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Seminormality Conditions in the Calculus of Variations for BV Solutions (*).

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In previous papers [4abc] we have proved existence theorems for the absolute minimum of simple and multiple integrals of the calculus of variations concerning solutions of bounded variation (BV) in the sense of CESARI ([1, 4a]). To this purpose we have replaced the usual Lebesgue integral $I(x) = \int_G F_0(t, x(t), Dx(t))\,dt$ by the corresponding Serrin functional, or integral $\mathfrak{I}$ (cfr. [4abc] and below). Thus we have dealt with classes Ω of BV trajectories $x: G \to \boldsymbol{R}^n$, $G \subset \boldsymbol{R}^\nu$, $\nu \geqslant 1$, $n \geqslant 1$, hence of class $(L_1(G))^n$ possibly discontinuous and not Sobolev's.

A relevant application has been discussed by CESARI in [3].

The conditions under which we have proved the existence of the absolute minimum for simple integrals ($\nu = 1$) and for multiple integrals ($\nu > 1$) are rather different. Indeed, in [4b], for simple integrals, we have required the usual seminormality condition (Q) and the equiboundedness of the total variations of the elements in Ω; while in [4c], for multiple integrals, we have strengthened property (Q) by an additional seminormality condition (F), without requiring any equiboundedness of the total variations. In both these situations we could prove closure, lower closure and semicontinuity properties. As an application we get the fundamental relation $I(x) \leqslant \mathfrak{I}(x)$, $x \in BV(G)$.

In the present paper we introduce a new and more general «weak condition F», or condition (wF), which subsumes the above mentioned assumptions of the simple and the multiple cases. In fact the couple af seminormality conditions (Q) and (wF) are implied by each set of requirements in [4b] and [4c]. We still prove closure, lower closure and semicontinuity theorems (Sections 1 and 2) which contain the analogous results of both [4b] and [4c]. As discussed in [4abc] and [3], these theorems imply existence theorems for the absolute minimum of the Serrin integral $\mathfrak{I}$.

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In Section 2 we remark, by an example, that condition (wF) alone does not imply semicontinuity.

In Section 3 we briefly restate, for the sake of clarity, the existence theorem of our papers [4*b*] and [4*c*] for problems of the calculus of variations in domains $G \subset \boldsymbol{R}^{\nu}$, $\nu \geq 1$, with the modifications and semplifications which derive from the analysis of the present paper. First, it is only one statement valid for both $\nu = 1$ and $\nu > 1$. Secondly, the properties (Q) and (wF) replace the different seminormality assumptions of papers [4*b*] and [4*c*]. On the other hand, the assumption of the equiboundedness of the total variations of the elements x of the given class Ω remains, since it is needed to garantee che compacteness in L_1 of any minimizing sequence. Such assumption can be disregarded in all cases in which the assumption itself, or at least the consequent compactness can be derived by the other hypotheses.

In Section 4 we then present, as an application, the problem discussed in [3]. By repeating the same process, and using condition (wF), instead of the assumptions (F) of [4*c*], we obtain an improvement in the assumptions which guarantee the existence of the absolute minimum for the Serrin integral taken into consideration; namely the Lipschitz conditions assumed in [3] are replaced here by mere continuity assumptions.

1. – The closure theorem.

Let $n \geq 1$, $\nu \geq 1$, $1 \leq N \leq n\nu$ be integers. We denote by $|E|$ the measure of a mesaurable subset E of $\boldsymbol{R}^{\nu}$. For the sake of brevity, we shall use the notations $t'^{j} = (t^1, \dots, t^{j-1}, t^{j+1}, \dots, t^{\nu}) = \tau$ and $(t^j, \tau) = (t^1, \dots, t^{\nu})$, $j = 1, \dots, \nu$. Moreover, for $E \subset \boldsymbol{R}^{\nu}$ and any $j = 1, \dots, \nu$, let E'^{j} denote the projection of E on the t'^{j}-space and let $E(t^j) = \{\tau \in \boldsymbol{R}^{\nu-1} \colon (t^j, \tau) \in E\}$.

Let $G \subset \boldsymbol{R}^{\nu}$ be a bounded open set and let A be a subset of the (t, x)-space $\boldsymbol{R}^{\nu+n}$ such that its projection on $\boldsymbol{R}^{\nu}$ contains G. For every $j = 1, \dots, \nu$ and for any $\tau \in G'^{j}$, let r_{τ} denote the straight line $t'^{j} = \tau$; then the intersection $G \cap r_{\tau}$ is the countable union of open disjoint intervals (α_s, β_s), or $G \cap r_{\tau} = \bigcup_s (\alpha_s, \beta_s)$.

As it is well-known a function $x \in L_1(G)$ is said to be of bounded variation in the sense of Cesari (BVC) [1] if there exists a set $E \subset G \subset \boldsymbol{R}^{\nu}$, with $|E| = 0$ such that, for every $j = 1, \dots, \nu$ and for almost all $\tau \in G'^{j}$, the total variations $V_{js} = V(x(\cdot, \tau), (\alpha_s, \beta_s))$, computed disregarding the values taken by x on E, are finite, $V_j(\tau) = \sum_s V_{js}$ is finite and $V_j(\cdot) \in L_1(G'^{j})$.

A BVC function has «generalized» partial derivatives $D^j x$ a.e. in G, $j = 1, \dots, \nu$, of class $L_1(G)$, obtained as limits a.e. in G of the usual incremental ratios when always we disregard the values taken by x on E. Furthermore, Krickeberg [6] proved that a function x on G is BVC if and only if $x \in L_1(G)$ and the first order partial derivatives of x in the sense of distributions are finite measures μ_j, $j = 1, \dots, \nu$.

For the sake of brevity we shall write $x \in BVC(G)$ if x is BVC in G, and the same notation will be used for vector valued functions $x = (x^1, \dots, x^n)$ if each component is in such class. In this case we put

$$V(x) = \sum_{i=1}^{n} \sum_{j=1}^{\nu} \int_{G^{ij}} V_j^i(\tau)\, d\tau .$$

As usual we shall denote by $W^{1,1}(G)$ the space of the functions $x \in (L_1(G))^n$ whose partial derivatives in the sense of distributions are again L_1-functions.

Let $x: G \to \boldsymbol{R}^n$ be a $BVC(G)$ vector function. For every $i = 1, \dots, n$, let $\{j\}_i$ denote any system of N_i integers $1 \leqslant j_1 < \dots < j_N \leqslant \nu$, and let Dx denote the system of $N = \sum_{i=1}^{n} N_i$ partial derivatives $D^j x^i$, with $j \in \{j\}_i$, $i = 1, \dots, n$.

Let $Q: A \to 2^{\boldsymbol{R}^N}$ be a given multifunction.

We shall consider problems of the calculus of variations concerning the Lebesgue integral

$$I(x) = \int_G F_0(t, x(t), Dx(t))\, dt$$

and relative Serrin functional $\mathfrak{I}$ (see below), with constraints represented by the following orientor field equation

$$(1) \qquad (t, x(t)) \in A , \qquad Dx(t) \in Q(t, x(t)) , \qquad \text{a.e. in } G .$$

In the present Section we shall take into consideration sequences of functions $(x_k)_{k \geqslant 0}$ which satisfy the conditions

$$(2) \qquad \begin{cases} x_k \in W^{1,1}(G) , \quad k \in \boldsymbol{N}, \quad x_0 \in BVC(G) , \quad x_k \to x_0 \ \text{in } (L_1(G))^n , \\ (t, x_k(t)) \in A , \quad Dx_k(t) \in Q(t, x_k(t)) , \quad k \in \boldsymbol{N}, \quad \text{a.e. in } G . \end{cases}$$

We shall need the following definition.

DEFINITION 0. – We say that the multifunction Q satisfies *condition* (wF) at the point $t_0 \in G$, with respect to a sequence $(x_k)_{k \geqslant 0}$ satisfying (2), provided

(wF) given any number $\rho > 0$, there exist a number $0 < \sigma = \sigma(\rho, t_0, (x_k)_{k \geqslant 0}) \leqslant \rho$ and a constant $0 < h_0 = h_0(t_0, (x_k)_{k \geqslant 0}, \rho) \leqslant \sigma$ such that for almost every $0 < h < h_0$ there exist a subsequence $(s_k)_k$ and two sequences of measurable functions $(\bar{x}_{s_k})_k$, $(\xi_{s_k})_k$ satisfying the conditions

$$(t, \bar{x}_{s_k}(t)) \in A , \qquad \xi_{s_k}(t) \in Q(t, \bar{x}_{s_k}(t)) , \qquad |\bar{x}_{s_k}(t) - x_0(t_0)| \leqslant \sigma$$

for a.e. $t \in q = [t_0 - h, t_0 + h]$, and moreover

$$\left| |q|^{-1} \int_q Dx_{s_k}(t)\, dt - |q|^{-1} \int_q \xi_{s_k}(t)\, dt \right| \leqslant \rho , \qquad k \in \boldsymbol{N} .$$

Following Cesari [2], we shall consider condition (Q) on the multifunction Q. We recall that the multifunction Q is said to have *property* (Q) at the point $(t_0, x_0) \in A$

(with respect to (t, x)) provided

$$(Q) \qquad Q(t_0, x_0) = \bigcap_{\sigma>0} \operatorname{cl} \operatorname{co} \cup [Q(t, x), |t-t_0| \leqslant \sigma, |x-x_0| \leqslant \sigma].$$

REMARK 1. – Property (Q) is trivial for $Q(t, x) = \boldsymbol{R}^N$ as in the classical problems of calculus of variations, and condition (wF) is trivially satisified for $Q(t, x) = \boldsymbol{R}^N$ and $A = G \times A_0$, $A_0 \subset \boldsymbol{R}^n$ a fixed set.

If the multifunction Q does not depend on x, i.e. $Q\colon G \to 2^{\boldsymbol{R}^N}$, then condition (wF) is trivially satisfied with respect to any sequence $(x_k)_{k\geqslant 0}$ such that $x_k \in W^{1,1}$ and $Dx_k(t) \in Q(t)$, a.e. in G, $k \in \boldsymbol{N}$.

Let us start by observing the following consequence of L_1-convergence.

LEMMA 2. – *Let $(x_k)_{k\geqslant 0}$ be a sequence of surfaces such that $x_k \in W^{1,1}(G)$, $k \in \boldsymbol{N}$, $x_0 \in BVC(G)$ and $x_k \to x_0$ in $(L_1(G))^n$. Then there exists a subsequence such that the following holds*

(3) *for a.e. $t_0 \in G$ there exists a set $H = H(t_0) \subset \boldsymbol{R}^+$ with $|H| = 0$ and the property that given any number $\rho > 0$, there exists a constant $h_0 = h_0(t_0, \rho) > 0$ such that, for every $h \in (0, h_0) \setminus H$, there exists an integer $k_0 = k_0(t_0, \rho, h)$ such that for every $k \geqslant k_0$*

$$\left| Dx_0(t_0) - |q|^{-1} \int_q Dx_k(t)\, dt \right| \leqslant \rho,$$

where $q = [t_0 - h, t_0 + h]$.

PROOF. – Let $E \subset G$ be a set of measure zero such that for every $t_0 \in G \setminus E$ we have (see the proof of Theorem 1 in [4c]), for a suitable subsequence and for $i = 1, \ldots, n$, $j = 1, \ldots, \nu$,

1) $$\lim_{k\to\infty} \int_{G(t_0^j)} |x_k^i(t_0^j, \tau) - x_0^i(t_0^j, \tau)|\, d\tau = 0,$$

2) $$\lim_{k\to 0^+} |q|^{-1} \int_q D^j x_0^i(t)\, dt = D^j x_0^i(t_0),$$

3) $$\int_{q^j} [x_0^i(t_0^j - h, \tau) - x_0^i(t_0^j + h, \tau)]\, d\tau = \int_q D^j x_0^i(t)\, dt + S_{ij}(t_0 - h, t_0 + h],$$

for a.e. $h > 0$,

4) $$\lim_{h\to 0^+} (2h)^{-\nu} S_{ij}(t_0 - h, t_0 + h] = 0,$$

where S_{ij} is the singular part of the measure μ_{ij}.

Let $t_0 \in G \setminus E$ be fixed and let $H = H(t_0) \subset \boldsymbol{R}^+$ be a set of measure zero in $\boldsymbol{R}^+$ such that for every $h \in \boldsymbol{R}^+ \setminus H$ condition 3) is satisfied and moreover condition 1) holds for the points $t_0^j - h$ and $t_0^j + h$, $j = 1, \dots, \nu$. Then from 2) and 4) it follows that, corresponding to $\rho > 0$, a constant $h_0 = h_0(t_0, \rho) > 0$ exists such that for every $h \in (0, h_0) \setminus H$ we have, for $i = 1, \dots, n$, $j = 1, \dots, \nu$,

$$5) \qquad \left| |q|^{-1} \int_q D^j x_0^i(t)\, dt - D^j x_0^i(t_0) \right| < \rho/2n,$$

$$6) \qquad (2h)^{-\nu} |S_{ij}(t_0 - h, t_0 + h]| < \rho/6n\nu,$$

and from 1), we derive that an integer $k_0 = k_0(t_0, \rho, h)$ exists such that for every $k \geq k_0$

$$7) \qquad (2h)^{-\nu} \int_{q'^j} |x_k^i(t_0^j - h, \tau) - x_0^i(t_0^j - h, \tau)|\, d\tau < \rho/6n\nu, \qquad i = 1, \dots, n, \ \ j = 1, \dots, \nu.$$

From 3), 7) and 6) it follows that for $i = 1, \dots, n$, $j = 1, \dots, \nu$,

$$8) \qquad \left| |q|^{-1} \int_q D^j x_0^i(t)\, dt - |q|^{-1} \int_q D^j x_k^i(t)\, dt \right| \leq$$

$$\leq (2h)^{-\nu} \int_{q'^j} |x_k^i(t_0^j - h, \tau) - x_k^i(t_0^j - h, \tau)|\, d\tau + (2h)^{-\nu} \int_{q'^j} |x_0^i(t_0^j + h, \tau) -$$

$$- x_k^i(t_0^j + h, \tau)|\, d\tau + (2h)^{-\nu} |S_{ij}(t_0 - h, t_0 + h]| \leq 3\rho/6n\nu = \rho/2n\nu.$$

Therefore from 8) and 5) we finally have

$$\left| Dx_0(t_0) - |q|^{-1} \int_q Dx_k(t)\, dt \right| < \rho$$

and Lemma 2 is thereby proved.

Now we can prove the main result.

THEOREM 3 (A closure theorem). – *Assume that A is closed and let $(x_k)_{k \geq 0}$ be a sequence satisfying* (2).

Suppose that the multifunction Q satisfies the conditions

(i) *has non empty, closed, convex values,*

(ii) *has property* (Q) (*with respect to* (t, x)) *at the point* $(t_0, x_0(t_0))$, *for a.e.* $t_0 \in G$,

(iii) *satisfies condition* (*wF*), *with respect to* $(x_k)_{k \geq 0}$, *at a.e. point* $t_0 \in G$.

Then for the limit function x_0 *we have*

$$(t, x_0(t)) \in A, \qquad Dx_0(t) \in Q(t, x_0(t)), \qquad \text{a.e. } in\ G.$$

PROOF. – Let $t_0 \in G \setminus E$, with $|E| = 0$, be fixed in such a way that $(t_0, x_0(t_0)) \in A$, condition (3) of Lemma 2 holds and all the assumptions are satisfied.

Then for every $\rho > 0$, two constants $0 < \sigma = \sigma(\rho) \leqslant \rho$ and $0 < h_0 = h_0(t_0, \rho) \leqslant \sigma$ exist such that, for almost every $0 < h < h_0$, there exist a subsequence $(s_k)_k$ and two sequences of measurable functions $(\bar{x}_{s_k})_k$, $(\xi_{s_k})_k$ such that $(t, \bar{x}_{s_k}(t)) \in A$, $\xi_{s_k}(t) \in Q(t, \bar{x}_{s_k}(t))$, $|\bar{x}_{s_k}(t) - x_0(t_0)| \leqslant \sigma$, for a.e. $t \in q = [t_0 - h, t_0 + h]$, and

$$1) \qquad \left| Dx_0(t_0) - |q|^{-1} \int_q \xi_{s_k}(t)\, dt \right| < \rho .$$

Thus

$$|q|^{-1} \int_q \xi_{s_k}(t)\, dt \in \text{cl co} \cup [Q(t', x') \colon |t' - t_0| \leqslant \sigma, |x' - x_0(t_0)| \leqslant \sigma] = \text{cl co } Q(t_0, x_0(t_0), \sigma)$$

and from 1) it follows that

$$2) \qquad Dx_0(t_0) \in [\text{cl co } Q(t_0, x_0(t_0), \sigma)]_\rho .$$

Note that for $\tau \geqslant \sigma$ we have

$$3) \qquad \text{cl co } Q(t_0, x_0(t_0), \sigma) \subset \text{cl co } Q(t_0, x_0(t_0), \tau) .$$

Now let $\tau > 0$ be fixed, then from 2) and 3) for every $0 < \rho \leqslant \tau$ we get

$$Dx_0(t_0) \in [\text{cl co } Q(t_0, x_0(t_0), \tau)]_\rho ;$$

thus from the arbitrariety of $\rho > 0$ we have

$$Dx_0(t_0) \in \text{cl co } Q(t_0, x_0(t_0), \tau)$$

and from the arbitrariety of $\tau > 0$ and property (Q) we finally have

$$Dx_0(t_0) \in \bigcap_{\tau > 0} Q(t_0, x_0(t_0), \tau) = Q(t_0, x_0(t_0)) .$$

Theorem 3 is thereby proved.

In order to compare the present closure result with the previous theorems proved in [4*bc*], we need the following propositions.

First we recall the definition of condition (F_1') given in [4*c*]:

the multifunction Q has *property* (F_1') at the point $(t_0, x_0) \in A$ provided

(F_1') given any number $\sigma > 0$ there are constants $C = C(t_0, x_0, \sigma) > 0$, $\delta = \delta(t_0, x_0, \sigma) > 0$ such that for any two measurable vector functions $x(t)$, $\xi(t)$, $t \in E$, on a measurable subset $E \subset G$, with $(t, x(t)) \in A$, $|x(t) - x_0| > \sigma$, $\xi(t) \in Q(t, x(t))$ for $|t - t_0| \leq \delta$, $t \in E$, there are two measurable vector functions $\bar{x}(t)$, $\bar{\xi}(t)$, $t \in E$, such that $(t, \bar{x}(t)) \in A$, $|\bar{x}(t) - x_0| \leq \sigma$, $\bar{\xi}(t) \in Q(t, \bar{x}(t))$ and $|\xi(t) - \bar{\xi}(t)| \leq C[|x(t) - \bar{x}(t)| + |t - t_0|]$ for $t \in E$, $|t - t_0| \leq \delta$.

Let us compare now the two conditions (F_1') and (wF).

PROPOSITION 4. – *Suppose that A is closed and let $(x_k)_{k \geq 0}$ be a given sequence satisfying condition* (2).

If the multifunction Q has property (F_1') at the point $(t_0, x_0(t_0))$, for a.e. $t_0 \in G$, then Q satisfies condition (wF), with respect to $(x_k)_{k \geq 0}$, at a.e. point $t_0 \in G$.

PROOF. – By the assumption $x_k \to x_0$ in $(L_1(G))^n$, there is a subsequence, that we still denote by $(x_k)_k$, such that $x_k \to x_0$ pointwise a.e. on G. Thus

1) $\quad (t, x_0(t)) \in A \quad$ for a.e. $t \in G$.

Moreover by Egoroff's and Lusin's theorems, for any given $\lambda > 0$ there is a compact set $K \subset G$, with $\operatorname{mis}(G \setminus K) < \lambda$, such that $x_{0/K}$ is uniformly continuous and $x_k \to x_0$ uniformly on K.

Furthermore, for a.e. $t \in K$, we have

2) $\quad \lim\limits_{h \to 0} (2h)^{-\nu} \operatorname{mis}(q \cap K) = 1, \qquad$ where $q = [t - h, t + h]$.

Consequently, corrisponding to any $0 < \sigma < 1$, there exist $0 < \delta_0 = \delta_0(\lambda, \sigma) \leq \sigma$ and an integer $k_0 = k_0(\lambda, \sigma)$ such that

3) $\quad$ for all $t', t'' \in K$ with $|t' - t''| < \delta_0$ then $|x_0(t') - x_0(t'')| < \sigma/2$,

4) $\quad$ for all $k > k_0$ and $t \in K$ then $|x_k(t) - x_0(t)| < \sigma/2$.

Since $x_0 \in (L_1(G))^n$, we have that for a.e. $t \in G$

5) $\quad \lim\limits_{h \to 0^+} |q|^{-1} \int\limits_q |x_0(\tau) - x_0(t)|\, d\tau = 0, \qquad q = [t - h, t + h] \subset G.$

Let $t_0 \in K$ be fixed in such a way that 1)-5) hold and property (F_1') is satisfied at the point $(t_0, x_0(t_0))$. Denote by δ_1 the distance of t_0 from ∂G and let $\delta > 0$, $C > 0$ be the constants given by the assumption (F_1'), corrisponding to $(t_0, x_0(t_0))$ and σ. For any given $\rho > 0$, there is $0 < h_0 = h_0(t_0, C, \rho) < \min(\delta_0, \delta_1, \delta)/2\sqrt{\nu}$ such that for all $0 < h < h_0$, put $q_0 = [t_0 - h, t_0 + h]$, the following relations hold (see 3), 2) and 5))

6) $\quad |x_0(t) - x_0(t_0)| < \sigma/2 \quad$ for all $t \in q_0 \cap K$,

7) $\quad (2h)^{-\nu} \operatorname{mis}(q_0 - K) < \rho/6C$,

8) $\quad |q_0|^{-1}\int_{q_0} |x_0(t) - x_0(t_0)|\, dt < \rho/3C.$

Let $0 < h < h_0$ be fixed. Because of the L_1-convergence of x_k to x_0, we can take an integer $k_1 = k_1(C, \rho, h) > k_0$ such that

9) $\quad |q_0|^{-1}\int_{q_0} |x_k(t) - x_0(t)|\, dt < \rho/3C \qquad \text{for all } k > k_1.$

Let $k > k_1$ be fixed. When $t \in q_0$, certainly $|t - t_0| < \delta_0 < \sigma$ and therefore (see 6) and 4)) for every $t \in q_0 \cap K$ we have $|x_k(t) - x_0(t_0)| < \sigma$.

Now we enlarge the set $q_0 \cap K$ into the maximal subset $K_k \subset q_0$ where $|x_k(t) - x_0(t_0)| \leq \sigma$; certainly K_k is measurable and for every $t \in q_0 - K_k$ we have $|x_k(t) - x_0(t_0)| > \sigma$.

By virtue of property (F_1'), corresponding to $(t_0, x_0(t_0))$, $\sigma > 0$ and the functions x_k, Dx_k over $q_0 - K_k$, we deduce that there are two measurable functions $\bar{x}_k$, ξ_k over $q_0 - K_k$ such that

10) $\quad (t, \bar{x}_k(t)) \in A\,, \quad \xi_k(t) \in Q(t, \bar{x}_k(t))\,, \quad |\bar{x}_k(t) - x_0(t_0)| \leq \sigma$

$$\text{and } \; |Dx_k(t) - \xi_k(t)| \leq C[|x_k(t) - \bar{x}_k(t)| + |t - t_0|]\,, \qquad t \in q_0 - K_k.$$

Moreover we can extend $\bar{x}_k$ and ξ_k to all q_0 by putting

11) $\quad \xi_k(t) = Dx_k(t)\,, \qquad \bar{x}_k(t) = x_k(t)\,, \qquad t \in K_k.$

Thus from 10) and the assumptions on x_k we have

12) $\quad (t, \bar{x}_k(t)) \in A\,, \quad \xi_k(t) \in Q(t, \bar{x}_k(t))\,, \quad |\bar{x}_k(t) - x_0(t_0)| \leq \sigma\,, \quad t \in q_0$

and from 10), 11), 9), 8) and 7) we get

$$\left| |q_0|^{-1}\int_{q_0} \xi_k(t)\, dt - |q_0|^{-1}\int_{q_0} Dx_k(t)\, dt \right| \leq (2h)^{-\nu} \int_{q_0 - K_k} |\xi_k(t) - Dx_k(t)|\, dt \leq$$

$$\leq C(2h)^{-\nu} \int_{q_0 - K_k} [|x_k(t) - \bar{x}_k(t)| + |t - t_0|]\, dt \leq$$

$$\leq C(2h)^{-\nu}\left[\int_{q_0} |x_k(t) - x_0(t)|\, dt + \int_{q_0} |x_0(t) - x_0(t_0)|\, dt + \right.$$

$$\left. + \int_{q_0 - K_k} |x_0(t_0) - \bar{x}_k(t)|\, dt + \int_{q_0 - K_k} |t - t_0|\, dt \right] \leq$$

$$\leq C[\rho/3C + \rho/3C] + C(2h)^{-\nu} \operatorname{mis}(q_0 - K_k)(\sigma + \sigma) \leq 2\rho/3 + \rho/3 = \rho$$

In conclusion we have proved: for every $\lambda > 0$ a compact set $K \subset G$, with $\operatorname{mis}(G - K) < \lambda$, exists such that condition (wF) is satisfied at a.e. $t_0 \in K$. By the arbitrariety of $\lambda > 0$ the thesis follows.

In force of Proposition 4, property (wF) can be considered as a «weak» form of condition (F_1') and this justifies the terminology weak$(F) = (wF)$. Note that, in an analogous way, weak forms of properties (F_2') and (F_3') in [4c] can be taken into consideration in order to obtain different versions of closure Theorem 3.

Let us consider now the particular case $\nu = 1$. Then the set G is replaced by a closed interval $[t_1, t_2]$ (see [4b]).

PROPOSITION 5. – *Let $\nu = 1$. Suppose that A is closed and let $(x_k)_{k \geq 0}$ be a sequence satisfying condition* (2).

In the functions x_k, $k \geq 0$ are equiBV, then the multifunction Q satisfies condition (wF), with respect to a subsequence $(x_k)_{k \geq 0}$ at a.e. $t_0 \in [t_1, t_2]$.

PROOF. – As in Proposition 4, assumptions assure that

1) $\quad (t, x_0(t)) \in A \qquad \text{for a.e. } t \in [t_1, t_2]$.

Moreover in force of what is shown in the proof of Theorem 1 of [4b], the following holds:

given any number $\sigma > 0$ and for a.e. $t_0 \in [t_1, t_2]$, a constant $h_0 = h_0(\sigma, t_0) > 0$ exists such that, for almost all $0 < h < h_0$, there is a subsequence $(n_k)_k$ such that

2) $\quad |x_{n_k}(t) - x_0(t_0)| < \sigma, \qquad t \in [t_0 - h, t_0 + h]$.

Thus condition (wF) is trivially verified, with $\bar{x}_{n_k} = x_{n_k}$ and $\xi_{n_k} = x'_{n_k}$.

REMARK 6. – Propostions 4 and 5 show that Theorem 2 improves and also unifies all the closure results proved in [4b] and [4c].

2. – The lower semicontinuity result.

Let us consider now a modified version of the closure theorem which is useful in order to prove lower semicontinuity results (see also [4c], Sect. 3).

Let $R_0 = [a_0, b_0] = \{a_0^j \leq t^j \leq b_0^j, j = 1, \dots, \nu\}$ be a closed interval such that $G \subset R_0$ and let $x^0 : R_0 \to \boldsymbol{R}$ be a given function. For any interval $R = [a, b] \subset R_0$, we consider the usual difference of order ν relative to the 2^ν vertices of R, say

$$\Delta_R x^0 = x^0(b) - x^0(a) \quad \text{if } \nu = 1,$$

$$\Delta_R x^0 = x^0(b^1, b^2) - x^0(b^1, a^2) - x^0(a^1, b^2) + x^0(a^1, a^2) \quad \text{if } \nu = 2,$$

and so on.

As it is well-known, the function x^0 is said to be of bounded variation in the sense of Vitali (VBV) [7] if the interval function $\Delta_R x^0$ is BV. A VBV function has a.e. superficial derivative, say $D^* x^0(t_0) = \lim_{h\to 0}(2h)^{-\nu}\Delta_q x^0$, $t_0 \in R_0$, where $q=[t_0-h, t_0+h]$; and $D^* x^0$ is in $L_1(R_0)$.

The function x^0 is said to be absolutely continuous in the sense of Vitali (VAC) [7] if the interval function $\Delta_R x^0$ is AC.

We consider now the following orientor field equation

$$(\tilde{1}) \qquad (t, x(t)) \in A\,, \qquad (D^* x^0(t), Dx(t)) \in \tilde{Q}(t, x(t)) \qquad \text{a.e. in } G$$

where $\tilde{Q}\colon A \to 2^{R^{N+1}}$ is a given multifunction.

In the present Section we shall consider sequences $(x_k^0, x_k)_{k\geq 0}$ of functions satisfying the conditions:

$$(\tilde{2}) \qquad \begin{cases} x_k^0 \in VAC, \quad x_k \in W^{1,1}(G), \quad k \in N, \quad x_0^0 \in VBV, \quad x_0 \in BVC(G), \\ (t, x_k(t)) \in A, \quad (D^* x_k^0(t), Dx_k(t)) \in \tilde{Q}(t, x_k(t)), \quad \text{a.e. in } G,\ k \in N, \\ \text{and } x_k^0 \to x_0^0 \text{ pointwise a.e.}, \ x_k \to x_0 \text{ in } (L_1(G))^n. \end{cases}$$

We shall need the following further definition.

DEFINITION $\tilde{0}$. – We shall say that the multifunction $\tilde{Q}$ satisfies condition $(w\tilde{F})$ at the point $t_0 \in G$, with respect to a given sequence $(x_k^0, x_k)_{k\geq 0}$ satisfying $(\tilde{2})$, provided

$(w\tilde{F})$ given any numbers $\rho > 0$, there exist a number $0 < \sigma = \sigma(t_0, \rho, (x_k^0, x_k)_{k\geq 0}) \leq \rho$ and a constant $0 < h_0 = h_0(t_0, (x_k^0, x_k)_{k\geq 0}, \rho) \leq \sigma$ such that for almost every $0 < h < h_0$ there exists a subsequence $(s_k)_k$ and three sequences of measurable functions $(\bar{x}_{s_k})_k$, $(\xi_{s_k})_k$, $(\eta_{s_k})_k$ such that

$$(t, \bar{x}_{s_k}(t)) \in A\,, \qquad (\eta_{s_k}(t), \xi_{s_k}(t)) \in \tilde{Q}(t, \bar{x}_{s_k}(t))\,, \qquad |\bar{x}_{s_k}(t) - x_0(t_0)| \leq \sigma$$

for a.e. $t \in q = [t_0 - h, t_0 + h]$, and moreover

$$\left| |q|^{-1}\int_q Dx_{s_k}(t)\,dt - |q|^{-1}\int_q \xi_{s_k}(t)\,dt \right| \leq \rho\,,$$

$$|q|^{-1}\int_q \eta_{s_k}(t)\,dt \leq |q|^{-1}\int_q D^* x_{s_k}^0(t)\,dt + \rho\,, \qquad k \in N\,.$$

Note that a result analogous to Lemma 2 holds for the sequence $(x_k^0)_{k\geq 0}$.

REMARK $\tilde{1}$. – If the multifunction $\tilde{Q}$ does not depend on x, i.e. $\tilde{Q}\colon G \to 2^{R^{N+1}}$, then condition $(w\tilde{F})$ is trivially satisfied with respect to any sequence $(x_k^0, x_k)_{k\geq 0}$ such that $x_k^0 \in VAC$, $x_k \in W^{1,1}$ and $(D^* x_k^0(t), Dx_k(t)) \in Q(t)$, a.e. in G, $k \in N$.

LEMMA 6. – *Let* $x_k^0: R_0 \to \boldsymbol{R}$, $k \in N$, *be a sequence of VAC functions such that* $x_k^0 \to x_0^0 \in VBV$ *pointwise* a.e. *Then the following holds*

(4) *for a.e.* $t_0 \in R_0$ *there exists a set* $H = H(t_0) \subset \boldsymbol{R}^+$ *with* $|H| = 0$ *and the property that, given any number* $\rho > 0$, *there exists a constant* $h_0 = h_0(t_0, \rho) > 0$ *such that, for every* $h \in (0, h_0) \setminus H$, *there exists an integer* $k_0 = k_0(t_0, \rho, h)$ *such that for every* $k \geq k_0$ *we have*

$$\left| D^* x_0^0(t_0) - |q|^{-1} \int_q D^* x_k^0(t)\, dt \right| \leq \rho,$$

where $q = [t_0 - h, t_0 + h]$.

PROOF. – Let $E \subset G$ be the null set such that for every $t_0 \in G \setminus E$ we have (see the proof of Theorem 1′ in [4c])

$$1) \qquad \lim_{h \to 0^+} |q|^{-1} \int_q D^* x_0^0(t)\, dt = D^* x_0^0(t_0),$$

$$2) \qquad \lim_{h \to 0^+} |q|^{-1} \left(\int_q D^* x_0^0(t)\, dt - \Delta_q x_0^0 \right) = 0,$$

and let $H = H(t_0) \subset \boldsymbol{R}^+$ be a set of measure zero in $\boldsymbol{R}^+$ such that for every $h \in \boldsymbol{R}^+ \setminus H$

$$3) \qquad \lim_{k \to \infty} \Delta_q x_k^0 = \Delta_q x^0.$$

Thus, given any $\rho > 0$, a constant $0 < h_0 = h_0(t_0, \rho)$ exists such that, for every $h \in (0, h_0) \setminus H$, an integer $k_0 = k_0(t_0, \rho, h)$ exists in such a way that for every $k \geq k_0$ we have

$$4) \qquad \left| |q|^{-1} \int_q D^* x_0^0(t)\, dt - D^* x_0^0(t_0) \right| < \rho/3,$$

$$5) \qquad |q|^{-1} \left| \int_q D^* x_0^0(t)\, dt - \Delta_q x_0^0 \right| < \rho/3,$$

$$6) \qquad |q|^{-1} |\Delta_q x_k^0 - \Delta_q x_0^0| < \rho/3.$$

From 6) and 5) it follows that

$$7) \qquad \left| |q|^{-1} \int_q D^* x_k^0(t)\, dt - |q|^{-1} \int_q D^* x_0^0(t)\, dt \right| < 2\rho/3$$

and from 7) and 4), we have

$$\left| |q|^{-1} \int_q D^* x_0^0(t)\, dt - D^* x_0^0(t_0) \right| < 3\rho/3 = \rho .$$

Lemma 6 is thereby proved.

By virtue of Lemmas 2 and 6 the following closure result for orientor field $(\tilde{2})$ can be proved.

THEOREM 7 (A closure theorem). – *Assume that A is closed and let $(x_k^0, x_k)_{k\geq 0}$ be a sequence satisfying $(\tilde{2})$. Suppose that the multifunction $\tilde{Q}$ satisfies the conditions*

(i) *has non empty, closed, convex values,*

(ii) *has property (Q) (with respect to (t,x)) at the point $(t_0, x_0(t_0))$, for a.e. $t_0 \in G$,*

(iii) *satisfies condition $(w\tilde{F})$, with respect to $(x_k^0, x_k)_{k\geq 0}$, at a.e. point $t_0 \in G$,*

(iv) *if $(\eta, \xi) \in \tilde{Q}(t,x)$ and $\eta' > \eta$ then $(\eta', \xi) \in \tilde{Q}(t,x)$, $(t,x) \in A$.*

Then for the limit function (x_0^0, x_0) we have

$$(t, x_0(t)) \in A, \qquad (D^* x_0^0(t), Dx_0(t)) \in \tilde{Q}(t, x_0(t)), \qquad a.e. \ \ in \ G .$$

PROOF. – Let $t_0 \in G \setminus E$, with $|E| = 0$, be fixed in such a way that $(t_0, x_0(t_0)) \in A$, condition (3) of Lemma 2, condition (4) of Lemma 6 hold and all the assumptions are satisfied.

Then for any given $\rho > 0$, two constants $0 < \sigma = \sigma(\rho) \leq \rho$ and $0 < h_0 = h_0(t_0, \rho) \leq \sigma$ exist such that, for almost every $0 < h < h_0$, there exist a subsequence $(s_k)_k$ and three sequences of measurable functions $(\overline{x}_{s_k})_k$, (ξ_{s_k}) and $(\eta_{s_k})_k$ such that $(t, \overline{x}_{s_k}(t)) \in A$, $(\eta_{s_k}(t), \xi_{s_k}(t)) \in \tilde{Q}(t, \overline{x}_{s_k}(t))$, $|\overline{x}_{s_k}(t) - x_0(t_0)| \leq \sigma$, for a.e. $t \in q = [t_0 - h, t_0 + h]$, and moreover

$$1) \qquad \left| Dx_0(t_0) - |q|^{-1} \int_q \xi_{s_k}(t)\, dt \right| < \rho ,$$

$$2) \qquad D^* x_0^0(t_0) \geq |q|^{-1} \int_q \eta_{s_k}(t)\, dt - \rho .$$

From 1), 2) and assumption (iv) it follows that

$$(D^* x_0(t_0), Dx_0(t_0)) \in [\text{cl co}\, \tilde{Q}(t_0, x_0(t_0), \sigma)]_\rho$$

and the assertion can be obtained immediately, as in Theorem 3.

We shall prove now a lower semicontinuity result as a consequence of Theorem 7.

Let $Q: A \to 2^{R^N}$ be a given multifunction and let M be its graphic, *i.e.* $M = \{(t, x, \xi): (t, x) \in A, \xi \in Q(t, x)\}$. We consider an integrand $F_0: M \to \boldsymbol{R}$ and let Ω denote a class of functions $x: G \to \boldsymbol{R}^n$ satisying

(i) $x \in BVC(G)$;

(ii) $(t, x(t)) \in A,\ Dx(t) \in Q(t, x(t))$, a.e. in G;

(iii) $F_0(\cdot, x(\cdot), Dx(\cdot)) \in L_1(G)$.

Then the usual integral functional $I: \to \boldsymbol{R}$ is defined by the Lebesgue integral

$$I(x) = \int_G F_0(t, x(t), Dx(t))\, dt. \tag{5}$$

Finally, given $x_0 \in BVC(G)$, let $\Gamma(x_0)$ denote the class of all the sequences $(x_k)_{k \in N}$ with $x_k \in W^{1,1}(G) \cap \Omega$ and such that $x_k \to x_0$ in $(L_1(G))^n$.

Then the Serrin functional $\mathfrak{J}$ associated to I is defined by

$$\mathfrak{J}(x_0) = \inf_{\Gamma(x_0)} \varliminf_{k \to +\infty} I(x_k) = \inf_{\Gamma(x_0)} \varliminf_{k \to +\infty} \int_G F_0(t, x_k(t), Dx_k(t))\, dt, \qquad \text{if } \Gamma(x_0) \neq \emptyset \tag{6}$$

$$\text{and } \mathfrak{J}(x_0) = +\infty, \qquad \text{if } \Gamma(x_0) = \emptyset.$$

The class Ω is said to be *closed* if it has the following property: for any sequence $(x_k)_{k \in N}$ in Ω such that $x_k \in W^{1,1} \cap \Omega$, $k \in \boldsymbol{N}$, and $x_k \to x$ in $(L_1(G))^n$, with x satisfying conditions (i), (ii) and (iii), then $x \in \Omega$.

Let us consider now the multifunction $\tilde{Q}: A \to 2^{R^{N+1}}$ defined by

$$\tilde{Q}(t, x) = \{(\eta, \xi): \xi \in Q(t, x), \eta \geq F_0(t, x, \xi)\}.$$

Moreover let $(x_k)_{k \geq 0}$ be a given sequence satisfying (2). Denoted by $R_0 = [a_0, b_0]$ a closed rectangle such that $R_0 \supset G$, we consider the functions $\varphi_k^0: R_0 \to \boldsymbol{R}$ defined by $\varphi_k^0(t) = \int_{[a_0, t]} F_k(\tau)\, d\tau$, $k \in \boldsymbol{N}$, where

$$F_k(t) = \begin{cases} F_0(t, x_k(t), Dx_k(t)), & t \in G \\ 0, & t \in R_0 \setminus G. \end{cases}$$

Note that φ_k^0 is VAC, $k \in \boldsymbol{N}$. Furthermore, the sequence $(\varphi_k^0)_k$ is equi VBV (see [4c] proof of Theorem 1″), thus by Helly's theorem for functions of bounded variation in the sense of Vitali (see [5], pg. 115), there is a subsequence, say still (k), and a VBV function $\varphi_0^0: R_0 \to \boldsymbol{R}$ such that $\varphi_k^0 \to \varphi_0^0$ pointwise on R_0.

Then the following lower semicontinuity statement can be proved by virtue of Theorem 7, analogously to Theorem 1″ in [4c].

THEOREM 8 (A lower semicontinuity theorem). – *Assume that A is closed and let $(x_k)_{k \geq 0}$ be a sequence satisfying* (2). *Suppose that the integrand F_0 is* *lower semicontinuous and there is a function* $\lambda \in L_1(G)$ *such that* $F_0(t, x, \xi) \geq \lambda(t)$ *on* M. *Moreover suppose that the multifunction* $\tilde{Q}$

(i) *has non empty, closed, convex values,*

(ii) *has property* (Q) *(with respect to* (t, x)*) at the point* $(t_0, x_0(t_0))$*, for a.e.* $t_0 \in G$,

(iii) *satisfies condition* $(w\tilde{F})$*, with respect to* $(x_k^0, x_k)_{k \geq 0}$*, at a.e. point* $t_0 \in G$*, where* $x_k^0 = \varphi_k^0$, $k \in \boldsymbol{N}$.

Then $x_0 \in \Omega$ *and*

$$\varliminf_{k \to \infty} I(x_k) \geq I(x_0).$$

REMARK 9. – Because of the position $x_k^0 = \varphi_k^0$, $k \in \boldsymbol{N}$, we are actually requiring here slightly less that the total condition $(w\tilde{F})$ above.

We shall show, by an example, that Theorem 8 may fail if property (Q) does not hold, even if condition $(w\tilde{F})$ and all the other assumptions are satisfied.

EXAMPLE 10. – Let $A = [0, 2\pi] \times [-1, 1]^2$, $Q(t, x) = \boldsymbol{R}^2$, $(t, x) \in A$ and let F_0: $[-1, 1]^2 \times \boldsymbol{R}^2 \to \boldsymbol{R}^+$ be defined by

$$F_0(x_1, x_2; \xi_1, \xi_2) = \exp(x_1 \xi_2 - x_2 \xi_1).$$

Let us consider the sequence $(x_k)_{k \geq 0}$ given by

$$x_k^1(t) = k^{-1/3} \sin kt, \quad x_k^2(t) = k^{-1/3} \cos kt, \quad t \in [0, 2\pi], \quad k \in \boldsymbol{N},$$

$$x_0^1(t) = x_0^2(t) = 0, \quad t \in [0, 2\pi]; \text{ which converges uniformly}.$$

Note that $\tilde{Q}(x) = $ epigraphic $F_0(x, \cdot)$, $x \in [-1, 1]^2$. Since F_0 is not seminormal, then property (Q) does not hold (see [2]). Moreover $F_0(x, \cdot)$ is convex, $x \in [-1, 1]^2$. Let us show that $\tilde{Q}$ satisfies condition $(w\tilde{F})$ with respect to the sequence $(\varphi_k^0, x_k)_{k \geq 0}$; in fact, by virtue of the uniform convergence, we can take

$$\bar{x}_{n_k}(t) = x_{n_k}(t), \quad \xi_{n_k}(t) = x'_{n_k}(t) \quad \text{and} \quad \eta_{n_k}(t) = F_0(x_{n_k}(t), x'_{n_k}(t)), \quad t \in [0, 2\pi], \quad k \in \boldsymbol{N}.$$

Finally observe that $I(x_k) = \int_0^{2\pi} F_0(x_k(t), x_k'(t))\, dt = 2\pi \exp(-k^{1/3})$ $k \in \boldsymbol{N}$, therefore we have

$$\varliminf_{k \to \infty} I(x_k) = 0 < I(0) = 2\pi.$$

3. – The existence result.

By the closure result and the lower semicontinuity statement of Sections 1 and 2 above, we can derive the following existence theorem (see [4bc]), for $\nu \geq 1$, $n \geq 1$.

THEOREM 11 (An existence theorem). – *Suppose that:*

(i) *G has the cone property, A is compact and M is closed;*

(ii) *the multifunction $\tilde{Q}$ has closed, convex values and satisfies property* (Q) *with respect to (t, x), at every point $(t_0, x_0) \in A$ with the exception perhaps of a set of points whose t-coordinate lie on a set H of measure zero in $\boldsymbol{R}$;*

(iii) *for any given sequence $(x_k)_{k\geq 0}$ satisfying* (2), *the multifunction $\tilde{Q}$ satisfies condition $(w\tilde{F})$ with respect to the sequence $(\varphi_k^0, x_k)_{k\geq 0}$, at a.e. point $t_0 \in G$;*

(iv) *the integrand F_0 is lower semicontinuous and there is a function $\lambda \in L_1(G)$ such that $F_0(t, x, \xi) \geq \lambda(t)$ on M.*

Assume that Ω is closed, $W^{1,1} \cap \Omega \neq \emptyset$ and

(v) *there is a constant W_0 such that $V(x) \leq W_0$, $x \in \Omega$.*

Then the functional $\mathfrak{J}$ has an absolute minimum $x \in \Omega$.

We sketch the proof which is the same as in [4c].

Let $i = \inf_{\Omega} \mathfrak{J}(x)$. Then $i = \inf_{W^{1,1}(G)\cap\Omega} I(x)$ and $0 \leq i < +\infty$. By definition there is a sequence $(x_k)_k$ in $W^{1,1}(G) \cap \Omega$ with $\lim_{k\to+\infty} I(x_k) = i$. Because of the assumptions we have $V(x_k) \leq W_0$ for all k, and, by a known compacteness theorem there is a subsequence, say still (k), with $x_k \to x_0 \in BVC(G)$.

But Ω is closed, hence $x_0 \in \Omega$ and $i \leq \mathfrak{J}(x_0)$. The Serrin integral is certainly lower semicontinuous, therefore

$$\mathfrak{J}(x_0) \leq \varliminf_{k\to+\infty} \mathfrak{J}(x_k), \qquad \text{where } \mathfrak{J}(x_k) = I(x_k), \quad k \in \boldsymbol{N}.$$

Finally

$$0 \leq I(x_0) \leq \mathfrak{J}(x_0), \qquad i \leq \mathfrak{J}(x_0) \leq \lim_{k\to+\infty} I(x_k) = i.$$

Since i is finite, we have $0 \leq I(x_0) \leq \mathfrak{J}(x_0) = i$.

REMARK 12. – The hypothesis (v) can be replaced by the lighter assumption

(v)′ the level sets $L_K = \{x \in W^{1,1} \cap \Omega: I(x) \leq K\}$ are equibounded in variation.

In fact condition (v) is used only to garantee that every minimizing sequence of $W^{1,1}$ surfaces is equibounded in variation.

REMARK 13. – By virtue of Propositions 4 and 5 in Section 1, Theorem 11 improves and also unifies the existence results given in [4b] and [4c].

4. – Application to a class of integrands without growth properties.

Note that no growth assumptions are explicitely requested on the integrand F_0 in the existence theorem (Theorem 11). Let us consider now a particular class of integrands, without growth, to which Theorem 11 can be applied. In more details we reconsider the problem discussed by CESARI in [3], and we show that the results of Sections 1 to 3 above lead to a reduction in the assumptions needed in [3] for the existence of a BVC solution of the optimization problem discussed there.

(*a*) Let G be a bounded open subset of the t-space $\boldsymbol{R}^\nu$ whose boundary ∂G has the cone property and let A be a compact subset of the (t, u)-space $\boldsymbol{R}^{\nu+m}$ whose projection on the t-space covers G. Let $F_{ij}\colon A \to \boldsymbol{R}$, $j \in \{j\}_i$, $i = 1, \dots, m$, be functions of class C^1 and let $F_i\colon A \to \boldsymbol{R}$, $i = 1, \dots, m$, be continuous functions. Let Ω_0 denote the class of all the surfaces $u = (u^1, \dots, u^m)\colon G \to \boldsymbol{R}^m$ with $x \in BVC(G)$, $(t, x(t)) \in A$ a.e. in G, satisfying some Dirichelet-type boundary condition $u(t) = w(t)$ on $D \subset \partial G$, and those total variations are equibounded. Thus let W_0 be a constant such that $V(u) \leq W_0$, $u \in \Omega_0$. Let $I\colon \Omega_0 \to \boldsymbol{R}$ denote the integral of the calculus of variations

$$I(u) = \int_G \sum_{i=1}^{m} \left| \sum_{j \in \{j\}_i} F_{ij}(t, u(t))_{t_j} + F_i(t, u(t)) \right| dt$$

and let $\mathfrak{I}$ denote the corresponding Serrin integral. We shall further assume that $\Omega_0 \cap W^{1,1}(G) \neq \emptyset$.

(*b*) As in CESARI [3] let Φ denote the transformation which maps any m-vector function $u\colon G \to \boldsymbol{R}^m$ of the class Ω_0 into the $(N+m)$-vector function $v\colon G \to \boldsymbol{R}^{N+m}$ defined by

$$v(t) = \Phi u(t) = (v'(t), v''(t)), \qquad t \in G$$

where $v'(t) = u(t) = (u_1(t), \dots, u_m(t))$ and

$$v''(t) = (v_{ij}(t), j \in \{j\}_i, i = 1, \dots, m), \qquad v_{ij}(t) = F_{ij}(t, u(t)).$$

Let $\Omega_0' = \Phi\Omega_0$ denote the set of all functions $v = (v', v'')$ obtained by the transformation of all $u \in \Omega_0$.

Then the integral I is transformed by Φ into the integral $H\colon \Omega_0' \to \boldsymbol{R}$ defined by

$$H(v) = \int_G F_0(t, v(t), Dv(t))\, dt, \tag{8}$$

with

$$F_0(t, v, Dv) = \sum_{i=1}^{m} \left| \sum_{j \in \{j\}_i} (v_{ij})_{t_j} + F_i(t, v') \right|.$$

If we denote by $F\colon A \to \boldsymbol{R}^{m+N}$ the vector function $F(t, u) = (u_i, i = 1, \dots, m, F_{ij}(t, u), j \in \{j\}_i, i = 1, \dots, m)$, then we can write $v(t) = F(t, u(t))$, $t \in G$. Moreover the condition $u(t) = w(t)$, $t \in D \subset \partial G$ is transformed into

$$v(t) = F(t, w(t)), \qquad t \in D \subset \partial G.$$

If for every $\bar{t} \in G$ we denote by $A(\bar{t}) = \{u: (\bar{t}, u) \in A\}$, and we take $B(\bar{t}) = \{v: v = F(\bar{t}, u), u \in A(\bar{t})\}$, and $B = \pi[B(\bar{t}), \bar{t} \in G]$, then the new problem has the natural constraint

$$(t, v(t)) \in B \quad \text{for } t \in G.$$

Finally, the integrand F_0 in (8) can also be written in the form

$$F_0(t, v, \xi) = \sum_{i=1}^{m} \left| \sum_{j \in \{j\}_i} \xi_{ij} + F_i(t, v) \right|$$

where $\xi = (\xi_{11}, \ldots, \xi_{1N_1}, \xi_{21}, \ldots, \xi_{2N_2}, \ldots, \xi_{m1}, \ldots, \xi_{mN_m})$, $\sum_{i=1}^{m} N_i = N$.

(*c*) We shall prove that Theorem 8 can be applied to the integral H. In order to do that let $\tilde{Q}: B \to 2^{\boldsymbol{R}^{N+1}}$ be the multifunction defined by

$$\tilde{Q}(t, v) = \{(\eta, \xi): \eta \geq F_0(t, v, \xi), \xi \in \boldsymbol{R}^N\}.$$

Let $(v_k)_{k \geq 0}$ be a sequence in Ω_0' such that $v_k \in W^{1,1}(G)$ and $v_k \to v_0$ in $(L_1(G))^{m+N}$. Consider the *VAC* functions $v_k^0 = \varphi_k$, $k \in \boldsymbol{N}$, (see Section 2) and suppose that $v_k^0 \to v_0^0 \in VBV$ a.e. in G.

Let us prove now that $\tilde{Q}$ satisfies condition $(w\tilde{F})$, with respect to the sequence $(v_k^0, v_k)_{k \geq 0}$.

Note that there is a subsequence, say still (k), such that $v_k \to v_0$ a.e. in G. Then, by Egoroff's and Lusin's theorems, given $\lambda > 0$, there is a compact set $K \subset G$ with $|G \setminus K| < \lambda$ such that $v_{0/K}$ is continuous and $v_k \to v_0$ uniformly on K.

Thus, given $\rho > 0$, there is a number $0 < \delta_1 = \delta_1(\rho) \leq \rho$ and an integer $k_1 = k_1(\rho)$ such that

$$|v_k(t) - v_0(t_0)| < \rho \quad \text{for every } t \in K \text{ and } k \geq k_1.$$

Note that almost all points of K are points of density one for K. Let $t_0 \in K$ be one of these points and let δ_2 be the distance of t_0 from ∂G. Thus for any given $\rho > 0$, we take $\sigma = \rho$ and $h_0 = h_0(t_0, \rho) = \min\{\delta_1(\rho), \delta_2\}$. Now for every $0 < h \leq h_0$ and $k \geq k_1(\rho)$ we consider the functions:

$$\bar{v}_k: q \to \boldsymbol{R}^{m+N}, \qquad \xi_k = (\xi_k^{ij}, j \in \{j\}_i, i = 1, \ldots, m): q \to \boldsymbol{R}^N \qquad \text{and} \qquad \eta_k: q \to \boldsymbol{R},$$

with $q = [t_0 - h, t_0 + h]$, defined by

$$\bar{v}_k(t) = \begin{cases} v_k(t), & t \in q \cap K \\ v_0(t_0), & t \in q \setminus K, \end{cases}$$

$$\xi_k^{ij}(t) = \begin{cases} D^j v_k^i(t), & t \in q \cap K \\ D^j v_k^i(t) + F_i(t, v_k(t)) - F_i(t, v_0(t_0)), & t \in q \setminus K, \end{cases}$$

$$\eta_k(t) = F_0(t, \bar{v}_k(t), \xi_k(t)) = \sum_{i=1}^{n} \left| \sum_{j \in \{j\}_i} \xi_k^{ij}(t) + F_i(t, \bar{v}_k(t)) \right| .$$

Finally note that

$$|q|^{-1} \int_q \eta_k(t)\, dt - |q|^{-1} \int_q F_0(t, v_k(t), Dv_k(t))\, dt = 0\,,$$

moreover we have

$$\left| |q|^{-1} \int_q Dv_k(t)\, dt - |q|^{-1} \int_q \xi_k(t)\, dt \right| \leqslant$$

$$\leqslant |q|^{-1} \int_{q \setminus K} \sum_{i=1}^{n} \left| \sum_{j \in \{j\}_i} F_i(t, v_k(t)) - F_i(t, v_0(t_0)) \right| dt \leqslant \frac{|q \setminus K|}{|q|} 2MN\,,$$

where $M = \max_{1 \leqslant i \leqslant n} \max_{B} |F_i(t, v)|$, and the last expression approaches zero, as h goes to 0^+.

Since t_0 denotes here almost any point of K, and $|G \setminus K| < \lambda$ with $\lambda > 0$ as small as we want, we have proved that for almost every $t_0 \in G$ the multifunction $\tilde{Q}$ satisfies condition $(w\tilde{F})$ with respect to the sequence $(v_k^0, v_k)_{k \geqslant 0}$.

Concerning *condition* (Q), this property was proved in [3] (cfr. the proof of Theorem B in Section 2 of [3]), and the same proof holds in the present more general situation.

Thus by force of Theorem 8 we conclude that, for every sequence $(v_k)_{k \geqslant 0}$ in Ω_0' with $v_k \in W^{1,1}(G)$ and $v_k \to v_0$ in $(L_1(G))^{m+N}$, we have

$$\varliminf_{k \to +\infty} H(v_k) \geqslant H(v_0)\,.$$

As we have seen in [4*bc*], this property implies that $H(v) \leqslant \mathcal{X}(v)$,, where $\mathcal{X}$ is the Serrin functional corresponding to H.

Moreover I is lower semicontinuous as H and $I(u) \leqslant \mathfrak{J}(u)$, for every $u \in \Omega_0$ and $I(u) = \mathfrak{J}(u)$, for every $u \in \Omega_0 \cap W^{1,1}(G)$.

(*d*) As an application of Theorem 11, under the assumptions listed in (*a*) there is an absolute minimum for the Serrin integral $\mathfrak{J}$, associated to the integral I, in any closed subclass $\Omega \subset \Omega_0$. We note that the present existence result in more general than the one stated in [3] since the functions $F_i(t, u)$ are assumed here only continuous and not necessarily Lipschitzian as in [3].

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