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Randomized System Trajectories

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Abstract. The notion of system trajectory of a time-varying input-output, dynamical system is reviewed. By introducing a probability measure on a class of such systems a stochastic system, the randomized system, is defined. The randomized system has a trajectory induced by the trajectories of the original systems. A theorem is proved giving fairly general conditions under which the randomized system trajectory is generated by a strongly continuous semigroup of bounded linear operators in a Banach space. An example is presented for a system represented by a quadratic integral operator.

1. Introduction

A theorem is stated and proved here giving sufficient conditions for the generation of a system trajectory for a stochastic system by a strongly continuous semigroup of bounded linear operators. The theorem appears in section 3 and is followed by an example in section 4. Since the problem considered (but not the analysis) is judged to be somewhat nonstandard, some intuitive and motivational background material is included in this introduction. Even the term "system trajectory" is presumed not to be immediately meaningful, so the concept is developed below in heuristic fashion.

We are concerned with nonanticipative "dynamical" systems, but no particular theory of nonanticipative systems is necessary. The system trajectories are not

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state trajectories for the system under consideration. Thus the rather extensive theory of state in causal systems, in particular involving abstract causality in the context of resolution spaces [9] and [10] is not used. However, the system trajectories can sometimes be interpreted as state trajectories for a "dominating" system (as pointed out below), so it is entirely possible this work could be tied to the theory mentioned, and perhaps generalized in one direction with the use of resolution spaces. The ideas developed in [6] and [7], particularly the latter, are conceptually useful here, but again none of the results of these papers is actually used. The theorem of this paper has been stated, in incomplete form and without proof, in [8].

For the purposes of this paper a system model (Y, F, U) is a mapping F: $U \to Y$ where U and Y are spaces to be specified. The interpretation is that each element $u \in U$ represents a possible input and the element $y \in Y$, $y = F(u)$, is the corresponding output. A compound system model (Y, f, X, U) is a mapping f: $X \times U \to Y$. The interpretation of U and Y as, respectively, input and output spaces remains the same, but the presence of X allows for consideration of a family of input-output maps, $F(\cdot) = f(x, \cdot)$, $x \in X$. A (compound) system model is said to be a dynamical (compound) system model if U, Y and (each) F satisfy the following conditions: (1) U and Y are spaces of functions (or equivalence classes of functions) defined on the real line R. Both U and Y are to be invariant under translation on R. (2) F is a nonanticipative mapping; i.e. if u_1, $u_2 \in U$ and $u_1(s) = u_2(s)$ for all $s \leqslant t$, then $[F(u_1)](s) = [F(u_2)](s)$ for all $s \leqslant t$.

The notations U, Y and F will be used consistently so that they have the sort of interpretation assigned, even though they carry affixes and represent various spaces and mappings. Any of the system models will be referred to simply as a "system," or if appropriate, as a "dynamical system." The output spaces Y are always taken to be linear spaces; the input spaces typically are not. Topological considerations concerning Y, F, U are deferred till later.

In any space of functions (or equivalence class of functions) defined on R, L_v, $v \in R$, denotes translation to the left by v; thus $[L_v(u)](s) = u(v+s)$. If a function space is translation invariant, L_v is a bijection. A system (Y, F, U) is time-invariant if $F(L_v u) = L_v(Fu)$ for all $u \in U$, $v \in R$; otherwise it is time-varying.

Suppose now that for each $t \in R$ a space U_t is somehow defined so that: (i) the elements u_t of U_t are equivalence classes of u's such that if $u(s) = u'(s)$, $s \leqslant t$, u and u' represent the same u_t (but not necessarily do two representatives of u_t satisfy this condition), and (ii) for all s, $t \in R$, L_{t-s} is a $1:1$ mapping of U_s into U_t that preserves all of the mathematical structure of U_s. This latter property will be possible because of the translation invariance of U. Suppose further that linear spaces Y_t can be defined satisfying the same requirements as just listed, but with respect to Y. Finally, suppose F is such that for each t it induces a mapping $\tilde{F}_t$ from U_t into Y_t. Let F_t: $U_0 \to Y_0$ be defined by

$$F_t = L_t \tilde{F}_t L_{-t}.$$

We call the mapping $t \to F_t$ the *trajectory* of (Y, F, U) with respect to U_0 and Y_0 and denote it by $(U_0, Y_0; F_t, \ t \in R)$. Sometimes only a positive trajectory is considered, in which case R is replaced by $R_+ = \{t \geqslant 0\}$. This concept can be extended at once to a compound dynamical system (Y, f, X, U). If the families of spaces $\{U_t\}$ and $\{Y_t\}$ satisfy the conditions just stated, and if each $f(x,\cdot)=F_x(\cdot)$ induces a mapping $\tilde{F}_{x,t}$ from U_t into Y_t, $t\in R$, there is a trajectory $(U_0, Y_0; F_{x,t}, t\in R)$ for each F_x. Again R may be replaced by R_+.

If the system (Y, F, U) is time-invariant, F_t is the same mapping for all t and the trajectory is trivial. The condition that F induces a mapping from U_t to Y_t of course implies that F is nonanticipative. In general, however, this condition implies more and so amounts to a further restriction one can place on the system; it may impose a restriction on the "memory" [7].

In addition to topologies for the spaces involved, including the space of mappings F_t, some more structure has to be specified before an investigation of trajectory properties becomes interesting. The introduction of more structure is tied to the question: why might one be interested in time-varying systems? Two possible answers to this question that concern us here are as follows. First, suppose there is a dominating or "master" time-invariant system that controls the dynamical system (X, F, U) in the sense that $\{F_t\}$ is determined entirely by the state of the dominating system. In particular suppose the dominating system is an autonomous linear system with state $\{x_t\}_{t\geqslant 0}$ generated by a semigroup of linear operators in some space, and x_t determines F_t continuously. Then a structured, predictable time-varying system (Y, F, U) is described (subject to the condition that the F_t's determine an F); (Y, F, U) may itself be a nonlinear system. This kind of example suggests the problem: give nontrivial conditions on (Y, F, U) such that $\{F_t\}_{t\geqslant 0}$ is generated by a, say, strongly continuous semigroup of bounded linear operators in a Banach space; the conditions will include "geometrical" conditions on the trajectory. A solution for such a problem is given in [7] which generalizes a result in [6].

Second, any nontrivial stochastic system for which sample paths are well-defined is time-varying if one regards it as a family of deterministic systems, no matter whether the statistical description is invariant under time shift or not. This fact suggests a preliminary problem: starting from a family of dynamical systems, (i.e. a compound dynamical system) introduce a probability measure in such fashion that a dynamical stochastic system results that relates naturally to the family of deterministic systems. Once this is done, the first problem stated above can be posed for the stochastic system in the form: find conditions on the underlying family of systems and on the probability measure such that the trajectory of the stochastic system with respect to some pair of spaces X_0 and Y_0 is generated by a strongly continuous semigroup. A solution to a problem of this kind is provided by the theorem of section 3.

2. Preliminaries

If U is a metric space and Y is a Banach space, the Banach space of all bounded continuous mappings from U into Y with norm $\|F\|=\sup_{u\in U}\|F(u)\|$ is denoted $\mathfrak{F}(U, Y)$. To avoid confusion subscripts are sometimes used with norms, as, for example

$$\|F\| = \|F\|_{\mathfrak{F}(U,Y)}.$$

Any compound system $S=(Y, f, X, U)$ has a *natural representation* $S_1=(Y, f_1, \mathcal{H}, U)$ defined as follows. Let $\psi(x)=F$ where F is the mapping from U into Y given by $F(u)=f(x,u)$. Then $\mathcal{H}=\psi(X)$ and $f_1(F,u)=F(u)$. The mapping ψ is called the *natural mapping*. In what follows we usually consider compound systems in terms of their natural representations. The symbol f_1 is consistently used as above, no matter what the spaces U, X and Y may be. The special case of interest here is that for which U and X are metric spaces, Y is a Banach space and f is continuous on $X \times U$ and bounded on U for each fixed x. Henceforth any compound system referred to will satisfy the conditions for this special case, either by assumption or by proof, as is appropriate. First we note that if S satisfies these conditions, so does S_1. In fact, $\psi(x)=F \in \mathcal{F}(U,Y)$ and we may regard $\mathcal{H}$ as a metric subspace of $\mathcal{F}(U,Y)$. It follows readily that f_1 is continuous on $\mathcal{H} \times U$, and it is of course bounded on U for each fixed F. It may also be noted that the map ψ: $X \to \mathcal{F}(U,Y)$ is continuous iff the mappings $f(\cdot, u)$ are equicontinuous; a sufficient condition is that U be compact [5].

In the terminology used here a system is *stochastic* if the output quantities can be interpreted as random variables, i.e. if they are measurable functions on a probability space. For the class of stochastic systems to be considered in the next section the output space Z is an L_2-space of strongly measurable Banach-space-valued functions on a probability space. In any stochastic system the output quantities, being random variables, cannot be "observed" or even approximately observed as can a deterministic quantity; so the interpretation of stochastic systems is obviously different from that of deterministic systems. However this fact, which means that in applications one is faced with problems of statistical inference, does not concern us here.

System trajectories are studied in [6] and [7]. In the first mentioned reference time-truncation projection operators are used to determine the spaces U_0 and Y_0 from U and Y; in the second the device called fitted families of normed linear spaces is used to do the same thing in a more general context. It is irrelevant for our purpose here what formalism is used to establish a trajectory. Indeed, it is sufficient to start with an *abstract trajectory* defined simply as an ordered family of mappings $(U_0, Y_0; F_t, t \in R)$, where each F_t is a mapping from U_0 into Y_0. Some additional conditions are imposed on U_0, Y_0 and the family $\{F_t\}$ later.

Remark 1. An abstract trajectory determines the system (Y, F, U), where $U=\Pi_{t \in R} U_t$, $Y=\Pi_{t \in R} Y_t$, and F: $U \to Y$ is defined by $y_t=\tilde{F}_t(u_t)$, $u_t \in U_t$, $\tilde{F}_t=L_{-t}F_tL_t$. However, this U and Y may well be "too large" to yield a useful interpretation of the system. If U and Y are only subsets of the spaces defined above, then one expects that a consistency condition must be imposed on the family of maps $\{F_t\}$ in order for them to determine a mapping F: $U \to Y$. For example, in the structure used in [7] involving fitted families of normed linear spaces, with the families of spaces $\{U_t\}$ and $\{Y_t\}$ related to U and Y as specified there, the mappings $\tilde{F}_t$ must satisfy the usually nontrivial condition: for each $u \in U$ the set

$$\bigcap_{t \in R} \{y \in Y : y \in \tilde{F}_t(u)\}$$

must be nonempty. Recall that $\tilde{F}_t(u)$ is well defined for all $u \in U$ since u is a representative of an equivalence class $u_t \in U_t$, and that $\tilde{F}_t(u)$ is an equivalence class of elements of Y.

Remark 2. As a sort of obverse to the previous remark, we note that if $(U_0, Y_0; F_t, t \in R)$ is a trajectory for (Y, F, U), it may not determine the system (Y, F, U), but rather a "smaller" system. A trajectory defined as in section 3 of [7] does determine (Y, F, U), but if Y_0 is redefined as a quotient space with a nontrivial kernel the resulting trajectory does not.

In dealing with a compound system in the natural form $(Y, f_1, \mathcal{H}, U)$ it is convenient and not very restrictive to assume that the set (metric space) $\mathcal{H}$ of mappings F is closed under translation; if $F \in \mathcal{H}$, so does $G = L_{-v} F L_v$ for all $v \in R$. Thus, even if we start with a single system (Y, F, U), it is converted to a compound system, and we deal with a family of trajectories. Under this assumption the set $\mathcal{H}$ determines a metric subspace $\mathcal{H}_0$ of $\mathcal{F}(U_0, Y_0)$ and each $F_t \in \mathcal{H}_0$, for all $F \in \mathcal{H}$, $t \in R$. It is thus appropriate to introduce families of abstract trajectories of the form $\{(U_0, Y_0; F_t^\alpha, t \in R), F_t^\alpha \in \mathcal{H}_0, \alpha \in \mathcal{Q}\}$ where $\mathcal{Q}$ is an index set and $\mathcal{H}_0 = \{F_0^\alpha, \alpha \in \mathcal{Q}\} = \{F_t^\alpha, \alpha \in \mathcal{Q}\}$ for all t. This family also forms a compound system $(Y_0, f_1, \mathcal{H}_0, U_0)$.

3. Stochastic Trajectories

Suppose there is given a family of trajectories $\mathcal{C} = \{(U_0, Y_0; F_t^\alpha, t \in R_+), F_t^\alpha \in \mathcal{H}_0\}$ restricted to R_+, where U_0 is a metric space, Y_0 is a real separable Banach space and $\mathcal{H}_0 = \{F_t^\alpha, \alpha \in \mathcal{Q}\}$, $t \in R_+$, is a metric subspace of $\mathcal{F}(U_0, Y_0)$. This family can represent the trajectories of a compound dynamical system. It is assumed the following condition is satisfied:

(1) There exists a semigroup of transformations $\{\tau_s, s \in R_+\}$ such that $F_{t+s}^\alpha = \tau_s F_t^\alpha$ for all $t, s \geq 0$ and all α.

A stochastic trajectory, a "randomization" of $\mathcal{C}$, is given by the following construction, described first heuristically. A probability measure μ with finite variance is imposed on $\mathcal{H}_0$. Each input $u \in U_0$ determines a mapping from $\mathcal{H}_0$ into Y_0; i.e., corresponding to each input there is an "output" in the form of a Y_0-valued random variable on $\mathcal{H}_0$. This is the situation for t fixed, say $t = 0$. Then as t increases and τ_t maps $\mathcal{H}_0$ into itself the measure μ is transformed, so that the same input produces an output random variable with a different probability law. Actually, instead of dealing with transformed probability spaces, we use the transformations of random variables induced by the $\{\tau_s\}$, so that the output random variables are all defined on the same probability space. A trajectory $\{\Phi_t\}_{t \geq 0}$ of mappings from inputs to stochastic outputs is thus generated. This construction is now made precise.

To start with we consider only the compound system $(Y_0, f_1, \mathcal{H}_0, U_0)$ formed by $\mathcal{C}$, without reference to trajectories per se. Let μ be a probability measure on $\mathcal{B}$ (the σ-algebra of Borel sets of $\mathcal{H}_0$) which satisfies the condition:

(2) $\int_{\mathcal{H}_0} \| F \|^2 d\mu(F) = k^2 < \infty$.

Let ρ be the mapping that carries U_0 into Y_0-valued functions on $\mathcal{H}_0$ defined by $\rho(u) = f_1(\cdot, u)$.

Lemma 1. *If μ satisfies condition* (2), *ρ has the properties*:

(i) $\rho(u)$: $\mathcal{H}_0 \to Y_0$ *is continuous for each u and hence is Borel measurable.*

(ii) $\rho(U_0) \subset L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0)$ *and is bounded.*

(iii) ρ: $U_0 \to L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0)$ *is continuous.*

Proof. (i) For any $F, F' \in \mathcal{H}_0$,

$$\|[\rho(u)](F) - [\rho(u)](F')\|_{Y_0} = \|f_1(F,u) - f_1(F',u)\|_{Y_0}$$
$$= \|F(u) - F'(u)\|_{Y_0} \leqslant \|F - F'\|_{\mathcal{F}(U_0, Y_0)}.$$

(ii) By (i), $[\rho(u)](\cdot)$ is Borel measurable. Then

$$\int_{\mathcal{H}_0} \|[\rho(u)](F)\|^2_{Y_0} d\mu(F) \leqslant \int_{\mathcal{H}_0} \|F\|^2_{\mathcal{F}(U_0, Y_0)} d\mu(F) = k^2.$$

(iii) $\|[\rho(u)](F) - [\rho(u')](F)\|^2_{Y_0} = \|F(u) - F(u')\|^2_{Y_0} \to 0$ as $u' \to u$. Also

$$\|[\rho(u)](F) - [\rho(u')](F)\|^2_{Y_0} \leqslant 4\|F\|^2_{\mathcal{F}(U_0, Y_0)},$$

so the Lebesgue dominated convergence theorem applies to yield the assertion. □

We note that since Y_0 is separable, strong, weak and Borel measurability of a mapping from $\mathcal{H}_0$ to Y_0 are equivalent. Also, if U_0 is compact, $\mathcal{F}(U_0, Y_0)$ is separable, so that $\mathcal{H}_0$, as a metric subspace of $\mathcal{F}(U_0, Y_0)$ is separable. It then follows that $L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0)$ is a separable L_2 space.

By the above lemma $(L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0), \rho, U_0)$ is a stochastic system with continuous and bounded ρ; it is called here an L_2-randomization of the system $(Y_0, f_1, \mathcal{H}_0, U_0)$. We now want a compound randomization of $(Y_0, f_1, \mathcal{H}_0, U_0)$ which keeps U_0 as input space and $L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0)$ as output space. One way to accomplish this, which fits the introductory description above, is as follows. Suppose $\{\tau_s\}$ is a parametrized set of transformations (it does not yet need to be a semigroup) on $\mathcal{H}_0$ such that for each s:

(3) τ_s: $\mathcal{H}_0 \to \mathcal{H}_0$ is Borel measurable.

(4) If $E \in \mathcal{B}$ and $\mu(E) = 0$, then $\mu\{\tau_s^{-1}(E)\} = 0$.

(5) $\sup \dfrac{\mu\{\tau_s^{-1}(E)\}}{\mu(E)} = M(s) < \infty$, where the supremum is over all $E \in \mathcal{B}$, $\mu(E) \neq 0$.

Now define operators T_s on $L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0)$ by

$$[T_s z](F) = z(\tau_s(F)), \qquad z \in L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0). \tag{1}$$

Each T_s is a continuous linear operator with linear operator norm $|T_s| = [M(s)]^{1/2}$ (see [1], VIII 5.7). The parameter space of the stochastic compound system being constructed is taken to be $\tilde{\mathcal{T}} = \{T_s\}$ regarded as a metric subspace of the Banach space of bounded linear operators on $L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0)$. Define g, a mapping from

$\tilde{\mathfrak{T}} \times U_0$ to Y_0-valued functions on $\mathcal{K}_0$, by

$$g(T_s, u) = T_s[\rho(u)] \tag{2}$$

so that $[g(T_s, u)](F) = [\rho(u)](\tau_s F)$. Clearly, g is a mapping into $L_2(\mathcal{K}_0, \mathcal{B}, \mu, Y_0)$ and $\| g(T_s, u)\|_{L_2} \leqslant k|T_s| = k[M(s)]^{1/2}$. Furthermore, if T, $T' \in \tilde{\mathfrak{T}}$ and u, $u' \in U_0$, one has

$$\| g(T, u) - g(T', u')\|_{L_2} \leqslant |T| \cdot \|\rho(u) - \rho(u')\|_{L_2} + |T - T'| \cdot \|\rho(u')\|_{L_2}.$$

From these considerations and Lemma 1 the lemma immediately below follows easily.

Lemma 2. *Let $S = (Y_0, f_1, \mathcal{K}_0, U_0)$ be a compound system with U_0 a metric space, Y_0 a real separable Banach space, $\mathcal{K}_0$ a metric subspace of $\mathfrak{F}(U_0, Y_0)$ and f_1: $\mathcal{K}_0 \times U_0 \to Y_0$ given by $f_1(F, u) = F(u)$. Let μ be a probability measure satisfying condition* (2), *and let $\{\tau_s\}$ be a parametrized set of transformations on $\mathcal{K}_0$ such that conditions* (3), (4), (5) *are satisfied. With T_s and g as defined in Equations* (1) *and* (2) *and $\tilde{\mathfrak{T}}$ as defined above*

$$\Sigma = \big(L_2(\mathcal{K}_0, \mathcal{B}, \mu, Y_0), g, \tilde{\mathfrak{T}}, U_0\big).$$

is a compound system with g continuous on $\tilde{\mathfrak{T}} \times U_0$ and bounded on U_0 for each $T \in \tilde{\mathfrak{T}}$. If $M(s) \leqslant M < \infty$ for all s and U_0 is compact, then g is uniformly continuous.

Remark 3. The system Σ has a natural representation

$$\Sigma_1 = \big(L_2(\mathcal{K}_0, \mathcal{B}, \mu, Y_0), f_1, \mathfrak{T}_0, U_0\big)$$

where $\mathfrak{T}_0 = \psi(\tilde{\mathfrak{T}})$ and f_1 is as previously defined. The elements of $\mathfrak{T}$ are the mappings $T_s \circ \rho$, and the natural mapping ψ: $\tilde{\mathfrak{T}} \to \mathfrak{F}(U_0, L_2(\mathcal{K}_0, \mathcal{B}, \mu, Y_0)$ is continuous. Put $\Phi_s := T_s \circ \rho$. If now the parameter $s \in R_+$, i.e. s again represents "time," the mapping $s \to \Phi_s$ is a stochastic trajectory $(U_0, L_2(\mathcal{K}_0, \mathcal{B}, \mu, Y_0); \Phi_s, s \in R_+)$. It is fair to call this trajectory a randomization of the family of trajectories $\mathcal{C}$ since a sample path is given by

$$\begin{aligned}[\Phi_s(u)](F_0) &= [T_s(\rho(u)](F_0) = \rho(u)(\tau_s(F_0)) \\ &= f_1(F_s, u) = F_s(u), \qquad s \in R_+, \quad F_0 \in \mathcal{K}_0.\end{aligned}$$

We note that establishing a stochastic trajectory in this fashion uses conditions (2), (3), (4), (5) but not (1). It remains to verify certain properties of this kind of stochastic trajectory.

Lemma 3. *Under conditions* (1),...,(5):

(i) $\{T_t\}_{t \geqslant 0}$ *is a semigroup of bounded linear operators on* $L_2(\mathcal{K}_0, \mathcal{B}, \mu, Y_0)$.

(ii) $\Phi_{t+s} = T_t \Phi_s$.

Proof. (i) An easy verification; using the fact $\{\tau_s\}_{s \geq 0}$ is a semigroup, shows that $\{T_t\}_{t \geq 0}$ is a semigroup. It has already been established that each T_t is a bounded linear operator.

(ii) This is immediate from (i) and the definition of Φ_t. □

Lemma 4. *Let $\{\tau_s\}_{s \geq 0}$ be a semigroup of transformations on $\mathcal{H}_0$ satisfying* (3), (4), (5). *Then, for any closed finite interval I on R_+ there is a constant $M > 0$ such that*

$$\mu(\tau_t^{-1}(E)) \leq M\mu(E)$$

for all $t \in I$ and all Borel sets E in $\mathcal{H}_0$.

Proof. The $M(t)$ defined in condition (5) is finite for all $t \geq 0$. We have

$$\begin{aligned} M(t+s) &= \sup_E \frac{\mu(\tau_{t+s}^{-1}(E))}{\mu(\tau_s^{-1}(E))} \cdot \frac{\mu(\tau_s^{-1}(E))}{\mu(E)} \\ &\leq \sup_E \frac{\mu(\tau_{t+s}^{-1}(E))}{\mu(\tau_s^{-1}(E))} \cdot \sup_E \frac{\mu(\tau_s^{-1}(E))}{\mu(E)} \\ &= \sup_E \frac{\mu(\tau_{t+s}^{-1}(E))}{\mu(\tau_s^{-1}(E))} \cdot M(s) \end{aligned}$$

where the suprema are over all Borel sets with $\mu(E) > 0$. Put $\tilde{E} = \tau_s^{-1}(E)$. Then the above inequality gives

$$\begin{aligned} M(t+s) &\leq \sup_{\tilde{E}} \frac{\mu(\tau_t^{-1}(\tilde{E}))}{\mu(\tilde{E})} \cdot M(s) \\ &\leq \sup_E \frac{\mu(\tau_t^{-1}(E))}{\mu(E)} \cdot M(s) = M(t)M(s), \end{aligned}$$

since each $\tilde{E}$ is a Borel set. Now let $\phi(t) := \log M(t)$. Then $\phi(t) \geq 0$ and is a subadditive function on R_+. By a property of subadditive functions (see [2], Theorem 7.4.1) $\phi(t)$ is bounded on any compact set. The assertion follows. □

One further condition is introduced.

(6) Each trajectory of $\mathcal{C}$ is continuous; i.e. $t \to F_t^\alpha$ is a continuous mapping from R_+ into $\mathcal{H}_0$ for each α.

Lemma 5. *If conditions* (1),...,(6) *are satisfied then $t \to \Phi_t$ is a continuous mapping from R_+ into $\mathcal{F}(U_0, L_2(\mathcal{H}_0, \mathcal{B}, \mu, Y_0))$.*

Proof. We show that $\|\Phi_{v+h} - \Phi_v\|_{\mathcal{F}} \to 0$ as $h \to 0$, where $\|\cdot\|_{\mathcal{F}}$ denotes the norm in $\mathcal{F}(U_0, L_2(\mathcal{K}_0, \mathcal{B}, \mu, Y_0))$. Note first that

$$\begin{aligned}
\|\Phi_{v+h} - \Phi_v\|_{\mathcal{F}} &= \|T_{v+h}\circ\rho - T_v\circ\rho\|_{\mathcal{F}} \\
&= \sup_{u\in U_0} \|(T_{v+h}\circ\rho - T_v\circ\rho)(u)\|_{L_2} \\
&= \sup_{u\in U_0}\left[\int_{\mathcal{K}_0} \|[T_{v+h}(\rho(u))](F_0) \right. \\
&\qquad \left. -[T_v(\rho(u))](F_0)\|_{Y_0}^2\, d\mu(F_0)\right]^{1/2} \\
&= \sup_{u\in U_0}\left[\int_{\mathcal{K}_0} \|[\rho(u)](\tau_{v+h}(F_0)) \right. \\
&\qquad \left. -[\rho(u)](\tau_v(F_0))\|_{Y_0}^2\, d\mu(F_0)\right]^{1/2} \\
&= \sup_{u\in U_0}\left[\int_{\mathcal{K}_0} \|[\tau_{v+h}(F_0)](u) - [\tau_v(F_0)](u)\|_{Y_0}^2 d\mu(F_0)\right]^{1/2} \\
&= \sup_{u\in U_0}\left[\int_{\mathcal{K}_0} \|[\tau_{v+h}(F_0) - \tau_v(F_0)](u)\|_{Y_0}^2 d\mu(F_0)\right]^{1/2} \qquad (3)
\end{aligned}$$

For any Borel set $E \subset \mathcal{K}_0$,

$$\begin{aligned}
&\int_E \|[\tau_{v+h}(F_0) - \tau_v(F_0)](u)\|_{Y_0}^2 d\mu(F_0) \\
&\quad \leqslant \int_E \|\tau_{v+h}(F_0) - \tau_v(F_0)\|_{\mathcal{F}}^2 d\mu(F_0) \\
&\quad \leqslant 2\int_E \|\tau_{v+h}(F_0)\|_{\mathcal{F}}^2 d\mu(F_0) + 2\int_E \|\tau_v(F_0)\|_{\mathcal{F}}^2\, d\mu(F_0). \qquad (4)
\end{aligned}$$

Consider

$$\int_{\{F_0:\ \|\tau_v(F_0)\|^2 \geqslant a\}} \|\tau_v(F_0)\|_{\mathcal{F}}^2 d\mu(F_0) = \int_{\{F':\ \|F'\|^2 \geqslant a\}} \|F'\|_{\mathcal{F}}^2 d\left(\mu\tau_v^{-1}\right)(F').$$

For a fixed finite interval $[t_1, t_2]$, there is $M > 0$ such that $\mu(\tau_v^{-1}(E)) \leqslant M\mu(E)$ for all Borel sets E and all $v \in [t_1, t_2]$ (by Lemma 4). Hence the integral on the right side of the equation above is dominated by

$$M\cdot\int_{\{\|F'\|^2\geq a\}}\|F'\|_{\mathcal{F}}^2\,d\mu(F'),$$

which approaches zero as $a\to\infty$. Thus, the functions $\|\tau_v(F_0)\|^2$ are uniformly integrable for $v\in[t_1,t_2]$. Consequently, by Inequality (4), the functions $\|\tau_{v+h}(F_0)-\tau_v(F_0)\|^2$ are uniformly integrable for $v\in[t_1,t_2]$ and $v+h\in[t_1,t_2]$. Since condition (6) says that $\|\tau_{v+h}(F_0)-\tau_v(F_0)\|_{\mathcal{F}}^2\to 0$ as $h\to 0$, we have

$$\lim_{h\to 0}\int_{\mathcal{H}_0}\|\tau_{v+h}(F_0)-\tau_v(F_0)\|_{\mathcal{F}}^2\,d\mu(F_0)=0.$$

for all $v\geq 0$ ($h>0$ if $v=0$). Then by Equation (3) and Inequality (4),

$$\lim_{h\to 0}\|\Phi_{v+h}-\Phi_v\|_{\mathcal{F}}=0$$

for all $v\in R_+$, where the limit is from the right if $v=0$. □

Theorem. *If conditions* (1),...,(6) *are satisfied, there exists a closed linear subspace* $\mathfrak{M}$ *of* $L_2(\mathcal{H}_0,\mathcal{B},\mu,Y_0)$ *such that the following are true.*

(i) $T_s(\mathfrak{M})\subset\mathfrak{M}$, $s\in R_+$.

(ii) *Let* T_s *restricted to* $\mathfrak{M}$ *be written* $T_s^{\mathfrak{M}}$. $\{T_s^{\mathfrak{M}}\}_{s\geq 0}$ *is a strongly continuous semigroup of bounded linear operators on* $\mathfrak{M}$.

(iii) $\Phi_{t+s}=T_t\Phi_s$.

Proof. (i) The subspace $\mathfrak{M}$ is defined as follows. Put $\mathcal{P}:=\rho(U_0)$; $\mathcal{R}:=\bigcup_{t\in R_+}T_t\mathcal{P}$; $\mathfrak{M}:=\overline{V\mathcal{R}}$ (closed linear span of $\mathcal{R}$). By Lemma 1 and the fact each T_t is a bounded linear operator on $L_2(\mathcal{H}_0,\mathcal{B},\mu,Y_0)$ it follows that $\mathfrak{M}$ is a closed linear subspace. If $g\in\mathcal{R}$, $g=T_s\rho(u)$ for some s and u. Then $T_tg=T_{t+s}\rho(u)$; so $T_t\mathcal{R}\subset\mathcal{R}$. It then follows from the linearity of T_t that $T_t(V\mathcal{R})\subset V\mathcal{R}$. Finally, since

$$\|T_tg-T_tg'\|_{L_2}\leq[M(t)]^{1/2}\cdot\|g-g'\|_{L_2},$$

$g\in\overline{V\mathcal{R}}$ implies $T_tg\in\overline{V\mathcal{R}}$.

(ii) All that remains to be proved is that $\{T_t^{\mathfrak{M}}\}_{t\geq 0}$ is strongly continuous. Suppose $g\in\mathcal{R}$, then $g=T_s(\rho(u))$ for some s and u, and

$$\|T_tg-g\|_{L_2}=\|T_{t+s}(\rho(u))-T_s(\rho(u))\|_{L_2}\to 0$$

as $t\to 0$ by Lemma 5 (stochastic trajectory continuity). This result extends to $V\mathcal{R}$ by virtue of the linearity of T_t. Finally, suppose $g\in\overline{V\mathcal{R}}$. One has

$$\|T_tg-g\|_{L_2}\leq\|T_tg-T_tg'\|_{L_2}+\|T_tg'-g'\|_{L_2}+\|g'-g\|_{L_2}$$

for any $g' \in V\mathfrak{R}$. Thus,

$$\|T_t g - g\|_{L_2} \leqslant \left[M(t)^{1/2} + 1\right] \cdot \|g - g'\|_{L_2} + \|T_t g' - g'\|_{L_2}$$
$$\leqslant [M^{1/2} + 1] \cdot \|g - g'\|_{L_2} + \|T_t g' - g'\|_{L_2}$$

for $t \leqslant$ constant. By choosing g' so that $\|g - g'\|_{L_2}$ is sufficiently small and then taking t sufficiently small, the right side of this inequality can be made less than an arbitrary $\epsilon > 0$.

(iii) This fact has already been noted. □

Remark 4. If a compound dynamical system satisfies the conditions of Proposition 7 of [7] it has trajectories generated by a strongly continuous semigroup, which will then satisfy the conditions for the Theorem with a suitable probability measure μ.

4. Example

The example chosen is simple, and covers what seems a small but reasonable class of system models. It is not quite as simple as it might be, however, because a couple of minor complications have been introduced in the specification of U_0 and Y_0 to illustrate some of the flexibility possible in the system modeling.

The space U_0 is a subset of an L_2-space, not of $L_2(-\infty, 0)$ (the L_2-space of real-valued functions with respect to Lebesgue measure on $(-\infty, 0)$), but rather of an L_2-space formed with respect to a measure that inhibits the "memory". Let ϕ: $R \to R$ be any function that satisfies the conditions: $\phi(t) = 0$, $t < 0$; $\phi(0) = 1$; ϕ is monotone nonincreasing on $[0, \infty)$; $\phi(t) \to 0$ as $t \to \infty$, and $\int_0^\infty \phi(t)\,dt = c$, $0 < c < \infty$. Let A_0 be the L_2-space formed from equivalence classes of real-valued functions f for which

$$\|f\|_0^2 := \int_0^\infty |f(-t)|^2 \phi(t)\,dt < \infty. \tag{5}$$

Take U_0 to be a bounded subset of A_0 with $\|u\|_0 \leqslant K < \infty$, regarded as a metric subspace of A_0.

The space Y_0 is $L_2(-b, 0)$ where b is a fixed positive constant. This choice implies that "observations" are not to be thought of as values of a real-valued output function at a point, but rather as segments of an output function. The norm in Y_0 is denoted $|||\cdot|||_0$.

The mappings F_t: $U_0 \to Y_0$, $t \in R$, are defined by

$$[F_t(u)](\sigma) = \int_0^\infty \int_0^\infty h(t + \sigma; v_1, v_2) u(\sigma - v_1) u(\sigma - v_2)\,dv_1 dv_2 \tag{6}$$

where $-b \leqslant \sigma \leqslant 0$, $u \in U_0$, and the kernel h of the integral operator is a real-valued function satisfying the conditions to follow. Let $\bar{v} = \sup\{t\colon \phi(t) > 0\}$; $\bar{v}$

may be ∞. Then it is required that

$$h(s; v_1, v_2) = 0 \qquad \text{if } v_1 > \bar{v} \quad \text{or} \quad v_2 > \bar{v} \tag{7}$$

and

$$\int_{-b}^{0}\int_{0}^{\bar{v}}\int_{0}^{\bar{v}} \frac{|h(t+\sigma; v_1, v_2)|^2}{\phi(b+v_1)\phi(b+v_2)} dv_1 dv_2 dt < \infty \tag{8}$$

for all $t \in R$.

Denote the integral on the left side of the Inequality (8) by $\|h_t\|_{\phi}^2$. One can readily verify, using no more than the Schwarz and triangle inequalities and the monotonicity of ϕ that:

$$|||F_t(u)|||_0 \leq \|h_t\|_{\phi}\|u\|_0^2 \leq K^2\|h_t\|_{\phi}, \tag{9}$$

$$\begin{aligned} |||F_t(u) - F_t(u')|||_0^2 &\leq \|h_t\|_{\phi}^2 \cdot \|u-u'\|_0^2(\|u\|_0^2 + \|u'\|_0^2) \\ &\leq | 2K^2\|h_t\|_{\phi} \cdot \|u-u'\|_0^2, \end{aligned} \tag{10}$$

and

$$\begin{aligned} |||F_t(u) - F_s(u)|||_0 &\leq \|h_t - h_s\|_{\phi} \cdot \|u\|_0^2 \\ &\leq K^2\|h_t - h_s\|_{\phi}. \end{aligned} \tag{11}$$

The first two of these inequalities imply that each F_t is indeed a mapping from U_0 to Y_0 and belongs to $\mathcal{F}(U_0, Y_0)$. Furthermore, each F_t is uniformly continuous.

Thus $(U_0, Y_0; F_t, t \in R)$ is a trajectory and $\mathcal{C}$, the family of all translates of this trajectory, meets the preliminary conditions laid down in section 3. Conditions (1), (3) and (6) for Theorem 1 can be satisfied by suitably restricting h, and possibly U_0. Consider the following very special case. Let

$$h(s; v_1, v_2) = w(s)g(v_1, v_2) \tag{12}$$

where,

(a) $\displaystyle\int_0^{\bar{v}}\int_0^{\bar{v}} \frac{|g(v_1, v_2)|^2}{\phi(b+v_1)\phi(b+v_2)} dv_1 dv_2$ exists,

(b) $\displaystyle\int_{-b}^{0} | w(t+\sigma)|^2 d\sigma < \infty$ for all t.

Clearly, Inequality (8) is satisfied, and hence (9), (10), and (11). Further suppose:

(c) there exists $u \in U_0$ such that

$$\int_0^{\bar{v}}\int_0^{\bar{v}} g(v_1, v_2)u(\sigma - v_1)u(\sigma - v_2)\, dv_1 dv_2 \neq 0$$

for a.e. $\sigma\in[-b,0]$,

(d) $\int_{-b}^{0}|w(t+\sigma)-w(s+\sigma)|^2 d\sigma \neq 0$ when $s\neq t$.

It then follows easily that $\|F_t-F_s\|_{\mathcal{F}(U_0,Y_0)}>0$ whenever $t\neq s$. Thus the map ξ: $R\to\mathcal{F}(U_0,Y_0)$ that carries $t\to F_t$ is injective and $\xi(R)=\mathcal{H}_0$.

We can now define a group of transformations $\{\tau_s,\ s\in R\}$ (and hence *a fortiori* a semigroup $\{\tau_s,\ s\in R_+\}$) on $\mathcal{H}_0$ by

$$\tau_s:=\xi\circ L_{-s}\circ\xi^{-1}. \tag{13}$$

Thus, $\tau_r(F_t)=F_{t+r}$. The mapping ξ is composed of maps indicated by the following chain: $t\to w_t\to h_t\to F_t$, where w_t is the element of $L_2(-b,0)$ given by $(w(t+\sigma),\ -b\leqslant\sigma\leqslant 0)$. The first of these is continuous because of the shift-continuity of $L_2(-b,0)$; the second is continuous by the definition of $\|\cdot\|_\phi$ and the fact $h_t(\sigma;v_1,v_2)=w_t(\sigma)\cdot g(v_1,v_2)$; the third is continuous by Inequality (11). Thus ξ is a continuous mapping and the trajectory continuity condition (6) is satisfied.

It remains to guarantee that τ_s is Borel measurable. This can be accomplished by requiring that $\mathcal{H}_0$ be separable. For then, since ξ is a continuous injective map from a complete separable metric space (R) into a separable metric space ($\mathcal{H}_0$), ξ^{-1} is Borel measurable by a theorem of Kuratowski (see [3], p. 22), and τ_s is a composition of a measurable mapping with continuous ones according to Equation (13). The separability of $\mathcal{H}_0$ can be achieved by requiring that U_0 be compact; for then $\mathcal{F}(U_0,Y_0)$ is separable since Y_0 is separable. This is a further restriction, as U_0 has previously only been assumed bounded. An alternative is to require that the set of w_t's be totally bounded in $L_2(-b,0)$, which also implies that $\mathcal{H}_0$ is separable.

Conditions (2), (4) and (5) for Theorem 1 involve the measure μ. For ν a Borel probability measure on R, define

$$\mu(E)=\nu(\xi^{-1}(E)),\qquad E\in\mathcal{B}. \tag{14}$$

For such a μ the second moment condition (2) becomes

$$\int_{\mathcal{H}_0}\|F\|^2 d\mu(F)=\int_R\|\xi(t)\|^2 d\nu(t)$$
$$=\int_R\left(\sup_{u\in U_0}|||[\xi(t)]u|||_0^2\right)d\nu(t).$$

This integral is dominated by,

$$\int_R K^4\|h_t\|_\phi^2 d\nu(t)$$
$$=K^4\int_0^{\bar v}\int_0^{\bar v}\frac{|g(v_1,v_2)|^2}{\phi(b+v_1)\phi(b+v_2)}dv_1dv_2\int_{-\infty}^{\infty}\int_0^{-b}|w(t+\sigma)|^2 d\sigma\, d\nu(t).$$

Thus, a sufficient condition is that

$$\int_{-\infty}^{\infty}\int_{-b}^{0} | w(t+\sigma) |^2 d\sigma\, d\nu(t) < \infty. \tag{15}$$

Note that this is satisfied for any ν if $\int_{-b}^{0}|w(t+\sigma)|^2 d\sigma$ is bounded.

Suppose the measure ν has a density p. If $p(t)\neq 0$ a.e., condition (4) is satisfied. Then,

$$\frac{\mu\left(\tau_s^{-1}(E)\right)}{\mu(E)} = \frac{\int_{\tilde{E}} p(t-s)\,dt}{\int_{\tilde{E}} p(t)\,dt}$$

where $\tilde{E} = \xi^{-1}(E)$, $E \in \mathfrak{B}$. Clearly, if there is $m(s)$ such that $p(t-s) \leqslant m(s)p(t)$ for all t, $M(s)$ exists and $M(s) \leqslant m(s)$. In this circumstance condition (5) is satisfied. A density p will have this behavior if, for example, it is continuous, not equal to zero at any t, differentiable with bounded derivative except possibly at a finite set of points, and monotonically decreasing to zero outside some finite interval at a rate not exceeding $(t)^{-n}e^{-at}$, $a > 0$.

In summary, the set of all translates of a trajectory $(U_0, Y_0;\ F_t,\ t \in R)$, with U_0 and Y_0 the L_2 spaces specified above and F_t defined by Equations (6) and (12), can be randomized by introducing a probability measure ν on R so that the conditions of Theorem 1 are satisfied. The resulting stochastic trajectory for $t \geqslant 0$ is then generated by a strongly continuous semigroup of bounded linear operators.

Remark 5. We started with a deterministic trajectory so as to make the example fit the theorem. The same thing can be done starting with a dynamical system (U, F, Y), with U and Y chosen as the bounding spaces (see [7] or [8]) for the fitted family of Banach spaces that yields the A_0 and Y_0 used here. See [8], where a similar example is treated. Proposition 6 of [7] then guarantees the deterministic trajectory is continuous if U_0 is totally bounded and h satisfies only the Inequality (8). Of course that theorem is not needed in this example because of the special properties of h.

In this example U_0 can be taken instead to be a bounded subset of $L_2(-\infty,0)$ and things go through in essentially the same way. One may then ask why U_0 was chosen to be a subset of the weighted L_2 space. The reason, perhaps not a very strong one, is that the A_0 used here belongs to a tapered fitted family [7] while $L_2(-\infty,0)$ does not. The trajectory theory is more satisfactory when a tapered fitted family is used for inputs. In particular if it is required that U_0 be compact the dynamical interpretation requires that U be a shift-invariant set of functions that determines a compact set in each A_t. Such sets are so strongly constrained as to be uninteresting if $A_t = L_2(-\infty, t)$, which is what the A_t are when $A_0 = L_2(-\infty,0)$.

Remark 6. It should be clear that the example can be extended to include n-power integral operators, $n = 1, 2, \ldots,$ for which the kernels are of the form

$$h(s; v_1, \ldots, v_n) = w(s) g(v_1, \ldots, v_n), \tag{16}$$

with only relatively trivial complications. Also, rather simple changes are required to allow the kernels h to be sums of terms of the general form of Equation (12), of not necessarily different degree. Finally, one can let the inputs u and the functions $g(v_1, \ldots, v_n)$ be vector-valued and use tensor products. Thus a fairly wide class of system models involving polynomic integral operators falls into the scheme of this example. It seems possible that known results on approximation by polynomic operators [4] and [5] could be applied to extend the range of the example, but that is not considered here.

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