Mathematical Programming 35 (1986) 17-31
North-Holland

A FINITE CHARACTERIZATION OF K-MATRICES IN DIMENSIONS LESS THAN FOUR

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The class of real $n\times n$ matrices M, known as K-matrices, for which the linear complementarity problem $w-Mz=q$, $w\geqslant 0$, $z\geqslant 0$, $w^{\mathrm{T}}z=0$ has a solution whenever $w-Mz=q$, $w\geqslant 0$, $z\geqslant 0$ has a solution is characterized for dimensions $n<4$. The characterization is finite and 'practical'. Several necessary conditions, sufficient conditions, and counterexamples pertaining to K-matrices are also given. A finite characterization of completely K-matrices (K-matrices all of whose principal submatrices are also K-matrices) is proved for dimensions <4.

Key words: Linear complementarity problem, K-matrix, Q_0-matrix, finite characterization, Q-matrix.

1. Introduction

Let E^n be the n-dimensional Euclidean space and let $E^{n\times n}$ be the set of real $n\times n$ matrices. For $M\in E^{n\times n}$, M_{ij} denotes the (i,j) entry in M, and for I, $J\subset \{1,\ldots,n\}$, $M_{I\cdot}$ is the submatrix of M consisting of the rows indexed by I; and $M_{\cdot J}$ consists of the columns indexed by J. The jth column of M is denoted either M_j or $M_{\cdot j}$, the ith row of M is denoted by $M_{i\cdot}$.

Given a matrix $M\in E^{n\times n}$ and vector $q\in E^n$, the linear complementarity problem, denoted by (q, M), is to find vectors w, $z\in E^n$ such that

$$\begin{aligned} &w-Mz=q,\\ &w\geqslant 0,\quad z\geqslant 0,\quad w^{\mathrm{T}}z=0. \end{aligned} \tag{LCP}$$

This problem arises in such diverse areas as economics, game theory, linear programming, mechanics, lubrication, numerical analysis, and nonlinear optimization. Generally in a particular application area the matrix M has a special structure (e.g.,

* Partially supported by NSF Grant MCS-8207217.
** Research partially supported by NSF Grant No. ECS-8401081.

symmetric positive definite), and moderately efficient algorithms have been developed to solve these special linear complementarity problems. Nevertheless considerable attention has been devoted to the general linear complementarity problem, and in particular to special classes of matrices related to the problem.

The matrices $M \in E^{n\times n}$ for which (q, M) has a unique solution for every vector $q \in E^n$ are known as P-matrices, and are characterized algebraically by having all their principal minors positive. The equivalence of (q, M) having a unique solution for all $q \in E^n$ and M being a P-matrix (all principal minors positive) has been proved in many different ways, most notably by Ingleton [10], Samelson, Thrall, and Wesler [17], Murty [16], Watson [19], and Kelly and Watson [11]. Another finite characterization for (q, M) to have a unique solution for every $q \in E^n$ (equivalently, for M to be a P-matrix) proved in Murty [15], Tamir [18] states that this property holds iff (q, M) has a unique solution whenever q is in the finite test set $\Gamma = \{I_{\cdot 1}, \ldots, I_{\cdot n}, \ldots, M_{\cdot 1}, \ldots, M_{\cdot n}, -M_{\cdot 1}, \ldots, -M_{\cdot n}, e\}$ where I is the unit matrix of order n and e is the column vector in E^n all of whose entries are 1. M is called a Q-matrix if (q, M) has at least one solution for all $q \in E^n$, and for over ten years there have been serious attempts to characterize Q-matrices in a manner similar to P-matrices. Numerous necessary conditions or sufficient conditions have been found (see, e.g., Watson [19], Doverspike [4], Doverspike and Lemke [5]), but simple algorithmic necessary *and* sufficient conditions have remained elusive. Progress is being made, though, as Cottle [3] recently characterized completely Q-matrices (matrices whose principal submatrices are all Q-matrices).

To investigate the situation further, we require more notation.

Given any matrix D, $C(D)$ denotes the cone $\{x \mid x = Dy, y \geqslant 0\}$. The cone generated by the set of vectors $V = \{V_1, \ldots, V_k\}$ is

$$C(V) = C(V_1, \ldots V_k) = \left\{ \sum_{i=1}^{k} \alpha_i V_i \,\middle|\, \alpha_i \geqslant 0, i = 1, 2, \ldots, k \right\}.$$

For $q \in E^n$, $V \subset E^n$, if $q \in C(V)$ we say that q is *covered by* $C(V)$. If $q \in E^n$, $V = \{V_1, \ldots V_n\} \subset E^n$, q is in the *interior* of $C(V)$ if $q \in C(V)$ and $q = \sum_{i=1}^{n} \alpha_i V_i$ imply $\alpha_i > 0$ for all i.

$C(A)$, where $A_i \in \{I_i, -M_i\}$, $i = 1, 2, \ldots, n$, is called a *complementary cone formed from M*. (q, M) has a solution if and only if q is in some complementary cone. $C(A)$, where $\{A_1, \ldots, A_n\} \subset \{I_j, -M_j \mid j = 1, 2, \ldots, n\}$, is called a *noncomplementary cone formed from M*. Note: The set of complementary cones formed from M is contained in the set of noncomplementary cones formed from M.

M is called a K-matrix (following Lemke [13]) if the existence of a solution to $w\text{-}Mz = q$, $w \geqslant 0$, $z \geqslant 0$ implies the existence of a solution to the linear complementarity problem (q, M). (The class of K-matrices is also referred to as the class of Q_0-matrices in some of the literature.) So a *K-matrix* is a matrix M such that (q, M) has a solution for every $q \in C(A)$, where $C(A)$ is any noncomplementary cone. A matrix M is called $\bar{K}$ (completely K) iff it and all its principal submatrices are K. Clearly a K-matrix M is a Q-matrix iff $C(M) \bigcap$ the interior of $C(I) \neq \emptyset$. Thus a finite characterization of K-matrices would give a characterization for Q-matrices. Attempts at an algorithmic characterization of K-matrices have been made by Garcia [8], Doverspike [4], Doverspike and Lemke [5], among others. This is a larger class of matrices than the Q-matrices, but for various reasons, it may be easier to characterize than the class of Q-matrices. A finite but impractical (even for $n=4$) algorithm to check whether a given matrix M is a Q-matrix attributed to D. Gale has been described in M. Aganagic and R.W. Cottle [1]. Here we briefly discuss how this algorithm can be readily extended into a finite algorithm to check whether a given square matrix M of order n is a K-matrix. Given M, a square matrix A of order n is called a *complementary submatrix* of $(I\,|-M)$ if $A_{.j} \in \{I_{.j}, -M_{.k}\}$ for $j=1$ to n. Let $A^1, \ldots, A^l$ be all the nonsingular complementary submatrices, where $l \leq 2^n$. Let $D^T = (A^t)^{-1}$, $t=1$ to l. Consider the following system of linear constraints in which the variables are $w \in E^n$, $z \in E^n$, $q \in E^n$,

$$
\begin{aligned}
&D^T_{i_t.}q < 0 \quad t=1 \text{ to } l,\\
&w - Mz - q = 0, \qquad\qquad (1)\\
&w \geq 0, \quad z \geq 0, \quad q \text{ unrestricted},
\end{aligned}
$$

where, for each $t=1$ to l, $1 \leq i_t \leq n$. Because of the possibility of the choice of i_t, there are n^l systems of the type (1). Clearly, M is a K-matrix iff each of the n^l systems of type (1) has no feasible solution (w, z, q). The feasibility or infeasibility of a system of type (1) can be established finitely using the well known theorems of the alternative and algorithms for linear programming. This provides a finite algorithmic approach for checking whether the given square matrix M of order n is a K-matrix, but, since this involves checking the infeasibility of n^l systems of type (1), where l could be as large as 2^n, this approach is impractical even for $n=4$. Since this approach, though finite, is very inefficient, in this paper we explore other finite characterizations for K-matrices. Using P-matrices as an ideal model, there are two types of desirable characterizations. One type would say that M is a K-matrix if and only if some linear algebraic statement about M holds, where the statement is in terms of eigenvalues, minors or some other algebraic quantity. The second type of characterization would say that M is a K-matrix if and only if (q, M) has a solution for all q in some finite 'test set', which is permitted to depend on M and q. The exhaustive numerical examples in Watson [19] and Kelly and Watson [11] indicate it is unlikely that a characterization of either type exists for Q-matrices. The present paper gives a characterization of the second type for K-matrices of dimension less than 4. The situation in dimension 4 is considerably more complex, and is being investigated.

Section 2 gives a characterization of K-matrices for dimension $n=2$, and a few other minor results. Section 3 contains the characterization of K-matrices for dimension 3, and Section 4 contains a characterization of completely K-matrices (M is completely K if all its principal submatrices are K-matrices) in dimension 3. Note that since every 1×1 matrix is a K-matrix, a 2×2 K-matrix is also completely

K. Hence the characterization of 2×2 K-matrices in Section 2 is also a characterization of completely K-matrices for dimension 2.

S^{n-1} denotes the unit sphere in E^n.

In the sequel, the word 'generator' refers to any column vector of $(I \vdots -M)$.

2. Two-dimensional theorems

Theorem 1. *For $M \in E^{2\times 2}$, M is $\bar{K}$ iff $-M_1+I_1$ and $-M_2+I_2$ are covered by the complementary cones of M.*

Proof. If M is $\bar{K}$ then by definition both $-M_1+I_1$ and $-M_2+I_2$ are covered by the complementary cones of M.

Assume $-M_1+I_1$ and $-M_2+I_2$ are covered by complementary cones. M is not $\bar{K}$ if and only if there is some portion of the cone generated by $-M_i$ and I_i, for $i=1$ or 2, which is not covered by the complementary cones.

Case 1. Assume that the cone $C(-M_i, I_i)$ does not contain any I_j or $-M_j$, $j\neq i$. $-M_j+I_i$ is covered and since there are no generators in the interior of the cone $C(-M_i, I_i)$, the cone that covers $-M_i+I_i$ must cover the entire cone $C(-M_i, I_i)$.

Case 2. Assume that the cone $C(-M_i, I_i)$ does contain I_j or $-M_j$, $j\neq i$, and let A_j be this generator. Then the two cones $C(A_j, I_i)$ and $C(A_j, -M_i)$ will cover all of the cone $C(-M_i, I_i)$. $\square$

Counterexample to Theorem 1 for $M \in E^{3\times 3}$: Let

$$M=\begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -1 & -1 \end{bmatrix} \quad \text{and} \quad q=\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}.$$

Then

$$3I_1+4I_2-M_1=q,$$

$$I_1-M_2=-2M_1-M_2+I_3,$$

$$I_2-M_2=2I_1+3I_2+I_3,$$

$$I_3-M_3=I_1+I_2-2M_3,$$

but q is not covered by any of the complementary cones of M.

Theorem 2. *Let $M \in E^{2\times 2}$. If* $\det(M)>0$, *then M is $\bar{K}$.*

Proof. $\mathrm{Det}(M)>0$ implies the arc from $-M_1$ to $-M_2$ is counterclockwise.

Case 1. $C(I_1, I_2)\cap C(-M_1, -M_2)\neq\{0\}$. If the cone $C(I_1, I_2)$ contains $-M_i$, for $i=1$ or 2, then $-M_i+I_i$ is covered by the cone $C(I_1, I_2)$. If the cone $C(I_1, I_2)$ does not contain $-M_i$, for $i=1$ or 2, then $-M_i+I_i$ is covered by the cone $C(-M_i, -M_2)$. Therefore, by Theorem 1, M is $\bar{K}$.

Case 2. $C(I_1, I_2) \cap C(-M_1, -M_2) = \{0\}$. The arc from I_1 to I_2 is also counterclockwise. Starting at I_1 and going counterclockwise the generators are encountered in the following order: I_1, I_2, $-M_1$, $-M_2$. Since the generators are found in an alternating order (alternating order refers to the subscripts of the generators), if the cone $C(I_i, -M_i)$ has a nonempty interior it must contain $A_j = I_j$ or $-M_j$, $j \neq i$. In this case the cone $C(I_i, -M_i)$ is covered by the two cones $C(A_j, I_i)$ and $C(A_j, -M_i)$. If the cone $C(I_i, -M_i)$ has an empty interior then $C(I_i, -M_i)$ is a straight line and is obviously covered by the complementary cones of M. □

Counterexample to Theorem 2 for $M \in E^{3\times 3}$. Let

$$M = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 0.2 & -0.05 \end{bmatrix}.$$

Then $\det(M) = 13.95 > 0$, but $I_2 - M_2 = (1, 4, -0.2)^t$ is not covered by any of the complementary cones of M. This matrix is from Kelly and Watson [11].

Theorem 3. *Let $M \in E^{2\times 2}$. If $M_{1\cdot} < 0$ or $M_{2\cdot} < 0$ then M is $\bar{K}$.*

Proof.

Case 1. $M_{1\cdot} < 0$. If $M_{1\cdot} < 0$ then both $-M_1$ and $-M_2$ lie in the first or fourth quadrant of $E^{2\times 2}$. In this case $I_2 - M_2$ is covered by $C(I_1, I_2) \cup C(I_1, -M_2)$.

Case 1a. $-M_1$ lies on I_1 or $-M_1$ is in the first quadrant. In this case $I_1 - M_1$ is covered by the cone $C(I_1, I_2)$. Then, by Theorem 1, M is $\bar{K}$.

Case 1b. $-M_1$ lies in the fourth quadrant. In this case I_1 is covered by the cone $C(I_2, -M_1)$, therefore $I_1 - M_1$ is covered by the same cone. Then, by Theorem 1, M is $\bar{K}$.

The case for $M_{2\cdot} < 0$ can similarly be proved. □

Counterexample to Theorem 3 for $M \in E^{3\times 3}$. Let

$$M = \begin{bmatrix} -1 & -2 & -1 \\ -1 & -3 & 2 \\ 3 & 5 & -1 \end{bmatrix}.$$

Then $M_{1\cdot} < 0$ but $I_3 - M_3 = (1, -2, 2)^t$ is not covered by any of the complementary cones of M.

Theorem 4. *Let $M \in E^{2\times 2}$. If $C(I_1, -M_1)$ covers I_2 or $C(I_2, -M_2)$ covers I_1 then M is $\bar{K}$.*

Proof. Assume $C(I_i, -M_i)$, $i = 1$ or 2, covers I_j, $j \neq i$. Then $I_i - M_i$ is covered by $C(I_i, I_j) \cup C(-M_i, I_j)$.

Case 1. $-M_j$ is covered by $C(I_i, -M_i)$. Since I_j is also covered by $C(I_i, -M_i)$, $I_j - M_j$ is covered by $C(I_i, I_j) \cup C(-M_i, I_j)$. Therefore, by Theorem 1, M is $\bar{K}$.

Case 2. $-M_j$ is not covered by $C(I_i, -M_i)$. Since I_j is covered by $C(I_i, -M_i)$, we know that if $C(I_j, -M_j)$ has a nonempty interior, then $C(I_j, -M_j)$ contains A, where $A = I_i$ or $A = -M_i$. So $I_j - M_j$ is covered by $C(A, I_j) \cup C(A, -M_j)$. Therefore by Theorem 1, M is $\bar{K}$. If $C(I_j, -M_j)$ has an empty interior then $I_j - M_j$ lies in either $C(I_j)$ or $C(-M_j)$ and is obviously covered by the complementary cones of M. Therefore, by Theorem 1, M is $\bar{K}$. $\square$

Theorem 5. *Let $M \in E^{2\times 2}$. If $M_{11}M_{22} > 0$ then M is $\bar{K}$.*

Proof. Case 1. $M_{ii} > 0$ for $i = 1, 2$. $M_{ii} > 0$ for $i = 1, 2$ implies that $-M_1$ lies in $C(-I_1)$ or the second or third quadrants and that $-M_2$ lies in $C(-I_2)$ or the third or fourth quadrants. For either $-M_1 \in C(-I_1)$ or $-M_2 \in C(-I_2)$, $\det(M) > 0$ so, by Theorem 2, M is $\bar{K}$. If $-M_1$ lies in the second quadrant, then $C(I_1, -M_1)$ covers I_2; if $-M_2$ lies in the fourth quadrant, then $C(I_2, -M_2)$ covers I_1. Therefore in either case, by Theorem 4, M is $\bar{K}$. If both $-M_1$ and $-M_2$ lie in the third quadrant, M is strictly positive, hence a Q-matrix [6, 16], hence $\bar{K}$.

Case 2. $M_{ii} < 0$ for $i = 1, 2$. $M_{ii} < 0$ for $i = 1, 2$ implies that $-M_1$ lies in $C(I_1)$ or the first or fourth quadrants and that $-M_2$ lies in $C(I_2)$ or the first or second quadrants. For either $-M_1 \in C(I_1)$ or $-M_2 \in C(I_2)$, $\det(M) > 0$ so, by Theorem 2, M is $\bar{K}$. If $-M_1$ or $-M_2$ lies in the first quadrant, then $M_{2.} < 0$ or $M_{1.} < 0$, respectively. Therefore, by Theorem 3, M is $\bar{K}$. For the remaining case, $-M_1$ lies in the fourth quadrant and $C(-M_1, I_2)$ covers I_1, so $C(-M_1, I_2)$ also covers $I_1 - M_1$. Also, since $-M_2$ lies in the second quadrant $C(I_1, -M_2)$ covers I_2 so $I_2 - M_2$ is covered by $C(I_1, -M_2)$. Therefore, by Theorem 1, M is $\bar{K}$. $\square$

Counterexamples to Theorem 5 for $M \in E^{3\times 3}$. Let

$$M = \begin{bmatrix} 1 & -3 & 3 \\ -3 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} -3 \\ -2 \\ -3 \end{bmatrix}.$$

Then $M_{ii} > 0$ for $i = 1, 2, 3$ but $I_3 - M_3 = q$ is not covered by any of the complementary cones of M. Let

$$M = \begin{bmatrix} -1 & 1 & 2 \\ -1 & -3 & -2 \\ 3 & -2 & -4 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

Then $M_{ii} < 0$ for $i = 1, 2, 3$ but $I_1 - M_1 = q$ is not covered by any of the complementary cones of M.

Theorem 6. *Let $M \in E^{2\times 2}$. For M to be $\bar{K}$ it is necessary that $M_{11}M_{22} > 0$ or $M_{21} \leqslant 0$ or $M_{12} \leqslant 0$.*

Proof. Assume $M_{11}M_{22} \leqslant 0$ and $M_{21} > 0$ and $M_{12} > 0$. $M_{11}M_{22} \leqslant 0$ implies that one of the following cases holds: $M_{11} = 0$, $M_{22} = 0$, $M_{11} < 0$ and $M_{22} > 0$, or $M_{11} > 0$ and $M_{22} < 0$.

Case 1. $M_{11} = 0$. In this case $-M_1 = -\alpha I_2$, $\alpha > 0$, and $-M_2$ lies in either the second or third quadrant. If $-M_2$ lies in the second quadrant or on $C(-I_1)$, then $I_1 - M_1$ is not covered by any of the complementary cones of M; if $-M_2$ lies in the third quadrant, then $I_2 - M_2$ is not covered by any complementary cones of M. In either case, by Theorem 1, M is not $\bar{K}$.

Case 2. $M_{22} = 0$. By symmetry, the proof of this case follows from Case 1.

Case 3. $M_{11} < 0$ and $M_{22} > 0$. In this case $-M_1$ lies in the fourth quadrant and $-M_2$ lies in the third quadrant. Then $I_2 - M_2$ is not covered by any of the complementary cones of M and hence, by Theorem 1, M is not $\bar{K}$.

Case 4. $M_{11} > 0$ and $M_{22} < 0$. By symmetry, this case follows from Case 3.

Therefore, in each case, M is not $\bar{K}$. □

Theorem 7. *Let $M \in E^{2\times 2}$. For M to be $\bar{K}$ it is necessary that* $\det(M) \geqslant 0$ *or not both $M_i \leqslant 0$ and $M_j > 0$ for $i \neq j$ hold.*

Proof. Assume $\det(M) < 0$ and $M_i \leqslant 0$ and $M_j > 0$ for $i, j \in \{1, 2\}$ and $i \neq j$. $-M_j$ lies in the third quadrant and $-M_i$ lies in the first quadrant or on $C(I_1)$ or $C(I_2)$. Since $\det(M) < 0$, $I_j - M_j$ is not covered by any of the complementary cones of M. Therefore, by Theorem 1, M is not $\bar{K}$. □

Theorem 8. *Let $M \in E^{2\times 2}$ have all* nonzero minors. *Then M is $\bar{K}$ if and only $M_{1\cdot} < 0$ or $M_{2\cdot} < 0$ or $M_{11}M_{22} > 0$ or* $\det(M) > 0$.

Proof. By Theorems 2, 3, and 5 the conditions

$$M_{1\cdot} < 0 \quad \text{or} \quad M_{2\cdot} < 0 \quad \text{or} \quad M_{11}M_{22} > 0 \quad \text{or} \quad \det(M) > 0$$

imply that M is $\bar{K}$.

The negation of the conditions of the theorem:

$$M_{11}M_{22} < 0 \quad \text{and} \quad \det(M) < 0 \quad \text{and} \quad M_{1\cdot} \not< 0 \quad \text{and} \quad M_{2\cdot} \not< 0$$

imply

$$[M_{11}M_{22} < 0 \quad \text{and} \quad M_{21} > 0 \quad \text{and} \quad M_{12} > 0] \quad \text{or}$$

$$[\det(M) < 0 \quad \text{and} \quad M_i < 0 \quad \text{and} \quad M_j > 0, \quad i \neq j],$$

which imply M is not $\bar{K}$ by Theorems 6 and 7. □

A referee has suggested an alternative proof for the results in this section using the signs of the linear dependence of I_1, I_2 and $-M_1$ (or $-M_2$).

It had been conjectured that a 3×3 matrix is $\bar{K}$ iff all of its 2×2 principal submatrices are $\bar{K}$. As a counterexample to this, let

$$M=\begin{bmatrix} -3 & -2 & 3 \\ -1 & -4 & -2 \\ 2 & 1 & -1 \end{bmatrix}.$$

Then all nine 2×2 submatrices of M are K-matrices but I_3-M_3 is not covered by any of the complementary cones of M.

A matrix can be a K-matrix and not be $\bar{K}$. Let

$$M=\begin{bmatrix} 4 & -4 & 0 \\ -1 & -1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

(Watson [19]). Then M is a Q-matrix so it is a K-matrix. The submatrix

$$\begin{bmatrix} 4 & 0 \\ 2 & -1 \end{bmatrix}$$

is not a K-matrix so M is not $\bar{K}$.

3. Three-dimensional theorems

The preceding section gave a finite characterization and several theorems for two dimensions which do not generalize to three dimensions. In this section a finite characterization for 3-dimensional K-matrices is given. For dimension <4 the spherical geometry approach of Kelly and Watson [11] is both convenient and rigorous, and will be employed here. For completeness, a few definitions from [11] are included here.

Definition. If G is a nonempty set in S^{n-1} (or in E^n) and P a point, then Vis(P, G) is the union of the half-open segments $[P, X)$ in S^{n-1} (or in E^n) which lie entirely in the complement of G.

Definition. If G is a nonempty set in S^{n-1} (or in E^n) and P a point, then St(P, G) is the union of closed segments $[P, X]$, $X\in G$.

When dealing with S^{n-1}, segments in these two definitions and in the rest of the paper refer, of course, to spherical segments in S^{n-1}, that is, great circle arcs of length less than π, and Euclidean segments in E^n. The segment between antipodal points is defined to be those two points.

Let $M=\{M_i\}$, $N=\{N_i\}$, $i=1, 2, \dots, r$, be two r-tuples of points on the unit sphere S^{n-1} in E^n. A spherical $(r-1)$-simplex with vertices $P_i\in\{M_i, N_i\}$, $i=1, 2, \dots, r$, is called a complementary simplex relative to M and N. The union of such simplices is denoted $C^r(M, N)$. $C_j^r(M, N)$ means $C^{r-1}(M', N')$ where $M' = M - \{M_j\}$, $N' = N - \{N_j\}$. The union of spherical simplices with vertices $P_i \in M \cup N$, $i = 1, 2, \ldots, r$, is denoted by $S^r(M, N)$. $S_j^r(M, N)$ means $S^{r-1}(M', N')$ where $M' = M - \{M_j\}$, $N' = N - \{N_j\}$. Note that $C^r(M, N) \subset S^r(M, N)$ always. Also note that no independence assumption is made for the sets M and N, hence some of the simplices may be degenerate.

Analogous to the definition of a Q-arrangement in [11] we have:

Definition. $C^n(M, N)$ is a K-arrangement on S^{n-1} if $C^n(M, N) = S^n(M, N)$.

We now set $r = 3$ and look at the three dimensional case.

Fix $i \in \{1, 2, 3\}$, let $A = M \cup N - \{M_i, N_i\}$, and denote the elements of A by A_h, $h = 1, 2, 3, 4$. If $\overline{A_jA_k} \cap \overline{A_lA_m} = S \neq \emptyset$ for distinct j, k, l, m, then let $R \in S$ and let P^i contain the elements $A_j + A_m + R$, $A_j + A_l + R$, $A_k + A_m + R$, $A_k + A_l + R$, and $M_i + N_i$. Else if $\overline{A_jA_k} \cap \overline{A_lA_m} = \emptyset$ for all distinct j, k, l, m, then let $P^i = \{A_j + A_k + A_l, A_j + A_k + A_m, A_j + A_l + A_m, A_k + A_l + A_m, M_i + N_i\}$.

Theorem 9. *$C^3(M, N)$ is a K-arrangement iff* $P^1 \cup P^2 \cup P^3 \subset \mathrm{St}(M_1, C_1^3(M, N)) \cup \mathrm{St}(N_1, C_1^3(M, N))$.

Proof. The necessity is trivial, since clearly $P^1 \cup P^2 \cup P^3 \subset S^3(M, N) = C^3(M, N) = \mathrm{St}(M_1, C_1^3(M, N)) \cup \mathrm{St}(N_1, C_1^3(M, N))$.

For the sufficiency, let $P^1 \cup P^2 \cup P^3 \subset \mathrm{St}(M_i, C_i^3(M, N)) \cup \mathrm{St}(N_i, C_i^3(M, N)) = S$, where i is arbitrary. We wish to show that all simplices of each $S_i^3(M, N)$ are contained in S. For $j \neq i$ assume, without loss of generality, that for some $D \in \{M_i, N_i\}$, $M_j + N_j \in \mathrm{St}(D, C_i^3(M, N))$. The line from D to $-D$ through $M_j + N_j$ intersects a point of $C_i^3(M, N)$ at $M_j + N_j$ or beyond $M_j + N_j$; let X be the first such point.

Case 1. $M_j \neq \pm N_j$ and X and D are separated by the great circle through M_j and N_j.

Case 1a. $X \neq -D$. For some $F \in \{M_k, N_k \mid k \neq i, k \neq j\}$ the point X lies on either $\overline{FM_j}$ or $\overline{FN_j}$. Since X and D are separated by the great circle through M_j and N_j, F is also separated from D by the great circle through M_j and N_j. Therefore the triangle ΔFM_jN_j does not contain either D or $-D$. Any line from D to $-D$ through a point on $\overline{M_jN_j}$ must also intersect either $\overline{FM_j}$ or $\overline{FN_j}$ of ΔFN_jM_j (Pasch's Theorem [9]). Therefore, since $\{\overline{FM_j}, \overline{FN_j}) \subset C_i^3(M, N)$, $\overline{M_jN_j} \subset \mathrm{St}(D, C_i^3(M, N))$.

Case 1b. $X = D$. There is no line segment from D to $-D$ so this case is impossible.

Case 2. $M_j = \pm N_j$.

Case 2a. $M_j = N_j$. In this case $\overline{M_jN_j} = \{N_j\} \subset C_i^3(M, N)$ so $\overline{M_jN_j} \subset \mathrm{St}(D, C_i^3(M, N))$.

Case 2b. $M_j = -N_j$. In this case $\overline{M_jN_j} = \{M_j, N_j\} \subset C_i^3(M, N)$ so $\overline{M_jN_j} \subset \mathrm{St}(D, C_i^3(M, N))$.

Case 3. $M_j \neq \pm N_j$ and D lies on the great circle through M_j and N_j. In this case M_j, N_j, M_j+N_j, D, and X all lie on the same great circle.

Case 3a. $X = M_j$ or $X = N_j$. If D lies on $\overline{M_jN_j}$ then $\overline{M_jN_j}$ is covered by the two lines $\overline{DM_j}$ and $\overline{DN_j}$, so $\overline{M_jN_j} \subset \mathrm{St}(D, C_i^3(M, N))$. If D does not lie on $\overline{M_jN_j}$ then without loss of generality assume $X = N_j$. The line from D to N_j covers $N_j + M_j$ so that line must also cover all of $\overline{M_jN_j}$. Therefore $\overline{M_jN_j} \subset \mathrm{St}(D, C_i^3(M, N))$.

Case 3b. $X \neq N_j$ and $X \neq M_j$. For some $F \in \{M_k, N_k \mid k \neq j, k \neq i\}$ the point X lies on either $\overline{FM_j}$ or $\overline{FN_j}$. Without loss of generality assume X lies on $\overline{FN_j}$. Since X and N_j lie on the great circle through M_j and N_j, the line $\overline{FN_j}$ must also lie on that great circle. Since X is the first point of $C_i^3(M, N)$ that the line from D to $-D$ through $M_j + N_j$ intersects after intersecting $N_j + M_j$, X lies on $\overline{M_jN_j}$. Therefore $X \in (\overline{FN_j} \cap \overline{M_jN_j} - \{N_j\}) \neq \emptyset$. Since F, N_j, and M_j are all on the same great circle, this implies that F is between N_j and M_j or M_j is between N_j and F, so $\overline{M_jN_j} \subset \overline{FN_j} \cup \overline{FM_j} \subset \mathrm{St}(D, C_i^3(M, N))$.

Case 4. $M_j \neq \pm N_j$ and D does not lie on the great circle through M_j and N_j, but X does lie on this great circle. In this case, $X = M_j + N_j$ and lies on $\overline{FM_j}$ or $\overline{FN_j}$ with F as before. Assume without loss of generality that $X \in \overline{FN_j}$. Since X, M_j, and N_j all lie on the same great circle, F must also lie on this great circle. Then $X = M_j + N_j$ on $\overline{FN_j}$ implies F is between N_j and M_j or M_j is between N_j and F. In both cases $\overline{M_jN_j} \subset \overline{FN_j} \cup \overline{FM_j} \subset \mathrm{St}(D, C_i^3(M, N))$.

This completes the proof that $S_i^3(M, N) \subset S$, $i = 1, 2, 3$. The simplices in $S_i^3(M, N)$ (or $C_i^3(M, N)$) create triangles on S^2 (some of which possibly degenerate to just line segments), refered to as *cells* of $S_i^3(M, N)$ (or $C_i^3(M, N)$). We now wish to show that any cell B in $S_i^3(M, N)$ is in $\mathrm{St}(M_i, C_i^3(M, N)) \cup \mathrm{St}(N_i, C_i^3(M, N))$.

The set P^i was defined so that any cell of $S_i^3(M, N)$ with nonempty interior contains a point of P^i in its interior. This can be seen by looking at the various different forms that $S_i^3(M, N)$ can have. Figure 1 shows the two cases where $\overline{A_jA_k} \cap \overline{A_lA_m} = \emptyset$ for all distinct j, k, l, m. In the one case there are three cells and in the other there are four cells, but in either case it is clear that P^i contains the midpoints of those cells. Figure 2 shows the other nondegenerate case where $\overline{A_jA_k} \cap \overline{A_lA_m} = S \neq \emptyset$ for distinct j, k, l, m. In this case there are four cells and again it is clear that P^i contains the midpoint of those cells. Figure 3 shows one of the

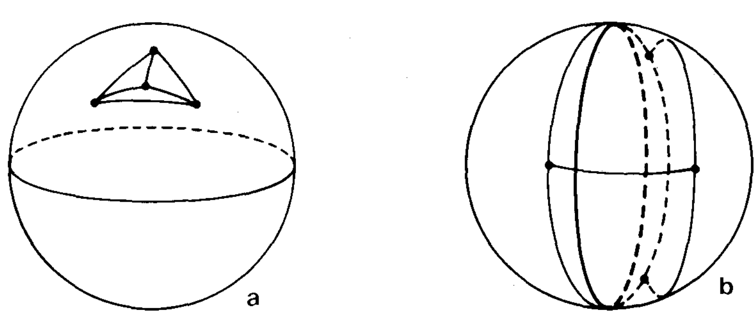

Fig. 1.

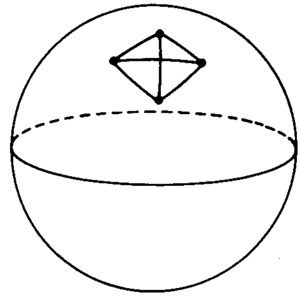

Fig. 2.

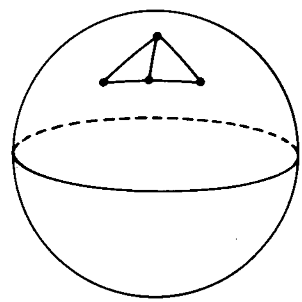

Fig. 3.

degenerate cases with only two cells. For the degenerate cases it is again easy to see that P^i contains points in the interior of the cells.

Take any cell B of $S_i^3(M, N)$.

Case 1. B is a cell of $C_i^3(M, N)$. If B contains both $-M_i$ and $-N_i$ in its interior then the interior of B would not be covered at all by $\mathrm{St}(N_i, C_i^3(M, N) \cup \mathrm{St}(M_i, C_i^3(M, N))$, a contradiction. So B does not contain both $-N_i$ and $-M_i$ in its interior, and $B \subset \mathrm{St}(M_i, C_i^3(M, N) \cup \mathrm{St}(N_i, C_i^3(M, N))$.

Case 2. B is not a cell in $C_i^3(M, N)$ and B only has one edge contained in an element of $\{\overline{M_jN_i} \mid j \neq i\}$. Let the cell B have one edge on $\overline{M_jN_j} \subset \mathrm{St}(D, C_i^3(M, N))$, $D \in \{M_i, N_i\}$, $j \neq i$ (it has already been shown that $M_j + N_j$ and $\overline{M_jN_j}$ are in the same star).

Case 2a. B contains D. A line from D through any point, say Y, in the interior of the cell B also intersects one side of B. If it intersects a side of B which lies on a part of $C_i^3(M, N)$, then $Y \in \mathrm{St}(D, C_i^3(M, N))$ clearly. Also if the line intersects the side of B lying on $\overline{M_jN_j}$, then $Y \in \mathrm{St}(D, C_i^3(M, N))$ because any line from D to $-D$ through a point of $\overline{M_jN_j}$ also intersects a point of $C_i^3(M, N)$ after intersecting $\overline{M_jN_j}$.

Case 2b. B does not contain D. Then since any line from D to $-D$ through a point of $\overline{M_jN_j}$ intersects a point of $C_i^3(M, N)$ after intersecting $\overline{M_jN_j}$, it follows that B does not contain $-D$. Any line from D to $-D$ through a point, say Y, in the interior of the cell intersects two sides of the cell. If the line doesn't intersect the side lying on $\overline{M_jN_j}$ second, then $Y \in \mathrm{St}(D, C_i^3(M, N))$. If it intersects the side lying on $\overline{M_jN_j}$ second, then $Y \in \mathrm{St}(D, C_i^3(M, N))$ since the line continues to a point of $C_i^3(M, N)$.

Case 3. B is not a cell of $C_i^3(M, N)$, and has one edge on $\overline{M_jN_j}$ and one edge on $\overline{M_kN_k}$, with i, j, k distinct. Since the side of B on $\overline{M_jN_j}$ intersects the side of B on $\overline{M_kN_k}$, $\overline{M_jN_j}$ intersects $\overline{M_kN_k}$. So the great circle through M_j and N_j either separates M_k and N_k or one of them lies on that great circle. Without loss of generality assume that the third side of B lies on $\overline{N_kN_j}$. Then B lies entirely on one side of the great circle through M_j and N_j, the same side as N_k. B is also contained in the triangle $\Delta N_jM_jN_k$, which does not contain a point of $C_i^3(M, N)$ in its interior. By Case 2 the interior of triangle $\Delta N_jM_jN_k$, and hence also B, is contained in $\mathrm{St}(D, C_i^3(M, N))$.

Therefore all the cells in $S_i^3(M, N)$ are covered by $\mathrm{St}(N_i, C_i^3(M, N)) \cup \mathrm{St}(M_i, C_i^3(M, N))$. It follows that $S^3(M, N)$ is the union of $C^3(M, N)$ and all the cells of $S_i^3(M, N)$ for $i = 1, 2, 3$, so $S^3(M, N) \subset \bigcup_{i=1}^3 \mathrm{St}(N_i, C_i^3(M, N)) \cup \mathrm{St}(M_i, C_i^3(M, N)) = C^3(M, N)$, and $C^3(M, N)$ is a K-arrangement. $\square$

The statement of Theorem 9 appears complicated, and a reasonable conjecture is that the following simpler formulation should suffice. Let

$$\Gamma = \left\{ \sum_{r=1}^{k} A_{j_r} \,\middle|\, 1 \leq j_1 < \cdots < j_k \leq 3,\ 1 \leq k \leq 3,\ A_{j_r} \in \{M_{j_r}, N_{j_r}\} \right\},$$

$$Y = \{y_1 + \cdots + y_r \mid \{y_1, \ldots, y_r\} \subset \Gamma,\ 1 \leq r \leq 3\}.$$

Then $C^3(M, N)$ is a K-arrangement if and only if $Y \subset C^3(M, N)$.

Unfortunately, this simple characterization is false, as illustrated by

$$M = \begin{bmatrix} -0.05 & 0.15 & -0.01 \\ 0.15 & -0.05 & -0.01 \\ 1 & 1 & -1 \end{bmatrix}.$$

817 of the 2951 points in Y are unique and all are covered by the complementary cones of M, but $q = (0, 0, -1)^t = -0.25M_1 - 0.75M_2 + 0.1I_1$ is not. Therefore M is not a K-matrix, showing that the complexity of the statement of Theorem 9 is justified.

4. Completely K-matrices

A simple, finite, geometric characterization is:

Theorem 10. *Let $M \in E^{3\times 3}$. Then M is $\bar{K}$ iff M is K,*

$$I_1 - M_1,\ I_2 - M_2 \in \mathrm{St}(I_3, C_3^3(-M, I)) \cup \mathrm{St}(-I_3, C_3^3(-M, I)),$$

$$I_1 - M_1,\ I_3 - M_3 \in \mathrm{St}(I_2, C_2^3(-M, I)) \cup \mathrm{St}(-I_2, C_2^3(-M, I)),$$

and $I_2 - M_2,\ I_3 - M_3 \in \mathrm{St}(I_1, C_1^3(-M, I)) \cup \mathrm{St}(-I_1, C_1^3(-M, I))$.

Proof. For $J = K = \{1, 2\}$, $N = M_{JK}$ is one of the three principal 2×2 submatrices of M. The first two elements of $I_1 - M_1$, $I_2 - M_2$ are equal to $I_1 - N_1$, $I_2 - N_2$, respectively. Since the first two elements of I_3 and $-I_3$ are zero,

$$I_1 - M_1,\ I_2 - M_2 \in \mathrm{St}(I_3, C_3^3(-M, I)) \cup \mathrm{St}(-I_3, C_3^3(-M, I)),$$

if and only if the first two elements of $I_1 - M_1$, $I_2 - M_2$ are complementary combinations of $-N_1$, $-N_2$ and the columns from the two dimensional identity matrix if and only if $I_1 - N_2$ and $I_2 - N_2$ are both covered by complementary cones of N if and only if (by Theorem 1) N is a K-matrix.

The equivalence of the last two conditions in the theorem to the other two principal 2×2 submatrices of M being K-matrices is proved similarly. Since all 1×1 matrices are K-matrices, the theorem follows. □

An efficient characterization of completely K-matrices would exploit the K-matrix property of the lower dimensional principal submatrices to help characterize the K-matrix property of the higher dimensional principal submatrices. Thus for the case $n = 3$, it is natural to investigate the relationship between the 2×2 principal submatrices being K-matrices and the whole 3×3 matrix being a K-matrix. There is a weak relationship, as shown by

Theorem 11. *Let $M \in E^{3\times 3}$. If all the principal 2×2 submatrices of M are not K-matrices, then M is not a K-matrix.*

Proof. Assume all three principal 2×2 submatrices of M are not K-matrices. From Theorem 5 we know there can be at most one positive and at most one negative diagonal element of M.

For $N \in E^{2\times 2}$ not a K-matrix, it easily follows from the theorems of Section 3 that the following must hold:

If $N_{11} < 0$ and $N_{22} \geqslant 0$, then $N_{12} > 0$ and N_{21} must be such that $N_{12}N_{21} \geqslant N_{11}N_{22}$. (By Theorem 3 we know $N_{12} \geqslant 0$. If $N_{12} = 0$ then $\{I_1 - N_1, I_2 - N_2\} \subseteq C(I_1, I_2) \cup C(I_1, -N_2) \cup C(-N_1, I_2)$ and by Theorem 1 N is $\bar{K}$, so $N_{12} > 0$. From Theorem 2 we have that $N_{12}N_{21} \geqslant N_{11}N_{22}$.)

If $N_{11} > 0$ and $N_{22} = 0$, then $N_{12} \geqslant 0$ and $N_{21} > 0$. (If $N_{12} < 0$ then $C(I_2, -N_2)$ covers I_1 and by Theorem 4, N is K. If $N_{21} < 0$ then, again by Theorem 4, N is K. Also if $N_{21} = 0$ then $I_2 - N_2$ and $I_1 - N_1$ are covered by $C(I_1, I_2) \cup C(I_2, -N_1)$ so by Theorem 1, N is K. Therefore, $N_{21} > 0$.)

If $N_{11} = N_{22} = 0$, then $N_{12} \geqslant 0$, $N_{21} \geqslant 0$, and $N_{12} + N_{21} > 0$. (Using Theorem 4 we have that $N_{12} \geqslant 0$ and $N_{21} \geqslant 0$. With $N_{12} \geqslant 0$ and $N_{21} \geqslant 0$ we have $N_{12} + N_{21} \geqslant 0$, but if $N_{12} + N_{21} = 0$ then N is K, so $N_{12} + N_{21} > 0$.)

Using these facts and symmetry to reduce the number of cases, it immediately follows that M has one of the forms

$$\begin{bmatrix} - & + & + \\ -,0,+ & + & 0,+ \\ 0,+ & + & 0 \end{bmatrix}, \begin{bmatrix} - & + & + \\ 0,+ & 0 & 0,+ \\ 0,+ & 0,+ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0,+ & + \\ 0,+ & 0 & + \\ 0,+ & 0,+ & + \end{bmatrix}, \begin{bmatrix} 0 & 0,+ & 0,+ \\ 0,+ & 0 & 0,+ \\ 0,+ & 0,+ & 0 \end{bmatrix}$$

where '+', '−', '0' indicate a positive, negative, or zero element respectively. For each of these forms if we take $q_3 > 0$, which we may do independent of what q_1 and q_2 are, then we force $w_3 > 0$ and $z_3 = 0$. This reduces the problem to a 2×2 subproblem which we know isn't K. Therefore, M is not a K-matrix. □

If all the 2×2 principal submatrices of M are K-matrices, it should be easier to verify that M is also a K-matrix than if nothing were known about the 2×2 submatrices. Interestingly, this is not true, as shown by the following sequence of examples. All the 2×2 principal submatrices of

$$M = \begin{bmatrix} -0.05 & 0.15 & -0.01 \\ 0.15 & -0.05 & -0.01 \\ 1 & 1 & -1 \end{bmatrix}.$$

are K-matrices, but $q = -0.25M_1 - 0.75M_2 + 0.1I_1 = (0, 0, -1)^t$ is not covered by complementary cones. Similarly for

$$M = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & 2 & 1 \end{bmatrix},$$

$q = I_2 - M_2 = (0, -1, -2)^t$ is not covered. These examples show that it is necessary to check *all* the points mentioned in Theorem 9, and thus knowing that the 2×2 principal submatrices are K-matrices is of no help (at least for this type of finite characterization). The algebraic signs of the minors of K-matrices, completely K-matrices, and non-K-matrices were investigated, and no patterns were apparent other than what have already been mentioned.

Acknowledgement

The authors greatfully acknowledge the referees careful reading of the manuscript and their constructive criticisms which improved the exposition.

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