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ON AXIALLY SYMMETRICAL PLATES OF VARIABLE THICKNESS

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ON AXIALLY SYMMETRICAL PLATES OF VARIABLE THICKNESS[1]

by F. Essenburg[2]

In the classical treatment of elastic isotropic plates of variable thickness, an equation governing the deflection of the middle plane is obtained by substituting the moment-curvature (i.e., the stress-strain or stress-displacement) relations appropriate for plates of uniform thickness into the differential equations of equilibrium, treating the thickness as a function of the middle plane coordinates. It should be noted that the classical formulation of plates of variable thickness neglects the effects of both transverse shear deformation and transverse normal stress and employs assumed forms for the stresses which do not satisfy the requirements for the prescribed surface tractions at the top and bottom surfaces of the plate. In this connection it is pertinent to recall the improvement of the classical theory of bending of plates of uniform thickness by E. Reissner (1,2)[3] which accounts for the effects of both transverse shear deformation and transverse normal stress, and which stipulates the satisfaction of three boundary conditions at each edge of the plate.

In a recent paper (3) a system of suitable stress-strain relations (and appropriate boundary conditions) was derived for elastic isotropic plates of variable thickness, which includes the effects of both transverse shear deformation and transverse normal stress as well as the effects of the thickness variation. In this derivation, which was carried out by means of a variational theorem due to E. Reissner (2), the assumed forms for the stresses were modified (from the forms appropriate for Reissner's theory of uniform plates) so as to meet the requirements for the prescribed surface tractions at the top and bottom surfaces of the plate. On application of these derived stress-strain relations to the problem of torsion of plates of variable thickness, exact agreement with the Saint-Venant torsion theory was obtained in two cases. However, an examination of the significance of the inclusion of the additional terms due to thickness variation in the derived stress-strain relations (as compared with stress-strain relations of Reissner's theory of uniform plates) indicated that from a practical point of view (particularly in cases where the

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3. Numbers in parentheses refer to the references listed in the Bibliography.

twisting moment is not present) the use of stress-strain relations appropriate to the Reissner theory of uniform plates seems adequate for the improvement of the classical treatment of plates of variable thickness. It should also be mentioned that for the axially symmetric case the stress-strain relations given in (3) may be obtained as the limiting case of the results given by Naghdi (4) for the axisymmetric shell of revolution.

In the present note axially symmetric plates of variable thickness are considered by employing the basic equations of the Reissner theory for uniform plates, with the thickness treated as a function of the middle plane coordinates. These equations (in polar cylindrical coordinates and under the assumptions of axial symmetry) are:

$$(rM_r)' - M_\theta = rV \qquad (rV)' = -pr \tag{1}$$

$$M_r = D\left[\beta' + \nu\frac{\beta}{r}\right] + \frac{\nu}{10(1-\nu)}h^2 p \qquad M_\theta = D\left[\frac{\beta}{r} + \nu\beta'\right] + \frac{\nu}{10(1-\nu)}h^2 p$$

$$w' = -\beta + \frac{12(1+\nu)}{5Eh}V \tag{2}$$

where $D = Eh^3/12(1-\nu^2)$ and prime denotes differentiation with respect to r. In Equations [1] and [2], M_r and M_θ, respectively the radial and circumferential bending moments, V, the radial shear stress resultant, p, the surface load, β and w, respectively the radial rotation and deflection of the middle plane, as well as h, the thickness of the plate are functions of the radial coordinate, r, only; and E and ν are Young's modulus and Poisson's ratio respectively. The coordinate system is chosen so that the deflection is positive when measured downward. While the theory given in (3) stipulates the satisfaction of three boundary conditions at each edge of the plate, it will be noted that one of these edge boundary conditions is automatically satisfied by the vanishing of both the twisting moment and circumferential rotation of the middle plane under the assumptions of axial symmetry. Thus the two edge boundary conditions consistent with the above set of equations are (i) either M_r specified or β specified, and (ii) either V specified or w specified.

$$\beta'' + \left[\frac{D'}{D} + \frac{1}{r}\right]\beta' + \left[\nu\frac{D'}{D} - \frac{1}{r}\right]\frac{\beta}{r} = \frac{V}{D} - \frac{6\nu(1+\nu)}{5Eh^3}(h^2 p)' \tag{3}$$

It should be noted that upon the neglect of transverse shear deformation, i.e., when $\beta = -w'$, the homogeneous differential equation associated with Equation

[3] reduces to the corresponding equation of the classical theory. Furthermore, as was observed by L. Föppl (5), the homogeneous part of this classical theory equation is mathematically identical to the homogeneous part of the differential equation governing the rotating disk of variable thickness. Thus the literature is rich in complementary solutions of Equation [3] for various assumed forms for the thickness. In this connection see (6,7,8) where other references may be found. The particular integral of Equation [3] is readily obtained by any of several well known methods. After substitution of the expression for β into the third of Equations [2], a single integration gives the expression for w. The bending moments (and hence the radial and circumferential components of stress) are obtained by substituting the expression for β into the first two of Equations [2].

The improvement over the classical theory provided by the present treatment is demonstrated by the results of a simple example. Let us consider a plate of outer radius a and inner radius b of which the thickness is given by $h = h_o(r/a)^m$. Let the plate be rigidly clamped at its outer radius and at the inner radius let it be clamped to a rigid circular shaft of radius b, which is loaded with an axial force P, as shown in Fig. 1. The edge boundary conditions then are

$$\beta\Big|_{r=a} = w\Big|_{r=a} = \beta\Big|_{r=b} = 0 \qquad V\Big|_{r=b} = -\frac{P}{2\pi b} \tag{4}$$

We note that the expression for V (and hence the shearing stress) is identical to that of the classical theory. The differential equation governing β is

$$r^2\beta'' + (3m+1)r\beta' + (3\nu m-1)\beta = -\frac{Pr}{2\pi D_o}\left(\frac{a}{r}\right)^{3m} \tag{5}$$

where $D_o = Eh_o^3/12(1-\nu^2)$. Since the expression for β, obtained by the solution of Equation [5] subject to appropriate boundary conditions, is identical in form to the expression for $-w'$ of the classical theory, it follows that the expression for the radial displacement as well as the expressions for the bending moments (and hence the radial and circumferential components of stress) will be identical to the predictions of the classical theory.[4] The expression for the deflection of the middle surface, w, is given by[5]

4. While for this example the expressions for the stresses and radial displacement are identical to the classical theory predictions, other examples can be cited (involving different loading and boundary conditions) in which the expressions for these quantities will differ slightly from the classical theory predictions.

5. The solution [6] (on the following page) is not valid for $\nu = 1$ $m = 2/3$ $m = 0$ in which cases the solution will contain logarithmic terms.

$$w = \frac{P a^2}{6\pi D_0 m(1-\nu)} \left\{ \frac{A}{1+\lambda-\frac{3m}{2}} \left[\left(\frac{r}{a}\right)^{1+\lambda-\frac{3m}{2}} - 1 \right] + \frac{B}{1-\lambda-\frac{3m}{2}} \left[\left(\frac{r}{a}\right)^{1-\lambda-\frac{3m}{2}} - 1 \right] - \frac{1}{2-3m} \left[\left(\frac{r}{a}\right)^{2-3m} - 1 \right] + \left(\frac{h_0}{a}\right)^2 \frac{3}{5} \left[\left(\frac{r}{a}\right)^{-m} - 1 \right] \right\} \qquad [6]$$

where

$$A = \frac{n^{2\lambda} - n^{\lambda+\frac{3m}{2}-1}}{n^{2\lambda} - 1} \qquad \lambda = \left[\frac{9}{4}m^2 - 3m\nu + 1\right]^{1/2}$$

$$B = \frac{n^{\lambda+\frac{3m}{2}-1} - 1}{n^{2\lambda} - 1} \qquad n = \frac{a}{b}$$

For the case where $m = 1$ and $\nu = 1/3$ the ratio of w to w_c (w_c being the deflection predicted by the classical theory) at $r = b$ is plotted versus h_o/a for values of n of 2, 3, and 10 in Fig. 1. The classical treatment of this case is discussed in (7). It is clear from Fig. 1 that for small values of h_o/a and large values of n the predictions of the classical theory are reliable. In fact, the modification of the classical theory prediction for w is the last term in Equation [6]. Thus as $h_o/a \to 0$ there is no improvement over the classical treatment. However, it is also clear from Fig. 1 that for large values of h_o/a and small values of n the predictions of the classical theory are substantially in error.

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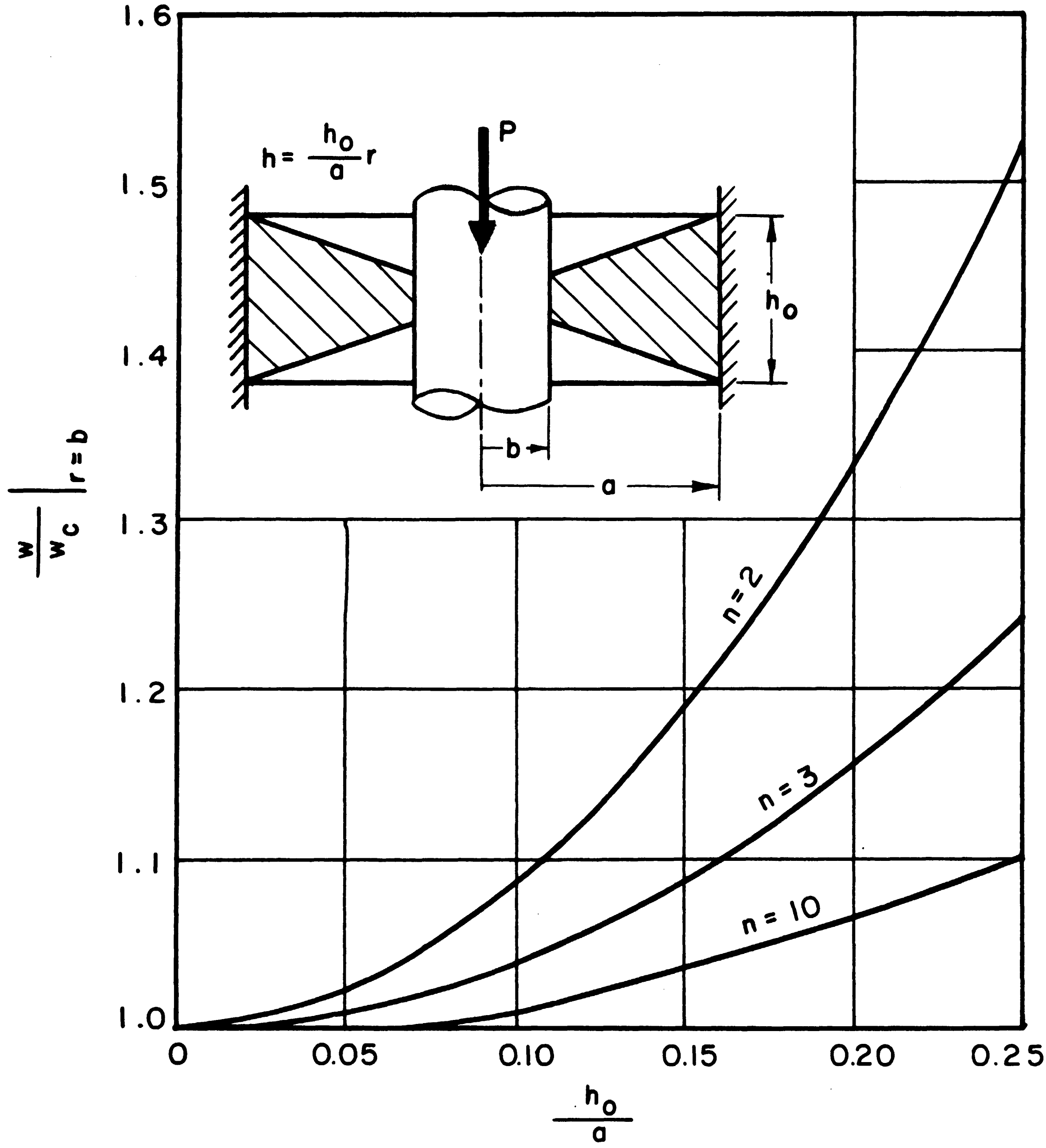

Fig. 1. Comparison of Deflection with that Predicted by the Classical Theory for Circular Plate with Linearly Varying Thickness.