SMOOTH QUASIREGULAR MAPPINGS WITH BRANCHING

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ABSTRACT

We give an example of a $\mathscr{C}^{3-\epsilon}$-smooth quasiregular mapping in 3-space with nonempty branch set. Moreover, we show that the branch set of an arbitrary quasiregular mapping in n-space has Hausdorff dimension quantitatively bounded away from n. By using the second result, we establish a new, qualitatively sharp relation between smoothness and branching.

1. Introduction

In this paper, we prove three theorems about quasiregular mappings in space. First we show, contrary to some expectations, that there exist continuously differentiable quasiregular mappings in 3-space with nonempty branch set. Then we show that the Hausdorff dimension of the branch set of a quasiregular mapping in a domain in $\mathbf{R}^n$ is bounded away from n by a constant that only depends on n and the dilatation of the mapping. Moreover, it turns out that these two problems are related, and we use the Hausdorff dimension estimate to provide a new relation between the smoothness of the mapping and its branching; the example shows that this relation is qualitatively sharp in dimension three. Precise formulations of the results will be given momentarily. First we require some notation and terminology.

Throughout this paper, G denotes a domain in $\mathbf{R}^n$, $n \geq 2$. A continuous and nonconstant mapping $f\colon \mathrm{G} \to \mathbf{R}^n$ in the local Sobolev space $\mathrm{W}^{1,n}_{\mathrm{loc}}(\mathrm{G};\mathbf{R}^n)$ is called *quasiregular*, or K-*quasiregular*, $\mathrm{K} \geq 1$, if

$$|f'(x)|^n \leq \mathrm{K}\,\mathrm{J}_f(x) \tag{1.1}$$

for almost every $x \in \mathrm{G}$. Here $f'(x)$ is the formal differential matrix of f and $\mathrm{J}_f(x)$ its Jacobian determinant. According to a deep theorem of Reshetnyak [13], quasiregular mappings are sense-preserving, discrete, and open. In particular, every quasiregular mapping is locally invertible outside a closed set of topological dimension at most $n-2$; the set where a quasiregular mapping f does not determine a local homeomorphism is called the *branch set*, and denoted by B_f. We refer to the monographs [14], [15], [5] for the basic theory of quasiregular mappings.

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If $n = 2$ and $K = 1$ in the preceding inequality (1.1), we recover, by Weyl's lemma, the (nonconstant) holomorphic functions of one complex variable. In particular, quasiregular mappings in $\mathbf{R}^2$ can be smooth without being locally invertible. In contrast, it has been known for some time that every quasiregular mapping $f\colon G \to \mathbf{R}^n$ that is $\mathscr{C}^3$-smooth if $n = 3$, and $\mathscr{C}^2$-smooth if $n \geq 4$, must be locally invertible. This result follows from a sharp Sard-type theorem due to Morse [4, 3.4.3], coupled with some basic properties of quasiregular mappings. See [15, p. 12] for this argument.

It has been a well-known open problem whether the aforementioned relation between smoothness and local invertibility, coming from the Morse-Sard theorem, can be improved upon for quasiregular mappings (see, e.g., [19]). The expected answer seemed to have been that the local invertibility follows from $\mathscr{C}^1$-smoothness; this was explicitly conjectured at least in [5, p. 419]. We resolve the conjecture in the negative by constructing an example as in the ensuing theorem.

We say that a mapping $g\colon G \to \mathbf{R}^m$ is *$\mathscr{C}^k$-smooth*, where $k \in \mathbf{R}$, $k \geq 1$, if g is $[k]$-times continuously differentiable and if for every compact set $F \subseteq G$ there exists a constant $C > 0$ such that

$$|\partial^\alpha g(x) - \partial^\alpha g(y)| \leq C|x - y|^{k-[k]}, \tag{1.2}$$

whenever $x, y \in F$ and α is a multi-index with $|\alpha| \leq [k]$.

1.1. *Theorem. — For every $\epsilon > 0$ and for every integer $d \geq 2$ there exists a $\mathscr{C}^{3-\epsilon}$-smooth quasiregular mapping $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ of degree d which has a branch set B_f homeomorphic to $\mathbf{R}$.*

In fact, the mapping f in Theorem 1.1 is quasiconformally conjugate to the *winding map* $(r, \theta, z) \mapsto (r, d \cdot \theta, z)$ in cylindrical coordinates.

By the earlier discussion, the smoothness of a map as in Theorem 1.1 cannot be improved to $\mathscr{C}^3$. The following theorem implies a yet sharper result; namely, that the dilatation K of maps as in Theorem 1.1 necessarily tends to infinity as ϵ tends to zero.

1.2. *Theorem. — Given $n \geq 3$ and $K \geq 1$, there exists $\delta = \delta(n, K) > 0$ such that every $\mathscr{C}^{n/(n-2)-\delta}$-smooth K-quasiregular mapping $f\colon G \to \mathbf{R}^n$ is locally invertible.*

Theorem 1.2 in turn will be obtained with the aid of a size estimate for the branch set of a quasiregular mapping.

1.3. *Theorem. — Given $n \geq 3$ and $K \geq 1$, there exists $\lambda = \lambda(n, K) > 0$ such that the branch set of every K-quasiregular mapping $f\colon G \to \mathbf{R}^n$ has Hausdorff dimension at most $n - \lambda$.*

This theorem solves another well-known open problem in the theory of quasiregular mappings; see [15, p. 74]. Earlier, Sarvas [17] had shown that for a given K-quasiregular mapping $f: G \to \mathbf{R}^n$, $n \geq 3$, and a compact set $F \subseteq G$, the Hausdorff dimension of $B_f \cap F$ has an upper bound which is strictly less than n, and which depends only on n, K, and the maximal local index of f on F (see Proposition 3.1 below). We prove Theorem 1.3 by using Sarvas's result together with estimates due to Rickman and Srebro [16] about the distribution of points where the local index of a quasiregular mapping is high in comparison to n and the dilatation K.

The conclusion of Theorem 1.3 is true for $n = 2$ as well, but this is obvious from the fact that every quasiregular map in dimension two is topologically equivalent to a holomorphic function. On the other hand, the branch set of a quasiregular map in dimensions $n \geq 3$ can have Hausdorff dimension arbitrarily close to n. We do not know a numerical estimate for the number $\lambda = \lambda(n, K)$ in Theorem 1.3 (cf. Remark 3.5).

We believe that Theorem 1.2 is sharp in all dimensions $n \geq 4$, in the sense that there exist $\mathscr{C}^{n/(n-2)-\epsilon}$-smooth quasiregular mappings with nonempty branch set in $\mathbf{R}^n$ for every $n \geq 4$ and $\epsilon > 0$.[1]

Our construction uses the fact that the dimension is equal to three in two crucial respects. We will comment on this in Remark 2.1 below.

Finally, we point out that if one assumes that f is a $\mathscr{C}^\infty$-smooth quasiregular mapping in a domain in $\mathbf{R}^n$, $n \geq 3$, then the local invertibility of f can be proved without the use of the Morse-Sard theorem. See [1, pp. 302–303] or [5, pp. 419–422]. But these arguments, in turn, rely on relatively deep analytic properties of quasiregular mappings that are not needed in the proof in [15, p. 12].

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2. The example

In this section, we construct an example as promised in Theorem 1.1. In the following we will denote by C various positive constants whose value may change from line to line.

[1] This assertion has recently been verified for $n = 4$ by Kaufman, Tyson, and Wu [6].

2.1. *Reduction.* — Let $1/3 < \alpha < 1/2$. Suppose that there exists a quasiconformal self-homeomorphism $g\colon \mathbf{R}^3 \to \mathbf{R}^3$ together with a constant $C \geq 1$ such that

$$(\mathbf{2.1}) \qquad |x-y|^\alpha \leq C|g(x)-g(y)|,$$

whenever $x, y \in \mathbf{R} \subseteq \mathbf{R}^3$ with $|x-y| \leq 1$. Then, by a result of Kiikka [7], there also exists a quasiconformal self-homeomorphism $\hat{g}\colon \mathbf{R}^3 \to \mathbf{R}^3$ which is a $\mathscr{C}^\infty$-smooth diffeomorphism when restricted to $\mathbf{R}^3 \setminus \mathbf{R}$, and satisfies $\hat{g}|\mathbf{R} = g|\mathbf{R}$. Let $h = \hat{g}^{-1}$ and define $\Gamma := g(\mathbf{R})$. Then $h\colon \mathbf{R}^3 \to \mathbf{R}^3$ is quasiconformal, $h|\mathbf{R}^3 \setminus \Gamma$ is $\mathscr{C}^\infty$-smooth, and

$$(\mathbf{2.2}) \qquad |h(x)-h(y)| \leq C|x-y|^{1/\alpha},$$

whenever $x \in \Gamma$ and $y \in \Gamma$ is near x, where $C \geq 1$ is a constant independent of the points x and y. From this information it is standard to deduce that

$$(\mathbf{2.3}) \qquad |h(x)-h(y)| \leq C|x-y|^{3-\epsilon}, \quad \epsilon = 3 - 1/\alpha,$$

whenever $x \in \Gamma$ and $y \in \mathbf{R}^3$ is near x, the constant $C \geq 1$ being independent of the points. It follows from the nature of the smoothening procedure in [7, Theorem 4] that h satisfies

$$(\mathbf{2.4}) \qquad |\partial^\beta h(z)| \leq C\frac{\operatorname{dist}(h(z), \mathbf{R})}{\operatorname{dist}(z, \Gamma)^{|\beta|}},$$

whenever $z \in \mathbf{R}^3 \setminus \Gamma$ and β is multiindex with $|\beta| > 0$. Here $C > 0$ is a constant that depends on $|\beta|$, but not on z. Inequalities (2.3) and (2.4) imply that h is $\mathscr{C}^{3-\epsilon}$-smooth in $\mathbf{R}^3$.

Next, let $d \geq 2$ be an integer and let $w\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the winding map, $w(r, \theta, z) = (r, d \cdot \theta, z)$ in cylindrical coordinates in $\mathbf{R}^3$. Then w is a quasiregular mapping with the z-axis as its branch set [15, p. 13]. By postcomposing the preceding map h with a rotation, we can assume that the image of Γ under h is the z-axis. Then

$$f = w \circ h$$

is a quasiregular mapping. From (2.3) and from the properties of w it is easy to see that, akin to (2.3), we have that

$$(\mathbf{2.5}) \qquad |f(x)-f(y)| \leq C|x-y|^{3-\epsilon}, \quad \epsilon = 3 - 1/\alpha,$$

whenever $x \in \Gamma$ and $y \in \mathbf{R}^3$ is near x, and that $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ is a $\mathscr{C}^{3-\epsilon}$-smooth quasiregular mapping of degree d with $B_f = \Gamma$.

We conclude that for the proof of Theorem 1.1 it suffices to construct a quasiconformal mapping $g\colon \mathbf{R}^3 \to \mathbf{R}^3$ as in (2.1).

2.2. *Notation.* — We let $\mathbf{Q}^k = [0, 1]^k$ denote the closed unit cube in $\mathbf{R}^k$, and $\mathbf{B}^k$ the closed unit ball in $\mathbf{R}^k$. For $s > 0$, we write $\mathbf{Q}^k(s)$ and $\mathbf{B}^k(s)$ for the images of $\mathbf{Q}^k$ and $\mathbf{B}^k$ under the map $x \mapsto sx$, $x \in \mathbf{R}^k$. The boundary of the square $\mathbf{Q}^2$ is denoted by $\mathbf{S}^1_\infty$, and the boundary of $\mathbf{B}^2$ by $\mathbf{S}^1$. Similarly, $\mathbf{S}^k_\infty(s) := \partial\mathbf{Q}^{k+1}(s)$ and $\mathbf{S}^k(s) := \partial\mathbf{B}^{k+1}(s)$. The standard basis of $\mathbf{R}^k$ is $\{e_i : i = 1, \ldots, k\}$.

The interior of a set E is denoted by int(E), the topology being understood from the context. A *similarity* is a map σ between subsets of $\mathbf{R}^k$ such that $\lambda\sigma$ is an isometry for some positive number λ. Two sets are said to be *similar* if there is a similarity between them. A homeomorphism φ between subsets of $\mathbf{R}^k$ is *bi-Lipschitz* if there exists a constant $L \geq 1$ such that

$$L^{-1}|a - b| \leq |\varphi(a) - \varphi(b)| \leq L|a - b|$$

for every pair of points a and b in the domain. All subsets of $\mathbf{R}^k$ are assumed to be equipped with the metric inherited from $\mathbf{R}^k$.

2.3. *Pipes and elbows.* — We begin by describing five geometric surfaces, one pipe, P, and four elbows, $E_1, \ldots, E_4$, in $\mathbf{R}^3$. All these surfaces are (bi-Lipschitz) homeomorphic to the cylindrical surface

$$C = \mathbf{S}^1 \times [0, 1] .$$

We define the *pipe* P to be the boundary of the unit cube $\mathbf{Q}^3$ with the interiors of the faces on $\{x_1 = 1\}$ and $\{x_1 = 0\}$ removed. To match the notation to follow, we write $\hat{P}$ for the *filled-in pipe*, which here simply means the unit cube; thus $\hat{P} = \mathbf{Q}^3$.

To describe the elbows, we first define the *filled-in elbow* $\hat{E}_1$ to be the union of $\mathbf{Q}^3$ and its translates $\mathbf{Q}^3 + e_1$ and $\mathbf{Q}^3 + e_2$. The corresponding *elbow* E_1 is the boundary of $\hat{E}_1$ with the interiors of the faces on $\{x_1 = 2\}$ and $\{x_2 = 2\}$ removed. The other filled-in elbows $\hat{E}_2$, $\hat{E}_3$, and $\hat{E}_4$ are obtained similarly by considering $\mathbf{Q}^3 + e_3$, $\mathbf{Q}^3 - e_2$, and $\mathbf{Q}^3 - e_3$, respectively, together with $\mathbf{Q}^3$ and $\mathbf{Q}^3 + e_1$. To obtain the corresponding elbows E_2, E_3, and E_4, we consider the boundaries of the filled-in elbows with pertinent face interiors removed. For example, E_2 is equal to the closure of the set

$$\partial\big(\mathbf{Q}^3 \cup \big(\mathbf{Q}^3 + e_1\big) \cup \big(\mathbf{Q}^3 + e_3\big)\big) \setminus (\{x_1 = 2\} \cup \{x_3 = 2\}) .$$

Each of the filled-in objects $\hat{P}, \hat{E}_i$ is (bi-Lipschitz) homeomorphic to the *filled-in cylinder*

$$\hat{C} = \mathbf{B}^2 \times [0, 1],$$

while each of the elbows E_i, as well as the pipe P, is (bi-Lipschitz) homeomorphic to C. Moreover, each of these sets has natural symmetry. For the pipe P the symmetry is determined by the plane $\{x_1 = 1/2\}$, for the elbow E_1 it is determined by the plane $\{x_1 = x_2\}$, and so on. By reflecting on the symmetry planes, we obtain isometric involutions on each of these pipes and elbows. Finally, on C and $\hat{C}$, too, we have a natural involution determined by the cross-section $\mathbf{B}^2 \times \{1/2\}$.

The two boundary components of each of the surfaces P and E_i are isometric to $\mathbf{S}^1_\infty$. In the following, we abuse notation and call all these boundary components $\mathbf{S}^1_\infty$. Similarly, we use the same notation for the isometric images of $\mathbf{Q}^2(s)$ and $\mathbf{S}^1_\infty(s)$. The specifics are clear from the context.

We understand that each boundary component $\mathbf{S}^1_\infty$ comes with a natural positive orientation and with four distinguished *corner points*. More precisely, we mark the points

$$(1, 1, 1), (1, 0, 1), (1, 0, 0), (1, 1, 0)$$

in this order in the boundary component $\mathbf{S}^1_\infty$ of P that is contained in $\{x_1 = 1\}$. By translating the points by e_1, and by applying the fixed involutions, we have four marked and ordered points in every boundary component $\mathbf{S}^1_\infty$.

2.4. *First maps.* — Fix bi-Lipschitz maps

$$\varphi_P : C \to P, \quad \varphi_{E_i} : C \to E_i, \quad i = 1, 2, 3, 4, \tag{2.6}$$

respecting the given involutions, and such that, by using the complex notation for the points on $\mathbf{S}^1$, the points

$$1, \sqrt{-1}, -1, -\sqrt{-1}$$

(in this order) are mapped to the distinguished ordered corner points. We further require that these bi-Lipschitz maps extend to bi-Lipschitz maps

$$\hat{\varphi}_P : \partial\hat{C} \to \partial\hat{P}, \quad \hat{\varphi}_{E_i} : \hat{\partial}C \to \partial\hat{E}_i, \quad i = 1, 2, 3, 4, \tag{2.7}$$

respecting the involutions, and that the restrictions $\mathbf{B}^2 \to \mathbf{Q}^2$ to the "end pieces" are in each case radial, and identical (modulo the obvious translations).

A further requirement for the maps $\hat{\varphi}_P$ and $\hat{\varphi}_{E_i}$ will be given below before (2.9).

2.5. *Subdivisions of cubes and broken line segments.* — Fix an odd integer $N \gg 1$, and subdivide the filled-in pipe $\hat{P} = \mathbf{Q}^3$ into N^3 essentially disjoint congruent subcubes; thus the side length of each of these subcubes is

$$l := 1/N.$$

Because N is odd, the point $(1, 1/2, 1/2)$ is the center of a face of a cube in the subdivision, and similarly for $(0, 1/2, 1/2)$. Between these two points, we choose a polygonal arc Γ inside $\hat{P}$ satisfying properties that are prescribed momentarily. First of all, we require that Γ can be deformed to the line segment connecting the points $(1, 1/2, 1/2)$ and $(0, 1/2, 1/2)$ through an isotopy of $\hat{P}$ that keeps the boundary $\partial\hat{P}$ pointwise fixed. (Thus, Γ does not form a knot inside $\hat{P}$.) The arc Γ determines an order on the collection of the subcubes it meets; the cube that has a face with center $(1, 1/2, 1/2)$ is declared to be the first cube. We refer to this order in the arguments to follow. Now we assume that Γ satisfies the following eight properties with respect to the cubes in the subdivision:

(i) Γ visits at least $10^{-6}N^3$ of the cubes in the subdivision,
(ii) Γ enters each cube in the subdivision at the center of a face,
(iii) the portion of Γ in each cube in the subdivision consists of either one straight line segment or two straight line segments concatenated at the center of the cube,
(iv) cubes in the subdivision meeting Γ only meet each other if they are consecutive in the order determined by Γ (in which case they have a common face),
(v) the portion of Γ in the first two and the last two cubes in the subdivision is a straight line segment,
(vi) the only cubes in the subdivision that meet both Γ and the boundary of $\hat{P}$ are the first and the last cube,
(vii) Γ is symmetric with respect to the plane $\{x_1 = 1/2\}$.

The union of the cubes in the subdivision that meet Γ is denoted by W_P; this set is homeomorphic to $\mathbf{Q}^3$. We denote by S_P the boundary of W_P minus the interiors of the two faces that meet $\partial\hat{P}$. Then S_P is homeomorphic to C. As the last property of Γ we require:

(viii) S_P can be expressed as a union of sets each similar either to the pipe P or to an elbow E_i.

Let us call these sets *the pipes and the elbows associated with* S_P. They are ordered in a natural way along Γ. Note that the first and the last set in this union is a pipe by condition (v). Only the consecutive pipes and elbows meet, and they meet on a set similar to $\mathbf{S}^1_\infty$ in such a way that the marked points correspond to each other. This correspondence together with the pertinent portion of Γ also determines which model of the four elbows is used as a successor for a given elbow, or pipe, in the ordered collection S_P. We denote by U_P the surface that is the boundary of the region $\mathrm{int}(\hat{P}) \setminus W_P$ in $\mathbf{R}^3$. Observe that U_P is homeomorphic to the 2-torus.

Next we perform an analogous construction for each of the filled-in elbows. Consider first $\hat{E}_1$. It consists of three closed essentially disjoint 3-cubes. Subdivide $\hat{E}_1$ into $3N^3$ essentially disjoint congruent subcubes of side length l, and choose a polygonal arc from $(2, 1/2, 1/2)$ to $(1/2, 2, 1/2)$ with properties as in (i)–(viii) above with obvious notational alterations; the symmetry requirement is with respect to the symmetry plane $\{x_1 = x_2\}$. We arrive at the collection of *pipes and elbows associated with the set* S_{E_1}, where S_{E_1} is the boundary of a (topological) 3-ball W_{E_1} (which is the union of the cubes that meet the constructed arc) minus the interior of the two faces that meet $\partial\hat{E}_1$. Denote by U_{E_1} the 2-surface that is the boundary of the region $\mathrm{int}(\hat{E}_1) \setminus W_{E_1}$ in $\mathbf{R}^3$. Then U_{E_1} is homeomorphic to the 2-torus.

The same construction can be done for each of the elbows and we obtain sets W_{E_i}, S_{E_i}, and U_{E_i}, for $i = 1, 2, 3, 4$. (Obviously, it suffices to do the construction once, for $\hat{E}_1$ say; the others are obtained by applying appropriate reflections.) Each W_{E_i} is homeomorphic to $\mathbf{Q}^3$, S_{E_i} is homeomorphic to C, and U_{E_i} is the boundary of the region $\mathrm{int}(\hat{E}_i) \setminus W_{E_i}$ in $\mathbf{R}^3$ and homeomorphic to the 2-torus.

Next, we stipulate that the total number of pipes and elbows that make up the sets S_P and S_{E_i} is, in each case, a fixed number M. This can be done if the number N is large enough. We observe that

$$\text{(2.8)} \qquad \frac{1}{3}10^{-6}N^3 \leq M \leq N^3$$

by condition (i).

The preceding understood, we further stipulate that the (radial) maps $\hat{\varphi}_P$ and $\hat{\varphi}_{E_i}$ in (2.7) map the 2-disk $\mathbf{B}^2(l')$ concentric with $\mathbf{B}^2 \subseteq \partial\hat{C}$ to the square $\mathbf{Q}^2(l)$ concentric with $\mathbf{Q}^2 \subseteq \partial\hat{P}$ or $\mathbf{Q}^2 \subseteq \partial\hat{E}_i$, as the case might be, where $l = 1/N$ and

$$\text{(2.9)} \qquad N^{-3} \leq l' = \frac{1}{M} \leq 3 \cdot 10^6 N^{-3}.$$

Recall that these maps respect the presiding symmetry, so that the requirements are well-defined.

2.6. *Second maps.* — Consider the topological 2-torus T_0 that is the boundary of the region $\mathrm{int}(\hat{C}) \setminus (\mathbf{B}^2(l') \times [0, 1])$. We describe bi-Lipschitz maps

$$\text{(2.10)} \qquad \psi_P\colon T_0 \to U_P, \quad \psi_{E_i}\colon T_0 \to U_{E_i},$$

consisting of gluings of similarity copies of the maps φ_P and φ_{E_i}. We describe the map ψ_P, the others being analogous. The *inner part* $\mathbf{S}^1(l') \times [0, 1]$ of the torus T_0 is the union of M cylinders, each similar to $\mathbf{S}^1(l') \times [0, l']$. The cylinders are naturally ordered; only the consecutive cylinders meet, and they meet on their end circles. Each cylinder can be mapped by a bi-Lipschitz map either to a pipe or an elbow, associated with S_P, and up to a similarity the map is one of the maps in (2.6). The pipes and elbows associated with S_P make up the *inner part* of U_P. The stipulations on the maps φ_P and φ_{E_i}, especially the fact that they respect the distinguished corner points, guarantee that they can be pieced together to make a bi-Lipschitz map ψ_P from the inner part of T_0 to the inner part of U_P. Moreover, by condition (vii) we can ascertain that ψ_P respects the given symmetry on P. The map ψ_P can be extended as a radial map on the part $T_0 \cap (\mathbf{B}^2 \times \{0, 1\})$, and to the rest of T_0 again by the aid of the map φ_P. Completely analogously, the maps φ_{E_i} can be used to find maps ψ_{E_i} from T_0 to the topological 2-torus U_{E_i}, $i = 1, 2, 3, 4$. We reiterate that the maps ψ_P and ψ_{E_i} each respect the given symmetries.

Denote by $\hat{C}^0$ the solid torus bounded by T_0, and similarly let $\hat{P}^0$ and $\hat{E}_i^0$ denote the solid tori bounded by U_P and U_{E_i}, respectively. Recall that Γ was chosen unknotted inside $\hat{P}$. We claim that each of the maps ψ_P and ψ_{E_i} can be extended as bi-Lipschitz maps

$$\Psi_P \colon \hat{C}^0 \to \hat{P}^0, \quad \Psi_{E_i} \colon \hat{C}^0 \to \hat{E}_i^0. \tag{2.11}$$

To see this, we only need to ascertain that the maps ψ_P, ψ_{E_i} map each of the paths $p \times [0, 1]$, $p \in \mathbf{S}^1(l')$, to a path with winding number zero on the topological annulus formed by the pipes and the elbows associated with S_P or S_{E_i} as the case may be. But this is clearly the case because of the stipulated symmetry in each of the maps ψ_P and ψ_{E_i}, as remarked in the previous paragraph.

2.7. *Final map.* — We are now able to define the map $g \colon \mathbf{R}^3 \to \mathbf{R}^3$ as in (2.1). The map will be defined first in the cylinder $A = \mathbf{B}^2 \times [0, 1] \subseteq \mathbf{R}^3$. The image of this cylinder will be the closed cube $\mathbf{Q}^3$.

To this end, consider first

$$A_1 = A \setminus (\text{int}(\mathbf{B}^2(l')) \times [0, 1]),$$

where l' is defined in (2.9), and $B_1 = \hat{P}^0$. Now with the earlier notation, $A_1 = \hat{C}^0$, and we define $g_1 \colon A_1 \to B_1$ to be the map Ψ_P as in (2.11). Next, let

$$A_2 = A \setminus (\text{int}(\mathbf{B}^2(l'^2)) \times [0, 1]) .$$

The closure of the region $A_2 \setminus A_1$ is a union of solid tori, each similar to $\hat{C}^0$. Using the maps in (2.7), the maps in (2.10), and ultimately the maps Ψ_P, Ψ_{E_i} in (2.11), it is clear how to define the extension $g_2 \colon A_2 \to B_2$ of g_1, where B_2 is the union of the solid torus $\hat{P}^0$ and all the solid tori that are similarity images of $\hat{\mathrm{P}}^0$ and $\hat{\mathrm{E}}_i^0$ corresponding to the pipes and elbows that are associated with $\mathrm{S_P}$. By exploiting the self-similarity of the situation, it is obvious how to continue to define the mapping g. At each stage we have a map $g_j\colon \mathrm{A}_j \to \mathrm{B}_j$, where

$$\mathrm{A}_j = \mathrm{A} \setminus (\mathrm{int}(\mathbf{B}^2(l'^j)) \times [0,1]) \,.$$

extending g_{j-1}. Each g_j is quasiconformal with a fixed dilatation because in the construction we only use finitely many bi-Lipschitz maps and their compositions by similarities.

In conclusion, we obtain a quasiconformal map that is defined in $\mathrm{A} \setminus \mathbf{R}$. Such a map extends by continuity to a quasiconformal map g of all of A by the well-known removability theorem [20, Theorem 35.1, p. 118]. By construction, the image of A under g is the unit cube $\mathbf{Q}^3$. Note that on the ends $\{0,1\} \times \mathbf{B}^2$ of A, g is a simple radial mapping, respecting the natural involutions on A and $\mathbf{Q}^3$. Consequently, by repeating the construction on translates of A, we get a quasiconformal map $g\colon \mathbf{R} \times \mathbf{B}^2 \to \mathbf{R} \times \mathbf{Q}^2$ which is equivariant under the translation $x \mapsto x + e_1$, $x \in \mathbf{R}^3$. It is clear from the construction that g can further be extended to a quasiconformal self-homeomorphism $g\colon \mathbf{R}^3 \to \mathbf{R}^3$.

2.8. *Expansion check.* — It remains to analyze the behavior of g on the real axis. To this end, let $x, y \in \mathbf{R} \cap \mathrm{A}$. By standard distortion theorems for quasiconformal maps [20, pp. 63–65], we may assume that $|x-y| = l'^j$ for some $j \geq 1$. It follows from the construction that $|g(x)-g(y)|$ is comparable to l^j. More precisely, also by observing the inequalities in (2.8) and (2.9), we compute

$$\begin{aligned} |g(x)-g(y)| &\geq c^j l^j = c^j \mathrm{N}^{-j} \geq (3 \cdot 10^6/\mathrm{N}^3)^{j\alpha} \\ &\geq (1/\mathrm{M})^{j\alpha} \geq l'^{j\alpha} = |x-y|^\alpha, \end{aligned}$$

where $c > 0$ is independent of x and y, $1/3 < \alpha < 1/2$ is a given number, and N is chosen large enough depending only on c and α.

The proof of Theorem 1.1 is thereby complete.

2.1. *Remark.* — (a) The construction in this section can be modified to obtain a quasiregular map from the 3-sphere onto itself with properties as in Theorem 1.1 (the branch set will be homeomorphic to a circle in this case).

(b) The dimensional restriction $n = 3$ was used in two respects that we do not know how to overcome in higher dimensions. First, it is easier to construct quasiconformal mappings g with expanding behavior as in (2.1) on a line, than on a codimension two plane in $\mathbf{R}^n$, $n \geq 4$, which is what an analogous higher dimensional construction would require. Second, in dimension $n = 3$ we do not have to take care of the smoothness of the map g as in (2.1) outside $\mathbf{R}$, for this is provided by Kiikka [7] (who ultimately relies on deep works of Moise [11] and Munkres [12]). Such general approximation is not available in dimensions $n \geq 5$ by results of Sullivan [18], and is an open problem in dimension $n = 4$ (cf. Donaldson and Sullivan [3]).

However, there is no obvious obstruction to the existence of an example as in Theorem 1.1 in all dimensions (within the limits imposed by Theorem 1.2). It is even conceivable that such an example can be constructed along the lines presented in this section; but in order to do so, the two difficulties explained in the previous paragraph need to be overcome. Maps with expanding behavior on hyperplanes were constructed by David and Toro in [2], and it is possible that the techniques in their paper help here.

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Thus, let $f\colon G \to \mathbf{R}^n$, $n \geq 3$, be a quasiregular mapping. Keeping with the notation in [15] and [16], we denote by

$$K_I = K_I(f)\,, \quad K_O = K_O(f)\,, \quad \text{and} \quad K = K(f)$$

the *inner*, *outer*, and *maximal dilatations* of f (see [15, p. 11] for the definitions). We denote by $B(x, r)$ the open ball in $\mathbf{R}^n$ with center x and radius $r > 0$. The closed ball is $\bar{B}(x, r)$.

Next, we denote by $i(x,f)$ the *local index* of f at $x \in G$. It can be defined by

$$i(x,f) = \lim_{r\to 0} \sup_{y\in B(f(x),r)} \#\{f^{-1}(y) \cap U(x,f,r)\},$$

where $U(f, x, r)$ denotes the x-component of $f^{-1}(B(f(x), r))$ for $r > 0$ small enough. The function $x \mapsto i(x,f)$ is upper semicontinuous. See [15, pp. 18–19] for more information.

The following result was proved by Sarvas [17] (see also [9]).

3.1. *Proposition. — If* $F \subseteq G$ *is compact, then the Hausdorff dimension of* $B_f \cap F$ *satisfies*

$$\dim_H(B_f \cap F) \leq c = c(n, K, \max_{x\in F} i(x,f)) < n.$$

We next formulate the main proposition of this section.

3.2. *Proposition. — Let*

$$\kappa := \kappa(n, K) = K_I(f)(n+1)^{n-1} + 2. \tag{3.1}$$

There exists $\alpha = \alpha(n, \mathrm{K}) > 0$ *such that the set*

$$\mathrm{E} = \{x \in \mathrm{G} : i(x,f) \geq \kappa\}$$

is α*-porous, i.e.,*

$$\liminf_{r \to 0} r^{-1} \sup\{\rho : \mathrm{B}(z, \rho) \subseteq \mathrm{B}(x, r) \setminus \mathrm{E}\} \geq \alpha \tag{3.2}$$

for every $x \in \mathrm{E}$.

It is well-known that α-porous sets in $\mathbf{R}^n$ have Hausdorff dimension bounded away from n, with a bound depending only on n and α (see [10, p. 156]). Since the set $\{x \in \mathrm{G} : i(x,f) < \kappa\} \subseteq \mathrm{G}$ is open by the upper semicontinuity of $i(x,f)$, it follows from Propositions 3.1 and 3.2 that the Hausdorff dimension of

$$\mathrm{B}_f = \{x \in \mathrm{G} : 2 \leq i(x,f) < \kappa\} \cup \{x \in \mathrm{G} : i(x,f) \geq \kappa\}$$

does not exceed a number $\lambda = \lambda(n, \mathrm{K}) < n$ as asserted in Theorem 1.3. Therefore, it suffices to prove Proposition 3.2.

This proposition is essentially contained in the proof of the main theorem in [16] by Rickman and Srebro. We only need to keep a more careful track of the dependence of the various parameters appearing in their proof. We begin by quoting [16, Theorem 1.1]:

3.3. *Proposition. — Fix* $x_0 \in \mathrm{G}$, *and write*

$$\mu = \left(\frac{i(x_0,f)}{\mathrm{K_I}}\right)^{1/(n-1)}. \tag{3.3}$$

Then there exist $t_0 > 0$ *and* $p_0 \in \mathbf{N}$ *such that the following holds: if*

$$0 < t \leq t_0, \quad 1 \leq \nu < \mu, \quad p_0 \leq p \leq m \leq p^\nu, \tag{3.4}$$

and if $x_1, \ldots, x_m \in \bar{\mathrm{B}}(x_0, t)$ *are points such that* $|x_0 - x_m| = t$ *and that* $|x_{j-1} - x_j| \leq t/p$ *for each* $j = 1, \ldots, m$, *then*

$$\min\{i(x_1,f), \ldots, i(x_m,f)\} < i(x_0,f).$$

As formulated, in the above result the integer p_0 depends (at least) on n, K, and the local index $i(x_0,f)$. We claim that p_0 can be chosen to depend only on n and K, provided both that the local index $i(x_0,f)$ is sufficiently large and that we only consider values $1 \leq \nu \leq n+1$ in (3.4). We also make a little adjustment to the condition $|x_0 - x_m| = t$. More precisely, we claim the following proposition:

3.4. *Proposition.* — *Let* $x_0 \in G$ *and assume that* $i(x_0, f) \geq \kappa$, *where* κ *is given in (3.1). Then there exists* $t_0 > 0$ *and* $p_0 = p_0(n, K) \in \mathbf{N}$ *such that the following holds: if*

$$0 < t \leq t_0, \quad 1 \leq \nu \leq n+1, \quad p_0 \leq p \leq m \leq p^\nu,$$

and if $x_1, \ldots, x_m \in \bar{B}(x_0, t)$ *are points such that* $|x_0 - x_m| \geq t/2$ *and that* $|x_{j-1} - x_j| \leq t/p$ *for each* $j = 1, \ldots, m$, *then*

$$\min\{i(x_1, f), \ldots, i(x_m, f)\} < i(x_0, f). \tag{3.5}$$

Proof. — We adhere to the notation and proof in [16, 3.1]. In particular, we denote $k = i(x_0, f)$ and assume $x_0 = 0 = f(x_0)$. We select $t_0 > 0$ as in the first paragraph of the proof in [16, 3.1]. The choice of t_0 is simply such that the ball $B(0, t_0)$ is reasonably well inside a normal neighborhood of 0. In particular, we have that $i(x, f) \leq k$ for every $x \in B(0, t_0)$. We let μ be as in (3.3). (Note that the latter notation is used but not explained in [16].)

Now fix numbers $0 < t \leq t_0$, $1 \leq \nu \leq n+1$, and $p_0 \leq p \leq m \leq p^\nu$, and assume that there are points $x_1, \ldots, x_m \in \bar{B}(0, t)$ with

$$|x_0 - x_m| \geq t/2, \quad |x_{j-1} - x_j| \leq t/p, \quad i(x_j, f) = k,$$

for each $j = 1, \ldots, m$. We will show that upon choosing p_0 large enough, but depending only on n and K, these stipulations lead to a contradiction.

In [16, (3.3)], the authors derive the estimate

$$|f(x_m)| \leq \rho_0 C^{*2\mu} p^{\nu - \mu}, \tag{3.6}$$

where $C^* \geq 1$ depends only on n and K, and $\rho_0 > 0$ is a number depending on f and x_0. On the other hand, in [16], a lower estimate

$$a_n \leq K_O k \omega_{n-1} \big(\log(\rho_0/s)\big)^{1-n} \tag{3.7}$$

for $s = |f(x_m)|$ is derived next, where ω_{n-1} and a_n are positive dimensional constants. We note here that our a_n is slightly smaller than the one in [16] due to the assumption $|x_0 - x_m| \geq t/2$ in place of $|x_0 - x_m| = t$ in [16]; the dimensional lower bound in our case follows from standard Teichmüller estimates as cited in [16].

By combining (3.6) and (3.7), we can cancel ρ_0 to obtain

$$e^{-bk^{1/(n-1)}} \leq C^{*2\mu} p^{\nu-\mu} \leq M^{k^{1/(n-1)}} p^{n+1-\mu}, \tag{3.8}$$

where

$$b = \left(\frac{K_O\, \omega_{n-1}}{a_n}\right)^{1/(n-1)}, \quad M = C^{*2(K_I)^{1/(1-n)}}, \tag{3.9}$$

both depending only on n and K. The choice $k \geq \kappa$ guarantees that

$$(\mathbf{3.10}) \qquad \frac{n+1-\mu}{k^{1/(n-1)}} \leq \frac{n+1}{(\mathrm{K}_{\mathrm{I}}(n+1)^{(n-1)}+2)^{1/(n-1)}} - \mathrm{K}_{\mathrm{I}}^{1/(1-n)} = -c < 0,$$

and hence (3.8) implies

$$(\mathbf{3.11}) \qquad p^c \leq e^b \mathrm{M}.$$

In conclusion, by (3.9) and (3.10), we find that a judicious choice of p_0, depending only on n and K, would lead to a contradiction with (3.11). Therefore, (3.5) holds and the proof of Proposition 3.4 is complete. □

3.5. *Remark.* — An explicit estimate for the number p_0 in Proposition 3.4 can be derived from the above proof and from [15, III 4.5]. Consequently, this estimate carries over to an explicit estimate for the porosity constant in Proposition 3.2. In contrast, no such quantitative estimate is known for the constant in Proposition 3.1. The proofs in [17] and [9] are indirect normal family arguments.

Proof of Proposition 3.2. — Let $x_0 \in \mathrm{G}$ be a point such that $i(x_0, f) = k \geq \kappa$, and let $t_0 > 0$ and $p_0 \in \mathbf{N}$ be the numbers provided by Proposition 3.3. Fix $0 < t \leq t_0$ and consider the n-cube Q_t of side length

$$l = l(\mathrm{Q}_t) = \frac{2t}{\sqrt{n}}$$

centered at x_0. Then subdivide Q_t dyadically into 2^{hn} essentially disjoint cubes, where $h = h(n, \mathrm{K})$ is the smallest integer satisfying

$$(\mathbf{3.12}) \qquad 2^h \geq \max\{4^{n+1}, 4p_0\}.$$

Then set $p = 2^h/4$, and observe that

$$p_0 \leq p \leq 2^{hn} \leq p^{n+1}.$$

It therefore follows from Proposition 3.4 that if $x_1, \ldots, x_m$, $m = 2^{hn}$, are distinct points in Q_t with $|x_0 - x_m| \geq t/2$ and $|x_{j-1} - x_j| \leq t/p$ for $j = 1, \ldots, m$, then for at least one point x_j we have that $i(x_j, f) < k$. This easily implies that there must be at least one cube of side length

$$2^{-h} l = 2^{-h} \frac{2t}{\sqrt{n}}$$

in the above dyadic subdivision of Q_t such that $i(x, f) < k$ for every x in the interior of the cube. To argue more precisely, suppose the contrary. Then we can find a sequence of points $x_1, \ldots, x_m$, one from the interior of each cube in the subdivision, such that $i(x_j, f) = k$ for $j = 1, \ldots, m$. Obviously, we can choose x_m such that $|x_0 - x_m| \geq t/2$, and that consecutive points lie in adjacent cubes; the latter implies that

$$|x_{j-1} - x_j| \leq 2\sqrt{n}2^{-h}l = 4 \cdot 2^{-h}t = t/p,$$

and we have a contradiction with the preceding remarks.

The proof of Proposition 3.2, and thereby that of Theorem 1.3, is complete. □

3.6. *Remark.* — The proof of Theorem 1.3, especially the proofs of Propositions 3.4 and 3.2, imply that the set $E = \{x \in G : i(x, f) \geq \kappa\}$ is porous along any smooth d-dimensional surface, $1 \leq d \leq n$ in $\mathbf{R}^n$. By choosing $d = n - 1$ and using [8, Theorem 3.1], we obtain that the Hausdorff dimension of B_f on every hypersurface is quantitatively bounded away from $n - 1$. This result merits a separate formulation.

3.7. *Theorem.* — *Given* $n \geq 3$ *and* $K \geq 1$, *there exists* $\lambda' = \lambda'(n, K) > 0$ *such that for every* K*-quasiregular mapping* $f : G \to \mathbf{R}^n$ *and for every smooth hypersurface* H *in* $\mathbf{R}^n$, *the Hausdorff dimension of* $B_f \cap H$ *is at most* $(n - 1) - \lambda'$.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Recall that G is a domain in $\mathbf{R}^n$, $n \geq 2$.

The following lemma surely appears somewhere in the literature, but rather than searching for a reference, we provide the straightforward proof.

4.1. *Lemma.* — *Suppose that* $u : G \to \mathbf{R}$ *is a* $\mathscr{C}^{k+\alpha}$*-smooth function, where* $k \in \{1, 2\}$ *and* $0 < \alpha < 1$. *Let* $N_u := \{x \in G : \nabla u(x) = 0\}$. *Then for every compact set* $F \subseteq G$ *there exists a constant* $C > 0$ *such that*

$$|u(x) - u(y)| \leq C|x - y|^{k+\alpha}, \qquad (\mathbf{4.1})$$

whenever $x, y \in F \cap N_u$.

Proof. — Fix $\delta > 0$ small enough so that the open δ-neighborhood F_δ of F has compact closure in G. It is enough to establish (4.1) under the additional assumption that $|x - y| \leq \delta$. Thus, let $x, y \in F \cap N_u$ with $|x - y| \leq \delta$ be given, and define $x_t := x + t(y - x)$ for $t \in [0, 1]$. Then $x_t \in F_\delta$ for all $t \in [0, 1]$.

Assume first that $k = 1$. Then u is $\mathscr{C}^{1+\alpha}$-smooth, and hence there exists a constant $C > 0$ such that

$$|\nabla u(a) - \nabla u(b)| \leq C|a - b|^{\alpha},$$

whenever $a, b \in F_\delta$. Therefore,

$$|\nabla u(x_t)| = |\nabla u(x_t) - \nabla u(x)| \leq C|x - y|^{\alpha}$$

for each t, and we conclude that

$$|u(x) - u(y)| = \left| \int_0^1 \nabla u(x_t) \cdot (y - x)\, dt \right| \leq C|x - y|^{1+\alpha}.$$

Now assume that $k = 2$. We let $H = D(\nabla u)$ be the Hessian matrix of u. Since u is $\mathscr{C}^{2+\alpha}$-smooth, there exists a constant $C > 0$ such that

$$|H(a) - H(b)| \leq C|a - b|^{\alpha},$$

whenever $a, b \in F_\delta$. Thus, the equation

$$\int_0^1 H(x_t)(y - x)\, dt = \nabla u(y) - \nabla u(x) = 0,$$

implies that

$$\begin{aligned} |H(x)(y - x)| &= \left| \int_0^1 \big(H(x_t) - H(x)\big)(y - x)\, dt \right| \\ &\leq |x - y| \int_0^1 |H(x_t) - H(x)|\, dt \\ &\leq C|x - y|^{1+\alpha}. \end{aligned}$$

Next, define $h(t) = u(x_t) - u(x)$ for $t \in [0, 1]$. Then

$$h(0) = 0, \qquad h'(0) = \nabla u(x) \cdot (y - x) = 0,$$

and

$$h''(t) = H(x_t)(y - x) \cdot (y - x)$$

for each t. The preceding understood, we apply Taylor's theorem with Lagrange remainder to find $\theta \in (0, 1)$ such that

$$\begin{aligned} |u(y) - u(x)| &= |h(1)| = |h''(\theta)/2| \\ &\leq |H(x_\theta) - H(x)||y - x|^2 + |H(x)(y - x) \cdot (y - x)| \\ &\leq C|x - y|^{2+\alpha}. \end{aligned}$$

The lemma follows. $\square$

4.2. *Corollary.* — *Suppose that* $f\colon G \to \mathbf{R}^n$ *is a* $\mathscr{C}^{k+\alpha}$*-smooth quasiregular mapping, where* $k \in \{1, 2\}$ *and* $0 < \alpha < 1$. *Then for every compact set* $F \subseteq G$ *there exists a constant* $C > 0$ *such that*

$$(\mathbf{4.2}) \qquad |f(x) - f(y)| \leq C|x-y|^{k+\alpha},$$

whenever $x, y \in F \cap B_f$.

Proof. — Since $f = (f_1, \ldots, f_n)$ is at least $\mathscr{C}^1$-smooth, it follows from the definition in (1.1) that $\nabla f_i(x) = 0$ for every $x \in B_f$ and for every $i = 1, \ldots, n$. By applying Lemma 4.1 to each of the coordinate functions f_i, we obtain (4.2) as required. □

Proof of Theorem 1.2. — Let $n \geq 3$ and $K \geq 1$, and suppose that $f\colon G \to \mathbf{R}^n$ is K-quasiregular and $\mathscr{C}^{n/(n-2)-\delta}$-smooth. We assume that $0 < \delta < 1$ if $n = 3$, and that $0 < \delta < 2/(n-2)$ if $n \geq 4$. Then f is $\mathscr{C}^{2+\alpha}$-smooth if $n = 3$, where $\alpha = 1 - \delta$, and f is $\mathscr{C}^{1+\alpha}$-smooth if $n \geq 4$, where $\alpha = 2/(n-2) - \delta$.

Assuming now that $B_f \neq \emptyset$, Corollary 4.2 applies, and for every compact subset $F \subseteq G$ there exists a constant $C > 0$ such that

$$(\mathbf{4.3}) \qquad |f(x) - f(y)| \leq C|x-y|^{n/(n-2)-\delta},$$

whenever $x, y \in F \cap B_f$. Together with Theorem 1.3, inequality (4.3) gives the Hausdorff dimension estimate

$$\dim_H(f(B_f)) \leq \frac{\dim_H(B_f)}{n/(n-2)-\delta} \leq \frac{n-\lambda}{n/(n-2)-\delta},$$

where $\lambda = \lambda(n, K) > 0$. In particular, if $\delta = \delta(n, K) > 0$ is small enough, then

$$\dim_H(f(B_f)) < n-2.$$

But this contradicts the fact that the image of the nonempty branch set of every discrete and open mapping has Hausdorff dimension at least $n-2$ (see [15, III.5.3]). The proof of Theorem 1.2 is thereby complete. □

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