text{-}44$$

and

$$Z = \frac{3\ \varphi'}{AT_1} (1 + \frac{N}{6\alpha^2}) \quad . \qquad (2\text{-}45)$$

Using Equations (2-43), the isothermal result, Equation (2-7), becomes

$$\varphi_o' = \frac{\psi}{3} \quad , \qquad (2\text{-}46)$$

where

$$\varphi_o' = \alpha_o\ \theta'$$

and

$$\alpha_o = \frac{Q_o}{\pi R V \Delta}$$

Substituting Equation (2-46) in Equation (2-44) gives

$$Y = \frac{\varphi'}{\varphi'_o}$$

Now

$$\lambda = \frac{Q}{Q_o} = \frac{\varphi}{\varphi_o} = \frac{\varphi'}{\varphi'_o} \quad . \tag{2-47}$$

Hence

$$Y = \lambda \quad . \tag{2-48}$$

Substituting Equations (2-44) and (2-45) in Equations (2-36) and (2-42) gives

$$T_2 - T_1 = \frac{\psi}{A}\left(1 + \frac{N}{6\alpha^2}\right) \tag{2-49}$$

and

$$T_2 - T_1 = \frac{1}{A} \ln \left[1 + 3\varphi' \left(1 + \frac{N}{6\alpha^2}\right) \right] \quad . \tag{2-50}$$

Let

$$\beta = \psi \left(1 + \frac{N}{6\alpha^2}\right), \tag{2-51}$$

and using Equations (2-44) and (2-51), Equations (2-49) and (2-50) become

$$T_2 - T_1 = \frac{\beta}{A} \tag{2-52}$$

and

$$T_2 - T_1 = \frac{1}{A} \ln (1 + \lambda\beta) \quad . \tag{2-53}$$

Eliminating $(T_2 - T_1)$ from Equations (2-52) and (2-53) and rearranging gives

$$\lambda = \frac{1}{\beta} (e^{\beta} - 1) \quad . \tag{2-54}$$

For particular values of the pressure number ψ, the velocity number θ and the attitude n, the adiabatic temperature rise can be determined from Equation (2-52) and the flow number φ can be determined from Equation (2-54).

These two equations can be reduced to the isothermal case under certain conditions. If β is very small, then according to Equation (2-52), $T_2 \approx T_1$. If Equation (2-54) is expanded in a power series, then

$$\lambda = \frac{1}{\beta} (\beta + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + - - - -)$$

$$= 1 + \frac{\beta}{2!} + \frac{\beta}{3!} + - - - - \quad .$$

Assuming β very small, gives

$$\lambda \approx 1.$$

According to Equation (2-47), this means

$$Q \approx Q_o \quad .$$

It may be concluded then, that the isothermal assumption is adequate if β is very small. This means, according to Equation (2-51), that whenever the pressure difference and the velocity are very small, and the rate of flow is relatively large, the isothermal results are good, a conclusion that could have been arrived at intuitively.

The temperature variation along the plates can be determined from Equation (2-40). Substituting Equations (2-45), (2-51), and (2-54) in Equation (2-40) and rearranging gives

$$A\ (T - T_1) = \ln [1 + \eta\ (e^{\beta} - 1)] \quad . \qquad (2\text{-}55)$$

Figure 4 is a plot of Equation (2-55) for $\beta = 2$.

The pressure variation along the plates can be determined by combining Equations (2-35), (2-44), (2-45) and (2-55), which gives

$$p = \frac{P}{P_1} = 1 - \frac{1}{\beta} \ln\ [1 + \eta\ (e^{\beta} - 1)]. \qquad (2\text{-}56)$$

Figure 5 is a plot of Equation (2-56) for $\beta = 2$.

The viscosity change along the plates can be determined in a similar manner. Combining Equations (2-39), (2-40), (2-45) and (2-54) gives

$$\sigma = \frac{\nu}{\nu_1} = \frac{1}{1 + \eta\ (e^{\beta} - 1)} \quad . \qquad (2\text{-}57)$$

Figure 6 is a plot of Equation (2-57) for $\beta = 2$.

Again these equations can be reduced to the isothermal case. If β is very small, then Equation (2-55) can be written as

$$A\ (T - T_1) = \ln\ [1 + \eta\ (e^{\beta} - 1)] \approx \ln\ (1 + \eta\ \beta) \approx 0,$$

which means that the temperature T along the plates would be constant and equal to T_1.

Similarly, Equation (2-56) can be written as

$$p = \frac{P}{P_1} = 1 - \frac{1}{\beta} \ln [1 + \eta\ (e^{\beta} - 1)] \approx 1 - \frac{1}{\beta} \ln\ (1 + \eta\ B) \approx 1 - \eta$$

which is the equation for the linear pressure variation in isothermal flow.

Finally, Equation (2-57) can be written as

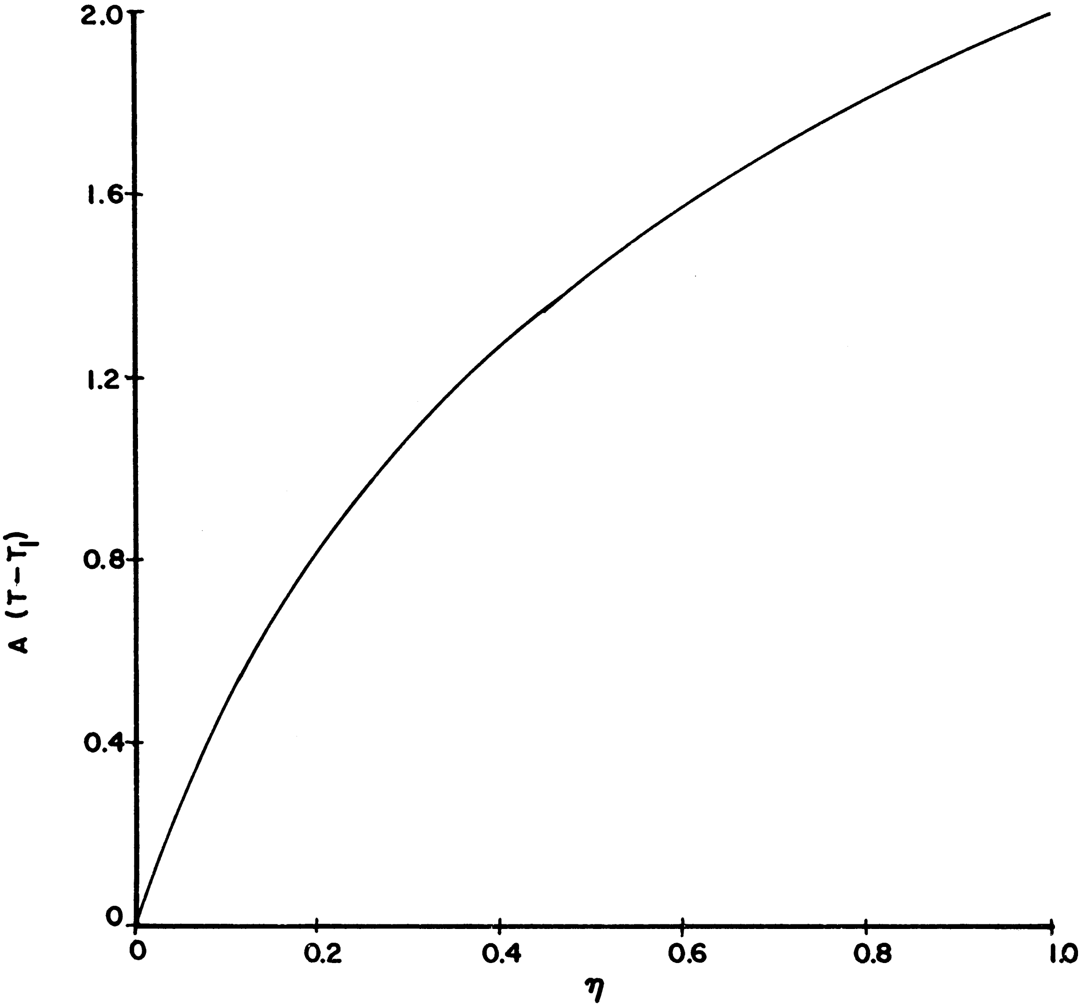

Figure 4. Temperature Variation Along the Plates for β = 2

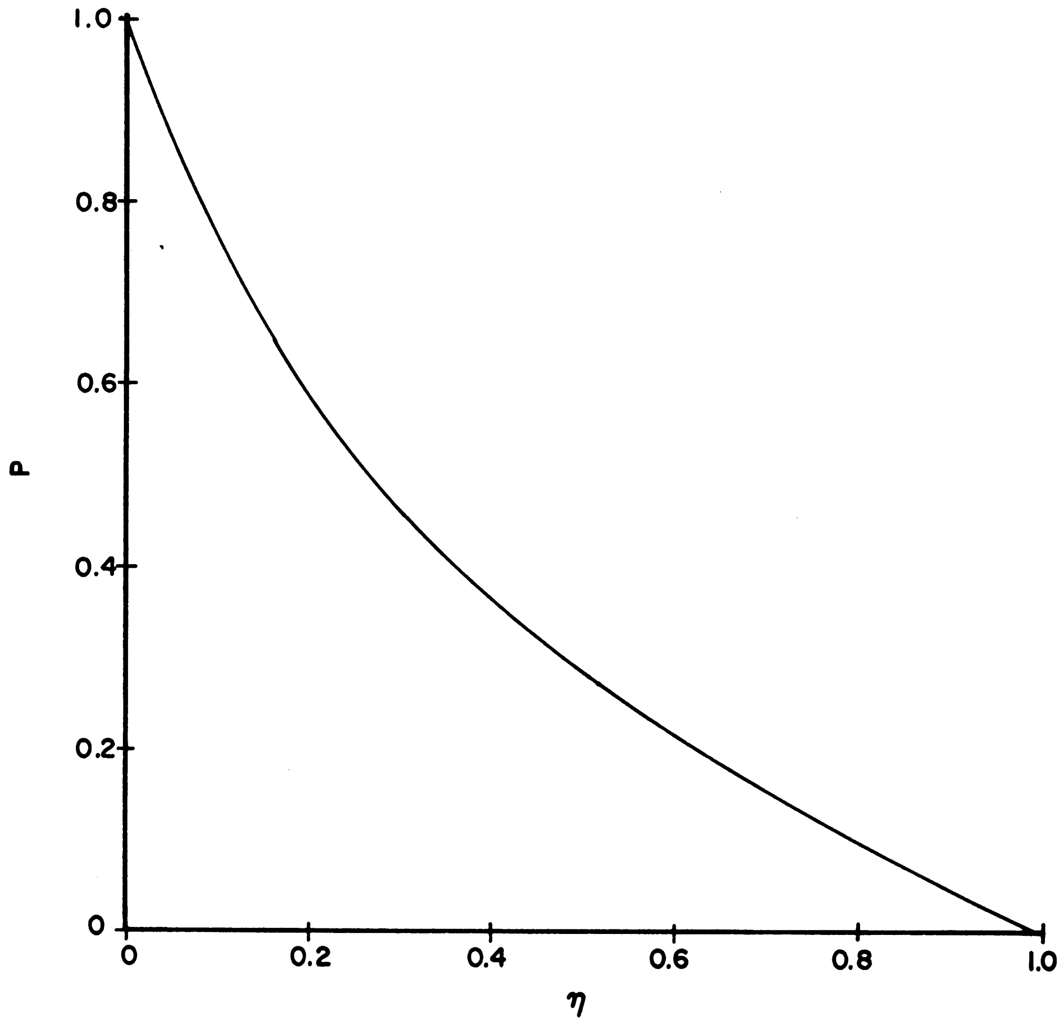

Figure 5. Pressure Variation Along the Plates for β = 2

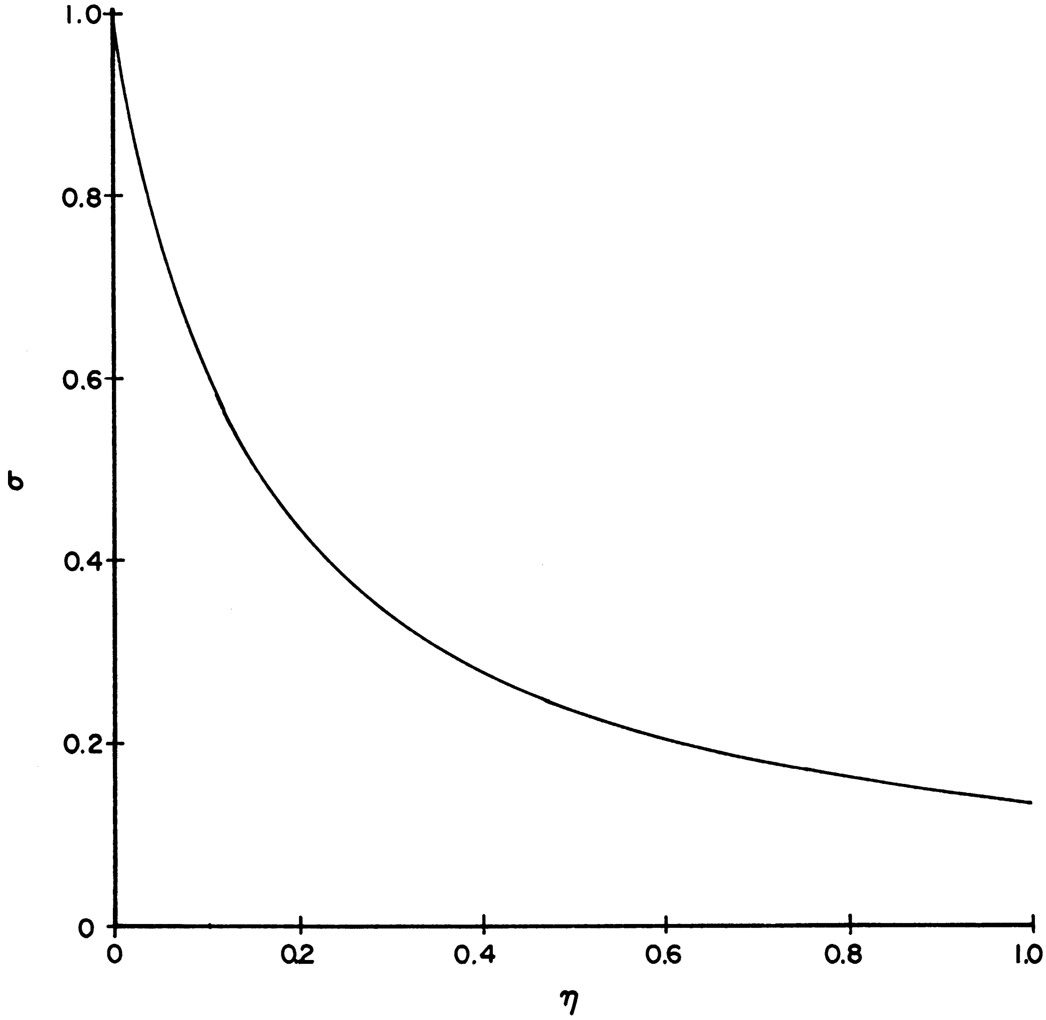

Figure 6. Viscosity Variation Along the Plates for β = 2

$$\sigma = \frac{\nu}{\nu_1} = \frac{1}{1 + \eta\,(e^{\beta} - 1)} \approx \frac{1}{1 + \eta\,\beta} \approx 1$$

showing that the kinematic viscosity would be constant, which is true in isothermal flow.

The average viscosity over the length of the plates can be obtained from Equation (2-57). Now

$$\sigma_{ave} = \frac{\int_0^1 \sigma\, d\,\eta}{1}$$

Substituting for σ from Equation (2-57) and integrating gives

$$\sigma_{ave} = \frac{\beta}{e^{\beta} - 1} \quad . \qquad (2\text{-}58)$$

Combining Equations (2-54) and (2-58) gives

$$\sigma_{ave} = \frac{\nu_{ave}}{\nu_1} = \frac{1}{\lambda} = \frac{Q_o}{Q} \quad . \qquad (2\text{-}59)$$

Eliminating Q_o from Equations (2-7) and (2-59) gives

$$Q = \frac{\pi R \Delta^3 P_1 (1 + 3/2\; n^2)}{6\;\mu_{ave}\; L} \quad . \qquad (2\text{-}60)$$

The comparison between Equation (2-7), for the isothermal flow rate Q_o, and Equation (2-60), for the adiabatic flow rate, is remarkable. The only difference between them is in the viscosity. Whereas the initial viscosity μ_1 is used in Equation (2-7) to determine the isothermal flow rate, the average viscosity μ_{ave} along the plates must be used in Equation (2-60) to determine the adiabatic flow rate. This confirms what has been speculated by earlier investigators.

However, the fact that the average viscosity is usually unknown makes this conclusion useful only after the above analysis has been taken into consideration. In fact it is more convenient to determine the adiabatic rate of flow from Equations (2-43), (2-51) and (2-54) rather than from Equation (2-60).

SECOND CASE:

In the relationship

$$\nu = G\, T^{-E} \quad ,$$

T is the temperature in degrees Fahrenheit. This equation gives a straight line when plotted on log-log paper. It is more convenient to modify the form of this equation as follows:

$$\nu = G\, T^{1-B} \qquad (2\text{-}61)$$

Using Equations (2-27) gives

$$\sigma = t^{1-B} \quad . \qquad (2\text{-}62)$$

Substituting in Equation (2-37) and integrating

$$\int_1^t t^{B-1}\, dt = Z \int_0^\eta d\eta \quad .$$

Hence

$$t = (1 + B\, Z\, \eta)^{1/B} \quad . \qquad (2\text{-}63)$$

Applying the boundary condition that when

$$\eta = 1,\ t = T_2/T_1 \quad ,$$

Equation (2-63) becomes

$$T_2/T_1 = (1 + BZ)^{1/B}$$

or

$$T_2 - T_1 = T_1 \; [(1 + B\,Z)^{1/B} - 1] \; . \tag{2-64}$$

Let

$$\alpha = \frac{Q}{\pi R V \Delta}$$

$$\theta = \frac{2\nu_1 VL}{g C_v \Delta^2 T_1} \; , \quad \theta' = \frac{\theta}{1 + 3/2\, n^2}$$

$$\psi = \frac{P_1}{\gamma C_v T_1}$$

$$\varphi = \alpha\,\theta, \qquad \varphi' = \alpha\,\theta' \tag{2-65}$$

$$N = \frac{1 + 3/2\, n^2}{\sqrt{1 - n^2}}$$

$$\beta = \psi\,(1 + \frac{N}{6\alpha^2})$$

$$\lambda = \frac{Q}{Q_o} = \frac{\varphi'}{\varphi_o'} = \frac{3\varphi'}{\psi}$$

Combining Equations (2-28) and (2-65) yields

$$Y = \frac{3\varphi'}{\psi} = \lambda \tag{2-66}$$

and

$$Z = 3\,\varphi'\,(1 + \frac{N}{6\alpha^2}) = \lambda\,\beta \quad . \tag{2-67}$$

Substituting in Equations (2-36) and (2-64) gives

$$T_2 - T_1 = \beta\, T_1 \tag{2-68}$$

and

$$T_2 - T_1 = T_1 \; [\; (1 + B \lambda \beta)^{1/B} - 1 \;] \; . \qquad (2\text{-}69)$$

Eliminating $(T_2 - T_1)$ from Equations (2-68) and (2-69) and rearranging

$$\lambda = \frac{1}{B\beta} \; [\; (1 + \beta)^B - 1 \;] \; . \qquad (2\text{-}70)$$

Equations (2-68) and (2-70) are sufficient to calculate the adiabatic temperature rise $(T_2 - T_1)$ and the flow number φ for particular values of the pressure number ψ, the velocity number Θ and the attitude n.

As in the First Case, Equations (2-68) and (2-70) reduce to the isothermal case, if β is very small.

The temperature variation along the plates can be determined from Equation (2-63). Combining Equations (2-63), (2-67) and (2-70) gives

$$t = \frac{T}{T_1} = \left\{1 + \eta \;\; [(1 + \beta)^B - 1]\right\}^{1/B} . \qquad (2\text{-}71)$$

The pressure variation along the plates, from Equations (2-35), (2-66), (2-67) and (2-71) is

$$p = \frac{P}{P_1} = 1 - \frac{1}{\beta} \; \left\{(1 + \eta \;\; [(1 + \beta)^B - 1])^{1/B} - 1\right\} . \; (2\text{-}72)$$

And similarly, the viscosity change along the plates, from Equations (2-62) and (2-71) is

$$\sigma = \frac{\nu}{\nu_1} = \left\{1 + \eta \;\; [(1 + \beta)^B - 1]\right\}^{1-B/B} \; . \qquad (2\text{-}73)$$

THIRD CASE:

The relation

$$\nu = e^{iT^{-j}}$$

holds fairly accurately, specifically for petroleum oils, but it is more complicated than the first two relationships. It gives a straight line when plotted on log log vs. log paper. The temperature here is in degrees absolute.

To make the results more convenient, this equation is put in the following form:

$$\nu = e^{(f/T)^{1/c}} \quad . \tag{2-74}$$

Combining Equations (2-27) and (2-74)

$$\sigma = \frac{1}{\nu_1} e^{t^{-1/c} \ln \nu_1} \tag{2-75}$$

Substituting in Equation (2-37) and integrating

$$\int_1^t \nu_1 \, e^{-t^{-1/c} \ln \nu_1} \, dt = Z \int_0^\eta d\eta \quad . \tag{2-76}$$

Let

$$t = \left(\frac{\ln \nu_1}{m}\right)^c \quad . \tag{2-77}$$

Hence

$$dt = - c \, \frac{(\ln \nu_1)^c}{m^{c+1}} \, dm \quad .$$

Substituting in Equation (2-76) and rearranging

$$\int_{\ln \nu_1}^{\ln \nu} e^{-m}\, m^{-c-1}\, dm = - \frac{Z}{c(\ln \nu_1)^c\, \nu_1}\, \eta \quad . \tag{2-78}$$

From Equation (2-74)

$$(\ln \nu_1)^c = \frac{f}{T_1} \quad .$$

Substituting in Equation (2-78) gives

$$\int_{\ln \nu_1}^{\ln \nu} e^{-m}\, m^{-c-1}\, dm = - \frac{ZT_1}{fc\nu_1}\, \eta . \tag{2-79}$$

The left-hand side of Equation (2-79) is an incomplete gamma function, so that

$$\Gamma(-c, \ln \nu) - \Gamma(-c, \ln \nu_1) = - \frac{Z\, T_1}{fc\nu_1}\, \eta \quad . \tag{2-80}$$

Applying the boundary condition that when

$$\eta = 1, \quad \nu = \nu_2 \ ,$$

then

$$\Gamma(-c, \ln \nu_2) - \Gamma(-c, \ln \nu_1) = - \frac{Z\, T_1}{fc\nu_1} \ . \tag{2-81}$$

Combining Equations (2-36), (2-74) and (2 81) gives

$$\Gamma(-c, \ln \nu_1) - \Gamma[-c, (1 + Z/Y)^{-1/c} \ln \nu_1] = \frac{Z\, T_1}{fc\nu_1} \ . \tag{2-82}$$

Let

$$\alpha = \frac{Q}{\pi R V \Delta}$$

$$\Theta = \frac{2\nu_1 VL}{g C_v \Delta^2 T_1} \quad , \quad \Theta' = \frac{\Theta}{1 + 3/2\ n^2}$$

$$\psi = \frac{P_1}{\gamma C_v T_1}$$

$$\varphi = \alpha\ \Theta, \qquad \varphi' = \alpha\ \Theta' \tag{2-83}$$

$$N = \frac{1 + 3/2\ n^2}{\sqrt{1 - n^2}}$$

$$\beta = \psi\ (1 + \frac{N}{6\alpha^2})$$

$$\lambda = \frac{T_1}{fc\nu_1} \frac{Q}{Q_o} = \frac{T_1}{fc\nu_1} \frac{\varphi'}{\varphi_o'}$$

Combining Equations (2-28) and (2-83) gives

$$Y = \frac{fc\nu_1}{T_1} \lambda \tag{2-84}$$

$$Z = \frac{fc\nu_1}{T_1} \lambda\ \beta\ . \tag{2-85}$$

Substituting in Equations (2-36) and (2-82)

$$T_2 - T_1 = \beta\ T_1 \tag{2-86}$$

and

$$\lambda = 1/\beta \left\{ \Gamma(-c, \ln \nu_1) - \Gamma[-c, (1 + \beta)^{-1/c} \ln \nu_1] \right\} . \tag{2-87}$$

Equations (2-86) and (2-87) are sufficient to determine the adiabatic temperature rise $(T_2 - T_1)$ and the flow number φ.

The viscosity variation alone the plates, from Equations (2-80), (2-83) and (2-87) is

$$\Gamma(-c, \ln \nu) = \Gamma(-c, \ln \nu_1) - \eta \{\Gamma(-c, \ln \nu_1) - \Gamma[-c, (1 + \beta)^{-1/c} \ln \nu_1]\} \ . \tag{2-88}$$

The temperature variation along the plates, from Equations (2-74) and (2-88) is

$$\Gamma[-c, (f/T)^{1/c}] = \Gamma(-c, \ln \nu_1) - \eta \{\Gamma(-c, \ln \nu_1) - \Gamma[-c, (1 + \beta)^{-1/c} \ln \nu_1]\} \ . \tag{2-89}$$

Similarly, the pressure variation along the plates, from Equations (2-27), (2-35), (2-84) and (2-85) is

$$p = \frac{P}{P_1} = 1 - \frac{T - T_1}{\beta\, T_1} \ . \tag{2-90}$$

where T is determined from Equation (2-89).

It is evident that the results of the third case are much more complicated than those of the first two cases. It also involves trial and error solutions, and the use of tables of Gamma functions. Also there is always the possibility that the desired functions corresponding to the particular parameters of the problem are not listed in the literature.

The results of the second case present an inconvenience.

Since the pressure number ψ is most advantageously used as a parameter in plotting φ vs. Θ, it would be more convenient to have ψ a function of the pressure difference P_1, and independent of the temperature T_1. The definition of ψ in the first case has this advantage, but not in the second case where ψ is a function of P_1 and T_1.

For these reasons, the first case is recommended over the other two, and will be used in the remainder of this work.

III. EXPERIMENTAL METHOD AND PROCEDURE

The experimental set-up is shown schematically in Figure 7. A 1 hp, 1725 rpm, 3 phase electric motor drives a 3/4 hp variable speed hydraulic transmission unit (speed reducer). The speed reducer is connected to the piston-rod assembly, which is inside the test cylinder, through a connecting rod and an eccentric plate. A storage tank feeds the oil side of a 3000 psi,5 gallon accumulator. Two oil filters are installed in this line to remove contaminants in the oils. Three 2000 psi nitrogen cylinders are connected to the other (gas) side of the accumulator to raise the pressure of the oil to the desired value. In order to maintain the pressure on the oil constant while leakage takes place, a pressure regulating valve was installed in the line connecting the nitrogen cylinders to the accumulator.

The test cylinder and the piston-rod assembly are shown in Figure 8. They are made of the same material, namely 4615 steel. The pistons and the bore of the cylinder are case hardened. The two outer pistons A-A are slightly larger in diameter than the inner ones B-B. With this arrangement, the outer pistons act as guides, suffer any possible wear and limit the maximum probable eccentricity of the inner pistons. The oil flows under constant pressure from the accumulator through the openings C-C into the pressure chambers. During reciprocation of the piston-rod assembly, the openings C-C remain open to the pressure chambers. This eliminates fluctuations of the pressure in the pressure chambers. Indeed, this was the case during operation. Leakage can take place from the pressure chambers either past the

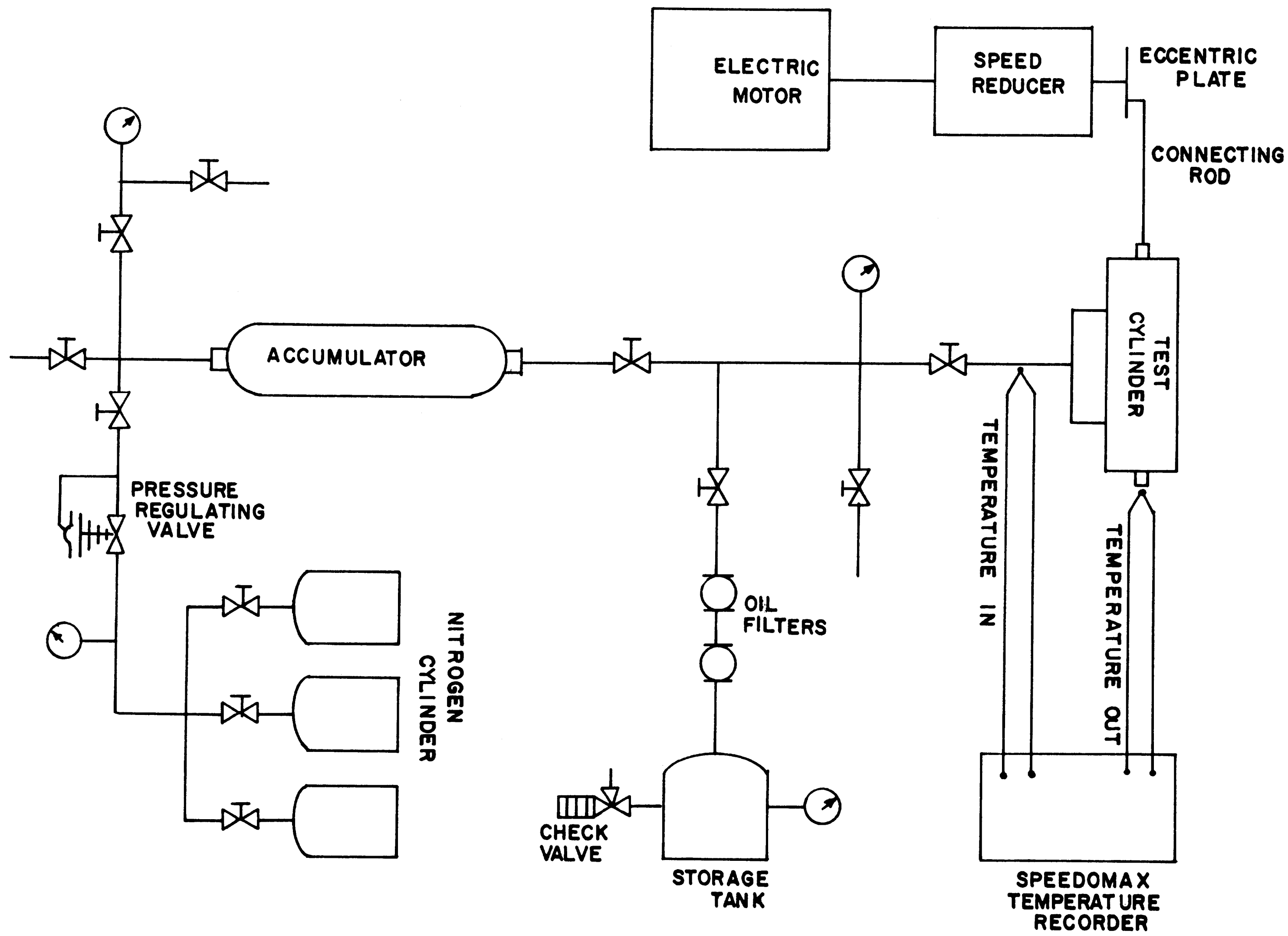

Figure 7. Schematic Diagram of the Experimental Set-up

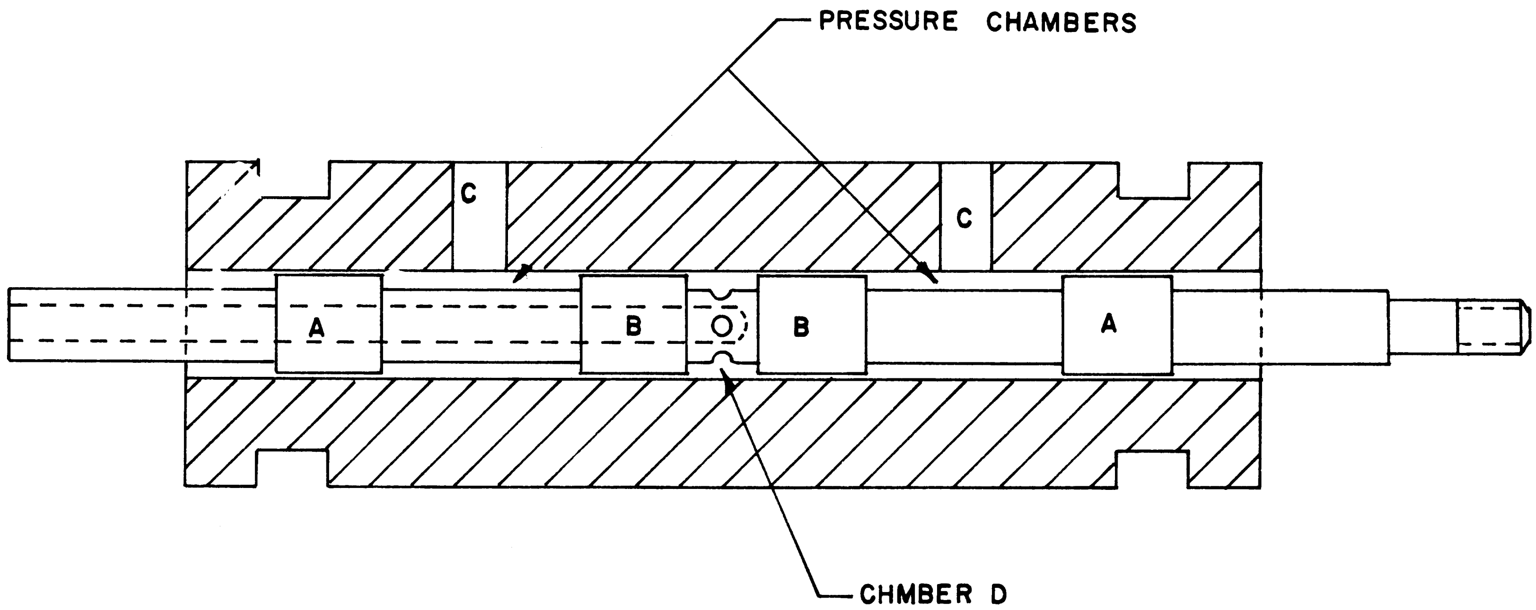

Figure 8. Test Cylinder with Piston-Rod Assembly

pistons A-A to the outside, which is uncontrolled leakage, or past the pistons B-B to the chamber D, which is the controlled leakage. The oil in the chamber D proceeds then through the radial holes of the rod to the axial hole of the rod and then out. It is then collected and measured. Two layers of spaghetti-type insulation were inserted snugly in the axial hole of the rod to minimize heat transfer from the oil after leaking to the fresh oil in the pressure chambers.

In this arrangement, the pressure of the oil in the pressure chambers, which is causing the controlled leakage, remains constant while the piston-rod assembly reciprocates. No compression of the oil is done by the pistons. This has the advantage of decreasing the size of the necessary motor. Drawings of the test cylinder and the piston-rod assembly are given in Appendix C.

Two Iron-Constantan thermocouples from an Iron-Constantan type G Speedomax were installed. One just ahead of the openings C-C to measure the temperature of the oil going in, and the other was installed at the bottom of the axial hole in the piston-rod assembly to measure the temperature of the oil after it has leaked.

The radius of the crank (eccentric) is 0.5 inch while the length of the connecting rod is 6.5 inches. Hence the length ratio of crank to connecting rod is 1:13 which is small enough to justify the assumption that the motion of the piston-rod assembly is a simple harmonic one. Photographs of the experimental set-up are shown in Figures 9, 10, and 11.

Tests were run on three oils. They are:

1. Gulfpride Motor Oil No. 50
2. Gulfpride Motor Oil No. 30
3. Univis J-43 put out by the Esso Standard Oil Co.

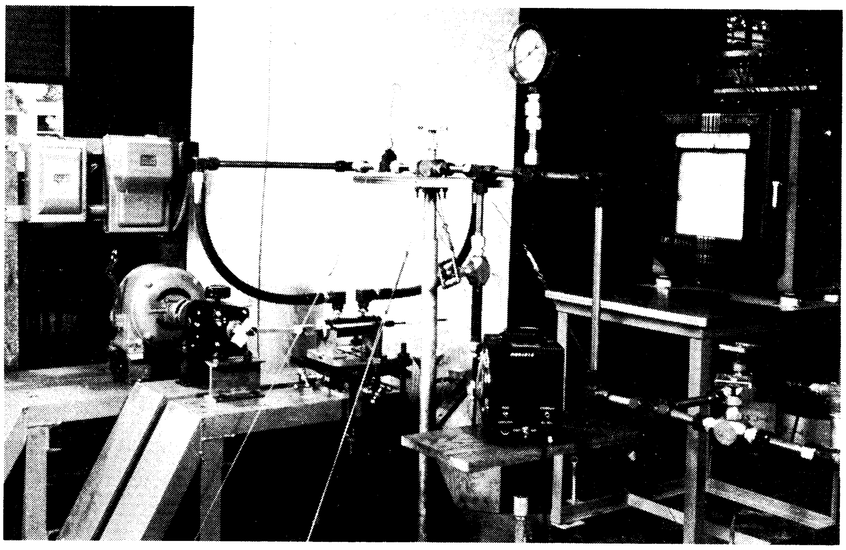

Figure 9. Part of the Experimental Set-up Showing the Motor, the Speed Reducer and the Test Cylinder

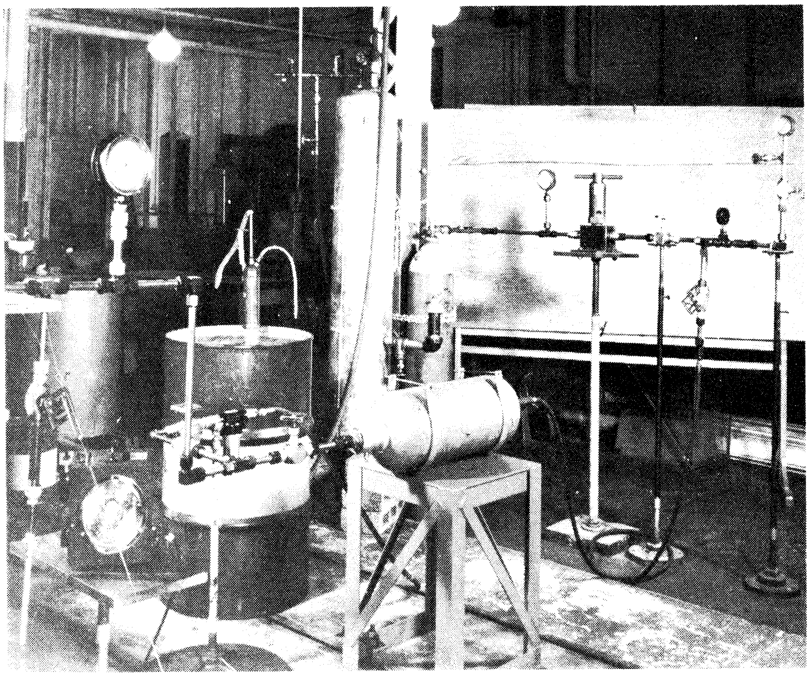

Figure 10. The Gas Side of the Experimental Set-up Showing the Nitrogen Cylinders, the Pressure Regulating Valve, the Accumulator and the Storage Tank

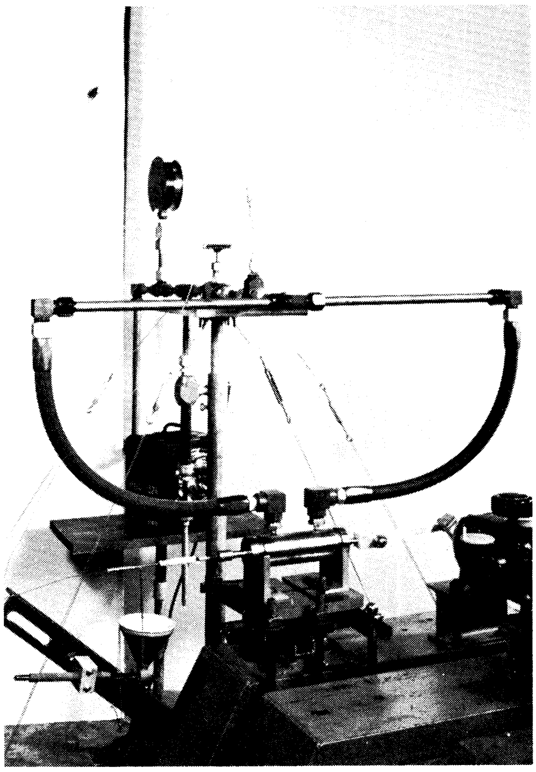

Figure 11. A Close-up Photograph of the Test Cylinder

The properties of these oils are given in Appendix D. In each run, the speed was set at a particular value, with the use of the speed reducer and a General Radio Strobotac, type 631-BL, which has a range of 60 to 14,500 rpm. The temperatures of the oil before and after leaking were recorded by the Speedomax. Leakage was measured over a reasonable period of time. The duration of each run was varied according to the rate of leakage, from one hour for the No. 50 oil, low pressure runs, to ten minutes for the Univis J-43, high pressure runs. The capacity of the accumulator was the reason for cutting down the duration of the runs, when high rate of leakage occurred.

By far, the most critical measurement was that of the clearances between the pistons and the cylinder. According to the isothermal results, the flow rate (leakage) is proportional to the cube of the clearance. Hence any error in its measurement would affect the predicted flow rate appreciably. The average size of the clearance Δ is one-half the difference between the diameter of the bore of the cylinder and the diameter of the piston.

The diameters of the four pistons were measured on a 48" Pratt and Whitney Universal Measuring Machine which reads to the hundred thousandth of an inch. For each piston, several readings were taken along two perpendicular planes to check for roundness and taper. Their average was taken as the diameter. The diameter of the bore of the cylinder was measured, before starting the tests, by the Sheffield Corporation of Dayton, Ohio, with their air gauges. These measurements were also taken along two perpendicular planes at half inch intervals along the length of the cylinder. The diameter of the bore was taken as the average of these readings over the operating length of the cylinder.

After the conclusion of the experimental work, the above measurements were repeated at the Gauging Laboratory of the Production Engineering Department of the University of Michigan, using the same type of equipment used earlier. All these measurements are given in Appendix C.

From these measurements, it is clear that although the diameters of the inner pistons and the cylinder bore were not the same before and after the experimental work, the average clearance remained essentially the same. The change in the diameters is expected, since the two sets of measurements were done at different temperatures. On the other hand, since the material of the pistons and the cylinder is the same, and hence the thermal coefficient of expansion is the same, it is to be expected that the average clearance would remain constant.

Since the first measurement of the bore diameter was made by Sheffield Corporation, and this is believed to be more accurate, the first set of measurements, which was taken before the start of the experimental work, was used in the calculation of the experimental results.

From this set of measurements, the maximum possible value for the attitude n was calculated to be 0.657.

The nitrogen cylinders used in this work limited the maximum pressure obtainable to about 1700 psi, while the maximum speed used was 1000 rpm for all oils.

IV. RESULTS

Equations (2-52) and (2-54) relate the temperature rise, $(T_2 - T_1)$, and the function λ to the function β. Equation (2-51) relates β to ψ and α, and from Equations (2-43), β can be related to ψ, φ and θ.

Figures 12 through 15, for φ vs. θ, and Figures 16 through 19, for $(T_2 - T_1)$ vs. θ, were constructed for the Gulfpride No. 50 and No. 30 oils, using the above equations. In each figure, two adiabatic curves are shown. One curve is plotted for n = o (no eccentricity) and the other for n = 0.657, which is the maximum possible value for n that can take place in the experimental set-up. The horizontal line in each figure represents the isothermal relationship between φ and θ for n = o as given by Equation (2-46). The flow rate obtained from the latter is the minimum rate of flow that is possible. The flow rate predicted from the adiabatic, n = 0.657, curve represents the maximum that can take place. These curves were plotted for one passage, that is for the flow past one piston.

The actual conditions during the experimental work were neither isothermal nor adiabatic, but a combination of the two, due to the fact that the cylinder wall was neither perfectly conducting for heat, nor insulated. Hence the experimental flow rate, and the corresponding flow number φ, as well as the temperature rise $(T_2 - T_1)$ should be expected to fall somewhere between the isothermal and the adiabatic curves.

The experimental data and results for all three oils are shown in Appendix E. The results of the Gulfpride No. 50 and No. 30

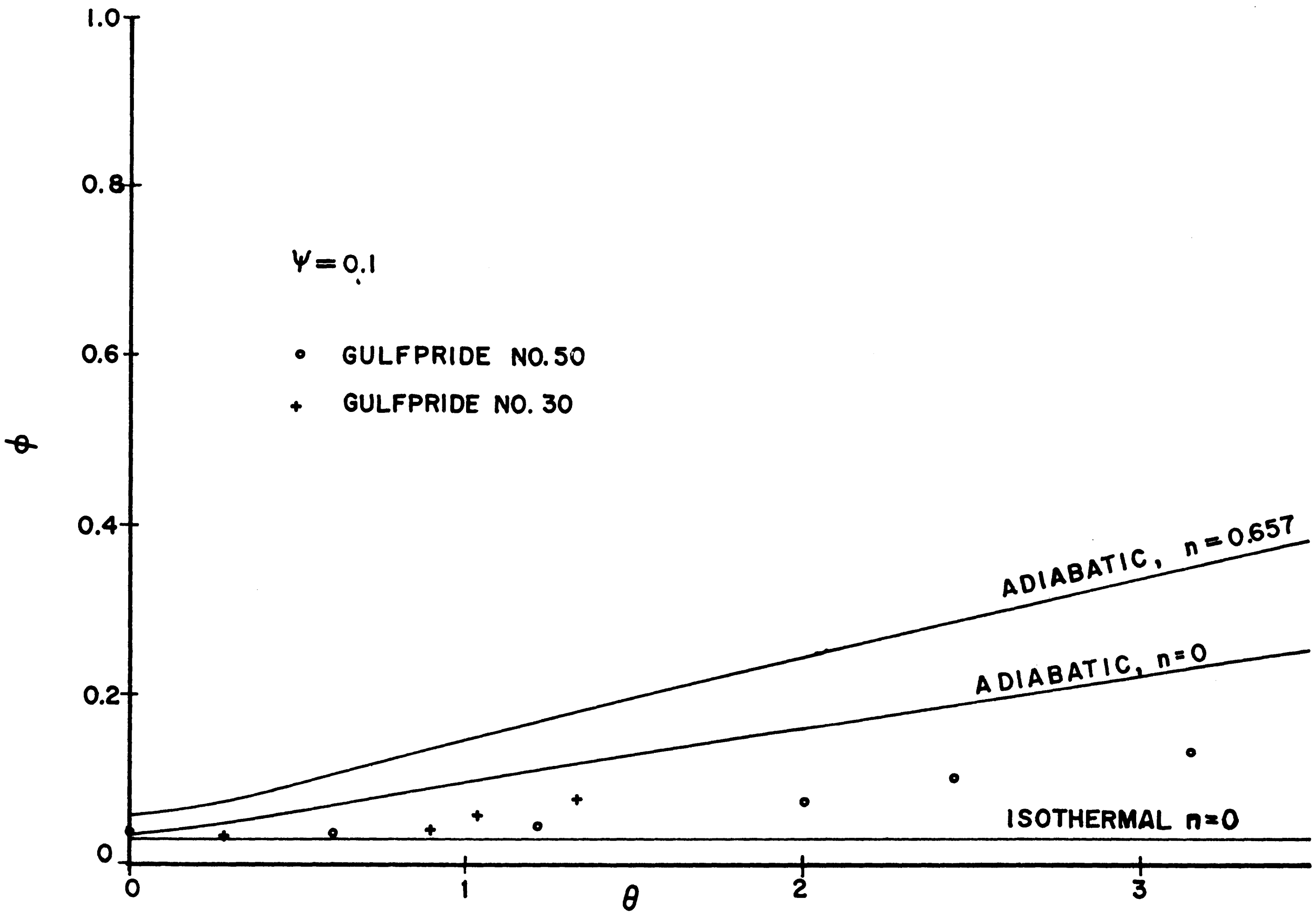

Figure 12. Flow Number vs. Velocity Number with $\psi = 0.1$ for the Gulfpride Motor Oils

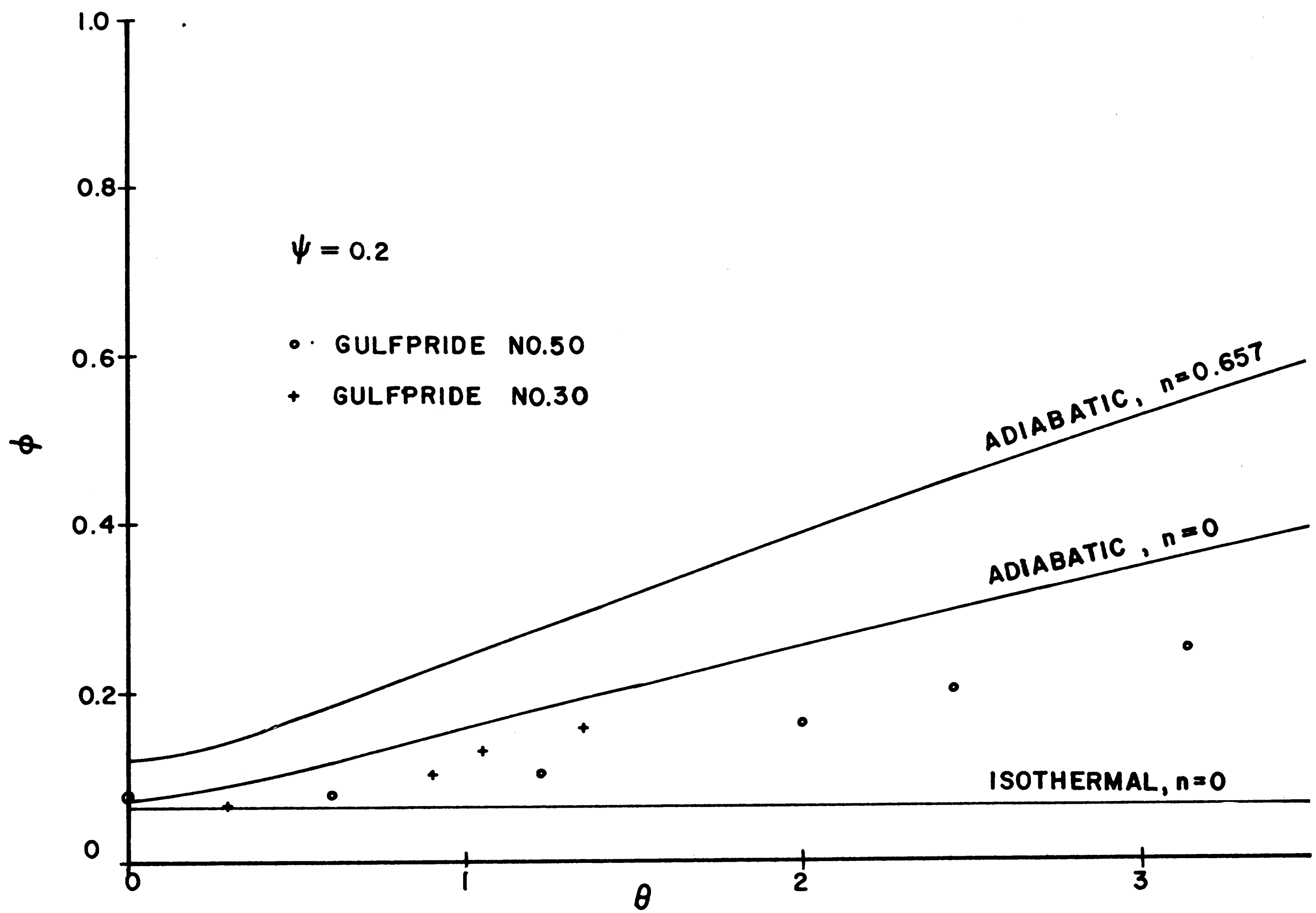

Figure 13. Flow Number vs. Velocity Number with $\psi = 0.2$ for the Gulfpride Motor Oils

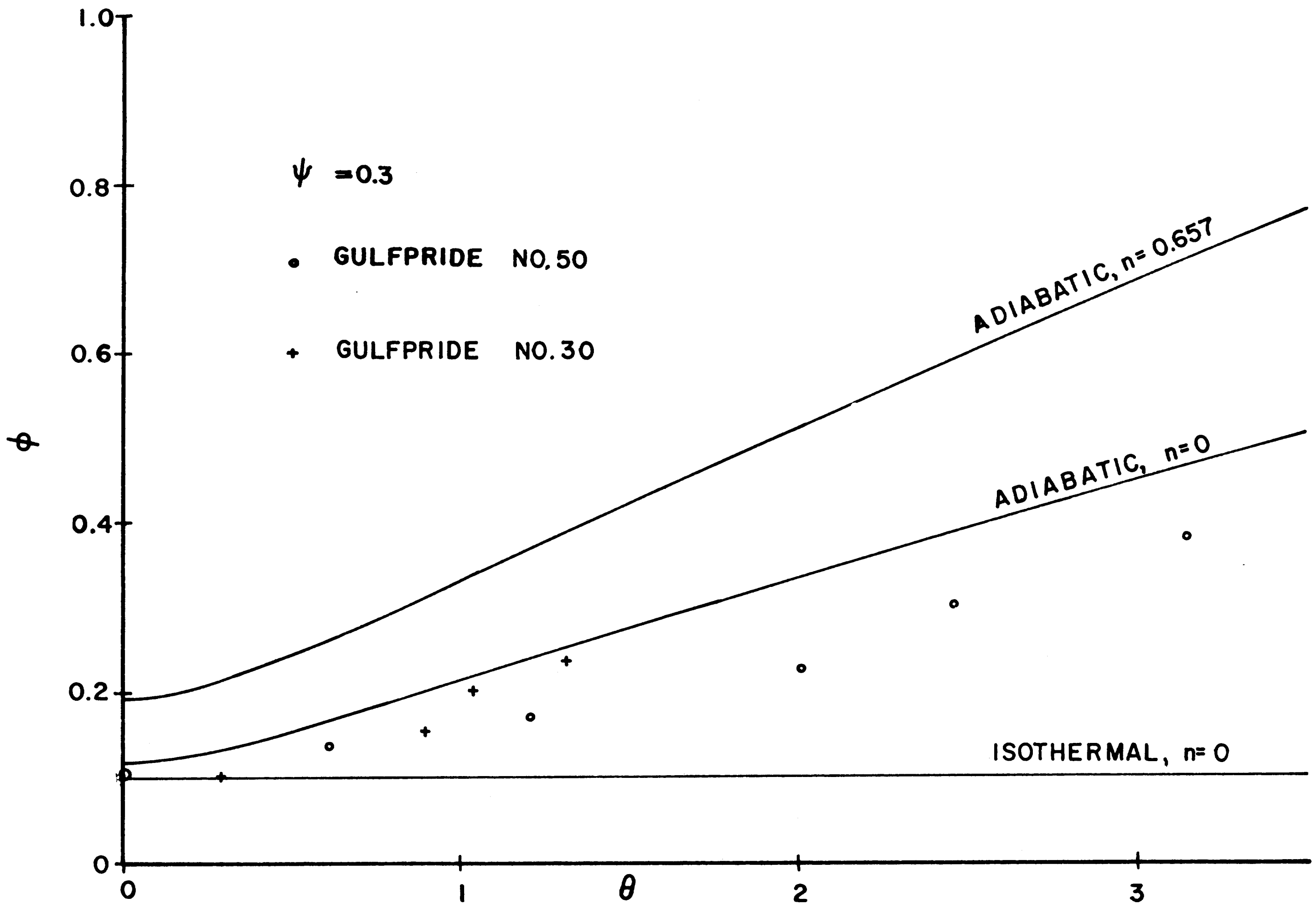

Figure 14. Flow Number vs. Velocity Number with ψ = 0.3 for the Gulfpride Motor Oils

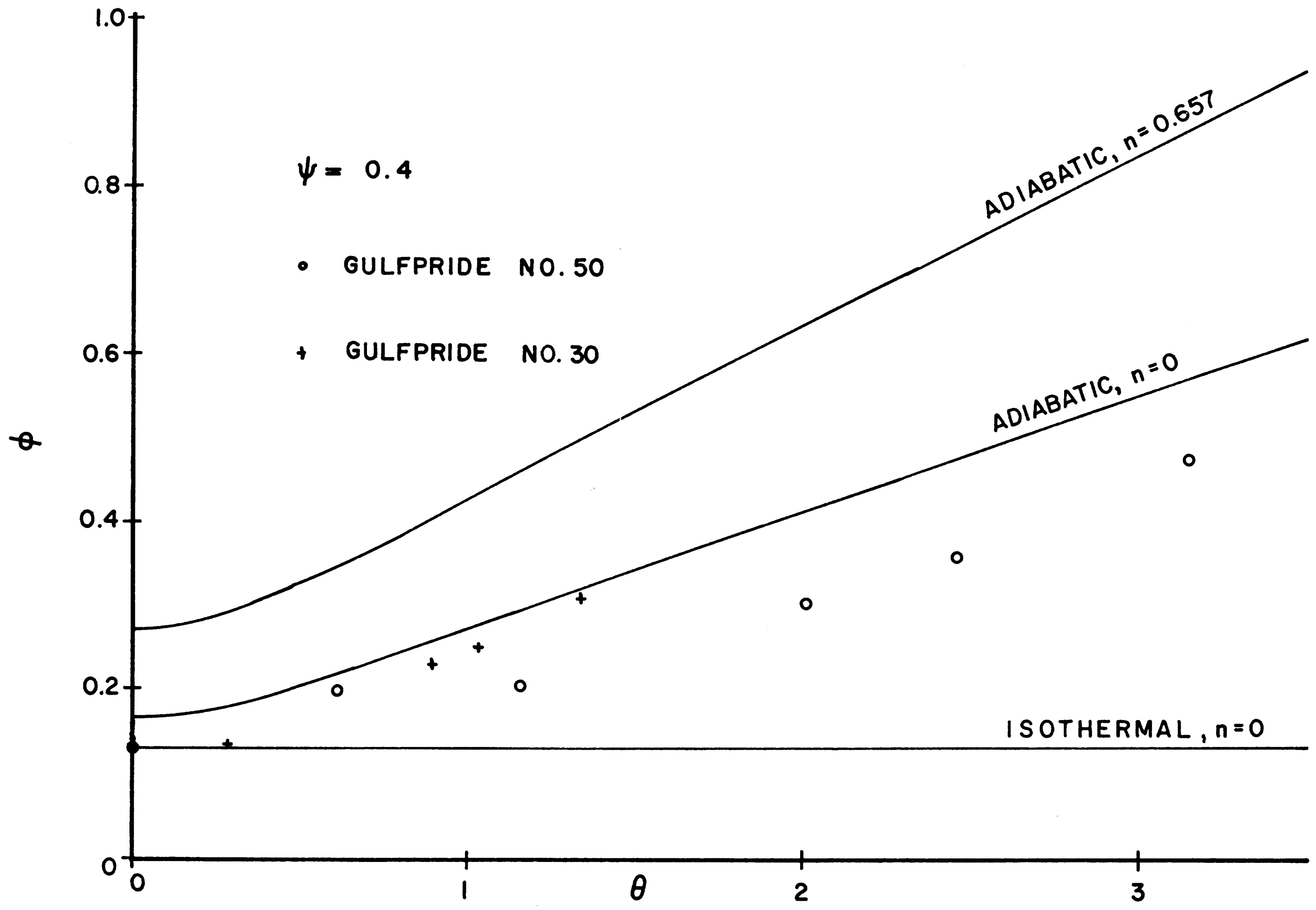

Figure 15. Flow Number vs. Velocity Number with $\psi = 0.4$ for the Gulfpride Motor Oils

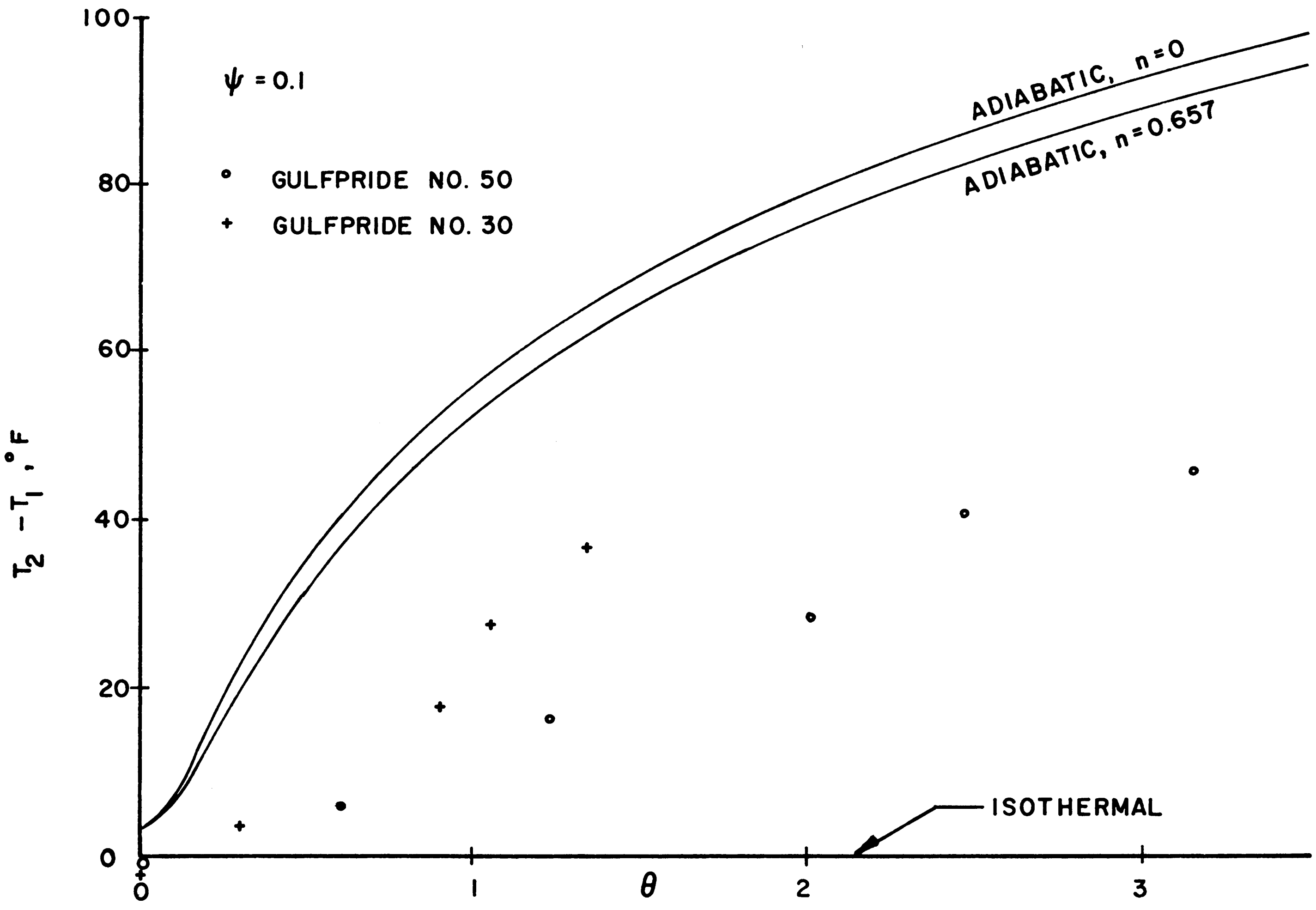

Figure 16. Temperature Rise vs. Velocity Number with ψ = 0.1 for the Gulfpride Motor Oils

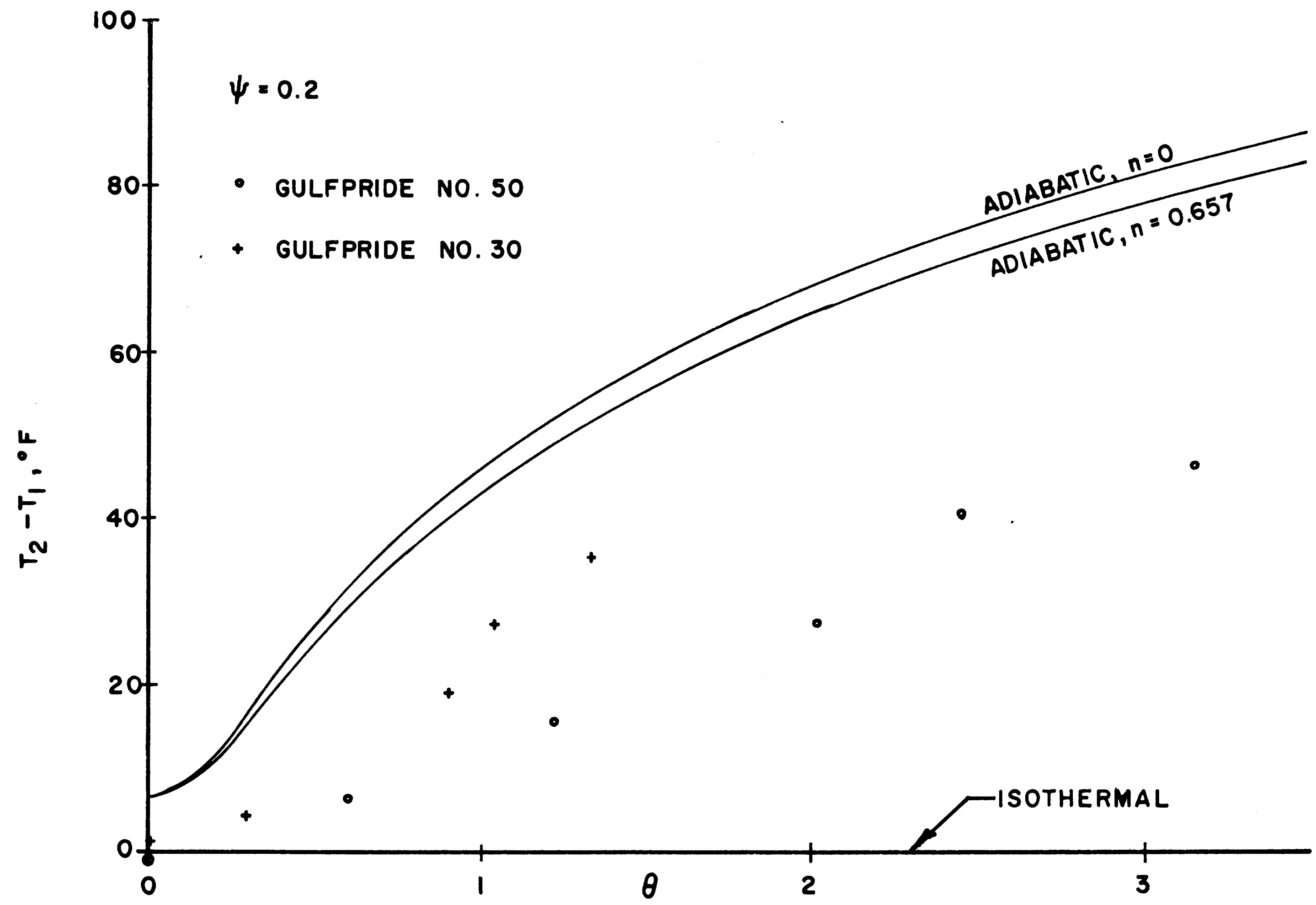

Figure 17. Temperature Rise vs. Velocity Number with ψ = 0.2 for the Gulfpride Motor Oils

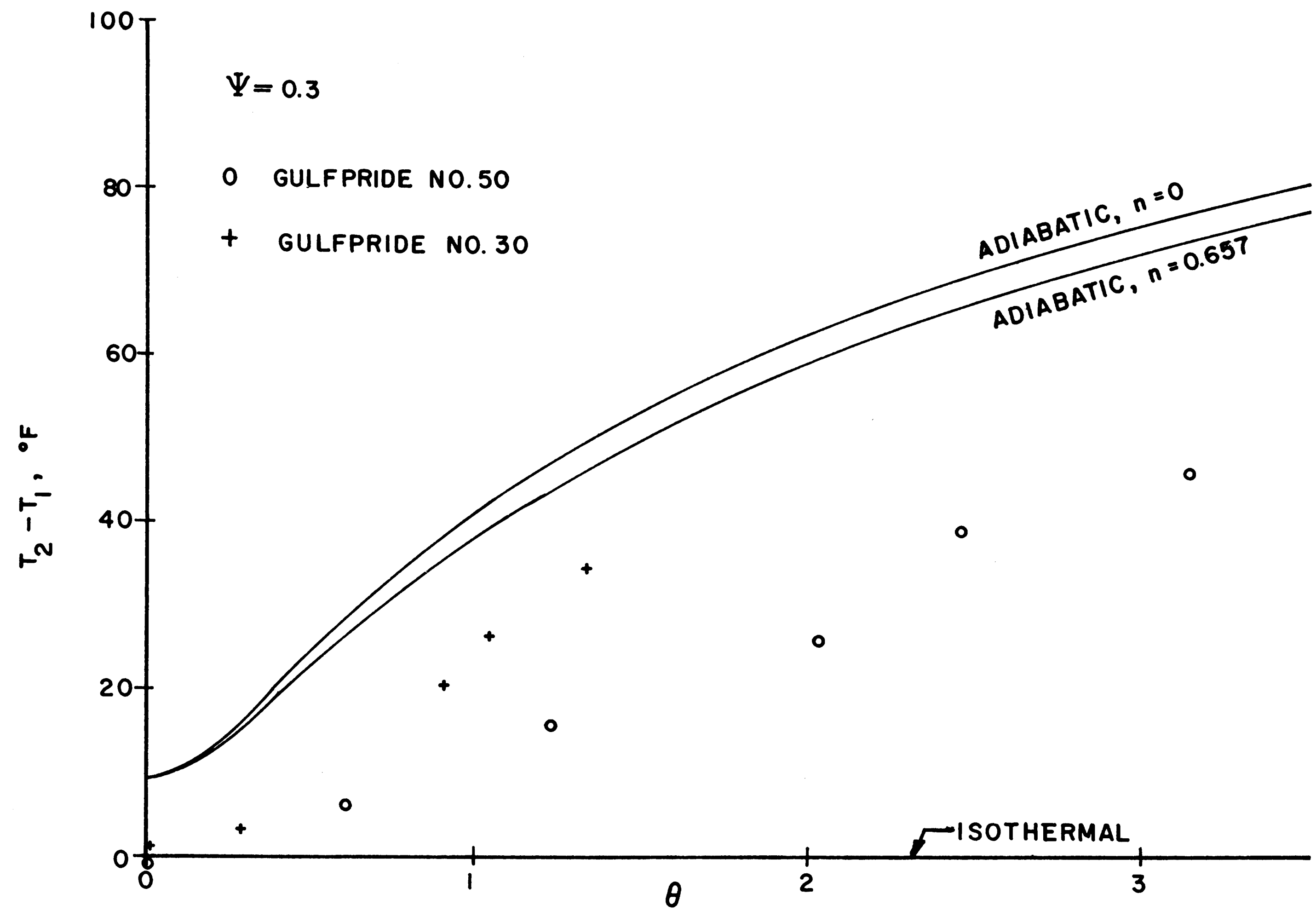

Figure 18. Temperature Rise vs. Velocity Number with $\psi = 0.3$ for the Gulfpride Motor Oils

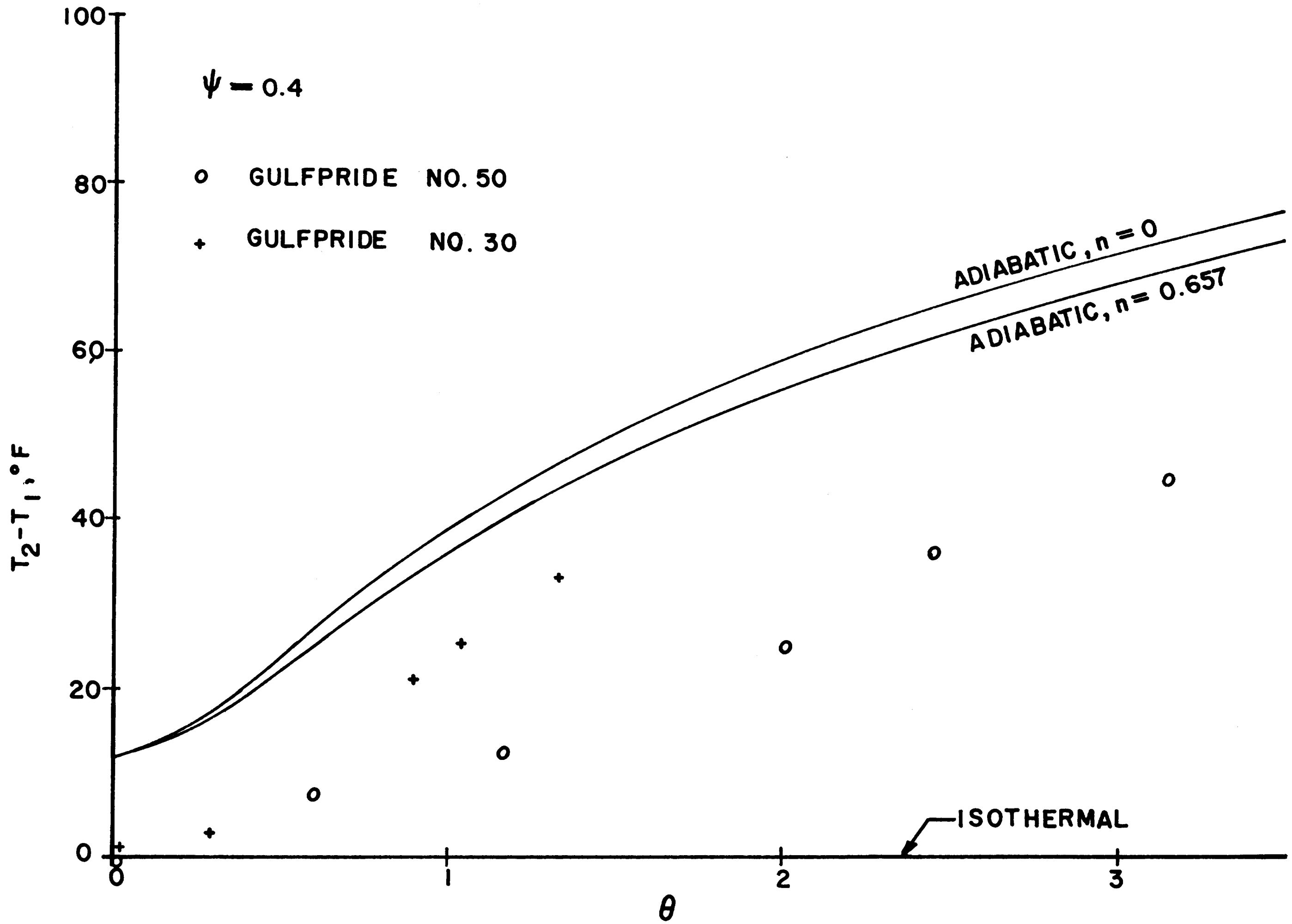

Figure 19. Temperature Rise vs. Velocity Number with ψ = 0.4 for the Gulfpride Motor Oils

oils were also plotted in Figures 12 through 19. It is apparent from these figures, that all the experimental results of these two oils lie between the isothermal, n = 0, curve and the adiabatic, n = 0, curve. This suggests that the piston remains concentric with respect to the cylinder during operation. Actually, the test cylinder and the piston-rod assembly were machined in such a manner as to facilitate concentricity of the pistons. The four pistons and the rod were made out of the same piece of material, and they were machined in one set-up. The test cylinder was bored through from one side. Hence, from the construction point of view, the possibility for eccentricity has been minimized. During operation, the oil flows between the pistons and the cylinder. When a piston is concentric, a continuous film of oil exists around it. The effect of eccentricity is to squeeze some of the oil out and this disturbs the balance of pressure forces around the piston, and the piston returns to a concentric position, so that the conclusion that the pistons remain concentric during operation seems reasonable. This conclusion will be incorporated in the analysis of the rate of leakage in reciprocating pumps discussed in Chapter V.

It is also apparent that the experimental results of the No. 50 and No. 30 oils do not follow a single trend. In fact, in most cases, the flow number and the temperature rise of the No. 50 oil are closer to the isothermal line than those of the No. 30 oil. This may be explained by referring back to Equation (2-30). From the expression of Z given in Equation (2-45), and using Equation (2-51), Equation (2-30) can be approximately written as

$$\frac{dt}{d\eta} + \frac{\Lambda}{\lambda}(t - 1) = \frac{\lambda\beta}{AT_1}\,\sigma \quad .$$

Here A and T_1 are practically the same for both oils. Also, for particular values of ψ and Θ, the theoretical values of λ and β are also the same. Hence the coefficient

$$\frac{\lambda\beta}{AT_1}$$

is the same for both oils, for the same ψ and Θ. However, the quantity Λ , from Equation (2-29) is

$$\Lambda = \frac{2\pi RUL}{\gamma C_v Q_o} \quad .$$

The quantities on the right hand side, with the exception of Q_o, are approximately the same for both oils. But, for the same ψ and Θ, Q_o for the No. 50 oil is less than that for the No. 30 oil, because its viscosity is higher. Hence the coefficient Λ/λ for the No. 50 oil is greater than that for the No. 30 oil. Since larger values of Λ/λ means greater heat transfer across the cylinder wall, it should be expected that the results of the No. 50 oil be closer to the isothermal line than those of the No. 30 oil.

Another way of explaining this is in terms of the length of time it takes the oil particles in moving through the clearance. Due to the higher viscosity of the No. 50 oil, its particles spend a longer time in the clearance. This means more heat transfer from the oil to the cylinder wall, and hence its results are closer to the isothermal line than those of the No. 30 oil.

This trend in the results agrees with the observations of Mahood and Littlefield[16] who investigated the flow through small capillary tubes. They found that when the flow was relatively small, greater heat loss through the tube wall occurred.

Another interesting observation is that the experimental results for the two oils moved closer to the adiabatic, $n = 0$, curve, as the pressure number ψ was increased. Again, increasing ψ, and hence the pressure, means greater velocities of flow and less time spent in the clearance. This decreases the heat transfer from the oils, thereby approaching the adiabatic condition.

The mathematical and experimental results of the Univis J-43 oil are plotted in Figures 20 through 25. They generally show the same trends as in the case of the first two oils. Here the Θ values are quite small, and the adiabatic and the isothermal curves are almost parallel. The experimental points in Figures 20, 21, and 22 fell outside the adiabatic, $n = 0$, curve but within the adiabatic, $n = 0.657$, curve. However, this may be attributed to the fact that the flow rate obtained with this oil was much larger than in the case of the other two oils. Since the accumulator was not too large, the time allowed for reaching steady state as well as the length of each run had to be shortened. This may be responsible for the small discrepancy of the results from the adiabatic, $n = 0$, curve.

The values of the Reynolds number in these tests were generally low. To get an approximate idea about it, the maximum velocity of the flow must be determined first.

From Equation (2-3)

$$u = \frac{v}{d} y^2 - \frac{1}{2\mu} \frac{dP}{dx} (yd - y^2).$$

For isothermal flow with $n = 0$, and $v = V$,

$$u = \frac{V}{\Delta} y + \frac{P_1}{2\rho \nu_1 L} (y\Delta - y^2).$$

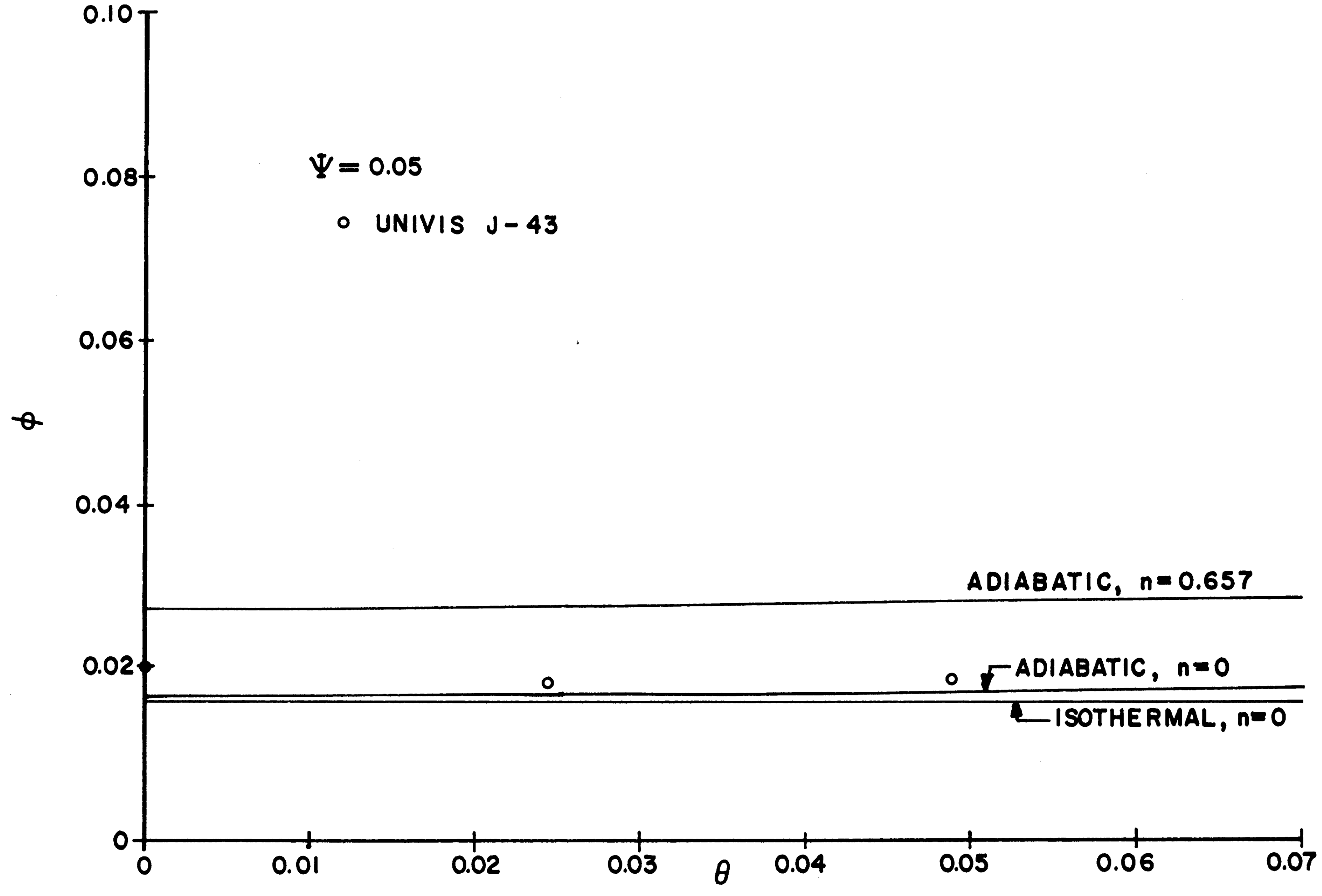

Figure 20. Flow Number vs. Velocity Number with $\psi = 0.05$ for the Univis J-43 Oil

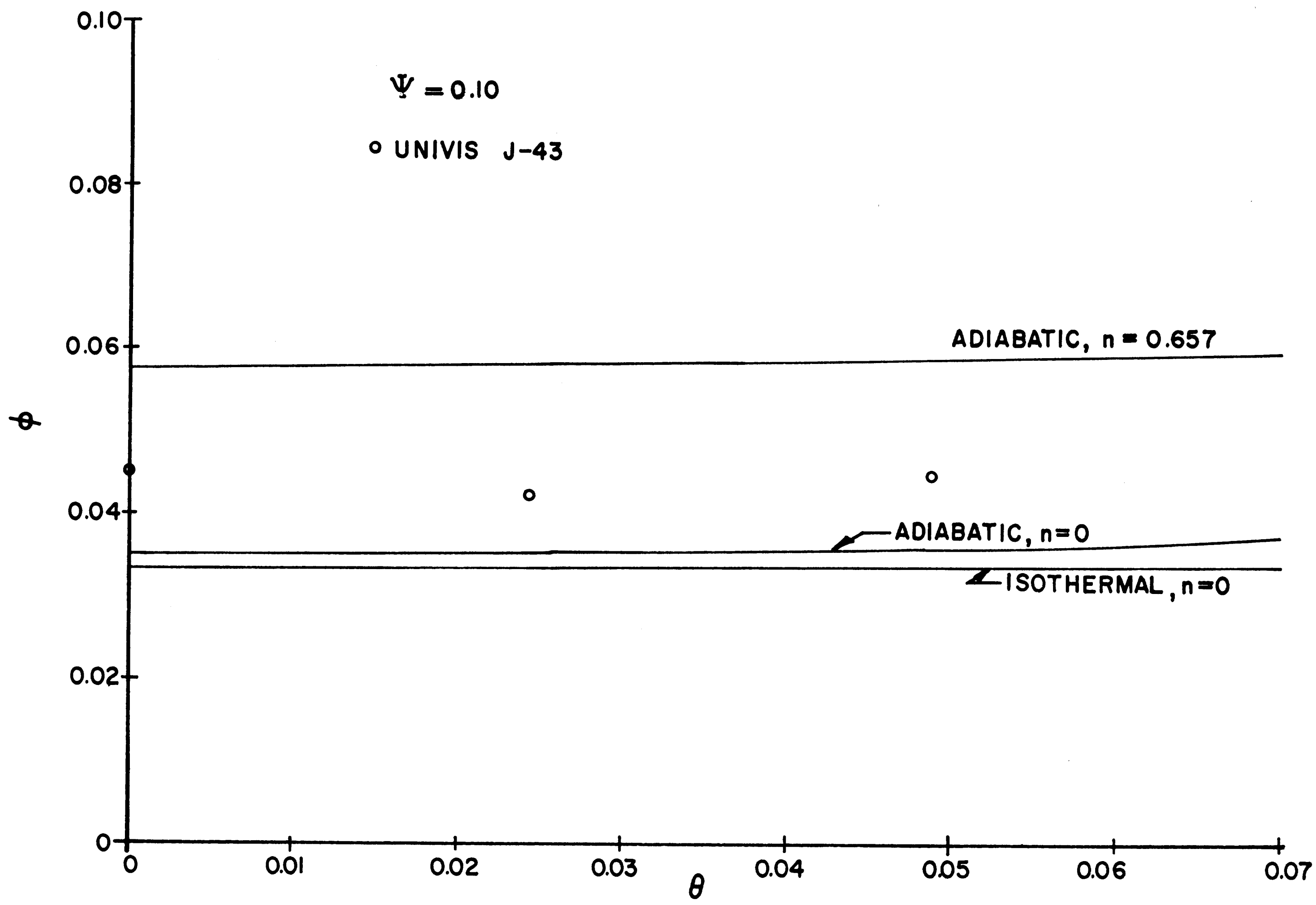

Figure 21. Flow Number vs. Velocity Number with ψ = 0.10 for the Univis J-43 Oil

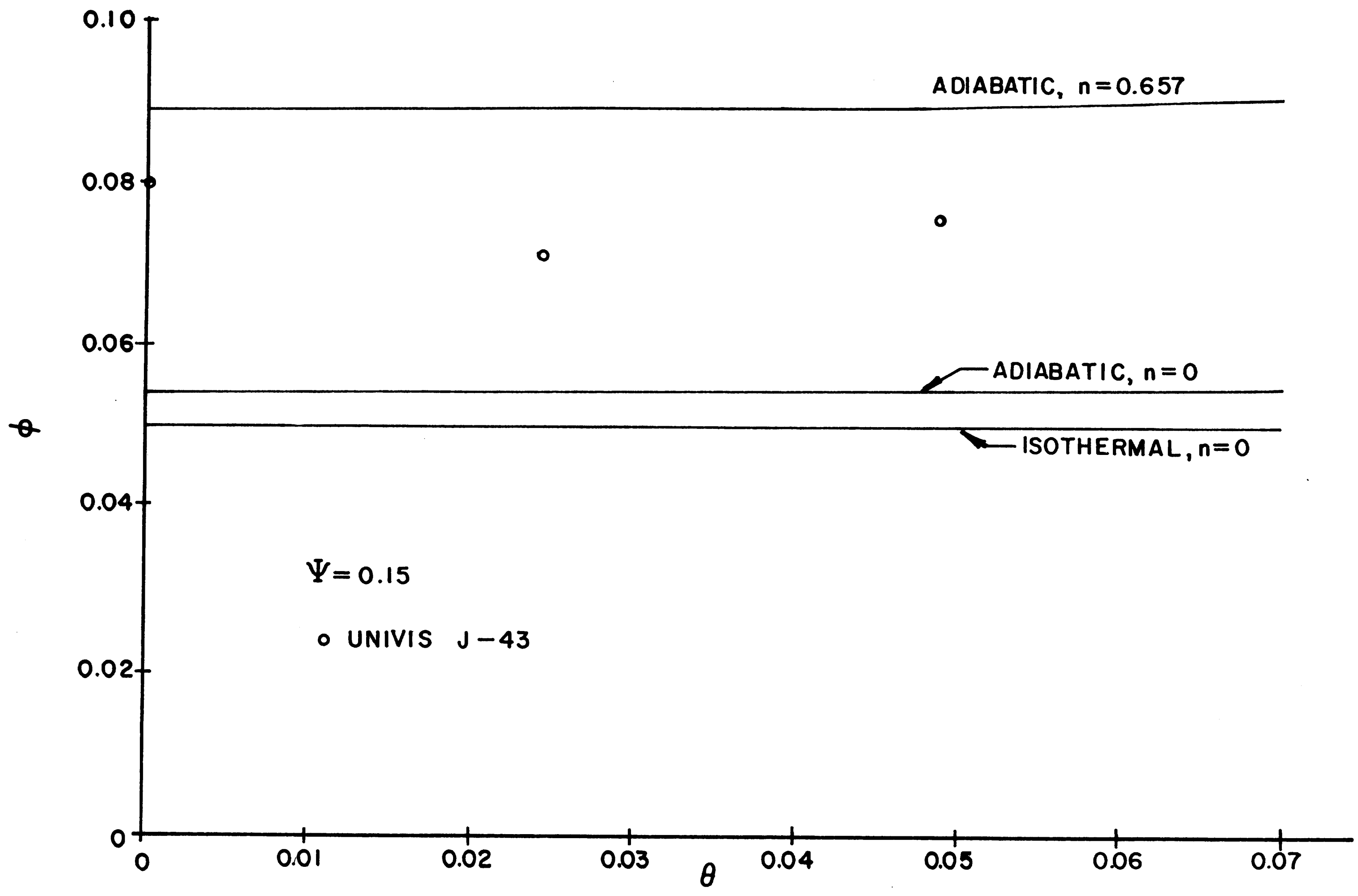

Figure 22. Flow Number vs. Velocity Number with ψ = 0.15 for the Univis J-43 Oil

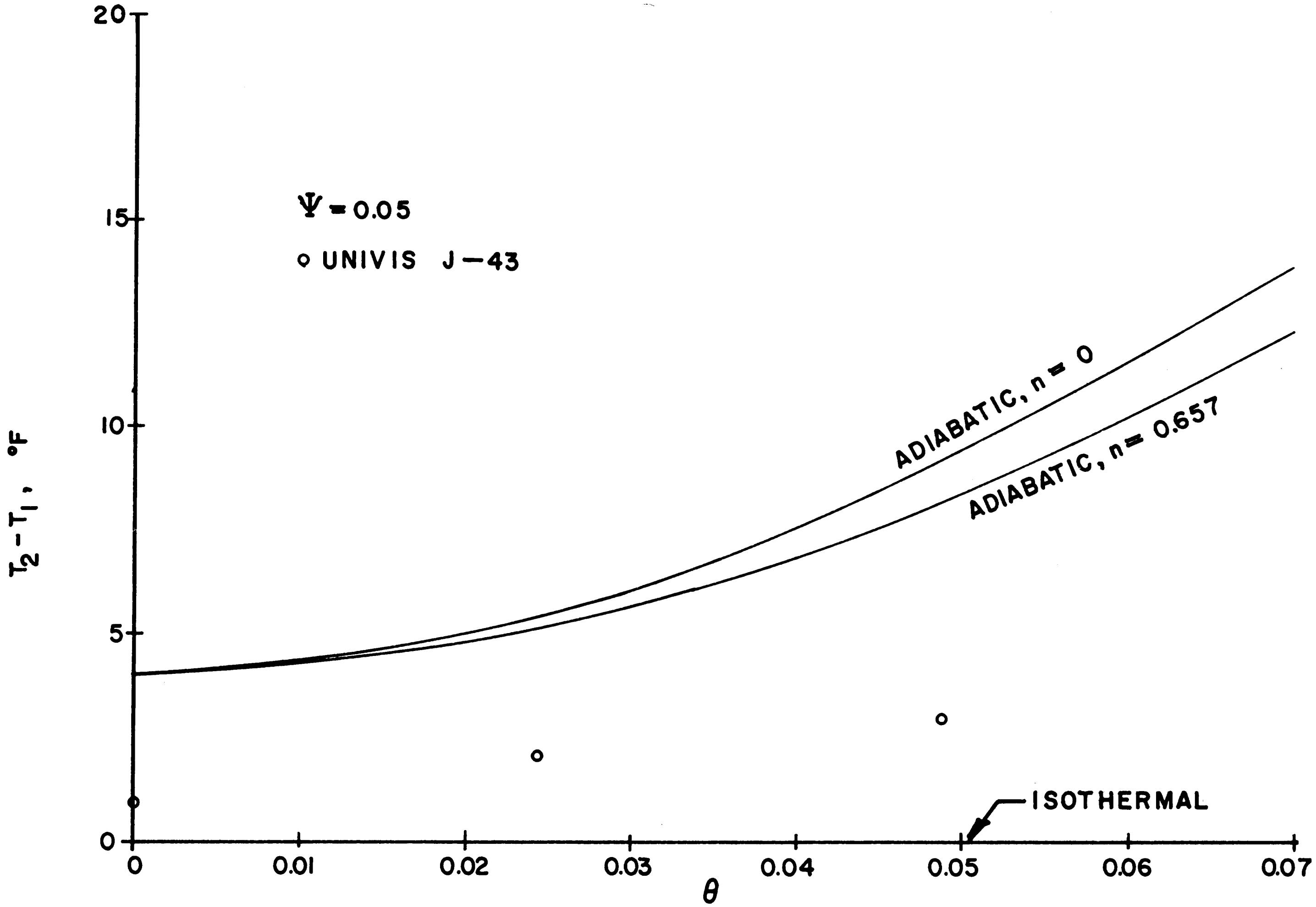

Figure 23. Temperature Rise vs. Velocity Number with $\psi = 0.05$ for the Univis J-43 Oil

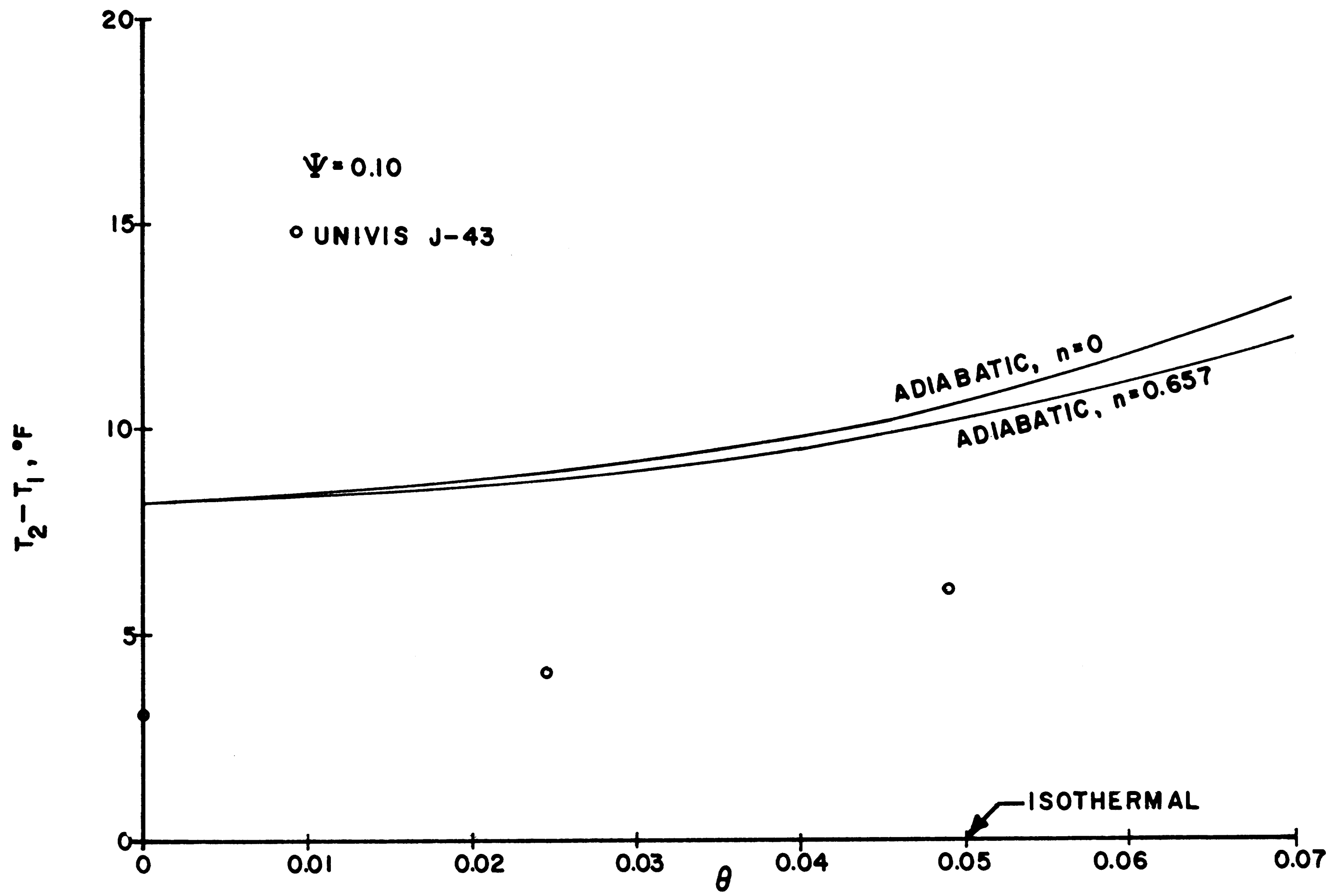

Figure 24. Temperature Rise vs. Velocity Number with ψ = 0.10 for the Univis J-43 Oil

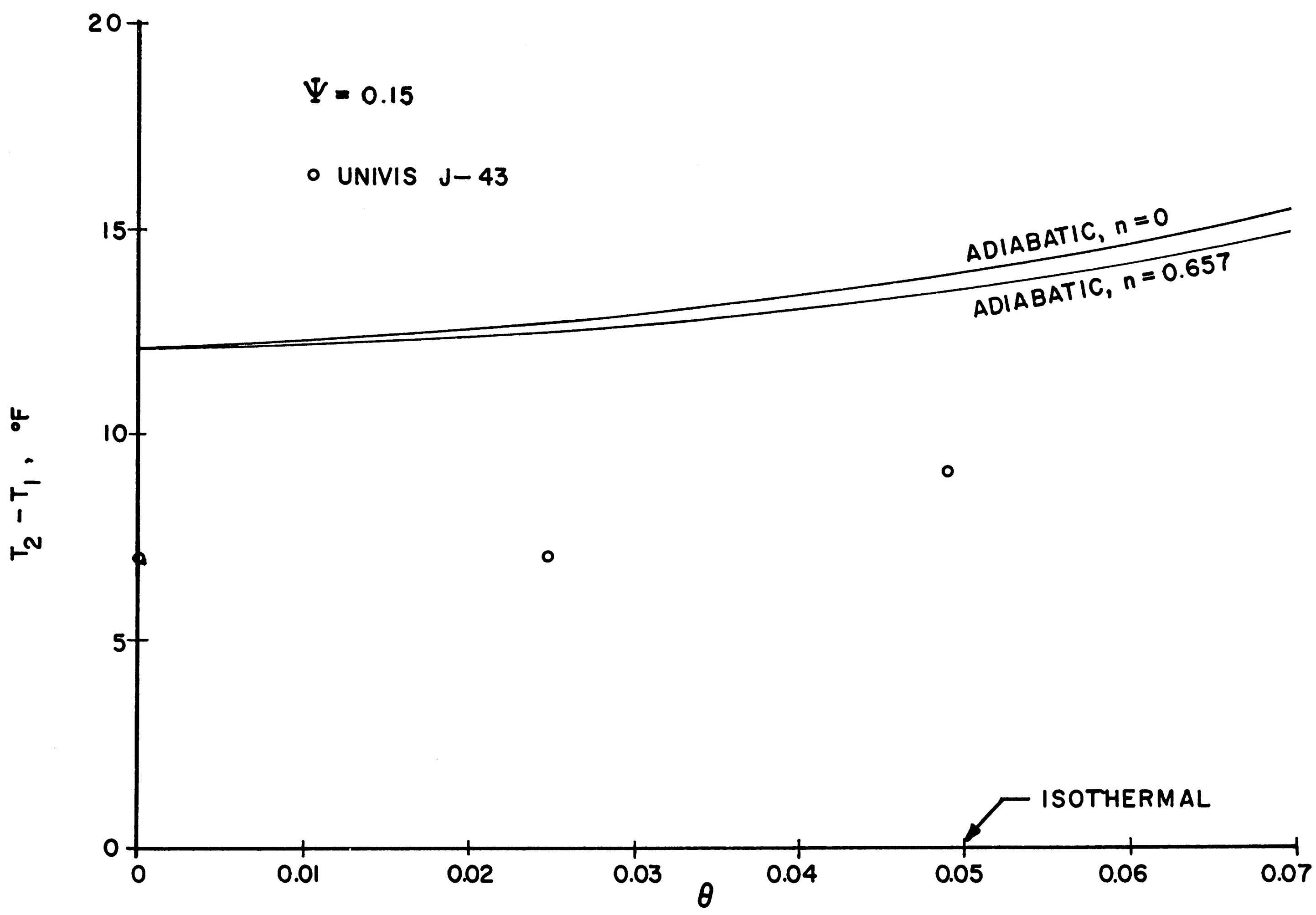

Figure 25. Temperature Rise vs. Velocity Number with ψ = 0.15 for the Univis J-43 Oil

In terms of the dimensionless variables of Equation (2-43),

$$\frac{u}{v} = \frac{y}{\Delta} + \frac{\psi}{\theta} \left[\frac{y}{\Delta} - \left(\frac{y}{\Delta}\right)^2\right]$$

and

$$\frac{d(u/V)}{d(y/\Delta)} = 1 + \frac{\psi}{\theta} \left(1 - 2\,\frac{y}{\Delta}\right) = 0$$

or

$$\frac{y}{\Delta} = \frac{1}{2} + \frac{\theta}{2\psi} \quad .$$

The 1000 rpm and 1625 psi run on the J-43 oil gives the maximum Reynolds number.

The values of θ and ψ for the various runs are given in Appendix E. For this particular run

$$\theta = 0.0487, \qquad \psi = 0.15 \quad .$$

Hence

$$\frac{y}{\Delta} = 0.662$$

and

$$\left(\frac{u}{V}\right)_{max} = 0.805$$

and

$$u_{max} = 0.805 \times \frac{1000}{60} \times 2\pi \times \frac{1}{24} = 3.51 \text{ ft/sec} \quad .$$

Now

$$N_R = \frac{4u}{\nu} R_H$$

where

N_R = Reynolds number

u = velocity (maximum)

ν = kinematic viscosity (initial)

R_H = the hydraulic radius = $\frac{\Delta}{2}$

Hence

$$N_R = \frac{2\, u_{max}}{\nu_1} \Delta = 2 \quad .$$

The Reynolds number, with the clearance as the characteristic length, is

$$N_R' = 1 \quad .$$

For all other runs, the approximate Reynolds number would be smaller. This means that in all the runs, the flow was highly laminar. According to Davies and White,[6] no disturbances survived in the clearance during the tests.

The experimental data and results are given in Appendix E. Since there were two controlled leakage passages in the experimental set-up (two pistons), and the mathematical results were plotted for one passage, the flow rate Q given in Appendix E is one-half the measured flow rate.

The values of A for the three oils were determined as follows:

For each oil, the kinematic viscosity was plotted against the absolute temperature on semi-log paper covering the operating range only. A fair straight line was passed through the plotted points. The value of A was determined as the slope of this straight line. These values are also given in Appendix E.

The specific heat at constant pressure, C_p, given in Appendix D, was assumed equal to the specific heat at constant volume, C_v, used in the expressions for the pressure and velocity numbers.

V. APPLICATION TO THE PROBLEM OF LEAKAGE IN RECIPROCATING PUMPS

The mathematical analysis based on the assumption of adiabatic flow, used in Chapter II, will be employed here in order to predict the maximum rate of leakage that may be expected in the operation of a reciprocating pump. The present problem differs from that of Chapter II in that the pressure difference here exists during one-half of the cycle only. The assumptions made in Chapter II regarding the properties of the fluid flowing will be repeated here. In addition, a few more assumptions will be introduced in order to simplify the mathematical analysis.

A. Idealization of the Problem

In the actual operation of a reciprocating pump, the fluid is admitted at a low pressure during the first half of each cycle. Compression of the fluid starts after the piston passes bottom-dead-center, and continues for a few degrees of crank rotation. During this period, the pressure of the fluid is increased to the desired maximum. After this maximum pressure has been reached, the discharge valve opens and the fluid is discharged during the remainder of the half cycle, at a constant pressure essentially equal to the maximum.

The motion of the piston is caused by the rotation of the crank. If the length ratio of the crank to the connecting rod is very small, then the motion of the piston would be close to a simple harmonic motion. If this ratio is fairly large, then the motion of the piston would consist of a simple harmonic one plus some higher harmonics.

The following assumptions are introduced to simplify the mathematical analysis for the above cycle of operation:

1. During the compression period, the pressure of the fluid changes, while during the discharge period, it is near maximum and constant. Since the compression period is much shorter than the discharge period, it will be assumed that the pressure of the fluid is constant and equal to the discharge pressure throughout the second half of the cycle.
2. Since in practice, the length ratio of the crank to the connecting rod is fairly small, it will be assumed that the motion of the piston is a simple harmonic one.
3. Since in all reciprocating pumps, the clearance between the piston and the cylinder is much smaller than the radii, it will be assumed that the problem is mathematically equivalent to that of the flow between two parallel plates, with no end effects.
4. In view of the conclusions of Chapter IV, it will be assumed that the piston remains concentric with the cylinder during operation. This means that $n = 0$.
5. It will be assumed that the axes of the piston and the cylinder coincide during operation.

According to the results of isothermal flow between two parallel plates, the rate of flow is proportional to the cube of the distance between the plates (the clearance in the present problem). Since the clearance is very small, any error in its determination would affect the predicted rate of flow considerably. This gives additional justification to neglect small changes (of the order of a few percent) in the properties of the fluid while leaking. Hence, in the operating range of temperature and pressure, it will be also assumed that:

6. The density of the leaking fluid does not change with temperature and pressure.
7. The viscosity of the leaking fluid depends on the temperature only. It does not change with the pressure.

Since eccentricity is being neglected here, the adoption of two parallel flat plates will be sufficient for the mathematical analysis.

The idealized problem then consists of an incompressible viscous fluid flowing between two parallel flat plates under the influence of a pressure difference. One plate is stationary and the other has a simple harmonic motion. The problem has two phases. The first phase consists of one-half of the cycle when the pressure difference exists and one plate is moving in the direction of decreasing pressure (discharge stroke). The second phase consists of the other half of the cycle when the plate is moving in the opposite direction with zero pressure difference (intake stroke). In a reciprocating pump, the flow in the first phase represents leakage, since it proceeds from the pressure chamber of the pump to the outside. In the second phase, the flow may be considered as "recovered leakage", since it proceeds from the outside to the pressure chamber of the pump. The net leakage per cycle then is equal to the flow in the first phase minus the flow in the second phase.

B. Mathematical Analysis

Referring to Figure 1, and employing the notation used there, the velocity distributions for n = 0 is

$$u_1 = \pm \frac{v}{\Delta} y \quad , \tag{5-1}$$

and

$$u_2 = - \frac{1}{2\mu} \frac{dP}{dx} (y\Delta - y^2) . \tag{5-2}$$

The combined velocity distribution for the first phase is

$$u = \frac{v}{\Delta} y - \frac{1}{2\mu} \frac{dP}{dx} (y\Delta - y^2), \tag{5-3}$$

and for the second phase

$$u = - \frac{v}{\Delta} y \quad . \tag{5-4}$$

The instantaneous rate of flow is

$$Q = R \int_0^{2\pi} \int_0^{\Delta} u \, dy \, d\xi \quad . \tag{5-5}$$

For the first phase,

$$Q_1 = R \int_0^{2\pi} \int_0^{\Delta} [\frac{v}{\Delta} y - \frac{1}{2\mu} \frac{dP}{dx} (y\Delta - y^2)] \, dy \, d\xi$$

$$= R \int_0^{2\pi} (\frac{v\Delta}{2} - \frac{\Delta^3}{12\mu} \frac{dP}{dx}) \, d\xi$$

$$= \pi R v \Delta - \frac{\pi R \Delta^3}{6\rho\nu} \frac{dP}{dx} \quad . \tag{5-6}$$

Its average during the first phase is

$$Q_{1_{ave}} = \frac{\int_0^{\pi} Q \, d(\omega t)}{\pi} \quad . \tag{5-7}$$

The velocity of the piston, from Appendix A, is

$$v = V \sin (\omega t) \ , \tag{5-8}$$

where

$$V = \omega r \quad . \tag{5-9}$$

Substituting Equations (5-6) and (5-8) in Equation (5-7) and integrating gives

$$Q_{1_{ave}} = 2 \ R \ V \ \Delta - \frac{\pi R \Delta^3}{6 \rho \nu} \frac{dP}{dx} \quad . \tag{5-10}$$

The instantaneous flow rate in the second phase is

$$Q_2 = R \int_0^{2\pi} \int_0^{\Delta} \frac{vy}{\Delta} \, dy \, d\xi$$

$$= \pi \ R \ V \ \Delta \quad . \tag{5-11}$$

Its average during the second phase is

$$Q_{2_{ave}} = 2 \ R \ V \ \Delta. \tag{5-12}$$

The average net rate of flow over the whole cycle is

$$Q_{ave} = \frac{Q_{1_{ave}} - Q_{2_{ave}}}{2}$$

$$= - \frac{\pi R \Delta^3}{12 \rho \nu} \frac{dP}{dx} \quad . \tag{5-13}$$

For isothermal flow $\nu = \nu_1$ and it is usually assumed that

$$\frac{dP}{dx} = - \frac{P_1}{L} ,$$

substituting in Equation (5-13), the net isothermal rate of flow Q_o is

$$Q_o = \frac{\pi R \Delta^3}{12 \rho \nu_1} \frac{P_1}{L} . \qquad (5\text{-}14)$$

Let

$$\alpha_o = \frac{Q_o}{\pi R V \Delta}$$

$$\theta = \frac{2\nu_1 VLA}{gC_v \Delta^2}$$

$$(5\text{-}15)$$

$$\varphi_o = \alpha_o \theta$$

$$\psi = \frac{AP_1}{\gamma C_v} .$$

Equation (5-14), in terms of these dimensionless variables, becomes

$$\varphi_o = \frac{\psi}{6} . \qquad (5\text{-}16)$$

When the flow is assumed to be adiabatic, the energy balance is represented by Equation (2-12) without the heat conduction term. Hence

$$\frac{dT}{dx} = \frac{1}{\gamma C_v Q} \frac{dW}{dx} - \frac{1}{\gamma C_v} \frac{dP}{dx} . \qquad (5\text{-}17)$$

At this point, the analysis for the two phases of the problem must be continued separately.

FIRST PHASE:

Here

$$\frac{dW}{dx} = v \frac{dF}{dx} \tag{5-18}$$

and

$$\frac{dF}{dx} = R \int_0^{2\pi} \tau \, d\xi \quad . \tag{5-19}$$

The shear stress τ in Equation (5-19) is that between the fluid and the flat plate (piston). Hence

$$\tau = \mu \left(\frac{du}{dy}\right)_{y=0} \tag{5-20}$$

Substituting Equations (5-3) and (5-20) in Equation (5-19) and integrating gives

$$\frac{dF}{dx} = 2 \pi \mu R \left[\frac{v}{\Delta} - \frac{\Delta}{2\rho\nu} \frac{dP}{dx} \right] \quad . \tag{5-21}$$

Substituting Equation (5-21) in Equation (5-18), and making use of

$$\nu = \frac{\mu}{\rho} \quad , \tag{5-22}$$

gives

$$\frac{dW}{dx} = 2 \pi \rho \nu R v \left[\frac{v}{\Delta} - \frac{\Delta}{2\rho\nu} \frac{dP}{dx} \right] \quad . \tag{5-23}$$

The average value of $\frac{dW}{dx}$ during the first phase is

$$\left(\frac{dW}{dx}\right)_{ave} = \frac{\int_0^{\pi} \frac{dW}{dx} \, d(\omega t)}{\pi} \quad . \tag{5-24}$$

Substituting from Equation (5-23) and using Equation (5-8) and integrating gives

$$\left(\frac{dW}{dx}\right)_{ave} = \frac{\pi\rho\nu RV^2}{\Delta} - 2\,R\,V\,\Delta\,\frac{dP}{dx} \quad . \tag{5-25}$$

Substituting in Equation (5-17) gives

$$\frac{dT}{dx} = \frac{\pi\nu RV^2}{gC_v Q_1 \Delta} - \left(\frac{2RV\Delta}{Q_1} + 1 \right) \frac{1}{\gamma C_v} \frac{dP}{dx} \quad . \tag{5-26}$$

From Equation (5-10)

$$\frac{dP}{dx} = \frac{6\rho Q_1 \left(\frac{2RV\Delta}{Q_1} - 1 \right)}{\pi\,R\,\Delta^3}\,\nu \quad . \tag{5-27}$$

Eliminating $\frac{dP}{dx}$ between Equations (5-26) and (5-27) gives

$$\frac{dP}{dx} = \left\{ \frac{\pi RV^2}{gC_v Q_1 \Delta} - \frac{6Q_1}{\pi R g C_v \Delta^3} \left[\left(\frac{2RV\Delta}{Q_1}\right)^2 - 1\right]\right\} \nu \quad . \tag{5-28}$$

Let

$$\begin{aligned} P &= p\,P_1 \\ T &= t\,T_1 \\ x &= \eta\,L \\ \nu &= \sigma\,\nu_1 \end{aligned} \tag{5-29}$$

In terms of the dimensionless variables of Equations (5-29), Equations (5-27) and (5-28) become

$$\frac{dP}{d\eta'} = \frac{6\rho\nu_1 L Q_1}{P_1 \pi R \Delta^3} \left(\frac{2RV\Delta}{Q_1} - 1 \right) \sigma \tag{5-30}$$

and

$$\frac{dt}{d\eta} = \left\{ \frac{\pi RV^2 \nu_1 L}{T_1 g C_v Q_1 \Delta} - \frac{6\nu_1 L Q_1}{T_1 \pi R g C_v \Delta^3} \left[\left(\frac{2RV\Delta}{Q_1}\right)^2 - 1 \right] \right\} \sigma . \quad (5\text{-}31)$$

Let

$$Y = \frac{6\rho \nu_1 L Q_1}{P_1 \pi R \Delta^3} \left(\frac{2RV\Delta}{Q_1} - 1\right)$$

$$(5\text{-}32)$$

and

$$Z = \frac{\pi RV^2 \nu_1 L}{T_1 g C_v Q_1 \Delta} - \frac{6\nu_1 L Q_1}{T_1 \pi R g C_v \Delta^3} \left[\left(\frac{2RV\Delta}{Q_1}\right)^2 - 1 \right] .$$

Equations (5-30) and (5-31) become

$$\frac{dp}{d\eta} = Y \sigma \quad (5\text{-}33)$$

and

$$\frac{dt}{d\eta} = Z \sigma \quad . \quad (5\text{-}34)$$

Eliminating σ from Equations (5-33) and (5-34) and integrating gives

$$\int_1^t dt = \frac{Z}{Y} \int_1^p dp \quad ,$$

or

$$t - 1 = \frac{Z}{Y} (p - 1) \quad . \quad (5\text{-}35)$$

Applying the boundary condition that when

$$\eta = 1, \; t_2 = \frac{T_2}{T_1} \text{ and } p_2 = 0 \; ,$$

Equation (5-35) becomes

$$T_2 - T_1 = - \frac{Z}{Y} T_1 \; . \quad (5\text{-}36)$$

Equation (5-36) is a general equation relating the temperature rise $(T_2 - T_1)$ to the various parameters of the problem.

Equation (5-34) can be written as

$$\frac{1}{\sigma}\frac{dt}{d\eta} = Z \quad . \qquad (5\text{-}37)$$

To integrate this equation, the viscosity-temperature relationship must be introduced. In Chapter II, it was decided to use the exponential form

$$\nu = D\ e^{-AT} \qquad (5\text{-}38)$$

where T is in degrees absolute. In terms of the dimensionless variables of Equations (5-39), Equation (5-38) becomes

$$\sigma = e^{-AT_1(t-1)} . \qquad (5\text{-}39)$$

Substituting in Equation (5-37) and integrating gives

$$\int_1^t e^{AT_1(t-1)}\ dt = Z \int_0^\eta d\eta \ ,$$

or

$$e^{AT_1(t-1)} = 1 + AT_1 Z\eta \ . \qquad (5\text{-}40)$$

Applying the boundary condition that when

$$\eta = 1, \quad t = \frac{T_2}{T_1} \ ,$$

Equation (5-40), after rearranging, becomes

$$T_2 - T_1 = \frac{1}{A} \ln\ (1 + AT_1 Z) \ . \qquad (5\text{-}41)$$

Let

$$\alpha_1 = \frac{Q_1}{2RV\Delta}$$

$$\theta = \frac{4\nu_1 VLA}{\pi g C_v \Delta^2}$$

$$\psi = \frac{AP_1}{\gamma C_v} \tag{5-42}$$

$$\varphi_1 = \alpha_1 \theta \quad .$$

Combining Equations (5-32) and (5-42) gives

$$Y = - \frac{3\varphi_1}{\psi} \frac{\alpha_1 - 1}{\alpha_1} \tag{5-43}$$

and

$$Z = \frac{3\varphi_1}{AT_1} \frac{1}{\alpha_1^2} [\alpha_1^2 - (1 - \frac{\pi^2}{24})] \quad . \tag{5-44}$$

Substituting in Equations (5-41) and (5-36) gives

$$T_2 - T_1 = \frac{1}{A} \ln \left\{ 1 + \frac{3\varphi_1}{\alpha_1^2} [\alpha_1^2 - (1 - \frac{\pi^2}{24})] \right\} \tag{5-45}$$

and

$$T_2 - T_1 = \frac{\psi}{A} \frac{1}{\alpha_1(\alpha_1 - 1)} [\alpha_1^2 - (1 - \frac{\pi^2}{24})] \quad . \tag{5-46}$$

Let

$$\beta = \frac{\psi}{\alpha_1(\alpha_1 - 1)} [\alpha_1^2 - (1 - \frac{\pi^2}{24})] \quad , \tag{5-47}$$

Equations (5-45) and (5-46) then become

$$T_2 - T_1 = \frac{1}{A} \ln [1 + \frac{3\varphi_1}{\psi} \frac{\alpha_1 - 1}{\alpha_1} \beta] \tag{5-48}$$

and

$$T_2 - T_1 = \frac{\beta}{A} \quad . \tag{5-49}$$

SECOND PHASE:

In this phase, the average rate of flow is

$$Q_2 = 2\,R\,V\,\Delta \quad , \tag{5-50}$$

so that

$$\alpha_2 = \frac{Q_2}{2RV\Delta} = 1 \quad . \tag{5-51}$$

NET FLOW:

The average net rate of flow is

$$Q = \frac{1}{2}\left(Q_1 - Q_2\right) \quad . \tag{5-52}$$

Let

$$\alpha = \frac{Q}{2RV\Delta}$$

$$\theta = \frac{4\nu_1 VLA}{\pi g C_v \Delta^2} \tag{5-53}$$

$$\varphi = \alpha\,\theta$$

$$\lambda = \frac{Q}{Q_o} = \frac{\varphi}{\varphi_o} = \frac{6\varphi}{\psi} \quad .$$

Then from Equations (5-42), (5-51), (5-52) and (5-53)

$$\alpha = \frac{1}{2}\left(\alpha_1 - 1\right) \tag{5-54}$$

and

$$\varphi = \alpha\,\theta = \frac{\theta}{2}\left(\alpha_1 - 1\right) = \frac{\varphi_1}{2}\,\frac{\alpha_1 - 1}{\alpha_1} \tag{5-55}$$

Substituting Equations (5-16), (5-53) and (5-55) in Equation (5-48) gives

$$T_2 - T_1 = \frac{1}{A} \ln (1 + \lambda\beta) \quad . \qquad (5\text{-}56)$$

Eliminating $(T_2 - T_1)$ from Equations (5-49) and (5-56) and rearranging gives

$$\lambda = \frac{1}{\beta} (e^{\beta} - 1) \quad , \qquad (5\text{-}57)$$

while the temperature rise $(T_2 - T_1)$ is, from Equation (5-49),

$$T_2 - T_1 = \frac{\beta}{A} \quad . \qquad (5\text{-}58)$$

Equations (5-57) and (5-58) are similar to those developed in Chapter II, but the definitions of the parameters in these equations are different. They also reduce to the isothermal case when β is very small.

The term β was given in terms of α_1 in Equation (5-47). It can be expressed in terms of α by combining Equations (5-47) and (5-54) which gives

$$\frac{\beta}{\psi} = \frac{1}{2\alpha(2\alpha+1)} [(2\alpha + 1)^2 - (1 - \frac{\pi^2}{24})] \quad . \qquad (5\text{-}59)$$

Equations (5-57), (5-58) and (5-59) are sufficient to plot the leakage number φ and the temperature rise $A(T_2 - T_1)$ against the velocity number θ.

Figure 26 is a plot of Equation (5-59) for $\frac{\beta}{\psi}$ vs. α. Figure 27 is a plot of φ vs. θ, while Figure 28 is a plot of $A(T_2 - T_1)$ vs. θ. The procedure used in plotting Figures 27 and 28 is as follows:

For $\psi = 0.8$ and $\beta = 2$,

$$\frac{\beta}{\psi} = 2.5 \quad .$$

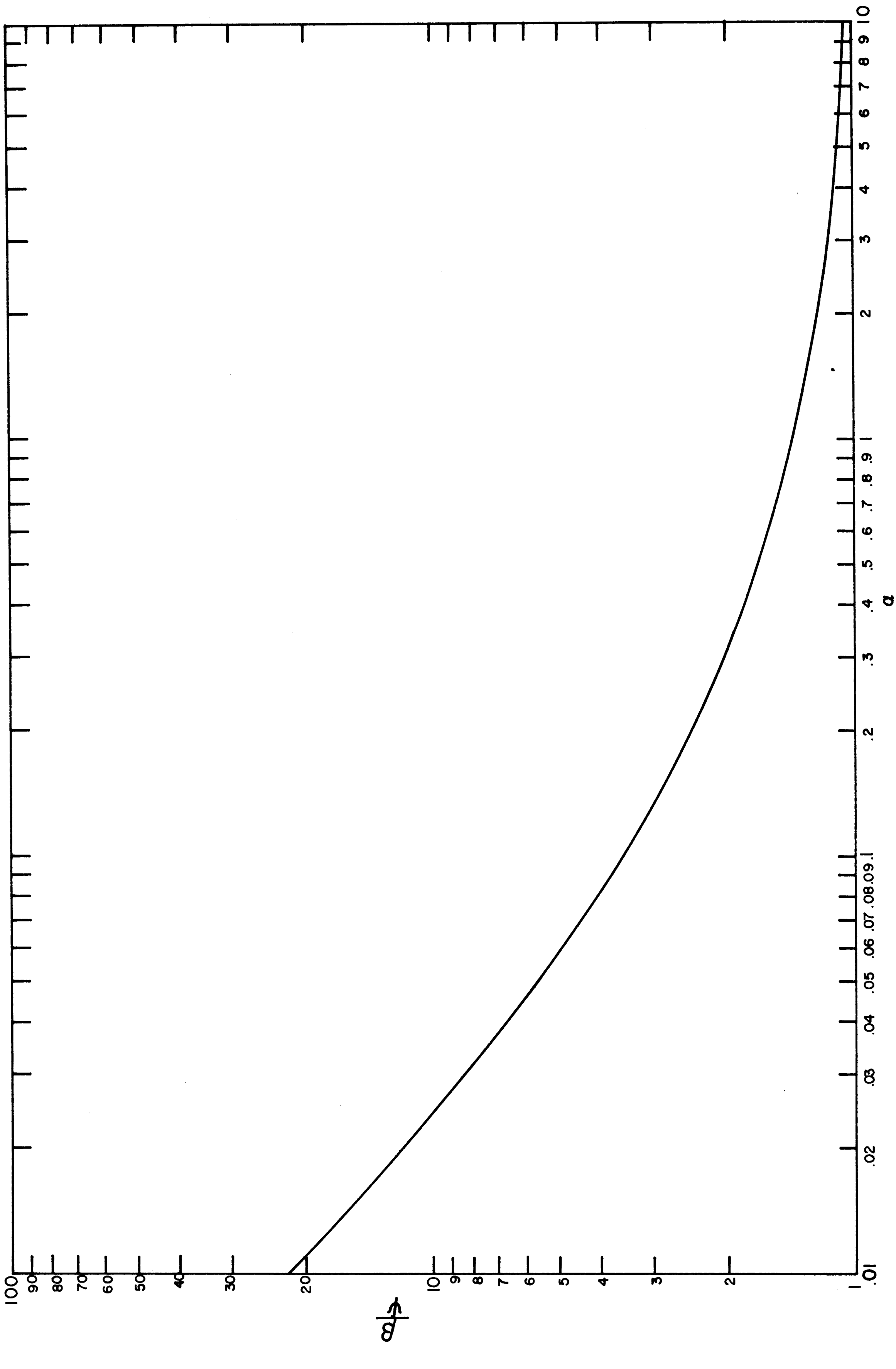

Figure 26. Plot of Equation (5-59)

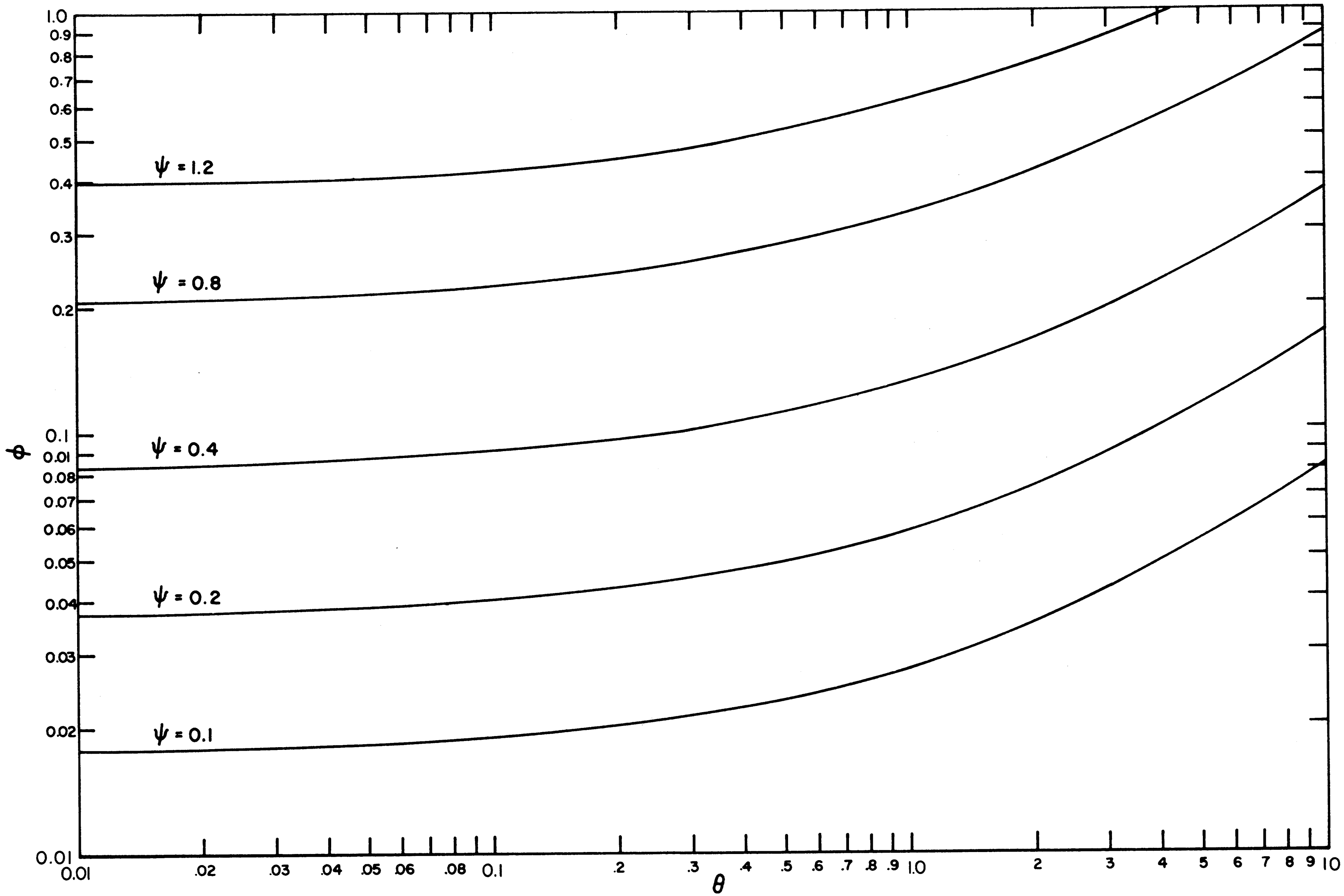

Figure 27. Leakage Number vs. Velocity Number

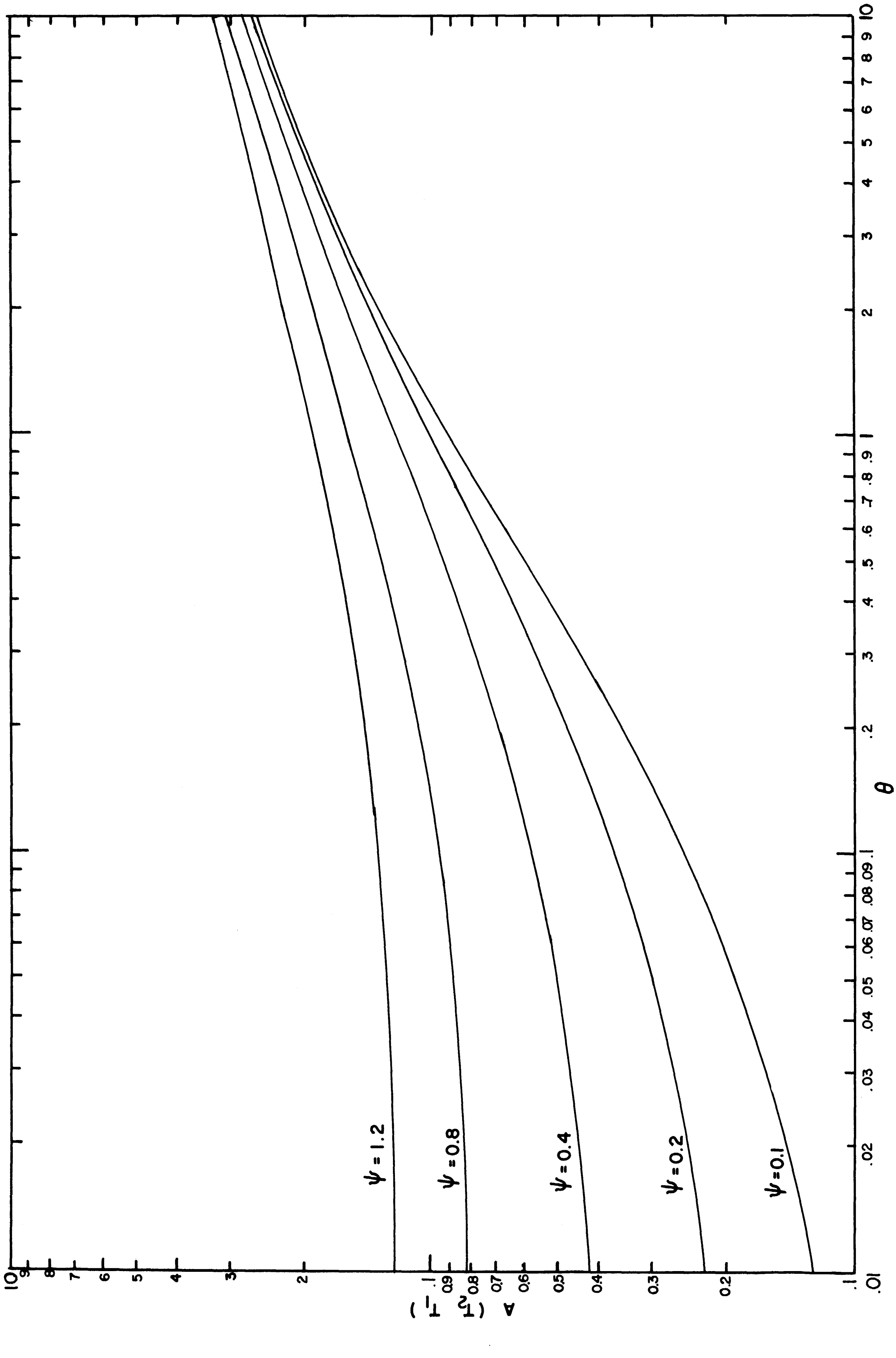

Figure 28. Temperature Rise vs. Velocity Number

From Figure 26, = 0.19.

From Equation (5-57)

$$\lambda = \frac{1}{2}(e^2 - 1) = 3.2.$$

Since

$$\lambda = \frac{6\varphi}{\psi},$$

then

$$\varphi = \frac{\psi}{6}\lambda = 0.426.$$

The corresponding value of θ is

$$\theta = \frac{\varphi}{\alpha} = 2.24.$$

And, from Equation (5-58),

$$A(T_2 - T_1) = 2 \text{ when } \theta = 2.24.$$

The rate of leakage and the temperature rise, under adiabatic conditions, can be determined from these results. The heat conduction through the cylinder wall has a cooling effect on the leakage, thereby decreasing its rate. Hence it would be desirable in practice to choose a material for the cylinder which has a high thermal conductivity.

As was stated once before, the clearance plays a major role in predicting the rate of leakage. Its initial value is equal to one-half the difference between the diameter of the cylinder bore and that of the piston. However, it may change during operation.

The pressure inside the cylinder has the effect of increasing the bore diameter. If the pressure is high and the material of the cylinder has a low modulus of elasticity, this increase in bore diameter may become significant and its consideration becomes necessary. The effect of this pressure on the diameter of the piston is smaller and may be neglected without affecting the accuracy of the results too much.

Another factor is the effect of the rise in temperature on the clearance. During steady state operation, the temperatures of the piston and that of the cylinder rise. If the thermal coefficients of expansion of the piston, α_p, and that of the cylinder, α_c, are equal, then this temperature rise would have little effect, if any, on the clearance. On the other hand, if α_c is different from α_p, then the clearance would change. It is clear, that in this case, α_c would have to be larger than α_p, if the risk of having interference between the piston and the cylinder is to be avoided. Hence, the temperature rise during operation would increase the clearance. It would be more appropriate then to base the leakage number φ and the velocity number θ on the actual clearance during operation rather than the initial clearance before operation.

VI. CONCLUDING REMARKS AND RECOMMENDATIONS FOR FURTHER WORK

The experimental results of this investigation give sufficient justification, within the investigated ranges of speed and pressure, for the various assumptions introduced in the analysis. Further work is necessary to cover larger ranges. Pressures of the order of 5000 psi are not uncommon, and speeds of 5000 rpm and more are being considered for reciprocating pumps.

The change of viscosity with pressure has been neglected. At high pressures, this change may become significant, and its consideration may improve the results presented here.

The heat transfer through the cylinder wall has a cooling effect on the leakage, thereby decreasing its rate. Investigation of this effect may shed some light on the advisability of outside cooling of pumps, such as water jacketing.

The eccentricity of the piston plays an important role in the prediction of the rate of leakage. Comparison of the mathematical and the experimental results show that, during operation, the pistons must have remained concentric. It is suspected that there exists some hydrodynamic forces which maintain the piston in a concentric position.

One such force may be due to the expansion of the fluid. If the piston is assumed eccentric, high rates of shear would prevail in the fluid on one side of the piston. These high rates of shear would increase the temperature of the fluid and cause it to expand. This expansion may then produce a centering force on the piston and restore it to its concentric position. Such a centering force would increase with the viscosity of the fluid. Accordingly, the piston operates more concentrically with high viscosity fluids. This may well be the reason why the experimental results of the Gulfpride oils (high viscosity) fell below, while those of the J-43 oil (low viscosity) fell above the adiabatic, n = 0, curves.

Another centering force may prevail from the pressure variation along the piston. In adiabatic flow, this variation is not linear, as shown in Figure 5, and depends, among other things, on the clearance. If the piston is assumed eccentric, the clearance would not be uniform around it. Now, if the pressure is integrated along and around the piston, a net force may result, which would help to restore the piston to its concentric position.

Mathematical and experimental work, to prove or disprove these arguments, is needed.

The effect of the temperature rise during operation on the clearance is another factor to be considered. If the thermal coefficients of expansion of the materials of the piston and the cylinder are the same, the change in the clearance would be small. However, if these coefficients are different, the change in the clearance may become significant. The extent of this change depends on the manner in which the piston and the cylinder are free to expand. This may differ from one pump to another.

APPENDIX A. VELOCITY OF THE PISTON

Referring to Figure 29, the position of the piston at any time is given by

$$x = \ell \cos \chi - r \cos \epsilon \quad . \qquad \text{(A-1)}$$

Since

$$\ell \sin \chi = r \sin \epsilon \quad , \qquad \text{(A-2)}$$

hence

$$\cos \chi = \left[1 - \left(\frac{r}{\ell}\right)^2 \sin^2 \epsilon\right]^{1/2} \quad .$$

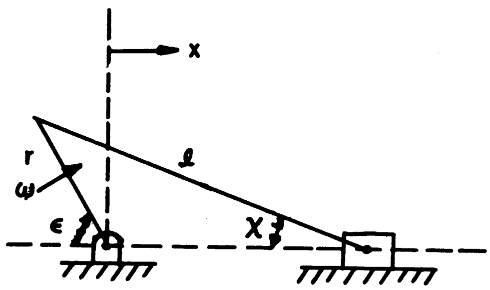

Figure 29. Coordinate System for the Motion of the Piston

Expanding this expression in a power series, gives

$$\cos \chi = 1 - \frac{1}{2} \left(\frac{r}{\ell}\right)^2 \sin^2 \epsilon - \frac{1}{8} \left(\frac{r}{\ell}\right)^4 \sin^4 \epsilon - - - -$$

If $\frac{r}{\ell}$ is small compared to one, then

$$\cos \chi \approx 1 \quad . \qquad \text{(A-3)}$$

Substituting in Equation (A-1) gives

$$x = \ell \left(1 - \frac{r}{\ell} \cos \epsilon\right) \quad . \qquad \text{(A-4)}$$

Differentiating with respect to time

$$\dot{x} = r \sin \epsilon \frac{d\epsilon}{dt} \quad . \tag{A-5}$$

Now

$$\epsilon = \omega t,$$

$$\frac{d\epsilon}{dt} = \omega$$

and

$$\dot{x} = v \quad .$$

Hence

$$v = \omega r \sin \epsilon$$

$$= V \sin (\omega t) \quad , \tag{A-6}$$

where

$$\omega r = V.$$

APPENDIX B. VARIATION OF THE CLEARANCE AROUND THE PISTON

Referring to Figure 30,

$$R_2^2 = e^2 + (R_1 + d)^2 + 2e\,(R_1 + d)\cos\xi \qquad \text{(B-1)}$$

or

$$2e\,(R_1 + d)\cos\xi = (R_2 - R_1 - d)(R_2 + R_1 + d) - e^2. \qquad \text{(B-2)}$$

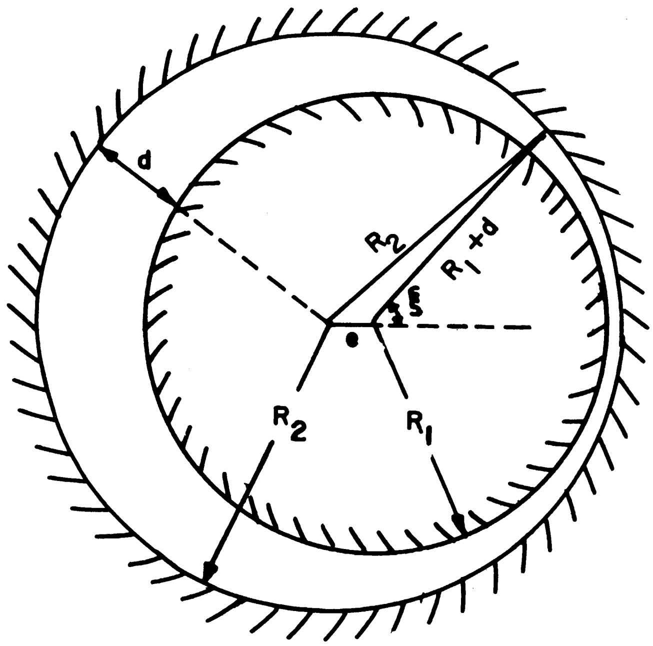

Figure 30. Variation of the Clearance Around the Piston

Let

$$\Delta = R_2 - R_1.$$

Hence

$$2e\,(R_1 + d)\cos\xi = (\Delta - d)(2R_1 + \Delta + d) - e^2 . \qquad \text{(B-3)}$$

Since d, Δ and e are small compared to R_1, Equation (B-3) can be written as

$$2\, e\, R_1 \cos \xi = 2\, R_1\, (\Delta - d)$$

or

$$d = \Delta \left(1 - \frac{e}{\Delta} \cos \xi\right) \quad . \qquad \text{(B-4)}$$

Let

$$n = \frac{e}{\Delta} \quad ,$$

Equation (B-4) becomes

$$d = \Delta\, (\, 1 - n \cos \xi\,) \, . \qquad \text{(B-5)}$$

APPENDIX C. CRITICAL DIMENSIONS

Drawings of the piston-rod assembly and the test cylinder are given in Figures 31 and 32. Tables I and II show the diameters of the pistons and the cylinder bore. For each piston, several measurements were taken at each end. The diameter of the cylinder bore was measured along two perpendicular planes at one-half inch intervals. These parts were machined by Vickers, Inc. of Detroit, Michigan.

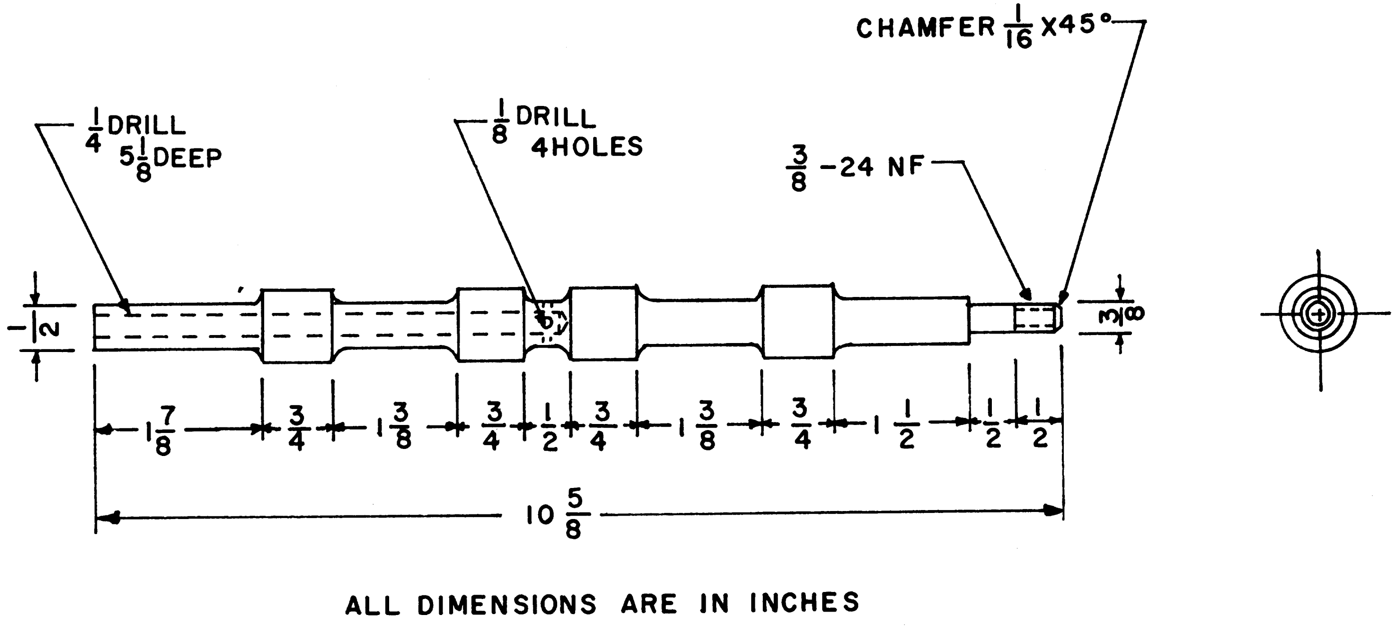

Figure 31. Piston-Rod Assembly

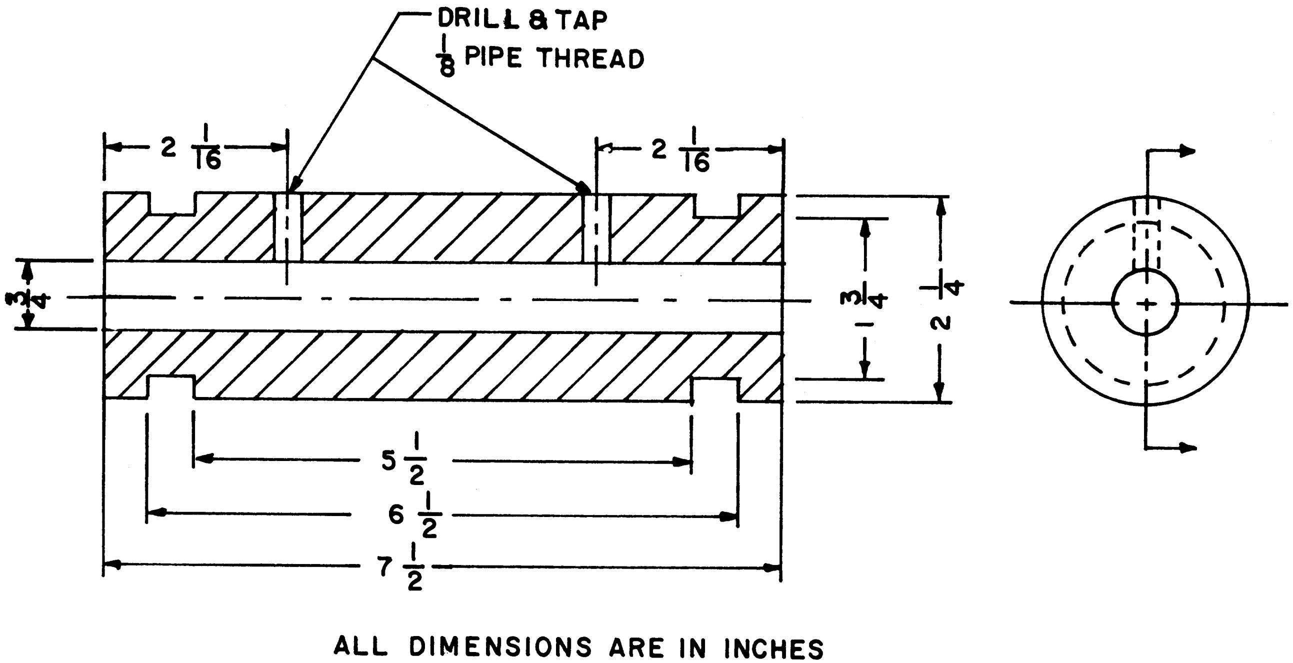

Figure 32. The Test Cylinder

TABLE I. DIAMETERS OF PISTONS

Starting from the threaded end of the piston-rod assembly.

	Before Experimental Work Temperature = 75°F		After Experimental Work Temperature = 85°F	
1-	0.74960	0.74955	0.74965	0.74960
	0.74960	0.74954	0.74968	0.74956
	0.74957	0.74955	0.74960	0.74958
	0.74957	0.74951	0.74964	0.74964
	Ave. = 0.74956		Ave. = 0.74962	
2-	0.74938	0.74940	0.74946	0.74942
	0.74938	0.74937	0.74947	0.74938
	0.74940	0.74938	0.74943	0.74940
	0.74938	0.74936	0.74944	0.74942
	Ave. = 0.74938		Ave. = 0.74943	
3-	0.74932	0.74934	0.74942	0.74941
	0.74932	0.74931	0.74938	0.74940
	0.74932	0.74934	0.74937	0.74938
	0.74932	0.74932	0.74941	0.74942
	Ave. = 0.74932		Ave. = 0.74940	
4-	0.74947	0.74953	0.74957	0.74960
	0.74942	0.74950	0.74956	0.74960
	0.74949	0.74953	0.74958	0.74962
	0.74943	0.74950	0.74956	0.74962
	Ave. = 0.74948		Ave. = 0.74959	

TABLE II. DIAMETER OF THE BORE OF THE TEST CYLINDER

The diameter was measured at half inch intervals starting from the end that is closer to the crank.

	Before Experimental Work Temperature = 75°F		After Experimental Work Temperature = 85°F	
1-	0.75026	0.75028	0.75026	0.75026
2-	0.75024	0.75025	0.75025	0.75024
3-	0.75031	0.75029	0.75036	0.75028
4-	0.75052	0.75046	0.75050	0.75048
5-	0.75068	0.75058	0.75070	0.75056
6-	0.75068	0.75057	0.75067	0.75058
7-	0.75054	0.75044	0.75054	0.75047
8-	0.75045	0.75039	0.75045	0.75039
9-	0.75040	0.75038	0.75042	0.75034
10-	0.75035	0.75027	0.75038	0.75024
11-	0.75025	0.75016	0.75020	0.75018
12-	0.75016	0.75014	0.75014	0.75014
13-	0.75018	0.75016	0.75015	0.75016
14-	0.75025	0.75020	0.75021	0.75020

APPENDIX D. PROPERTIES OF THE OILS

The oil properties which entered the analysis are:

1. The specific heat
2. The density or specific gravity
3. The kinematic viscosity

These properties are reported here in Figures 33, 34, and 35, as furnished by the manufacturers. The specific heat and the specific gravity were assumed constant in the mathematical analysis. The kinematic viscosity was allowed to vary with temperature only. The encircled points in Figure 35 were determined experimentally to check the accuracy of the viscosity curves.

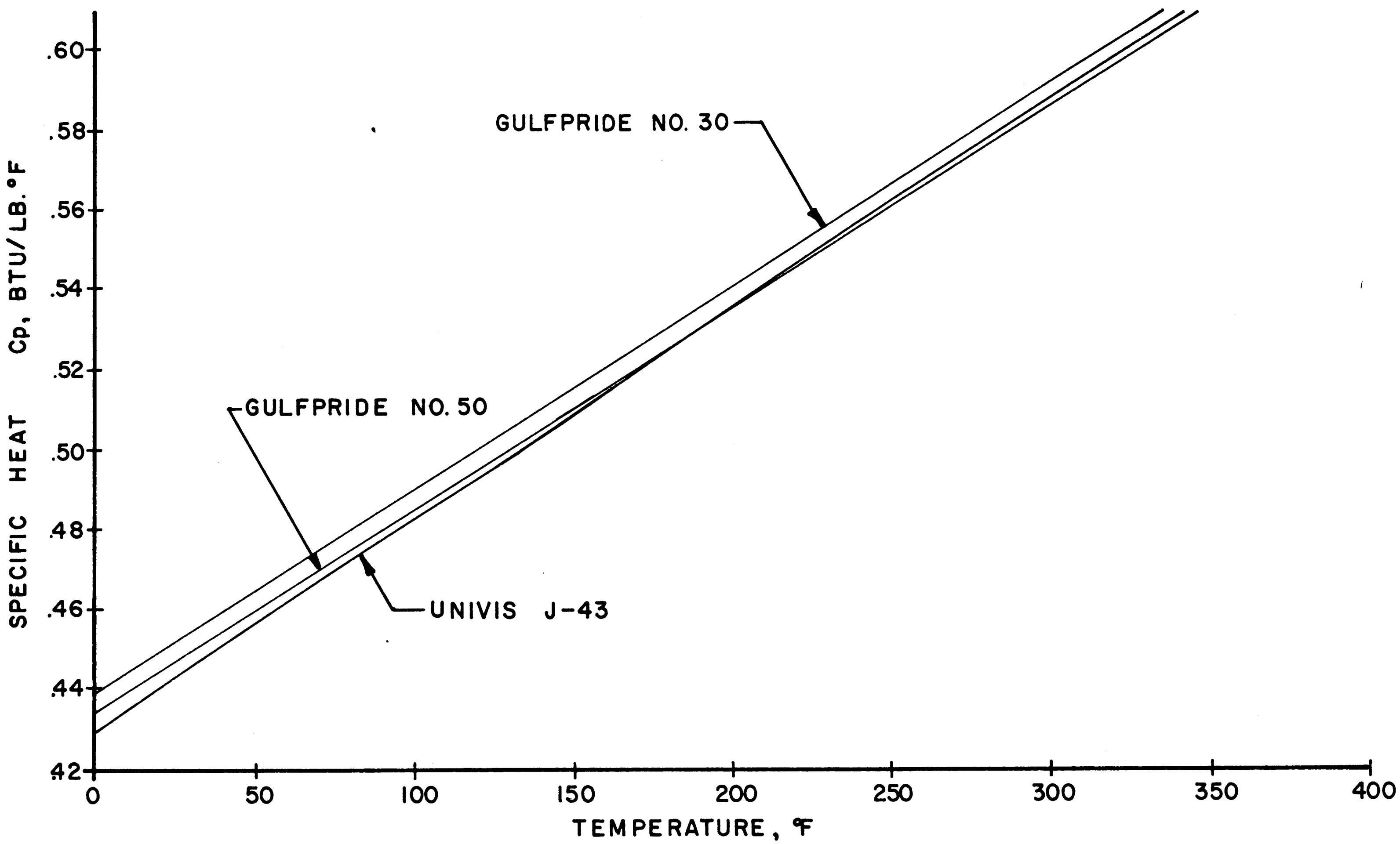

Figure 33. Variation of Specific Heat with Temperature

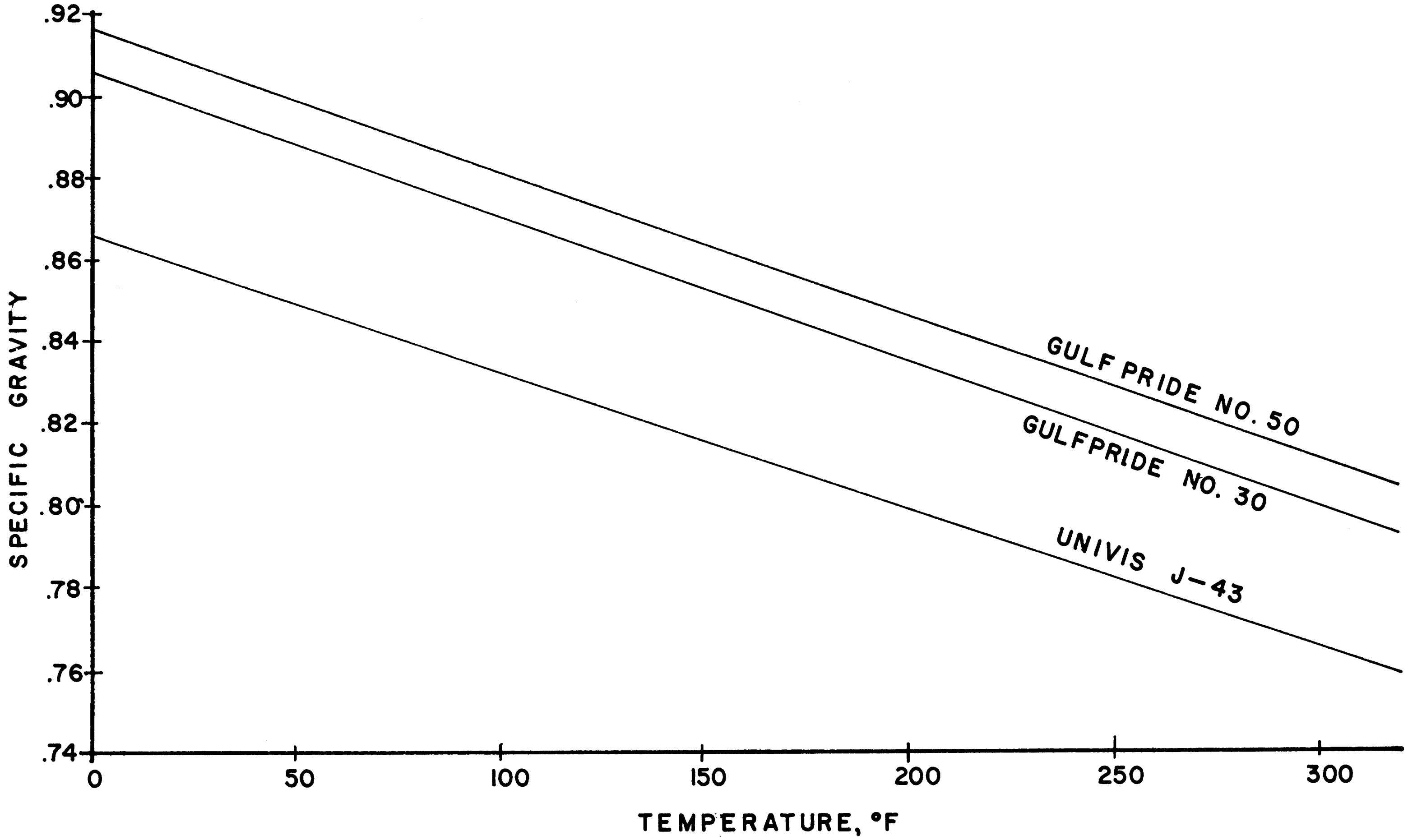

Figure 34. Variation of Specific Gravity with Temperature

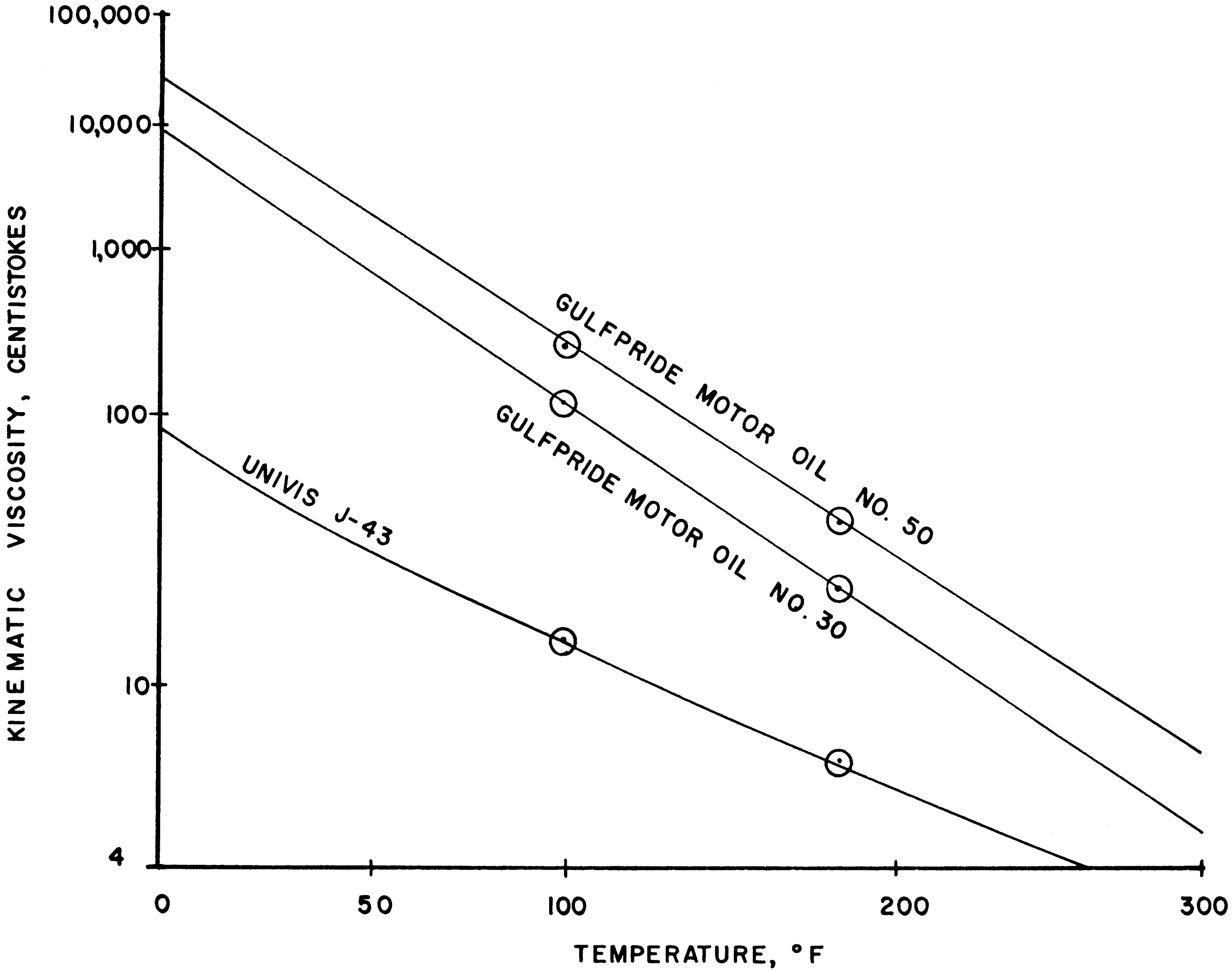

Figure 35. Variation of Kinematic Viscosity with Temperature

APPENDIX E. EXPERIMENTAL DATA

The experimental data are given in Tables III, IV, and V. The experimental values of ψ, Θ and φ were calculated from these data and shown in the same tables. Since there were two controlled leakage passages in the experimental set-up, the rate of flow Q reported in these tables is one-half the actual measured rate of flow. The values of φ in these tables are also for one leakage passage. The pressure difference is the same as the initial pressure P_1, since the final pressure P_2 was atmospheric.

TABLE III. EXPERIMENTAL DATA FOR GULFPRIDE MOTOR OIL NO. 50

A = 0.0333/°F

R = 0.375 inch

Δ = 5.6 x 10^{-4} inch

r = 0.500 inch

L = 0.750 inch

P_1 PSI	RPM	T_1 °F	T_2 °F	$Q \times 10^6$ cu ft/sec	ψ	θ	φ
445	0	83	82	0.187	0.105	0	0.0338
440	170	82	87	0.206	0.103	0.612	0.0372
430	340	83	98	0.255	0.101	1.224	0.0460
428	500	80	108	0.373	0.100	2.010	0.0751
428	750	85	125	0.588	0.100	2.460	0.0970
430	1000	87	131	0.823	0.101	3.150	0.1300
860	0	83	82	0.353	0.201	0	0.0638
860	170	83	89	0.442	0.201	0.612	0.0800
865	340	83	98	0.568	0.202	1.224	0.1025
850	500	80	107	0.755	0.199	2.010	0.1520
845	750	85	125	1.19	0.198	2.460	0.1960
855	1000	86	132	1.55	0.200	3.150	0.2440
1280	0	83	82	0.545	0.300	0	0.0985
1305	170	83	89	0.755	0.305	0.612	0.1365
1320	340	83	98	0.943	0.309	1.224	0.1700
1285	500	80	106	1.13	0.301	2.010	0.2280
1320	750	85	123	1.81	0.309	2.460	0.2980
1310	1000	86	131	2.43	0 307	3.150	0.384

TABLE III (CONT'D)

P_1 PSI	RPM	T_1 °F	T_2 °F	Q x 10^6 cu ft/sec	ψ	θ	φ
1725	0	83	82	0.767	0.404	0	0.1385
1720	170	83	90	1.11	0.402	0.612	0.2015
1700	340	84	96	1.175	0.398	1.170	0.2020
1712	500	80	104	1.52	0.400	2.010	0.3055
1715	750	85	121	2.19	0.401	2.460	0.3605
1685	1000	86	130	3.005	0.395	3.150	0.4750

TABLE IV. EXPERIMENTAL DATA FOR GULFPRIDE MOTOR OIL NO. 30

A = 0.0333/°F R = 0.375 inch

Δ = 5.6 x 10^{-4} inch r = 0.500 inch

L = 0.750 inch

P_1 PSI	RPM	T_1 °F	T_2 °F	Q x 10^6 cu ft/sec	ψ	θ	φ
420	0	86	84	0.555	0.102	0	0.0381
430	200	83	86	0.428	0.104	0.297	0.0319
425	500	78	95	0.462	0.103	0.900	0.0416
430	750	85	112	0.883	0.104	1.040	0.0611
440	1000	86	122	1.150	0.106	1.335	0.0767
860	0	84	85	0.965	0.208	0	0.0720
845	200	85	89	0.915	0.205	0.285	0.0652
860	500	78	97	1.130	0.208	0.900	0.102
880	750	85	112	1.88	0.213	1.040	0.130
875	1000	86	121	2.38	0.212	1.335	0.159
1235	0	84	85	1.36	0.299	0	0.101
1250	200	85	88	1.42	0.303	0.285	0.101
1245	500	78	98	1.76	0.301	0.900	0.159
1290	750	85	111	2.92	0.312	1.040	0.202
1320	1000	86	120	3.57	0.319	1.335	0.238
1680	0	84	85	1.84	0.406	0	0.137
1625	200	85	88	1.84	0.393	0.290	0.131
1630	500	78	99	2.59	0.394	0.900	0.234
1650	750	85	110	3.68	0.399	1.040	0.255
1660	1000	86	119	4.63	0.401	1.335	0.309

TABLE V. EXPERIMENTAL DATA FOR THE UNIVIS J-43 OIL

$A = 0.0124/°F$ $R = 0.375$ inch

$\Delta = 5.6 \times 10^{-4}$ inch $r = 0.500$ inch

$L = 0.750$ inch

P_1 PSI	RPM	T_1 °F	T_2 °F	$Q \times 10^6$ cu ft/sec	ψ	θ	φ
555	0	90	91	9.95	0.0512	0	0.0208
555	500	91	93	8.80	0.0512	0.0244	0.0185
545	1000	91	94	9.25	0.0503	0.0487	0.0194
1080	0	91	94	21.6	0.100	0	0.0454
1085	500	91	95	20	0.100	0.0244	0.0420
1095	1000	91	97	21.2	0.101	0.0487	0.0445
1615	0	91	98	38	0.150	0	0.0798
1585	500	91	98	33.95	0.1465	0.0244	0.0713
1625	1000	92	101	36.25	0.15	0.0487	0.0760

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