Recursive Integral Equations for the Detection of Counting Processes*

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ABSTRACT

A recursive stochastic integral equation for the detection of counting processes is derived from a previously known formula [5] of the likelihood ratio. This is done quite simply by using a result due to Doléans–Dade [4] on the solution of stochastic integral equations.

1. Introduction. Recently modern martingale theory has been used to describe Counting Processes (hereafter abbreviated CP) in a way specially appropriate to the problems of detection and filtering. This has given rise to the notion of Integrated Conditional Rate (ICR) [5], which generalizes the notion of random rate.

Expressions for likelihood ratios (involving ICR's) for the detection of CP's have been obtained in [5] using a three-step technique introduced by Kailath [9] and Duncan ([6], [7]) in their works on detection of a stochastic signal in white noise. The three steps are the Likelihood Ratio Representation Theorem ([2], [5], [6]), the Girsanov Theorem ([5], [8], [13]) and the Innovation Theorem ([2],[5], [9]). By this method likelihood ratios for a large class of CP's can be found. These expansions represent a generalization of the formulas given in [1] and [12]

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in the context of Poisson processes and [2] in the context of CP's which admit a conditional rate.

The purpose of this paper is not to present a proof of the likelihood ratio formula (for that see [5]) but to derive from this formula stochastic integral equations by which the likelihood ratio can be computed recursively. This can be done quite simply using a result due to Doléans–Dade [4] on the solution of stochastic integrals equations involving semimartingales. These recursive equations are most useful in applications as they give a way of implementing the computation of the likelihood ratio continuously in time.

2. Preliminaries. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space. By (X_t) we denote a real valued stochastic process defined on $\mathbf{R}_+$, the positive real line and by a Counting Process (CP) we mean

Definition 2.1. A CP is a stochastic process having sample paths which are zero at the time origin and consisting of right-continuous step functions with positive jumps of size one.

The *time of n^{th} jump* J_n of a CP (N_t) is the stopping time defined by

$$J_n = \begin{cases} \inf\{t : N_t \geqslant n\} \\ \infty \text{ if the above set is empty.} \end{cases}$$

Let $(\mathfrak{F}_t)$ be a right-continuous increasing family of σ-subalgebras of $\mathfrak{F}$ with $\mathfrak{F}_0$ containing all the P negligible sets, and suppose (N_t) is a CP, adapted to $\mathfrak{F}_t$, with the sole assumption that EN_t is finite for each t. Then, as a consequence of the Doob–Meyer decomposition for supermartingales we can associate to (N_t) a unique *natural increasing process* (A_t), dependent on the family $(\mathfrak{F}_t)$, which makes the process $(M_t \overset{\Delta}{=} N_t - A_t)$ a martingale (see [11]). This decomposition $(N_t = M_t + A_t)$ is intuitively a decomposition into the part (M_t) which is not predictable and (A_t) which can be perfectly predicted. This unique process (A_t) is called the *Integrated Conditional Rate* (ICR) of (N_t) with respect to $(\mathfrak{F}_t)$ ("the $(\mathfrak{F}_t)$ ICR of (N_t)") and has been studied in [5]. The terminology ICR is motivated by the fact that when (N_t) satisfies some sufficiency conditions its ICR takes on the form $(\int_0^t \lambda_s ds)$ where (λ_t) is a nonnegative process called the *conditional rate* (with respect to $(\mathfrak{F}_t)$) satisfying $\lambda_t = \lim_{h\to 0} E[h^{-1}(N_{t+h} - N_t)|\mathfrak{F}_t]$ ([5], Section 2.5). The existence of CP's possessing a bounded conditional rate with respect to the family of σ-algebras generated by the process itself has been first shown in [2] and in [5]. Sufficiency conditions for the existence of a conditional rate have been given in [5]. By a change of time we can show similar results (i.e., existence (see [5], Corollary 3.1.3) and sufficiency conditions) for $(\mathfrak{F}_t)$ ICR's of the form $(\int_0^t \lambda_s dm_s)$ where (λ_t) is a *locally bounded predictable process* and m_t a *deterministic increasing right-continuous function* with $m_0 = 0$. Denote by $\mathfrak{K}(\mathfrak{F}_t)$ the class of all locally bounded predictable (with respect to $(\mathfrak{F}_t)$) processes (see [3], p. 98). For example, processes adapted to $(\mathfrak{F}_t)$ and having left-continuous sample paths belong to $\mathfrak{K}(\mathfrak{F}_t)$.

Remark 2.2. Let the ICR (A_t) be of the form $(\int_0^t \lambda_s dm_s)$ and denote by Λ the union of all intervals of $\mathbf{R}_+$ on which the function m_t is constant. Observe that the ICR (A_t) is not affected by a change of values of (λ_t) for $t \in \Lambda$ and we may well have $\lambda_t = \infty$ for $t \in \Lambda$. To avoid problems due to this indeterminacy we adopt the following convention: for $t \in \Lambda$ we set λ_t equal to unity.

We assume here that modern martingale theory ([11], [3]) is known. Recall that a *semimartingale* (X_t) is a process which can be written as a sum $(X_t = X_0 + L_t + A_t)$ where X_0 is $\mathcal{F}_0$-measurable, (L_t) is a $(\mathcal{F}_t)$ local martingale and (A_t) is a right-continuous process adapted to $(\mathcal{F}_t)$ having sample paths of bounded variation on every finite interval and with $A_0 = 0$ a.s. (see [3]). A result basic to this study and due to Doléans-Dade [4] is the following: the stochastic integral equation

$$Z_t = 1 + \int_0^t Z_{s-} dX_s$$

where (X_t) is a semimartingale has a unique solution, which is a semimartingale given by†

$$Z_t = \exp\left(X_t - \frac{1}{2}\langle X^c \rangle_t\right) \prod_{s \leqslant t} (1 + \Delta X_s)\exp(-\Delta X_s)$$

where the product in the right hand side converges a.s. for each t. Here we define $(\langle X^c \rangle_t)$ as the *unique natural increasing process* (see [3]) associated to the continuous part of the *local martingale* (L_t); $(\langle X^c \rangle_t)$ is identically zero when (X_t) is a semimartingale with sample paths of bounded variation on every finite interval (see [3]).

3. The Detection Problem. Let P_0 and P_1 be two measures carried on $(\Omega, \mathcal{F})$. Suppose that (N_t) is a CP defined on $(\Omega, \mathcal{F})$ and denote by $\mathcal{N}_t$ the minimal σ-algebra generated by (N_t) up to and at time t. The notation $E_i(\cdot)$ for $i = 0, 1$ is intended for the expectation operator with respect to the measure P_i.

Definition 3.1. For a $(\mathcal{N}_t)$ stopping time R (possibly infinite) denote by $\bar{P}_i^R$ for $i = 0, 1$ the restriction of the measure P_i to the σ-algebra $\mathcal{N}_R$.

We have the inclusion $\mathcal{N}_R \subset \mathcal{F}$ so that if $P_0 \ll P_1$* then $\bar{P}_0^R \ll \bar{P}_1^R$ and the Radon-Nikodym derivative $d\bar{P}_0^R / d\bar{P}_1^R$ is well defined. We examine now the meaning of this Radon-Nikodym derivative. In the case where the stopping time R is equal to a constant a then $\mathcal{N}_R = \mathcal{N}_a = \sigma(N_u, 0 \leqslant u \leqslant a)$ so that $d\bar{P}_0^a / d\bar{P}_1^a$ is the likelihood ratio for testing the two hypotheses H_i for $i = 0, 1 : P_i$ is the probability measure on $(\Omega, \mathcal{F})$, by observations on the CP (N_t) for $t \leqslant a$. The detection scheme then consists in selecting H_0 or H_1 according as $d\bar{P}_0^a / d\bar{P}_1^a$ is above or below a given threshold. Now in the case where R is a stopping time

†When f_t is a right-continuous function with left-hand limits Δf_t denotes the jump $f_t - f_{t-}$.

*$P_0 \ll P$ means that the measure P_0 is absolutely continuous with respect to P while $P_0 \sim P$ indicates that the two measures are equivalent.

which is not a constant we know that $\mathcal{N}_R \supset \sigma(N_{u\wedge R}, 0 \leqslant u)$ (this follows from the fact that $N_{u\wedge R}$ is $(\mathcal{N}_R)$ measurable by Theorem 49-IV of [11]) but the reverse inclusion is not necessarily true. For this reason $d\bar{P}_0^R / d\bar{P}_1^R$ is not the likelihood ratio for our detection problem when the time of observation is the stochastic interval $[0, R]$, as one could have conjectured. But one can interpret $d\bar{P}_0^R / d\bar{P}_1^R$ as a likelihood ratio if we assume that the information accessible to the observer is $\mathcal{N}_R$ and not simply $\sigma(N_{u\wedge R}, 0 \leqslant u)$. For $i = 0, 1$ with the measure P_i carried on $(\Omega, \mathcal{F})$ suppose that the CP (N_t) has the process $(\int_0^t \lambda_s^i dm_s)$ for $(\mathcal{F}_t^i)$ ICR, where $(\mathcal{F}_t^i)$ is a family of σ-algebras with $\mathcal{F}_t^i \supset \mathrm{N}_t$, $(\lambda_t^i) \in \mathcal{H}(\mathcal{F}_t^i)$ is a positive process, and m_t is an increasing deterministic function with $m_0 = 0$.

It is known that we can make a change of measure under which (N_t) is a CP of independent increments with mean $m_t = EN_t$ under the new measure P (Theorem 2.6.1 of [5]). Using this fact and the three-step technique of Duncan and Kailath (see Introduction) the likelihood ratio for detecting CP's has been obtained according to

Theorem 3.2 (Theorem 3.4.4 of [5]). *For $i = 0, 1$ let (N_t) be, under the measure P_i, the* CP *described above. Assume*

(a) *$P_0 \ll P$ and $P \sim P_1$ and define for $i = 0, 1$ the $(P, \mathcal{N}_t)$ martingale*

$$L_t^i = E\left[\frac{d\bar{P}_i^\infty}{d\bar{P}^\infty} \,|\, \mathcal{N}_t\right];$$

(b) *For $i = 0, 1$, the stopping times T^i are such that there exists increasing sequences of stopping times (T_n^i) for which $T^i = \lim_n T_n^i$* a.s. *and $E(ln^- L_{T_n^i}^i)^2 < \infty$ for each n. Let $T = T^1 \wedge T^0$;*

(c) *For $i = 0, 1$ $E_i \int_0^t \lambda_s^i dm_s < \infty$.*

Then

$$\frac{d\bar{P}_0^{t\wedge T}}{d\bar{P}_1^{t\wedge T}} = \prod_{J_n \leqslant t\wedge T} \left[\frac{\hat{\lambda}_{J_n}^0}{\hat{\lambda}_{J_n}^1}\right] \exp\left[\int_0^{t\wedge T} (\hat{\lambda}_s^1 - \hat{\lambda}_s^0) dm_s\right] \tag{1}$$

where $\hat{\lambda}_t^i \triangleq E_i(\lambda_t^i | \mathcal{N}_t)$ for $i = 0, 1$ and J_n is the time of n^{th} jump of (N_t). By convention the product $\Pi(\cdot) = 1$ for $J_1 > t \wedge T$.

Remark 3.3. (a) The stopping time T^i which is the first time after which the martingale (L_t^i) can behave badly may take the value $+\infty$. It is in fact desirable for T^i to be as large as possible.

(b) By our convention (Remark 2.2) condition (c) above insures that the process $(\hat{\lambda}_t^i)$ is well defined.

4. Recursive Integral Equations for Likelihood Ratios We show here that the likelihood ratio (1) of our detection problem can be obtained as the unique solution of a stochastic integral equation. This stochastic integral equation can be mechanized by a feedback scheme tantamount to a recursive filter, as shown in Figure 1.

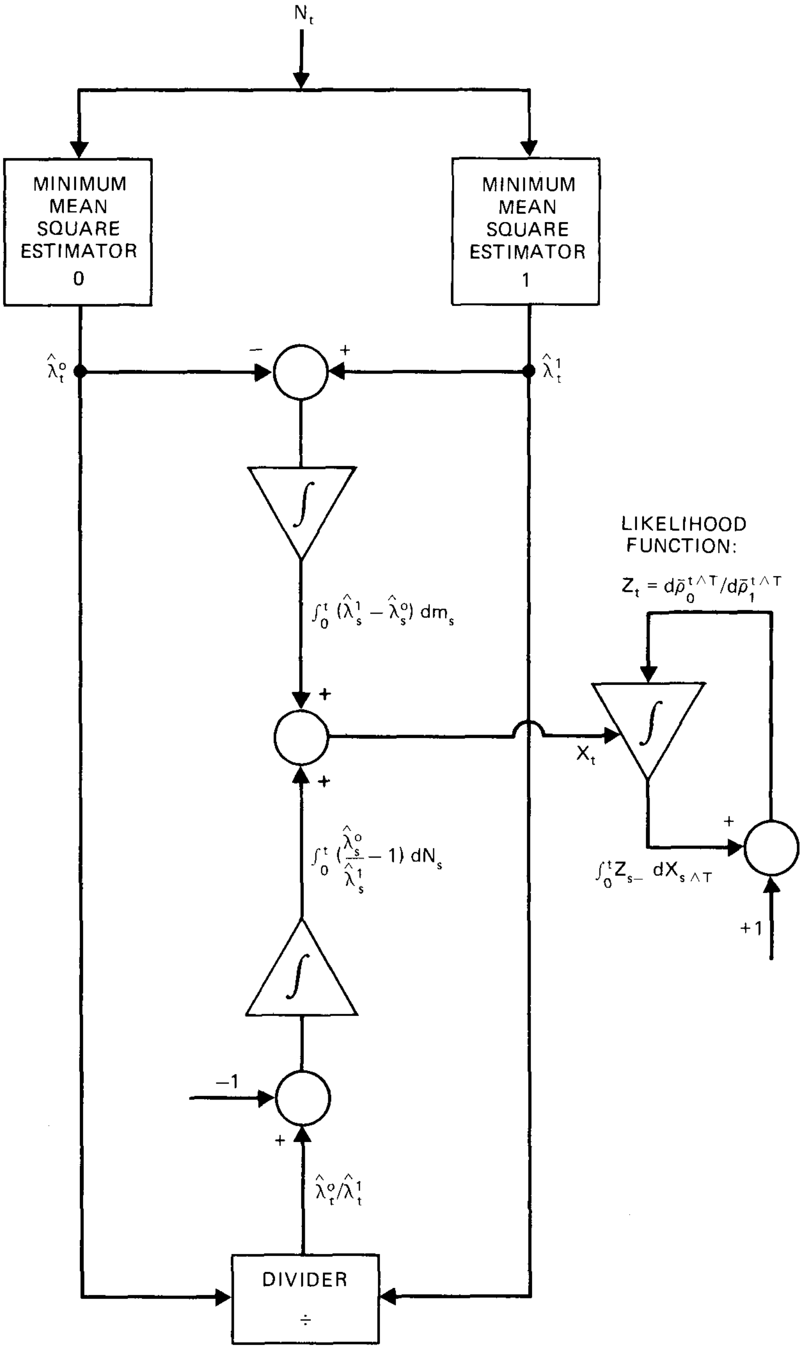

Figure 1. Recursive Scheme for Obtaining the Likelihood Function Z^t

Theorem 4.1. *The likelihood ratio $d\bar{P}_0^{t\wedge T}/d\bar{P}_1^{t\wedge T}$ of Theorem 3.2 is the unique solution of the following stochastic integral equation*:

$$Z_t = 1 + \int_0^t Z_{s-}\, dX_{s\wedge T} \tag{2}$$

where

$$X_t = \int_0^t \left\{ \left[\frac{\hat{\lambda}_s^0}{\hat{\lambda}_s^1} \right] - 1 \right\} dN_s + \int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\, dm_s \tag{3}$$

Proof. By assumption $(\lambda_t^i), i=0,1$, is positive a.s. finite for all t (by condition (c) of Theorem 3.2 and Remark 2.2). The process (N_t) has a finite number of jumps in any finite interval so that the process $(\int_0^{t\wedge T}[(\hat{\lambda}_s^0/\hat{\lambda}_s^1) - 1]dN_s)$ has sample paths of bounded variation on any finite interval; and so does the process $(\int_0^{t\wedge T}(\hat{\lambda}_s^1 - \hat{\lambda}_s^0)dm_s)$ by assumption (c) of Theorem 3.2. Hence $(X_{t\wedge T})$ is a semi-martingale with sample paths of bounded variation on any finite interval so that $(\langle X^c \rangle_{t\wedge T}) \equiv 0$ (see the remark, on p. 90, following proposition 3 of [3]). Then by Theorem 1 of [4] the unique solution of (2) is given by

$$Z_t = \exp(X_{t\wedge T}) \prod_{s \leqslant t} (1 + \Delta X_{s\wedge T}) \exp(-\Delta X_{s\wedge T}) \tag{4}$$

Now $\Delta X_{s\wedge T} = ((\hat{\lambda}_s^0/\hat{\lambda}_s^1) - 1)\Delta N_{s\wedge T}$ and hence the product in (4) becomes

$$\begin{aligned}
\prod_{s<t}(\cdot) &= \prod_{s \leqslant t} \left[1 + \left[\frac{\hat{\lambda}_s^0}{\hat{\lambda}_s^1} - 1 \right] \Delta N_{s\wedge T} \right] \exp\left[\sum_{s \leqslant t\wedge T} - \left[\frac{\hat{\lambda}_s^0}{\hat{\lambda}_s^1} - 1 \right] \Delta N_{s\wedge T} \right] \\
&= \prod_{J_n \leqslant t\wedge T} \left[\frac{\hat{\lambda}_{J_n}^0}{\hat{\lambda}_{J_n}^1} \right] \exp\left[-\int_0^{t\wedge T} \left[\frac{\hat{\lambda}_s^0}{\hat{\lambda}_s^1} - 1 \right] dN_s \right]
\end{aligned}$$

Substituting the above relation and expression (3) in (4) gives the desired result (compare with (1))

$$Z_t = \prod_{J_n \leqslant t\wedge T} \left[\frac{\hat{\lambda}_{J_n}^0}{\hat{\lambda}_{J_n}^1} \right] \exp\left[\int_0^{t\wedge T} (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\, dm_s \right] = \frac{d\bar{P}_0^{t\wedge T}}{d\bar{P}_1^{t\wedge T}}. \quad \square$$

Observe that if under the measure P_1 the CP (N_t) is a process of independent increments with mean m_t then $P \equiv P_1, \lambda_t^1 = 1$ and Eq. (3) becomes

$$X_t = \int_0^t (\hat{\lambda}_s^0 - 1)\, d(N_s - m_s) \tag{5}$$

The process $(M_t \overset{\Delta}{=} N_t - m_t)$ is a $(P, \mathcal{N}_t)$ martingale. Hence (5) shows that the process $(X_{t\wedge T})$ is a local martingale. In turn, (2) then implies that the process (Z_t) is a local martingale. In this case we in fact have $Z_t = E_1[(d\bar{P}_0^{\infty}/dP_1^{\infty})|\mathcal{N}_{t\wedge T}]$, i.e. the likelihood function is a uniformly integrable martingale.

In applications, Eqs. (2) and (3) give a way of implementing the computation of the likelihood ratio continuously in time. They represent recursive equations if one also obtains the best estimates $(\hat{\lambda}_t^i)$ in a recursive manner. The block diagram of this implementation is given in Figure 1.

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