Extension of Some Results for Channel Capacity Using a Generalized Information Measure

Aharon Ben-Tal[1]* and Marc Teboulle[2]

[1] Technion, Israel Institute of Technology, Haifa, Israel, and Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI 48109, USA

[2] Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

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Abstract. A new formulation for the channel capacity problem is derived by using the duality theory of convex programming. The simple nature of this dual representation is suitable for computational purposes. The results are derived in a unified way by formulating the channel capacity problem as a special case of a general class of concave programming problems involving a generalized information measure recently introduced by Burbea and Rao [10].

1. Introduction

Channel capacity, a basic concept in information theory, was introduced by Shannon [27] to specify the maximum rate at which information can be reliably conveyed by the channel. Roughly speaking, the basic theorem of information theory, the so-called "noisy channel coding theorem," states that if a given noisy channel has capacity C, it is possible to transmit, over this channel, messages of sufficiently large length and still be able to decode them with an arbitrary small probability for error, provided the rate of transmission is less than C. Methods for computing the capacity C of a discrete channel have been studied by Muroga [21], Cheng [12], and Takano [28]. The best-known algorithm, however, is the one introduced independently by Arimoto [2] and Blahut [7]. A somewhat similar

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iterative procedure based on the method of quasi-concave programming was proposed by Meister and Oettli [20]. In these previously mentioned works, the computational schemes are derived using the *primal formulation* of the channel capacity problem.

In this paper the classical channel capacity problem is embedded in a family (P_φ) of linearly constrained concave programming problems, each member of which is determined by a choice of a convex function φ; the classical case corresponding to $\varphi(t) = t \log t$. The objective function in (P_φ) is the *generalized average mutual information measure*, recently introduced by Burbea and Rao [10], and the optimal value of (P_φ) is our *generalized channel capacity* C_φ.

A duality theory is developed for (P_φ) resulting in a dual representation of C_φ. As a special case, a new formulation of the classical channel capacity is obtained. The dual of (P_φ) (denoted (D_φ)) is a specially structured unconstrained minimax problem, thus rendering itself to efficient computational methods. The dual formulation is also very useful for obtaining upper bounds for C_φ. The paper is organized as follows. In Section 2 the formulation of the classical channel capacity problem is given and the iterative method of Arimoto [2] and Blahut [7] is briefly reviewed. In Section 3 we formulate the generalized capacity C_φ and develop the theory leading to two different dual representations, the first (Theorem 3.1) is suitable for computations and the second (Corollary 3.1) is useful for deriving upper bounds. The bounds are given in Section 4, it is also shown there that the bound is attained for symmetric channels, i.e., an explicit formula for C_φ is obtained. Section 5 contains concluding remarks and a brief discussion on possible extensions.

2. The Channel Capacity Problem

Consider a communication channel described by an input alphabet $A = \{1, 2, \ldots, m\}$, an output alphabet $B = \{1, 2, \ldots, n\}$, and by a probability transition matrix $Q = \{Q_{kj}\}$, where Q_{kj} is the probability of receiving the output letter $k \in B$ when input letter $j \in A$ was transmitted, i.e., $\sum_{k=1}^{n} Q_{kj} = 1$ for all $j \in A$ and $Q_{kj} \geq 0$ for all $k \in B, j \in A$. The capacity of the channel is defined as

$$C := \max_{p \in \mathbb{P}_m} I(p, Q) := \max_{p \in \mathbb{P}_m} \sum_{j=1}^{m} \sum_{k=1}^{n} p_j Q_{kj} \log \frac{Q_{kj}}{\sum_{l=1}^{m} p_l Q_{Kl}}, \qquad (2.1)$$

where

$$\mathbb{P}_m := \left\{ p \in \mathbb{R}^m : p_j \geq 0, \forall j \in A, \sum_{j=1}^{m} p_j = 1 \right\} \qquad (2.2)$$

is the set of all discrete finite probability measures on the channel input, and $I(p, Q)$ is known as the *average mutual information* between the channel input and channel output, considered here as a function of p. The utility of the concept of capacity is widely discussed in the literature and for more details the reader is referred to Shannon [27], Gallager [15], and Jelineck [16], and to the more recent book of Csiszar and Korner [14]. For a given probability transition matrix

Q, it is shown in Gallager [15] that $I(\cdot, Q)$ is a concave function of p and therefore problem (2.1) is a *concave programming problem* over the simplex $\mathbb{P}_m$, then any of a number of readily available nonlinear programming codes can be used to compute C. However, as reported by Blahut [7], computational experience with nonlinear programming codes applied to problem (2.1) have proved to be inefficient even for small alphabet sizes and to be impractical for the larger alphabet sizes. This motivated Arimoto [2] and Blahut [7] to develop independently a systematic iterative method for computing the capacity. This was done by exploiting the special structure of the objective function $I(\cdot, Q)$. More specifically, let $P = (P_{jk})$ denote a transition matrix from the channel output alphabet to the channel input alphabet, then

$$I(P, Q) = \max_{P \in T} \left\{ J(p, P; Q) := \sum_{j=1}^{m} \sum_{k=1}^{n} p_j Q_{kj} \log \frac{P_{jk}}{p_j} \right\}, \tag{2.3}$$

where

$$T := \left\{ P \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{j=1}^{m} P_{jk} = 1, \text{ all } k = 1, \ldots, n, P_{jk} \geq 0 \text{ all } j \text{ and } k \right\}.$$

This can be verified by noting that the maximum of J is attained at

$$P_{jk}^* = \frac{Q_{kj} p_j}{\sum_{l=1}^{m} Q_{kl} p_l}.$$

The Arimoto–Blahut algorithm can be summarized as follows:

    (O)   *Choose* an initial probability vector $p^{(0)} \in \mathbb{P}_m$.
           At iteration r, where $p^{(r)}$ is given:

        (i) *Compute* $P^{(r)} = \text{argmax}_{P \in T} J(p^{(r)}, P; Q)$.
        (ii) *Update* $p^{(r+1)} = \text{argmax}_{p \in \mathbb{P}_m} J(p, P^{(r)}; Q)$. (2.4)
        (iii) *Iterate* $r \leftarrow r + 1$.

The solution of (2.4) is explicitly given by

$$p_j^{(r+1)} = p_j^{(r)} \frac{c_j(p^{(r)})}{\sum p_j c_j(p^{(r)})}, \tag{2.5}$$

where for any $p \in \mathbb{P}_m$

$$c_j(p) = \exp \left\{ \sum_k Q_{kj} \log \frac{Q_{kj}}{\sum_l p_l Q_{kl}} \right\}.$$

That the method (2.5) converges, i.e., $\lim_{r \to \infty} I(p^{(r)}, Q) = C$, see Arimoto [2] and Blahut [7]. The amount of computation involved depends upon the size of the channel matrix. Some conditions under which the amount of computation can be reduced are discussed in Cheng [12] and Takano [28].

    In this paper we suggest a dual formulation to the channel capacity problem. The simple nature of this dual problem opens the possibility of applying many of the recent numerical schemes available in the mathematical programming literature, in particular for a large alphabet size (see e.g., [26]). This will be discussed in the next sections.

3. The Generalized Capacity Problem and Its Duals

In this section we derive, in a unified way, a dual representation for the channel capacity problem. Recall that the channel capacity C is given as the optimal value of the following optimization problem:

$$\sup_{p \in \mathbb{P}_m} \sum_j \sum_k p_j Q_{kj} \log \frac{Q_{kj}}{\sum_{l=1}^m p_l Q_{kl}}. \tag{3.1}$$

We denote by q_k the output probabilities, then $q_k := \sum_{j=1}^m p_j Q_{kj}$, $q_k \geq 0$ for all $k = 1, \ldots, n$, $\sum_{k=1}^n q_k = 1$, i.e., with our notation $q \in \mathbb{P}_n$. In the decision variables (p_j, q_k), problem (3.1) can be written equivalently as

$$(P) \quad \sup_{p \in \mathbb{P}_m} \sup_{q \in \mathbb{P}_n} \sum_j \sum_k p_j Q_{kj} \log Q_{kj} - \sum_{k=1}^n q_k \log q_k \tag{3.2}$$

$$\text{s.t.} \quad \sum_{j=1}^m p_j Q_{kj} = q_k, \quad \forall k = 1, \ldots, n. \tag{3.3}$$

The objective function in (3.2) is concave in (p, q), therefore problem (P) is a linearly constrained concave program. Also note that the feasible set of (P) is a compact convex polyhedron in $\mathbb{R}^{m+n}$, hence the sup in (P) is actually attained. The special structure of the objective function formulated in (3.2) (linear in p minus strictly convex in q) motivated us to consider the following general class of concave programming problems:

$$(P_\varphi) \quad \max_{p \in \mathbb{P}_m} \max_{q \in \mathbb{P}_n} \sum_{j=1}^m p_j \sum_{k=1}^m \varphi(Q_{kj}) - \sum_{k=1}^n \varphi(q_k) \tag{3.4}$$

$$\text{s.t.} \quad \sum_{j=1}^m p_j Q_{kj} = q_k, \quad k = 1, \ldots, n.$$

Throughout the rest of this paper, we assume that φ is a given twice continuously differentiable strictly convex function defined on an interval containing $(0, 1]$, normalized such that $\varphi(0) = \varphi(1)$, $\varphi'(1) < \infty$, and satisfying the additional assumption $\lim_{t \to 0^+} \varphi'(t) = -\infty$. Note that the latter assumption holds if φ is essentially smooth in $[0, +\infty)$ (see, e.g., Rockafellar [24]). We denote the class of such φ by Φ, accordingly, C_φ will denote the optimal value of problem (P_φ). An important example of functions $\varphi \in \Phi$ is provided by the family Φ_α of functions (parametrized by α):

$$\varphi_\alpha(t) = \begin{cases} \dfrac{1}{1 - \alpha}(t - t^\alpha) & \text{for } 0 < \alpha < 1, \\ t \log t & \text{for } \alpha = 1. \end{cases}$$

Clearly, with $\varphi_1(t) = t \log t$, problem (P_φ) is just the classical channel capacity problem (P).

The objective function used in (P_φ) is exactly the generalized average mutual information measure introduced and studied by Burbea and Rao [10]. A related generalized measure of information was also recently introduced by Ben-Tal and

Teboulle [6] and the associated rate distortion function was studied. For additional generalizations and applications of generalized information measures the reader is referred to Aczel [1], Arimoto [3], Burbea [8], [9], Burbea and Rao [11], Csiszar [13], Rao and Yayak [22], Renyi [23], and Ziv and Zakai [30].

The dual representation of (P_φ) will be derived via Langrangian duality. Before stating the main result of this section we introduce the following notations and definitions. For any $\varphi \in \Phi$, let $f: \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(v) = \inf_{\eta \in \mathbb{R}} \left\{ \eta + \sum_{k=1}^n \varphi^*(v_k - \eta) \right\}, \tag{3.5}$$

where $\varphi^*(y) = \sup_t \{ ty - \varphi(t) \}$ denotes the usual convex conjugate of φ. Also, let $l: \mathbb{R}^n \to \mathbb{R}$ be defined by

$$l(v) = \max_{1 \le j \le m} \{ l_j(v) + b_j \}, \tag{3.6}$$

where $\{ l_j \}$ are the linear functions given by $l_j(v) = -\sum_{k=1}^m v_k Q_{kj}$ and b_j denotes the constants $b_j = \sum_{k=1}^m \varphi(Q_{kj})$.

Theorem 3.1. *The dual problem of* (P_φ), *for* $\varphi \in \Phi$, *is given by*

$$(D_\varphi) \quad \inf_{v \in \mathbb{R}^n} \{ f(v) + l(v) \}. \tag{3.7}$$

If (P_φ) *is feasible the minimum in* (D_φ) *is attained and the optimal values coincide:*

$$C_\varphi = \max(P_\varphi) = \min(D_\varphi).$$

Proof. The Langrangian for problem (P_φ) is

$$L(p, q, v) = -\sum_{k=1}^n \varphi(q_k) + \sum_{j=1}^m p_j \sum_{k=1}^n \varphi(Q_{kj}) + \sum_{k=1}^n q_k v_k - \sum_{j,k} p_j v_k Q_{kj}$$

and is separable in the decision variables (p, q). The dual objective function is then

$$h(v) = \max_{p \in \mathbb{P}_m} \sum_{j=1}^m p_j \left(\sum_{k=1}^n \varphi(Q_{kj}) - v_k Q_{kj} \right) + \max_{q \in \mathbb{P}_n} \sum_{k=1}^n q_k v_k - \varphi(q_k) \tag{3.8}$$

and the dual problem is defined as $\min_{v \in \mathbb{R}^n} h(v)$. The first "max" in (3.8) is easily computed:

$$\max_{p \in \mathbb{P}_m} \sum_j p_j \left(\sum_k \varphi(Q_{kj}) - v_k Q_{kj} \right) = \max_{1 \le j \le m} \left\{ \sum_{k=1}^n \varphi(Q_{kj}) - v_k Q_{kj} \right\}. \tag{3.9}$$

To evaluate the second "max" in (3.8) we note that a Langrangian dual of

$$\alpha := \max_{q \in \mathbb{P}_n} \sum_{k=1}^n q_k v_k - \varphi(q_k) \tag{3.10}$$

is given by

$$\beta := \min_{\eta \in \mathbb{R}} \left\{ \eta + \sum_k \max_{q_k \ge 0} \{ q_k(v_k - \eta) - \varphi(q_k) \} \right\}, \tag{3.11}$$

where η is the Lagrange multiplier for the equation $\sum q_k = 1$. Problem (3.10) is a linearly constrained concave program trivially satisfying the Slater constraint qualification and hence by standard duality arguments [23] we have $\alpha = \beta$.

To compute β, we consider the problem $e(y) = \sup_{t \geq 0}\{ty - \varphi(t)\}$. By the monotonicity of φ', it follows that

$$
e(y) = \begin{cases} \sup_{t \in \mathbb{R}} \{ty - \varphi(t)\} = \phi^*(y) & \text{if } y \geq \varphi'(0), \\ -\xi(0) & \text{if } y < \varphi'(0). \end{cases}
$$

But since we assumed that $\varphi'(0) = -\infty$, we conclude in fact that $e(y) = \varphi^*(y)$. Using this result in (3.11) we get

$$
\beta = \min_{\eta \in \mathbb{R}} \left\{ \eta + \sum_{k=1}^{n} \varphi^*(v_k - \eta) \right\}. \tag{3.12}
$$

Substituting (3.9) and (3.12) in (3.8) and using the notations (3.5) and (3.6) we thus obtain $h(v) = f(v) + l(v)$ and hence the dual representation (3.7) is proved. Further, for $\varphi \in \Phi$, φ is convex and then $\eta + \sum_k \varphi^*(v_k - \eta)$ is jointly convex in $(\eta, v) \in \mathbb{R} \times \mathbb{R}^n$, hence (by Theorem 1 of Rockafellar [25]) $f(v)$ is convex. Finally, since (P_φ) is assumed feasible and is linearly constrained, the Slater regularity condition holds trivially and then it follows from standard duality results (see, e.g., [24]) that the $\inf(D_\varphi)$ is attained and $\max(P_\varphi) = \min(D_\varphi)$. $\square$

The next result shows how to obtain the optimal solution of the primal problem (P_φ) from an optimal solution of the dual.

Theorem 3.2. *Let $\bar{v}$ be the optimal solution of (D_φ). Then the optimal solution $(\bar{p}, \bar{q})$ of the primal problem (P_φ) is computed as follows*:

$$
\bar{q}_k = (\varphi')^{-1}(\bar{v}_k - \bar{\eta}), \qquad k = 1, \ldots, n, \tag{3.13}
$$

where $\bar{\eta}$ is the unique solution of the equation

$$
\sum_k (\varphi')^{-1}(\bar{v}_k - \bar{\eta}) = 1 \tag{3.14}
$$

and $\bar{p}$ is the optimal solution of the linear program

$$
(L_\varphi) \quad \max \sum_j p_j \left(\sum_k \varphi(Q_{kj}) \right)
$$

$$
\text{s.t.} \quad \sum_j p_j Q_{kj} = \bar{q}_k, \qquad k = 1, \ldots, n,
$$

$$
\sum_j p_j = 1, \qquad p_j \geq 0, \quad j = 1, \ldots, m.
$$

Proof. The expression (3.13) for $\bar{q}_k$ is just the optimality conditions for $q_k = \bar{q}_k$ to solve the inner maximization in (3.11) (recall that the optimal q_k cannot be zero as explained in the proof of Theorem 3.1). The optimality condition for $\eta = \bar{\eta}$ to solve the convex unconstrained problem (3.12) is

$$
\sum_k (\varphi^*)'(v_k - \eta) = 1. \tag{3.15}
$$

But $(\varphi^*)' = (\varphi')^{-1}$ (see, e.g. Section 26 of [24]) and thus (3.15) is exactly (3.14). Now, since $\varphi \in \Phi$, $\varphi'(0) = -\infty$ and $\varphi'(1) < \infty$ implying $(\varphi')^{-1}(-\infty) = 0$ and $(\varphi')^{-1}(\infty) > 1$. Thus equation (3.14) has a solution $\bar{\eta} = \eta(\bar{v})$ for all $\bar{v}$, which is also unique since φ' (and hence $(\varphi')^{-1}$) is strictly monotone. The statement concerning $\bar{p}$ follows immediately from (3.4).   □

The dual problem (3.7) is an unconstrained discrete minimax problem. Many algorithms for such problems have been proposed in the nonsmooth optimization literature, see, e.g., Wolfe [29], Lemarechal [18], [19], and, more recently, Kiwiel [17]. Alternatively, the dual problem (3.7) can be reformulated as a linearly constrainted convex program in $\mathbb{R}^{n+1}$. Indeed, by defining $V_{n+1} := \max_{1 \le j \le n} \{l_j(v) + b_j\}$, problem (3.7) is equivalent to

$$\min_{v \in \mathbb{R}^{n+1}} f(v) + v_{n+1}$$
$$\text{s.t.} \quad l_j(v) + b_j - v_{n+1} \le 0 \qquad \text{for all} \quad j = 1, \ldots, m, \tag{3.16}$$

in which case many nonlinear programming codes are readily available for solving the dual formulation (3.16) (see, e.g., Schittowsky [26] or the recent generalized reduced gradient code of Lasdon).

The special structure of the dual problem

$$\min_{v \in \mathbb{R}^n} \left\{ f(v) + \max_{1 \le j \le m} \{l_j(v) + b_j\} \right\}$$

suggests a method in which f is approximated at the rth iteration by a *polyhedral function* (i.e., pointwise maximum of finitely many affine functions) $\pi_r(v)$, and the next iteration point v^{r+1} is the optimal solution of

$$\min_v \{\pi_r(v) + \max\{l_j(v) + b_j\}\}. \tag{3.17}$$

Since (3.17) is a linear minimax problem it can be efficiently solved with simplex-like algorithms (e.g., [4] and [5]). The new polyhedral approximation $\pi_{r+1}(v)$ is obtained by

$$\pi_{r+1}(v) = \max\{\pi_r(v), s_{r+1}(v)\},$$

where $s_{r+1}(\cdot)$ is the affine support of f at v^{r+1}.

Let us derive the dual representation of the classical channel capacity C. This is done simply by substituting $\varphi(t) = t \log t$ in problem (D_φ). The conjugate is $\varphi^*(t^*) = e^{t^* - 1}$ and so

$$f(v) = \min_{\nu \in \mathbb{R}} \left\{ \eta + \sum_k e^{v_k - 1} \right\} = \log \sum_{k=1}^n e_k^v$$

by simple calculus.

Then, using Theorem 3.1, a dual representation of C is given by

$$C = \min_{v \in \mathbb{R}^n} \left\{ \log \sum_{k=1}^n e^{v_k} + \max_{1 \le j \le m} \left\{ \sum_k Q_{kj} \log Q_{kj} - \sum_k Q_{kj} v_k \right\} \right\}. \tag{3.18}$$

In the last part of this section we present an alternative dual problem to (P_φ) which is given directly in terms of the problem's data.

Corollary 3.1. *Let $\varphi \in \Phi$, then*

$$C_\varphi = \min_{y \in \mathbb{P}_n} \max_{1 \leq j \leq m} \left\{ \sum_k \varphi'(y_k)(y_k - Q_{kj}) + \varphi(Q_{kj}) - \varphi(y_k) \right\}. \tag{3.19}$$

Proof. From the proof of Theorem 3.2 it follows that $f(v)$ defined in (3.5) is given by

$$f(v) = \eta(v) + \sum_k \varphi^*(v_k - \eta(v)), \tag{3.20}$$

where $\eta(v)$ is the unique solution of the equation

$$\sum_k (\varphi^*)'(v_k - \eta) = 1.$$

Define new variables $y_k = (\varphi^*)'(v_k - \eta(v))$. Then $\sum_k y_k = 1$ and $y_k \geq 0$ since $(\varphi^*)'(-\infty) = 0$ and $(\varphi^*)'$ is increasing, hence $y \in \mathbb{P}_n$. From the definition of y_k, and using $(\varphi^*)' = (\varphi')^{-1}$,

$$v_k - \eta(v) = \varphi'(y_k). \tag{3.21}$$

Substituting (3.21) in (3.7) and using (3.20) we obtain

$$C_\varphi = \min_{v,y} \left\{ \eta(v) + \sum_k \varphi^*(\varphi'(y_k)) + \max_{1 \leq j \leq m} \sum_k \varphi(Q_{kj}) - Q_{kj}(\eta(v) + \varphi'(y_k)) \right\}.$$

But $\sum_k Q_{kj} = 1$, hence

$$C_\varphi = \min_y \left\{ \sum_k \varphi^*(\varphi'(y_k)) + \max_{1 \leq j \leq m} \sum_k \varphi(Q_{kj}) - Q_{kj}\varphi'(y_k) \right\}. \tag{3.22}$$

Finally, using in (3.22) a simple fact concerning conjugate functions,

$$\varphi^*(\varphi'(t)) = t\varphi'(t) - \varphi(t),$$

we get the desired result (3.19). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Applying Corollary 3.1 to the classical case $\varphi(t) = t \log t$, a little algebra shows that a dual representation of C is

$$C = \min_{y \in \mathbb{P}_n} \max_{1 \leq j \leq m} \sum_k Q_{kj} \log \frac{Q_{kj}}{y_k}$$

and we recover here a result given in Meister and Oettli [20].

4. Upper Bound for C_φ

The dual representation derived in Corollary 3.2 is also useful for deriving an upper bound on C_φ.

Theorem 4.1. *For any $\varphi \in \Phi$,*

$$C_\varphi \leq \sum_k \left\{ \varphi'\left(\sum_j \frac{Q_{kj}}{m}\right) \sum_j \frac{Q_{kj}}{m} - \varphi\left(\sum_j \frac{Q_{kj}}{m}\right) \right\} + \max_{1 \leq j \leq m} \sum_k \varphi(Q_{kj}) - Q_{kj}\varphi'\left(\sum_j \frac{Q_{kj}}{m}\right). \tag{4.1}$$

Proof. From the dual representation (3.19)

$$C_\varphi \leq \max_{1 \leq j \leq m} \left\{ \sum_{k=1}^n \varphi'(y_k)(y_k - Q_{kj}) + \varphi(Q_{kj}) - \varphi(y_k) \right\} \tag{4.2}$$

for every y satisfying $\sum_{k=1}^n y_k = 1$ and $y_k \geq 0$, $k = 1, \ldots, n$, in particular, for $y_k = (\sum_{j=1}^m Q_{kj})/m$. Substituting this special choice in (4.2), and rearranging the terms, we obtain the upper bound in the theorem. □

Example 4.1. Let us consider the family Φ_α with functions φ_α given by

$$\varphi_\alpha(t) = \begin{cases} \dfrac{1}{1-\alpha}(t - t^\alpha), & 0 < \alpha < 1, \\ t \log t, & \alpha = 1. \end{cases}$$

We denote respectively by C_α and U_α the corresponding capacity and its upper bound. Using Theorem 4.1 we get

$$C_\alpha \leq U_\alpha$$

with

$$U_\alpha = \begin{cases} \dfrac{1}{m^\alpha} \sum_k \left(\sum_j Q_{kj}\right)^\alpha + \dfrac{1}{1-\alpha} \max_{1 \leq j \leq m} \dfrac{\alpha}{m^\alpha - 1} \sum_k Q_{kj}\left(\sum_j Q_{kj}\right)^{\alpha-1} - \sum_k Q_{kj}^\alpha \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 < \alpha < 1, \\ \\ \log m + \max_{1 \leq j \leq m} \sum_k Q_{kj} \log \dfrac{Q_{kj}}{\sum_l Q_{kl}} \qquad\qquad\qquad\quad \text{if } \alpha = 1. \end{cases}$$

In particular, we see that the classical lower bound derived in Arimoto [2] is recovered, i.e.,

$$C \equiv C_1 \leq U_1.$$

An interesting special case for which the upper bound is attained is in the case of a *symmetric* channel, i.e., one with the same set of entries in columns and rows of Q, with possible permutations. In that case we have $\sum_{j=1}^m Q_{kj} = \text{const.} = \delta$ for all k.

Theorem 4.2. *If the channel is symmetric, then, for any $\varphi \in \Phi$, C_φ is equal to the upper bound, i.e.,*

$$C_\varphi = -n\varphi\left(\frac{1}{n}\right) + \frac{1}{m} \sum_{j,k} \varphi(Q_{kj}). \tag{4.3}$$

Proof. From the primal formulation of C_φ (see (3.4)) a lower bound is given by

$$C_\varphi \geq -\sum_k \varphi(q_k) + \sum_{j=1}^m p_j \sum_{k=1}^n \varphi(Q_{kj}) \tag{4.4}$$

for any $(p, q) \in \mathbb{P}_m \times \mathbb{P}_n$ satisfying

$$\sum_j p_j Q_{kj} = q_k, \qquad k = 1, \ldots, n. \tag{4.5}$$

Since the channel is symmetric, $\sum_j Q_{kj} = \delta$ for all k and this implies that $m = n\delta$. Thus, $p_j^* = 1/m$ and $q_k^* = 1/n$ satisfy (4.5). Substituting (p^*, q^*) in (4.4) we obtain the lower bound

$$C_\varphi \geq -n\varphi\left(\frac{1}{n}\right) + \frac{1}{m} \sum_{j,k} \varphi(Q_{kj}). \tag{4.6}$$

From Theorem 4.1, using the fact $m = n\delta$, we have the upper bound

$$C_\varphi \leq -n\varphi\left(\frac{1}{n}\right) + \max_j \sum_k \varphi(Q_{kj}). \tag{4.7}$$

Since the channel is symmetric, it has the same set of entries in each column, thus $\sum_k \varphi(Q_{kj}) = \gamma_j = \text{const.}$ for all j and, hence,

$$\max_{1 \leq j \leq m} \sum_k \varphi(Q_{kj}) = \frac{1}{m} \sum_{j,k} \varphi(Q_{kj}). \tag{4.8}$$

Therefore, substituting (4.8) in (4.7), we see that the upper bound for C_φ coincides with its lower bound given in (4.6).   □

Example 4.2. Consider the Binary Symmetric Channel (BSC) defined by

$$Q = \begin{bmatrix} 1-\beta & \beta \\ \beta & 1-\beta \end{bmatrix}.$$

Using Theorem 4.2 we obtain, for any $\varphi \in \Phi$,

$$C_\varphi^{\text{BSC}} = -2\varphi(\tfrac{1}{2}) + \varphi(1-\beta) + \varphi(\beta).$$

In particular, for $\varphi(t) = t \log t$ we get the well-known result (see, e.g., [15] and [16])

$$C^{\text{BSC}} = \log 2 + (1-\beta) \log(1-\beta) + \beta \log \beta.$$

5.   Conclusions and Extension

A new formulation of the channel capacity problem has been obtained by using the duality theory of convex programming. This new dual representation seems useful for computational purposes and the derivation of bounds. Furthermore, the results in this paper demonstrate that the new information measure of Burbea and Rao [10] can be successfully used to develop a generalized channel capacity theory.

Finally, we remark that our duality framework can be easily extended to the multiple constrained channel capacity problem (i.e., which include additional linear inequality constraints on the input probability p, see [5]) to produce a simple dual formulation. Also, at the price of some additional technicalities, the continuous alphabet channel problem [7] [15] can be considered, via a duality theory for infinite-dimensional optimization problems framework, to obtain the continuous version of Theorem 3.1 and its corollary.

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