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Determination of optimal vertices from feasible solutions in unimodular linear programming

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In this paper we consider a linear programming problem with the underlying matrix unimodular, and the other data integer. Given arbitrary near optimum feasible solutions to the primal and the dual problems, we obtain conditions under which statements can be made about the value of certain variables in optimal vertices. Such results have applications to the problem of determining the stopping criterion in interior point methods like the primal-dual affine scaling method and the path following methods for linear programming.

Key words: Linear programming, duality theorem, unimodular, totally unimodular, interior point methods.

1. Introduction

We consider the following linear programming problem:

$$\begin{aligned} \text{Minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geqslant \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{A}$ is an unimodular matrix (i.e., the determinant of each Basis matrix of $\boldsymbol{A}$ is $-1, 1$, or 0), and $\boldsymbol{b}$, $\boldsymbol{c}$ are integers. For such linear programs, it is well known that all extreme vertices are integers.

In this paper we consider the problem of determining optimal solutions of this linear program from information derived from a given pair of primal and dual near

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optimum feasible solutions. An example of such a result is the strong duality theorem which asserts that if the objective function value of the given primal solution is equal to the objective function value of the given dual solution, then we can declare the pair to be optimal for the respective problems. Here we investigate the problem of determining optimal vertices of the two problems given that the difference in the objective function values (i.e., the duality gap) is greater than zero. For the special case of unimodular systems, under the hypothesis that the duality gap is small (not necessarily zero), we obtain results that assert the integrality of variables in optimal solutions. An example of such a result (Corollary 3) is that if the duality gap is less than $\frac{1}{2}$, and the optimum solution of the program is unique, then the optimum vertex can be obtained by a simple rounding routine.

These results have applications in determining stopping rules in interior point methods. The study of these methods was initiated by the seminal work of Karmarkar [3]. Our results are particularly applicable to the methods which work in both the primal and the dual feasible regions. These include the methods of Choi, Monma and Shanno [1], Kojima, Mizuno and Yoshise [4-6], Monteiro and Adler [9], Saigal [10], and Ye [12]. These results can also be used in the primal methods where a lower bound on the objective function value is available; and, in the dual methods where an upper bound on the objective function is available. In case the data of the linear program is integral (i.e., $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ are integers) it can be shown that an optimum solution of the linear program can be readily identified when the duality gap becomes smaller than $2^{-O(L)}$, where L is the size of the binary string needed to code all the integer data of the linear program. Compare this to the result just quoted above for the unimodular systems, which include the transportation and assignment problems. For such systems, the first such results were obtained by Mizuno and Masuzawa [8] in the context of the transportation problem, Masuzawa, Mizuno and Mori [7] in the context of the minimum cost flow problem, and Saigal [11] in the context of the assignment problem.

After presenting the notation and assumptions in Section 2, in Section 3 we prove our main results that show how to identify an optimal solution from the duality gap; and, in Section 4, we present the concluding remarks.

2. Notation and definitions

We consider the following primal and dual linear programming problems.

(P) Minimize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

subject to $\boldsymbol{x} \in \boldsymbol{G}_x = \{\boldsymbol{x}\colon \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geqslant \boldsymbol{0}\}$.

(D) Maximize $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$

subject to $(\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{F}_{yz} = \{(\boldsymbol{y}, \boldsymbol{z})\colon \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c}, \boldsymbol{z} \geqslant \boldsymbol{0}\}$.

Throughout the paper, we impose the following assumptions on (P) and (D).

Assumption 1. The vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are integral and the matrix $\boldsymbol{A}$ is unimodular.

Assumption 2. The feasible regions $\boldsymbol{G}_x$ and $\boldsymbol{F}_{yz}$ are nonempty.

From Assumption 1, all the vertices of the polyhedral sets $\boldsymbol{G}_x$ and $\boldsymbol{F}_{yz}$ are integral. From Assumption 2 and the duality theorem of linear programming, the problems (P) and (D) have optimal solutions and their optimal values are the same, say v^*. Let $\boldsymbol{S}_x$ and $\boldsymbol{S}_{yz}$ denote the optimal solution sets of (P) and (D), respectively:

$$\boldsymbol{S}_x = \{\boldsymbol{x} \in \boldsymbol{G}_x : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = v^*\},$$

$$\boldsymbol{S}_{yz} = \{(\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{F}_{yz} : \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} = v^*\}.$$

We define the orthogonal projective sets of $\boldsymbol{F}_{yz}$ and $\boldsymbol{S}_{yz}$ onto the space of $\boldsymbol{z}$:

$$\boldsymbol{F}_z = \{\boldsymbol{z} : (\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{F}_{yz}\},$$

$$\boldsymbol{S}_z = \{\boldsymbol{z} : (\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{S}_{yz}\}.$$

We also define

$$\boldsymbol{G}_x(\boldsymbol{x}^0) = \{\boldsymbol{x} \in \boldsymbol{G}_x : \lfloor x_j^0 \rfloor \leq x_j \leq \lceil x_j^0 \rceil \text{ for each } j\}$$

for each $\boldsymbol{x}^0 \in \boldsymbol{G}_x$ and

$$\boldsymbol{F}_z(\boldsymbol{z}^0) = \{\boldsymbol{z} \in \boldsymbol{F}_z : \lfloor z_j^0 \rfloor \leq z_j \leq \lceil z_j^0 \rceil \text{ for each } j\}$$

for each $\boldsymbol{z}^0 \in \boldsymbol{F}_z$, where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the largest integer smaller than or equal to x and the smallest integer larger than or equal to x, respectively.

3. The main results

Suppose $\boldsymbol{x}'$ denotes some primal feasible solution, $\boldsymbol{z}'$ denotes some dual feasible solution, and v^* denotes their common optimum objective function value. Theorem 2 develops a relationship between $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}' - v^*$, (the measure of non-optimality of $\boldsymbol{x}'$), and the distance of x_j' from $\lfloor x_j' \rfloor$ and $\lceil x_j' \rceil$. In particular, one part of the theorem asserts that if $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}' - v^*$ is less than $x_j' - \lfloor x_j' \rfloor$ then there is an optimal integer solution $\boldsymbol{x}^*(j)$ such that $x_j^*(j) = \lceil x_j' \rceil$. These results are used in Corollary 3 to establish that in the case the optimal solution is unique, and $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}' - v^* < \frac{1}{2}$, rounding $\boldsymbol{x}'$ to the nearest integer gives the optimal solution.

Except in the case that the optimal solution is unique, Theorem 2 and Corollary 3 do not guarantee that there is a solution $\boldsymbol{x}^*$ obtained by rounding each component of x_j' to the nearest integer. However, Theorem 4 gives conditions under which some components of a feasible solution $\boldsymbol{x}'$ can be simultaneously rounded. In particular, Theorem 4 states that there is an optimal solution $\boldsymbol{x}^*$ with the property that x_j^* is the closest integer to x_j', for all j such that x_j' is within $(1-(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}' - v^*))/(1+\dim(\boldsymbol{S}_x))$ of an integer. Theorem 6 gives bounds on a dual feasible solution z_j' under which an optimal solution x^* must have $x_j^* = 0$. It also gives bounds on a primal feasible solution x_j' under which an optimal solution to the dual must satisfy $z_j^* = 0$. We now prove these theorems.

Assume that feasible solutions $x^0 \in G_x$ and $(y^0, z^0) \in F_{yz}$ are available. Since the feasible region $G_x(x^0)$ is a bounded polyhedral convex set and $x^0 \in G_x(x^0)$, there exist vertices u^i $(i = 1, \ldots, m)$ of $G_x(x^0)$ such that

$$x^0 = \sum_{i=1}^{m} \lambda_i u^i,$$

$$\sum_{i=1}^{m} \lambda_i = 1, \qquad \lambda_i \geq 0 \quad \text{for } i = 1, 2, \ldots, m,$$

where $m \leq 1 + \dim(G_x(x^0))$; $\dim(S)$ denotes the dimension of the set S. By Assumption 1, each vertex u^i is integral (this follows from the fact that the inverse of the basis matrix is integer, and that a basic feasible solution for a bounded variable linear program has non-basic variables at either upper or lower bounds), so

$$u_j^i = \lfloor x_j^0 \rfloor \quad \text{or} \quad \lceil x_j^0 \rceil$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Some of the vertices u^i are optimal, but the others are not. We divide the vertices into two index sets I_O and I_N (possibly $I_O = \emptyset$ or $I_N = \emptyset$) and rewrite the relation above as follows:

$$\begin{aligned} & x^0 = \sum_{i \in I_O} \lambda_i u^i + \sum_{i \in I_N} \lambda_i u^i, \\ & \sum_{i \in I_O} \lambda_i + \sum_{i \in I_N} \lambda_i = 1, \qquad \lambda_i \geq 0 \quad \text{for } i \in I_O \cup I_N, \end{aligned} \tag{1}$$

where

$$I_O = \{i \colon u^i \in S_x,\ i = 1, 2, \ldots, m\}, \qquad I_N = \{i \colon u^i \notin S_x,\ i = 1, 2, \ldots, m\}.$$

Similarly, there exist optimal vertices $w^i \in S_z \cap F_z(z^0)$ $(i \in J_O)$ and nonoptimal vertices $w^i \in F_z(z^0) \backslash S_z$ $(i \in J_N)$ (integrality of these vertices is established in Theorem 1 of Hoffman and Kruskal [2]) such that

$$\begin{aligned} & z^0 = \sum_{i \in J_O} \mu_i w^i + \sum_{i \in J_N} \mu_i w^i, \\ & \sum_{i \in J_O} \mu_i + \sum_{i \in J_N} \mu_i = 1, \qquad \mu_i \geq 0 \quad \text{for } i \in J_O \cup J_N. \end{aligned} \tag{2}$$

Theorem 1. *Let $x^0 \in G_x$ and $(y^0, z^0) \in F_{yz}$, and let v^* be the optimal value of (P). Suppose that x^0 and z^0 are expressed as (1) and (2), respectively. Then we have*

$$c^T x^0 - v^* \geq \sum_{i \in I_N} \lambda_i, \qquad v^* - b^T y^0 \geq \sum_{i \in J_N} \mu_i, \qquad (x^0)^T z^0 \geq \sum_{i \in I_N} \lambda_i + \sum_{i \in J_N} \mu_i.$$

Proof. We easily see that

$$\begin{aligned} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v^* &= \sum_{i\in I_{\mathrm{O}}} \lambda_i \boldsymbol{c}^{\mathrm{T}}\boldsymbol{u}^i + \sum_{i\in I_{\mathrm{N}}} \lambda \boldsymbol{c}^{\mathrm{T}}\boldsymbol{u}^i - v^* \\ &= \sum_{i\in I_{\mathrm{O}}} \lambda_i(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{u}^i - v^*) + \sum_{i\in I_{\mathrm{N}}} \lambda_i(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{u}^i - v^*) \\ &\geqslant \sum_{i\in I_{\mathrm{N}}} \lambda_i, \end{aligned}$$

where the last inequality follows from $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{u}^i = v^*$ for each $i \in I_{\mathrm{O}}$ and $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{u}^i \geqslant v^* + 1$ for each $i \in I_{\mathrm{N}}$.

In the same way, we also have the second inequality of the theorem. The third inequality follows from the first two inequalities and

$$(\boldsymbol{x}^0)^{\mathrm{T}}\boldsymbol{z}^0 = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^0 = (\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v^*) + (v^* - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^0). \qquad \square$$

The above theorem can be used to obtain some information about an optimal solution.

Theorem 2. *Let $\boldsymbol{x}^0 \in \boldsymbol{G}_x$, and let v' be a lower bound on the optimal value v^* of* (P).

(a) *If x_j^0 is integral and $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v' < 1$ then there exists an optimal solution $\boldsymbol{x}^* \in \boldsymbol{S}_x$ such that $x_j^* = x_j^0$.*

(b) *If $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v' < x_j^0 - \lfloor x_j^0 \rfloor$ then there exists an optimal solution $\boldsymbol{x}^*(j) \in \boldsymbol{S}_x$ such that $x_j^*(j) = \lceil x_j^0 \rceil$.*

(c) *If $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v' < \lceil x_j^0 \rceil - x_j^0$ then there exists an optimal solution $\boldsymbol{x}^*(j) \in \boldsymbol{S}_x$ such that $x_j^*(j) = \lfloor x_j^0 \rfloor$.*

Proof. Suppose that $\boldsymbol{x}^0$ is expressed as (1).

If x_j^0 is integral then $u_j^i = x_j^0$ for each i. If $I_{\mathrm{O}} = \emptyset$ then Theorem 1 implies

$$\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v' \geqslant \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v^* \geqslant \sum_{i\in I_{\mathrm{N}}} \lambda_i = 1.$$

Hence, if $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v' < 1$ then $I_{\mathrm{O}} \neq \emptyset$, i.e., there exists an optimal solution $\boldsymbol{u}^i$ $(i \in I_{\mathrm{O}})$ such that $u_j^i = x_j^0$.

Under the condition of (b) x_j^0 cannot be integral. Thus

$$\lceil x_j^0 \rceil = \lfloor x_j^0 \rfloor + 1.$$

If $x_j^* \neq \lceil x_j^0 \rceil$ for each $\boldsymbol{x}^* \in \boldsymbol{S}_x$, we have

$$u_j^i \neq \lceil x_j^0 \rceil \quad \text{for each } i \in I_{\mathrm{O}},$$

or equivalently

$$u_j^i = \lfloor x_j^0 \rfloor \quad \text{for each } i \in I_{\mathrm{O}}.$$

Then we see

$$\begin{aligned} x_j^0 &= \sum_{i\in I_O} \lambda_i u_j^i + \sum_{i\in I_N} \lambda_i u_j^i \\ &\leqslant \sum_{i\in I_O} \lambda_i \lfloor x_j^0 \rfloor + \sum_{i\in I_N} \lambda_i (\lfloor x_j^0 \rfloor + 1) \\ &= \lfloor x_j^0 \rfloor + \sum_{i\in I_N} \lambda_i \\ &\leqslant \lfloor x_j^0 \rfloor + (\boldsymbol{c}^T \boldsymbol{x}^0 - v^*) \quad \text{(by Theorem 1)} \\ &\leqslant \lfloor x_j^0 \rfloor + (\boldsymbol{c}^T \boldsymbol{x}^0 - v'). \end{aligned}$$

Hence we have (b).

In the same way, we can prove (c). □

As a special case of the above theorem, we get the following useful result.

Corollary 3. *Let $\boldsymbol{x}^0 \in \boldsymbol{G}_x$ and $(\boldsymbol{y}^0, \boldsymbol{z}^0) \in \boldsymbol{F}_{yz}$. If $(\boldsymbol{x}^0)^T \boldsymbol{z}^0 < \frac{1}{2}$, for each j, there exists an $\boldsymbol{x}^*(j) \in \boldsymbol{S}_x$ such that*

$$x_j^*(j) = \begin{cases} \lfloor x_j^0 \rfloor & \text{if } x_j^0 - \lfloor x_j^0 \rfloor \leqslant \frac{1}{2}, \\ \lceil x_j^0 \rceil & \text{if } x_j^0 - \lfloor x_j^0 \rfloor > \frac{1}{2}. \end{cases} \tag{3}$$

In case the problem (P) *has a unique optimal solution $x^* \in \boldsymbol{S}_x$, we can compute each coordinate of the optimal solution by* (3).

Proof. If x_j^0 is integral, Theorem 2(a) implies that $x_j^* = \lfloor x_j^0 \rfloor$ for an $\boldsymbol{x}^* \in \boldsymbol{S}_x$. So we only consider the case where x_j^0 is not integral. If $x_j^0 - \lfloor x_j^0 \rfloor \leqslant \frac{1}{2}$, we have

$$\lceil x_j^0 \rceil - x_j^0 \geqslant \tfrac{1}{2} > (\boldsymbol{x}^0)^T \boldsymbol{z}^0 = \boldsymbol{c}^T \boldsymbol{x}^0 - \boldsymbol{b}^T \boldsymbol{y}^0.$$

Since $\boldsymbol{b}^T \boldsymbol{y}^0$ is a lower bound of v^*, by Theorem 2(c), there exists $\boldsymbol{x}^* \in \boldsymbol{S}_x$ such that $x_j^* = \lfloor x_j^0 \rfloor$. If $x_j^0 - \lfloor x_j^0 \rfloor > \frac{1}{2}$, we have

$$x_j^0 - \lfloor x_j^0 \rfloor > \tfrac{1}{2} > (\boldsymbol{x}^0)^T \boldsymbol{z}^0 = \boldsymbol{c}^T \boldsymbol{x}^0 - \boldsymbol{b}^T \boldsymbol{y}^0.$$

Hence, by Theorem 2(b), there exists $\boldsymbol{x}^* \in \boldsymbol{S}_x$ such that $x_j^* = \lceil x_j^0 \rceil$. □

Theorem 2 gives information about an element of an optimal solution. The next theorem shows a relation between a feasible solution and coordinates of an optimal solution.

Theorem 4. *Let $\boldsymbol{x}^0 \in \boldsymbol{G}_x$, and let v^* be the optimal value of* (P). *If $\boldsymbol{c}^T \boldsymbol{x}^0 - v^* < 1$, there exists an optimal solution $\boldsymbol{x}^* \in \boldsymbol{S}_x$ such that*

$$x_j^* = \begin{cases} \lfloor x_j^0 \rfloor & \text{for each } j \in \left\{ j\colon x_j^0 - \lfloor x_j^0 \rfloor < \dfrac{1 - (\boldsymbol{c}^T \boldsymbol{x}^0 - v^*)}{1 + \dim(\boldsymbol{S}_x)} \right\}, \\ \lceil x_j^0 \rceil & \text{for each } j \in \left\{ j\colon \lceil x_j^0 \rceil - x_j^0 < \dfrac{1 - (\boldsymbol{c}^T \boldsymbol{x}^0 - v^*)}{1 + \dim(\boldsymbol{S}_x)} \right\}. \end{cases} \tag{4}$$

Proof. Suppose that x is expressed as (1), where we may assume without loss of generality that the number of optimal vertices is less than or equal to $1+\dim(S_x)$:

$$\# I_O \leq 1+\dim(S_x).$$

By Theorem 1, we have

$$\sum_{i\in I_O} \lambda_i = 1 - \sum_{i\in I_N} \lambda_i \geq 1-(c^T x^0 - v^*) > 0.$$

Hence $0 < \# I_O$ and there is an index $i' \in I_O$ such that

$$\lambda_{i'} \geq \frac{1-(c^T x^0 - v^*)}{1+\dim(S_x)}.$$

From (1), we see

$$x_j^0 - \lfloor x_j^0 \rfloor = \sum_{i\in I_O} \lambda_i (u_j^i - \lfloor x_j^0 \rfloor) + \sum_{i\in I_N} \lambda_i (u_j^i - \lfloor x_j^0 \rfloor),$$

$$\lceil x_j^0 \rceil - x_j^0 = \sum_{i\in I_O} \lambda_i (\lceil x_j^0 \rceil - u_j^i) + \sum_{i\in I_N} \lambda_i (\lceil x_j^0 \rceil - u_j^i).$$

Hence we obtain

$$u_j^{i'} - \lfloor x_j^0 \rfloor \leq \frac{x_j^0 - \lfloor x_j^0 \rfloor}{\lambda_{i'}} \leq \frac{1+\dim(S_x)}{1-(c^T x^0 - v^*)} (x_j^0 - \lfloor x_j^0 \rfloor),$$

$$\lceil x_j^0 \rceil - u_j^{i'} \leq \frac{\lceil x_j^0 \rceil - x_j^0}{\lambda_{i'}} \leq \frac{1+\dim(S_x)}{1-(c^T x^0 - v^*)} (\lceil x_j^0 \rceil - x_j^0).$$

Since the vertex $u^{i'}$ is integral, $x^* = u^{i'}$ satisfies (4). □

Corollary 5. *Let* $x^0 \in G_x$ *and* $(y^0, z^0) \in F_{yz}$. *If* $\lfloor c^T x^0 \rfloor = \lceil b^T y^0 \rceil$ ($=v^*$; *the optimal value) and*

$$x_j^0 z_j^0 < \frac{(1-(c^T x^0 - v^*))(1-(v^* - b^T y^0))}{(1+\dim(S_x))(1+\dim(S_z))} \quad \textit{for each } j, \tag{5}$$

the following system has a solution and each solution is an optimal solution of (P):

$$Au = b, \quad u \geq 0, \tag{6}$$

$$u_j = 0 \quad \textit{for each } j \in K = \left\{ j:\ x_j^0 < \frac{1-(c^T x^0 - v^*)}{1+\dim(S_x)} \right\}. \tag{7}$$

Proof. Suppose that $\lfloor c^T x^0 \rfloor = \lceil b^T y^0 \rceil$. Then we have $c^T x^0 - v^* < 1$ and $v^* - b^T y^0 < 1$. In the same way as Theorem 4, there exists a $(y^*, z^*) \in S_{yz}$ such that

$$z_j^* = \lfloor z_j^0 \rfloor \quad \text{for each } j \in \left\{ j:\ z_j^0 - \lfloor z_j^0 \rfloor < \frac{1-(v^* - b^T y^0)}{1+\dim(S_z)} \right\}. \tag{8}$$

From (5) and the definition of $\boldsymbol{K}$, we see that

$$x_j^0 < \frac{1-(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v^*)}{1+\dim(\boldsymbol{S}_x)} \quad \text{for each } j \in \boldsymbol{K},$$

$$z_j^0 < \frac{1-(v^* - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^0)}{1+\dim(\boldsymbol{S}_z)} \quad \text{for each } j \notin \boldsymbol{K}.$$

From (4) and (8), there exist $\boldsymbol{x}^* \in \boldsymbol{S}_x$ and $(\boldsymbol{y}^*, \boldsymbol{z}^*) \in \boldsymbol{S}_{yz}$ such that

$$x_j^* = 0 \quad \text{for each } j \in \boldsymbol{K}, \tag{9}$$

$$z_j^* = 0 \quad \text{for each } j \notin \boldsymbol{K}. \tag{10}$$

Since $\boldsymbol{x}^* \in \boldsymbol{G}_x$ and (9) holds, $\boldsymbol{x}^*$ is the solution of the system (6) and (7).

Let $\boldsymbol{u}^*$ be any solution of the system (6) and (7), then (7) and (10) imply $\boldsymbol{u}^{*\mathrm{T}}\boldsymbol{z}^* = 0$, or $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{u}^* = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^*$, from which it follows that $\boldsymbol{u}^*$ is an optimal solution of (P). □

Now we show that some coordinates of all the optimal vertices can be fixed when feasible vertices of (P) and (D) are available.

Theorem 6. *Let* $\boldsymbol{x}^0 \in \boldsymbol{G}_x$ *and* $(\boldsymbol{y}^0, \boldsymbol{z}^0) \in \boldsymbol{F}_{yz}$, *and let* v' *and* v'' *be a lower bound and upper bound of the optimal value* v^* *of* (P), *respectively.*

(a) *If* $z_j^0 > v'' - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^0$, *then* $x_j^* = 0$ *for any optimal vertex* $\boldsymbol{x}^* \in \boldsymbol{S}_x$.

(b) *If* $x_j^0 > \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0 - v'$, *then* $z_j^* = 0$ *for any optimal vertex* $(\boldsymbol{y}^*, \boldsymbol{z}^*) \in \boldsymbol{S}_{yz}$.

Proof. Let $\boldsymbol{x}^* \in \boldsymbol{S}_x$ be any optimal vertex of (P), then we see

$$x_j^* z_j^0 \le \boldsymbol{x}^{*\mathrm{T}}\boldsymbol{z}^0 = \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{c} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}^0) = v^* - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^0 \le v'' - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^0.$$

If $z_j^0 > v'' - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^0$, we have

$$x_j^* \le \frac{v'' - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^0}{z_j^0} < 1.$$

Since x_j^* is integral, we obtain (a).

In the same way, we also have (b). □

4. Concluding remarks

In this paper we obtained results under the assumption that the linear program (P) is in the standard form. In the case the problem is given in inequality from, we can derive all the results of Section 3 with the added assumption that the matrix be totally unimodular. Similar results can also be derived for other forms of the problem, i.e., problems with upper and lower bounds on variables, etc.

These results have implications for solving integer programming problems via interior point methods. This will be a topic of a subsequent paper.

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