Homing in a Vacuum with Minimum Fuel Consumption

EXTERNAL MEMORANDUM NO. 18

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I. INTRODUCTION AND SUMMARY

The problem treated here is that of homing with a point mass. It is assumed that all action takes place in an isolated system such that there are no adjacent bodies to "push" against, no aerodynamic forces, etc.; the only outside forces are body forces, such as gravity. The homing particle, a rocket, hereafter called the craft, will be subjected to an extra force, called thrust, gained by emitting part of its own mass. The impulse, the time integral of the magnitude of the thrust, is assumed to be proportional to the mass of the consumed part[1], called the fuel.

A second point mass, henceforth called the target, is assumed to be in the neighborhood of the first, near enough so that the body forces are the same on each. By homing is meant the application of thrust to the craft in such a manner that its position shall at some time, called the homing time,

[1] This assumption is not necessary to most of the argument which follows; most of the discussion holds under more general conditions. Since the above case is the important one for rocketry, we limit ourselves to it.

coincide with the position of the target. The motion of the craft during homing will describe a curve called the homing path.

The homing problem arises in the following manner. Information will be received that a missile, the target, is on its way and approximate information as to its path will be given. The craft will be launched to go to some point on the anticipated path. As subsequent, more accurate information is received, it will be seen that the craft is going to miss the target.

The question then arises: is it possible to apply the available thrust of the craft in such a manner that homing can be effected? We would like to know the answer to the following questions:

1. At what times (if any) is homing possible?
2. For a chosen homing time, what is the fuel requirement?
3. What is the best way to apply thrust to cause homing at a given time?
4. What is the least amount of fuel required to cause homing?

The first two questions are treated in detail in a second report[1]. The last questions are important because homing will probably be effected not in a single stage, but as a

[1] UMM-19, issued simultaneously with this report.

series of corrections, each based on more accurate information than the previous. Only a limited amount of fuel will be available, and we wish to gain the utmost in correction from it.

The study of the above questions brings up the problems of minimum paths. In this report we define: a minimum path is a homing path that requires less fuel than any other allowable[1] homing path. In general, paths in the neighborhood of minimum paths require but little more fuel than the minimum paths.

Let r be the burned fuel ratio, that is, the ratio of the weight of fuel burned to the initial gross weight. Let r'' be the upper critical fuel ratio[2], defined as the amount of fuel necessary to accelerate the craft from rest to the initial relative velocity. If r can exceed r'', then homing is always possible for a suitably chosen path.

But if $r < r''$, then homing is possible only for minimum paths and for paths near the minimum path. This is an extremely important property. For project Wizard the burned fuel ratio in the homing stage is less than the probable upper

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[1]An allowable path is a path which the craft is able to follow. This path must not, for example, require greater thrust than the craft can exert.

[2]There is also a lower critical burned fuel ratio to be taken up later. These quantities are defined more explicitly in the text.

critical ratio[1]; hence we are required to use paths near minimum paths. The proof of this property falls naturally into a later report and is not given here.

It may also be asked: what is the information needed to completely specify the homing problem? The problem is completely determined if one knows:

1. The relative position of the target with respect to the craft initially.

2. The relative velocity of the target with respect to the craft initially[2].

3. The performance of the craft; that is, its thrust-to-weight ratio, its rate of fuel consumption per unit weight, and the amount of fuel available per unit weight[3].

The first two of these will sometimes be referred to as the <u>initial conditions</u>. Homing may in many cases be achieved with less information, but the problem is indeterminate in general. Because of the importance of minimum paths and the fact that a minimum path is possible only for incoming targets, emphasis will be placed on incoming targets. A target is said to be incoming if the distance between target and craft is decreasing

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[1]The upper critical fuel ratio is a function of the situation with which the craft is confronted.

[2]It should be noted that the actual velocity of the craft (and the target) are not required. This is in contrast to the usual homing problem in air.

[3]Note that these are all ratios.

initially. The analysis of homing holds for all target situations except where specifically mentioned.

Insofar as is possible, dimensionless terms have been used. For example, the mass of a rocket is unimportant in itself. The significant quantity is the ratio of the mass of fuel to the mass of the craft. This reduces the number of variables required to state a problem.

Before starting on the problem of homing, some formulas of rocket motion are developed. It is shown also that the integral

$$\int_0^t \frac{|\bar{T}|}{M} dt ,$$

is monotonic with respect to fuel consumption; $\bar{T}$ is the thrust vector and M is the mass of the craft. This result allows a simple attack on the problem of minimizing fuel consumption.

The difference between the problem taken up in this report and the one in UMM-2 is this: in UMM-2 the final specifications of the craft are assumed. The attack here is the one which must be treated in the design of equipment for the field where it is necessary to consider the craft as it exists at the instant that homing starts; the initial craft, not the final craft, is specified.

In general, the problem is simpler for this case: it is easier to solve the equations when all values are given initially than when some are given initially, and some are given at the (unknown) time corresponding to end of burning.

The coordinate system is chosen so that the origin describes the path that the craft would follow with respect to the earth if no correction were applied. In this coordinate set the target travels in a straight line with constant speed, and the entire motion of the craft is due to the thrust. This, in effect, achieves a separation of variables; the initial conditions determine the target path, and the thrust determines the craft path.

This choice of coordinates makes most of the theorems on minimum paths "obvious". An appendix is included to show that the results are independent of the choice of coordinate system, and to show how to transform from one system to another. To eliminate Coriolis forces and centrifugal forces, we consider only coordinate sets whose axes are not rotating in inertial space[1]. That is, they are fixed in direction like the axis of a free gyroscope. Strictly speaking, the earth does not afford such a system naturally because of its rotation.

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[1]For a more detailed discussion of coordinate systems see G. Joos, Theoretical Physics, particularly Chapter 10. He calls a coordinate set such as that above an inertial frame, that is, one in which the Law of Inertia (Newton's Law) holds.

The outstanding characteristic of the entire problem is its simplicity. It is entirely a problem in relative motion, since gravity, etc., act alike on target and craft.

In this memorandum we have not taken up the problem of finding minimum paths. Fortunately they can be determined easily and quickly. The method of solution has so many other interesting and useful properties that it is put out as a separate report.

As a final word I would say that the philosophy of this report is that expressed by von Neumann and Morgenstern[1]: "Economists frequently point to much larger, more 'burning' questions, and brush aside everything which prevents them from making statements about these. The experience of more advanced sciences, for example physics, indicates that this impatience merely delays progress, including that of the treatment of the 'burning' questions."

It is our hope than an exact and painstaking analysis of the component problems will be valuable in treating the overall problem. For this reason, this report is somewhat more detailed than may seem necessary. To simplify the reading, the most important conclusions are drawn up as theorems and those

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[1] J. von Neumann and O. Morgenstern, The Theory of Games and Economic Behavior, 2nd Edition (1947), 7.

theorems vital to the arguments which follow are indicated by an asterisk (*).

Conclusions

1. For the most efficient use of fuel, the thrust must be fixed in direction during homing. This direction depends only upon the initial conditions and the homing time. Any component of thrust not in this direction is wasteful of fuel.

2. For the most efficient use of fuel in homing, one should apply full thrust in the proper direction as soon as the error is known until the error is reduced to zero, then coast in to interception. An impulse[1] represents the ideal way to burn fuel. Any delay in starting wastes fuel.

3. The problem of homing, and particularly the problem of minimum paths, can be reduced

[1] An impulse is understood to be similar to that defined by H. Lamb, Hydrodynamics, Article 11, as the product of force and time, Ft where $F \to \infty$, $t \to 0$, and the product remains finite, giving a jump in velocity. Account must be taken of the variation of mass of a rocket during the impulse. Physically an impulse is not possible, but certain properties of the case of an impulse are approximated very closely if burning time $t_1 << t_2$ the homing time.

to a two-dimensional problem by a proper choice of coordinate systems. For minimum paths, it can be reduced to a one-dimensional problem as a result of conclusion (1).

4. The <u>lower bound to fuel consumption</u>[1] would be obtained by applying thrust as an impulse at 90° to the initial line of sight if the target is incoming. This is an immediate consequence of conclusions (1), (2), (3), and the equations of homing. There is no possible way for any rocket (with the same specific impulse) to home with less fuel.

5. There are three types of initial target conditions for a given rocket:

   a. Definitely incoming: here the target is incoming, the relative velocity is small enough, the initial relative range is great enough, and the acceleration due to thrust of the craft is great enough that a minimum path is possible.

[1] This is the lower critical burned fuel ratio referred to in an earlier footnote.

b. Essentially outgoing: in this case the target is either outgoing, or is incoming but is so close in when homing starts that the craft cannot move to cause interception before the target overshoots.

c. Impossible: where the given craft cannot intercept the target (when $r < r''$ and a minimum path does not exist). By delaying the start of homing or by poor maneuvering, a type (a) target may become a type (b) or a type (c) target.

6. For a definitely incoming target (and a finite burning time) the minimum path satisfies the following relations:

   a. The fuel consumption is less than that required to accelerate the craft to the initial relative velocity.

   b. The direction of application of thrust is greater than 90° from the initial line of sight.

7. For an essentially outgoing target there is no minimum path. The lower bound to fuel consumption is given by the quantity of fuel necessary to accelerate the craft to a

velocity equal to the initial relative velocity. This lower bound corresponds to an infinite homing time; more fuel is needed for the practical case.

8. The necessary and sufficient conditions for a given homing path to be a minimum path are these:
   a. Thrust must be applied as indicated in conclusions (1) and (2).
   b. The burning time $\underline{t_1}$ must be less than the homing time $\underline{t_2}$.
   c. The thrust vector must be perpendicular to the vector of relative position at the end of burning.

9. The necessary and sufficient conditions for homing to occur without further thrust, are that the vector of relative position be parallel to the vector of relative velocity and opposite in sense.

10. There are other cases of minimum fuel consumption. If the acceleration is constant during homing, the constant being chosen a function of $\underline{t_2}$ to cause homing, the lowest fuel consumption occurs if thrust is applied at 90° from the initial line of sight. The same is true if thrust is a constant all during the homing flight, the

constant value being properly chosen to effect homing and we have seen (conclusion 4) that it is also true if the thrust is applied as an impulse. These are all examples of a more general case: if the velocity due to thrust can be expressed in the form $v = f(A, t/t_2)$ where _f_ is an increasing function of _A_ for any fixed value of _t/t_2_, then minimum fuel consumption is attained when thrust is applied at 90° from the initial line of sight. It is shown that the impulse corresponds to the case in which $f(A, t/t_2) \equiv A$, constant acceleration to the case in which $f(A, t/t_2) \equiv At/t_2$ and constant thrust to the case in which $f(A, t/t_2) \equiv -c \ln(1 - At/t_2)$. In general these are not true minima but represent relative minima subject to the above restrictions.

The minimum referred to in paragraph 8 above is a relative minimum. There are some cases when there are two relative minima; the smaller of these is also an absolute minimum. In one other case there is a single isolated path which requires less fuel than the relative minimum. With these exceptions the relative minimum is also an absolute minimum; there is no possible way that the given craft can home with less fuel in the given situation[1].

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[1]See UMM-19, p 20.

II. THE BASIC EQUATIONS AND FORMULAS

A. General Formulas

In this section most of the basic formulas and equations needed later are developed, including the equations for velocity and distance, and the fundamental equations expressing fuel consumption in terms of acceleration due to thrust.

The equation of motion of a rocket with no outside forces, derived from Newton's second law, is

(2.1) $$M(t) \frac{d\bar{u}}{dt} = \bar{T} ,$$

where M is the mass, $\bar{u}$ is the velocity vector, and $\bar{T}$ is the thrust force, of magnitude T. The thrust is gained by the rocket by the ejection of part of its own mass, the fuel, and if c is the effective velocity of the emitted particles, here considered constant[1], then

(2.2) $$T = -\dot{M}c .$$

[1] The effective gas velocity c is actually a function of many variables including rate of burning and outside or "atmospheric"

We shall use the dot over a variable to indicate its time derivative. We shall use subscripts $_0$, $_1$, $_2$, to denote values of variables at time $t_0 = 0$, t_1, t_2 where $t_0 \leqq t_1 \leqq t_2$.

Let $\underline{m}$ be the mass of fuel consumed $m = M_0 - M(t)$ and define the __burned fuel ratio__, $\underline{r}$,

$$r = \frac{m}{M_0} .$$

Then it follows that

$$T = M_0 c\dot{r}$$

and

$$M = M_0(1 - r) .$$

We can now write equation (2.1) in the form

$$\bar{a} = \frac{c\dot{r}}{1 - r}\bar{\tau} . \tag{2.3}$$

Here $\underline{\bar{a}}$ is the acceleration vector, and $\underline{\bar{\tau}}$ is the unit orientation vector of the thrust. We see that only the ratios of masses appear in this equation. It has the scalar form

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pressure. Engineers frequently use the term __"specific impulse"__, denoted by "$\underline{I}$", defined by the relation

$$I = \frac{T}{\dot{W}} .$$

It follows from the definitions that

$$Ig = c .$$

$$(2.4) \qquad \left|\frac{d\bar{v}}{dt}\right| = \frac{c\dot{r}}{1 - r} = a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

where a_x, a_y, a_z are the components of the vector of acceleration due to thrust along three mutually perpendicular axes, x, y, z.

We can integrate equation (2.4) with respect to time to obtain

$$(2.5) \qquad \int_0^t \sqrt{a_x^2 + a_y^2 + a_z^2}\, dt = - c \ln (1 - r) .$$

This is the basic equation connecting the burned fuel ratio with the acceleration due to thrust. Since r is the ratio of the mass of fuel consumed to the initial mass and $\ln (1 - r)$ is a monotonic function of r, the problem of determining paths of minimum fuel consumption may be reduced to an equivalent problem of minimizing the integral in equation (2.5).

Because of its importance we state this as a theorem.

THEOREM I. To minimize the fuel consumption required for homing it is necessary and sufficient to follow a path such that the integral $\int_0^{t_2} a\, dt$ is a minimum, where t_2 is the homing time.

Equations of particular interest are those for velocity and for distance. These are obtained from equation (2.3) by integration.

$$(2.6) \qquad \bar{u} = \int_0^t \bar{a}\, dt = \int_0^t \frac{\bar{\tau}\, c\dot{r}}{1 - r}\, dt$$

and

$$(2.7) \qquad \bar{s} = \int_0^t \bar{u}\, dt = \int_0^t \int_0^{\tau} \bar{a}(\sigma)\, d\sigma\, d\tau$$

if the rocket starts at the origin with zero velocity. The last equation can also be written

$$(2.8') \qquad \bar{s} = \int_0^t (t - \tau)\, \bar{a}(\tau)\, d\tau$$

$$= t \int_0^t \bar{a}(\tau)\, d\tau - \int_0^t \tau\, \bar{a}(\tau)\, d\tau ,$$

or

$$(2.8) \qquad \bar{s} = \bar{u}t - \bar{s}' ,$$

obtained by changing the order of integration and integrating. This defines $\underline{\bar{s}'}$. Equation (2.8') is useful since it expresses distance as a single integral.

An important case is the one when thrust is fixed in direction. Equation (2.6) can then be integrated to give the velocity, whose magnitude is

$$q = -c \ln (1 - r) . \tag{2.9}$$

This equation allows a second expression of Theorem I. We shall make frequent use of this form so we state it as a second theorem.

*THEOREM II. If one consideres only paths for which thrust is fixed in direction, then to minimize fuel consumption it is necessary and sufficient to minimize the velocity acquired from thrust.

This follows from relations (2.5) and (2.9)

$$\begin{aligned} q &= -c \ln (1 - r) \\ &= \int_0^{t_2} a \, dt . \end{aligned}$$

We see that the velocity in this case depends only on the amount of fuel burned and not on the rate of burning. The relation for distance (2.7) has several forms, among them

$$s = -c \int_0^{t} \ln (1 - r) \, dt , \tag{2.10}$$

and

$$s = c \int_0^t \frac{(t - \tau)\, \dot{r}(\tau)\, d\tau}{1 - r(\tau)} ;$$

these expressions, however, cannot be integrated until r is specified as a function of time.

B. Thrust Constant and Fixed in Direction

The case when the thrust vector is constant (fixed in direction and in magnitude) is of particular interest for two reasons:

1. It affords a good approximation to many rocket problems.
2. It is frequently the simplest from the engineering and design viewpoint.

This case is characterized by a constant rate of burning; that is,

$$\dot{r} = \text{constant} = \dot{r}_0 ,$$

(2.11)

$$r = \dot{r}_0 t .$$

The formula for velocity (2.9) becomes

(2.12)

$$q = - c \ln (1 - \dot{r}_0 t)$$

during burning. Let us define the burning time t_1 by the relations

$$\dot{r} = \dot{r}_0 = \text{constant} \qquad \text{for } t < t_1 ,$$

(2.13)

$$\dot{r} = 0 \qquad \text{for } t > t_1 .$$

Then we have

(2.14) $$q = -c \ln (1 - \dot{r}_0 t_1)$$

for all time $\underline{t}$ such that $t \geqq t_1$.

Since $\underline{r}$ is now a specified function of time, we can integrate equation (2.10) to get distance,

(2.15) $$s = \frac{c}{\dot{r}_0} \left[r + (1 - r) \ln (1 - r) \right] ,$$

while burning is continuing. This has the alternate form

(2.16) $$s = ct \left[1 + (\frac{1}{\dot{r}_0 t} - 1) \ln (1 - \dot{r}_0 t) \right] .$$

If $\underline{t_1}$ is the burning time defined by the relation (2.13), the formula for distance has the form

(2.17) $$s(t,t_1) = (t - t_1)q_1 + s_1$$

$$= q_1 t - s_1' .$$

for time $t \geqq t_1$, where

$$q_1 = q(t_1)$$

$$s_1 = s(t_1)$$

$$s_1' = q_1 t_1 - s_1 .$$

We can combine relations (2.11), (2.14), and (2.16) and substitute into equation (2.17) to get

$$(2.18) \qquad s(t,t_1) = -ct \ln(1 - \dot{r}_0 t_1) + ct_1 \left[1 + \frac{1}{\dot{r}_0 t_1} \ln(1 - \dot{r}_0 t_1)\right]$$

If we take $t_1 = t$ during burning, then equation (2.18) includes equation (2.16). Equations (2.16) and (2.18) completely describe the one-dimensional motion of a rocket whose motion is due to prescribed constant thrust for a specified time.

III. THE CONDITION OF HOMING[1]

Let us denote the position vector of the craft by $\bar{x}$, whose components will be $\underline{x}$, $\underline{y}$, and $\underline{z}$; the position vector of the target by $\bar{X} = (\underline{X},\underline{Y},\underline{Z})$; and the vector of relative position by $\bar{\xi} = (\underline{\xi},\underline{\eta},\underline{\zeta}) = \bar{X} - \bar{x}$. We shall consider only rectangular cartesian coordinate sets whose axes are not rotating in inertial space except where it is specifically stated otherwise.

It is shown in the appendix that

$$(3.1) \qquad \bar{\xi} = \bar{\xi}_0 + \dot{\bar{\xi}}_0 t - \iint^t \bar{a}\, dt^2 .$$

The vector $\underline{\bar{a}}$ is the acceleration of the craft due to thrust, $\bar{a} = \bar{T}/M$. By $\iint^t \bar{a}\, dt^2$ is meant $\int_0^t \int_0^\tau \bar{a}(\sigma)\, d\sigma\, d\tau$; we shall use this notation throughout when no confusion is likely to arise.

[1] In the introduction we defined homing: by homing is meant the application of thrust to the craft in such a manner that its position will at some time coincide with that of the target.

We can interchange the order of integration in equation (3.1) to get

$$\iint^t \bar{a}\, dt^2 = \int_0^t (t - \tau)\, \bar{a}(\tau)\, d\tau \tag{3.2}$$

$$= t\bar{u}(t) - \bar{x}'(t)\,.$$

We may consider $\underline{\bar{u}}$ as the velocity due to thrust; $\bar{x}' = \int_0^t t\, \bar{a}\, dt$ has the dimensions of a displacement.

Mathematically, the definition of homing is the following:

$$\bar{\xi}\,(t_2) = \bar{0}\,, \qquad \text{for some } t_2 \geqq 0\,; \tag{3.3'}$$

that is,

$$\bar{\xi}_0 + \dot{\bar{\xi}}_0 t_2 - t_2\bar{u}_2 + \bar{x}_2{}' = 0\,, \text{ for some } t_2 \geqq 0\,, \tag{3.3}$$

where $\bar{u}_2 = \bar{u}(t_2)$, etc.

Since $a = |\bar{a}|$ is zero for time $t > t_1$, the burning time, we have the relations

$$\begin{aligned} u_1 &= \bar{u}(t) = \bar{u}_2 = \overline{\text{constant}}\,, \\ \bar{x}_1{}' &= \bar{x}'(t) = \bar{x}_2{}' = \overline{\text{constant}}, \text{ for } t_1 \leqq t \leqq t_2\,. \end{aligned} \tag{3.4}$$

From equations (3.1), (3.2), and (3.4) we get the relation

$$\dot{\bar{\xi}}(t) = \dot{\bar{\xi}}_0 - \bar{u}_1 = \overline{\text{constant}}, \text{ for } t_1 \leqq t \leqq t_2 ;$$

the relative velocity is constant for time greater than t_1. In the same way we can prove the lemma.

LEMMA. The relative velocity $\dot{\bar{\xi}}$ is constant during any time interval during which thrust is zero.

We can now prove the Theorem of Homing.

*THEOREM III. The necessary and sufficient conditions that homing occur without further corrective thrust are these:

At the end of burning, either

(1) the vector $\bar{\xi}_1$ of relative position must be parallel to and opposite in sense to the vector $\dot{\bar{\xi}}_1$ of relative velocity; or

(2) the vector of relative position must be the zero vector.

Proof. We can write $\bar{\xi}_1$ in the form

$$(3.5) \qquad \bar{\xi}_1 = - k \dot{\bar{\xi}}_1 + \bar{\lambda}$$

where $\bar{\lambda}$ is a vector perpendicular to $\dot{\bar{\xi}}_1$ and k is a constant with the dimension of time.

Now

(3.6) $$\dot{\bar{\xi}} = \dot{\bar{\xi}}_1 , \qquad \text{for } t \geqq t_1 ,$$

by the lemma above. Hence we can integrate to get

(3.7) $$\bar{\xi} - \bar{\xi}_1 = (t - t_1)\dot{\bar{\xi}}_1 , \text{ for } t \geqq t_1 ,$$

or, combining equations (3.5), (3.6), and (3.7),

(3.8) $$\bar{\xi} = (t - t_1 - k)\, \dot{\bar{\xi}}_1 + \bar{\lambda} .$$

It is necessary and sufficient for $\bar{\xi}$ to be zero at some time $t_2 \geqq t_1$, that

(3.9') $$\bar{\lambda} = 0 , \quad k \geqq 0 ;$$

that is, that

(3.9) $$\bar{\xi}_1 = - k\, \dot{\bar{\xi}}_1 , \quad k = t_2 - t_1 \geqq 0 ;$$

the theorem is proved.

COROLLARY III. The vector of relative velocity is parallel and opposite in sense to the vector of relative position for all times t greater than the burning time t_1 and less than the homing time t_2 if the craft is on a homing course.

Proof. Substitute relation (3.9') into equation (3.7).

Expressions (3.9) are called the conditions of homing[1]. From them we see that the conditions $k > 0$ and $k = 0$ correspond respectively to Cases (1) and (2) of the theorem when burning time is greater than homing time and when homing equals burning time. The Case (2) is an extreme case, and we shall be primarily concerned with Case (1) when $t_2 > t_1$.

We can write the conditions of homing as

$$(3.10) \quad \frac{X_0 + Ut_1 - u_1 t_1 + x_1'}{U - u_1} = \frac{Y_0 + Vt_1 - v_1 t_1 + y_1'}{V - v_1} = \frac{Z_0 + Wt_1 - w_1 t_1 + z_1'}{W - w_1} = t_1 - t_2 \leqq 0 .$$

Here (X_0, Y_0, Z_0) is the initial vector of relative position, and (U, V, W) is the initial vector of relative velocity; the other terms have been defined. We preclude terms in (3.10) which have zero as both numerator and denominator. By subtracting t_1 from all terms in relation (3.10), we obtain another form

$$(3.11) \quad \frac{X_0 + x_1'}{U - u_1} = \frac{Y_0 + y_1'}{V - v_1} = \frac{Z_0 + z_1'}{W - w_1} = - t_2 < - t_1 .$$

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[1]Relation (3.3) is sometimes called the condition of homing, but it seems desirable to consider it as the definition of homing.

It is shown in the appendix that by a proper choice of orientation of axes, we can always choose our coordinate set so that

$$Y_o = Z_o = 0 \qquad (3.12)$$
$$W = 0$$

or so that

$$Z_o = 0 \qquad (3.12')$$
$$U = W = 0 .$$

There is no real loss of generality if we consider the plane problem; for three dimensions, equations are replaced by systems of equations, and the algebra becomes laborious. In the next sections we shall see that the most important cases are the plane cases. Hence, we shall be concerned almost entirely with the plane problem.

The conditions of homing then become [relations (3.10) and (3.11)]

$$\frac{X_o + (U - u_1)\, t_1 + x_1'}{U - u_1} = \frac{(V - v_1)t_1 + y_1'}{V - v_1} = t_1 - t_2 \leqq 0 \qquad (3.13)$$

$$\frac{X_o + x_1'}{U - u_1} = \frac{y_1'}{V - v_1} = - t_2 \leqq - t_1 . \qquad (3.14)$$

The conditions of homing always consist of two parts, an equality which states that the two vectors $\bar{\xi}_1$ and $\dot{\xi}_1$ are parallel and an inequality which states they are opposite in sense. The equality of expression (3.14) becomes

$$(3.15) \qquad (X_o + x_1')(V - v_1) - y_1'(U - u_1) = 0 ;$$

this is called the equation of homing.

A case of particular interest is the case when thrust is fixed in direction at an angle $\underline{\theta}$ with the $\underline{x}$-axis. If we let

$$(3.16) \qquad q = \int_0^t a\,dt = -c \ln(1 - r), \qquad s' = \int_0^t \tau a(\tau)\,d\tau$$

then

$$(3.17) \qquad u = q \cos\theta, \quad v = q \sin\theta, \quad x' = s' \cos\theta, \quad y' = s' \sin\theta .$$

The equation of homing (3.15) then has the form

$$(3.18) \qquad X_oV + s_1' V \cos\theta - q_1 X_o \sin\theta - U s_1' \sin\theta = 0 .$$

Other forms of the conditions of homing: it seems worthwhile to point out some other forms of the conditions of homing.

If we denote the direction cosines of $\bar{\xi}_1$ by (λ, μ, ν), we get the conditions of homing as

(3.19)
$$\left.\frac{d|\bar{\xi}|}{dt}\right|_{t = t_1} < 0 ,$$
$$(\lambda, \mu, \nu) = (\lambda_1, \mu_1, \nu_1), \text{ for } t \geqq t_1 .$$

In two dimensions we can write this as

(3.19')
$$\left.\frac{d|\bar{\xi}|}{dt}\right|_{t_1} < 0 ,$$
$$\alpha = \alpha_1 \qquad \text{for } t \geqq t_1 ;$$

here $\alpha = \arctan (Y - y)/(X - x)$.

They can also be expressed as

(3.20)
$$\bar{\xi}_1 \wedge \dot{\bar{\xi}}_1 = 0$$
$$\bar{\xi}_1 \cdot \dot{\bar{\xi}}_1 < 0 ,$$

where the symbol $\wedge$ denotes the vector product.

Still another form is the following: if we denote by φ the angle between $\bar{\xi}_1$ and $\dot{\bar{\xi}}_1$, the conditions of homing become

(3.21) $$\varphi = \pi .$$

Numerous other expressions can be found, but they can all be reduced to expression (3.9). As stated previously, they always consist of two parts: an expression of the equation of homing and inequality. In equation (3.21) the inequality is implicit in the definition of φ .

IV. THE FIRST FUNDAMENTAL PRINCIPLE OF HOMING WITH MINIMUM FUEL CONSUMPTION

Consider the homing problem in the following form. Let the craft start at the origin and move only under its own thrust. Let the target be originally on the x-axis at a distance X_0 from the craft and let it move with constant velocity whose components are U, V, 0 (zero). It is shown in the appendix how the coordinate set can always be chosen to put the problem into this form.

The position of the craft is given by

$$\bar{x} = \iint^{t} \bar{a}\, dt^2 = \int_0^t (t - \tau)\, \bar{a}(\tau)\, d\tau , \tag{4.1}$$

where $\bar{a} = \bar{T}/M$ is the acceleration vector resulting from thrust. The position of the target is given by

$$\begin{aligned} X &= X_0 + Ut \\ Y &= Vt \\ Z &= 0 . \end{aligned} \tag{4.2}$$

The following analysis could be carried out for any coordinate system, but the above choice makes the results and the interpretation clear.

By definition, homing is achieved if there is a time $\underline{t_2}$ such that

$$\int_0^{t_2} (t_2 - t)\, a_x \, dt = X_0 + Ut_2 \,,$$

(4.3)
$$\int_0^{t_2} (t_2 - t)\, a_y \, dt = Vt_2 \,,$$

$$\int_0^{t_2} (t_2 - t)\, a_z \, dt = 0 \,.$$

The first fundamental principle of homing with minimum fuel consumption is that thrust must be fixed in direction. We prove this as a theorem.

*THEOREM IV. If one is given any homing path for which direction of thrust is varied, one can find a better path, and in this better path, thrust is fixed in direction.

By a <u>better path</u> is meant a path such that homing is achieved at the same time, hence the same place, such that less fuel is required and such that neither the thrust nor the acceleration required exceeds that required by the given path at corresponding times.

Proof. Assume that a homing path be given for which angle of thrust varies. The homing time t_2 will be specified, and the functions a_x, a_y, a_z, which describe the path. They will satisfy equations (4.3).

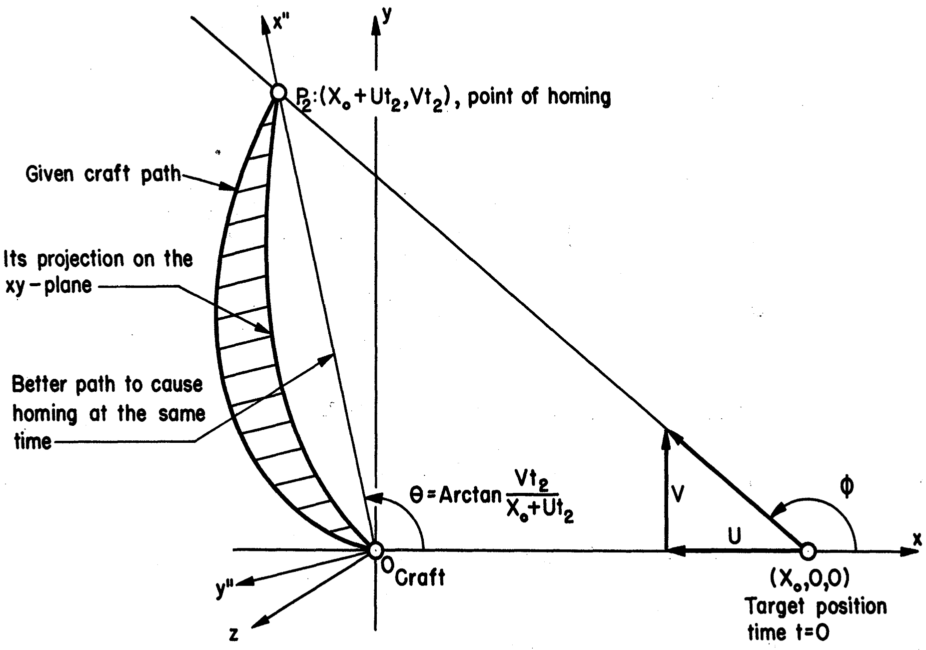

Figure I. Homing paths

Let us rotate coordinates through the angle

$$\theta = \arctan \frac{Vt_2}{X_0 + Ut_2}$$

so that the point of interception is on the new $\underline{x}''$-axis (see Figure I).

The result is now apparent from Figure I. It is required that the craft move from $\underline{0}$ to $\underline{P_2}$ under its own thrust. If thrust is applied with a component not along $\underline{OP_2}$, the craft will move off the line $\underline{OP_2}$ and more thrust will be required later to bring it back onto the line. Both components were wasted; since thrust represents fuel, fuel was wasted. The straight line represents a better path.

The analytic proof follows. We have

$$
\begin{aligned}
x'' &= x \cos \theta + y \sin \theta \\
y'' &= - x \sin \theta + y \cos \theta \\
z'' &= z ,
\end{aligned}
\tag{4.5}
$$

with corresponding expressions for $\underline{X}$, $\underline{Y}$, $\underline{Z}$ and $\underline{a_x}$, $\underline{a_y}$, $\underline{a_z}$. The conditions of homing (4.3) then become

$$
\begin{aligned}
\int_0^{t_2} (t_2 - t)\, a_x''\, dt &= X_2'' = \sqrt{(X_0 + Ut_2)^2 + (Vt_2)^2} , \\
\int_0^{t_2} (t_2 - t)\, a_y''\, dt &= 0 , \\
\int_0^{t_2} (t_2 - t)\, a_z''\, dt &= 0 ,
\end{aligned}
\tag{4.6}
$$

and a_y'' and a_z'' are not both identically zero since thrust is not fixed in direction.

Now consider thrust applied so that the accelerations are as follows:

$$(4.7)\qquad \begin{aligned} \alpha_x'' &= \sqrt{a_x''^2 + a_y''^2 + a_z''^2} \\ \alpha_y'' &= 0 \\ \alpha_z'' &= 0 . \end{aligned}$$

This set of accelerations satisfies the last two of equations (4.6) but not the first, since

$$(4.8)\qquad \int_0^{t_2} (t_2 - t)\, \alpha_x''\, dt > \int_0^{t_2} (t_2 - t)\, a_x''\, dt = X_2'' .$$

Now let us reduce the burning time as follows: choose the burning time t_1'' to satisfy the equation

$$(4.9)\qquad \int_0^{t_1''} (t_2 - t)\, \alpha_x''\, dx = X_2'' .$$

The accelerations (α_x'', α_y'', α_z'') satisfy the conditions of homing with burning time $t_1'' < t_1$ the original burning time.

But the thrust is identical in magnitude for the two cases for $t < t_1''$; therefore the accelerations are the same in magnitude at corresponding times. We have decreased the fuel consumption by decreasing the burning time.

Therefore we have found a better path, and in this better path, thrust is fixed in direction.

Many conclusions can be drawn from this; let us summarize some of the more important ones.

COROLLARY IV.1. For a minimum path, the angle of thrust is fixed.

COROLLARY IV.2. The study of minimum paths is a plane problem.

By our choice of coordinates the target motion is constrained to a plane[1]. By the theorem above and by Corollary IV.1, $a_z = 0$ for a minimum path. Therefore the craft motion is also confined to the xy-plane.

COROLLARY IV.3. The study of minimum paths can be treated as a linear problem.

By Corollary IV.1, thrust is fixed in direction for a minimum path. In the coordinate set used in this section

[1]The plane is moving with respect to the earth under the acceleration of the earth's gravitational field. The plane also rotates with respect to the earth with an angular velocity equal to the negative of the earth's angular velocity in inertial space.

a minimum path is a straight line. Hence we can drop subscripts in equation (4.9) and write the equation of homing as

$$(4.10) \qquad \int_0^{t_2} (t_2 - t)\, a\, dt = \int_0^{t_1} (t_2 - t)\, a\, dt = X_2 \;;$$

we treat $\underline{t_2}$ as a parameter and $\underline{X_2}$ is a function of $\underline{t_2}$.

*COROLLARY IV.4. By a proper choice of coordinate set the problem of determining minimum paths is reduced to the problem of effecting homing with minimum velocity.

The integral to be minimized is

$$(4.11) \qquad \int_0^{t_2} a\, dt = q_2$$

$$= -c \ln (1 - r) \,.$$

V. THE SECOND FUNDAMENTAL PRINCIPLE OF HOMING WITH MINIMUM FUEL CONSUMPTION

We have just seen that in order to minimize fuel consumption thrust must be fixed in direction; consequently the problem of minimum paths was shown to be a linear problem. For a chosen homing time, the problem now is to determine the best way to move along a straight line in inertial space.

The equation of homing can be written (see the last paragraph of Section IV)

$$\int_0^{t_1} (t_2 - t)\, a\, dt = D_2 \, . \tag{5.1}$$

Here $D_2 = \sqrt{(X_0 + Ut_2)^2 + (Vt_2)^2}$, and $\underline{a}$ is the magnitude of $\underline{\bar{a}}$, the vector of acceleration due to thrust. The thrust must be directed to accelerate the craft in the direction of $\theta = \arctan Vt_2/(X_0 + Ut_2)$. We must choose $\underline{a}$ to satisfy equation (5.1) and to minimize the integral

(5.2) $$\int_0^{t_2} a\,dt = q_2$$

in equation (4.11).

*THEOREM V. If the homing time is chosen, the best way to apply thrust is this: apply full thrust in the proper direction until the right velocity is reached, then coast to interception.

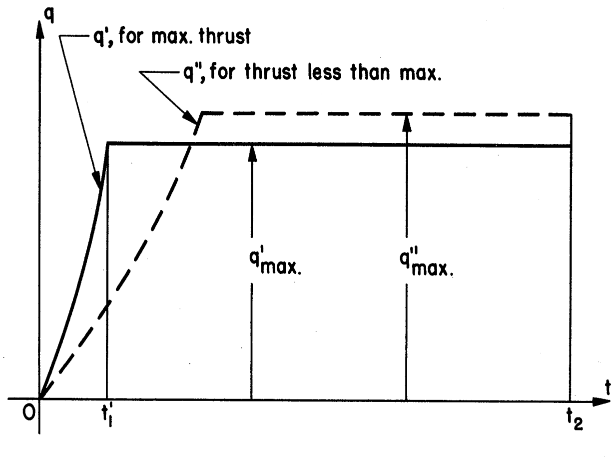

Figure II. Velocity curves for acceleration a' and a''

Proof. Consider a', the maximum acceleration which the thrust can effect; it is a function of time. Let q' be the velocity defined if the acceleration is applied for a time t_1', then the craft is allowed to coast to interception.

Let q'' be the velocity defined by any other acceleration a'' (see Figure II). Now $a'' < a'$ for part of the time interval $(0, t_1')$. Hence $q''(t_1') < q'(t_1')$.

Now if q'' is to cause homing at the time t_2 it must satisfy the relation

$$\int_0^{t_2} q'' \, dt = D_2$$

$$= \int_0^{t_2} q' \, dt \, .$$

But

$$\int_0^{t_1'} q'' \, dt < \int_0^{t_1'} q' \, dt \, ;$$

hence

$$\int_{t_1'}^{t_2} q'' \, dt > \int_{t_1'}^{t_2} q' \, dt$$

$$= q_2' \, (t_2 - t_1) \, .$$

Hence

$$q''_{max} > q'_{max} = q_2' \, . \tag{5.3}$$

This is obvious in Figure II if we interpret acceleration as the slope, velocity as the ordinate, and distance as the area under the curve.

Therefore, the path defined by a' requires less fuel.

This theorem has several interesting corollaries.

COROLLARY V.1. Any delay in the start of homing is wasteful of fuel.

This is a direct consequence of the theorem since the curve for q" lies below the above curve for q' when $t < t_1$. Hence it must rise above it later as in Figure II. This represents greater final velocity, hence greater fuel consumption.

COROLLARY V.2. Thurst applied as an impulse is most economical of fuel.

Proof. If thrust is applied as an impulse, q reaches its maximum value $q_2 = D_2/t_2$ at time $t = 0$ (see Figure III) so that the q-curve extends to the q-axis. Now if the thrust a" is finite, the curve for velocity q" defined by it lies below the line $q = D_2/t_2$ in a neighborhood of $t = 0$. Since the area under each curve must be D_2, the latter curve must rise over the line $q = D_2/t_2$ later; hence this path requires more fuel.

The analytic proof follows. It consists of two parts, the proof that for an impulse $q_2 = D_2/t_2$, and the proof that if the acceleration has an upper bound, then $q_2 > D_2/t_2$.

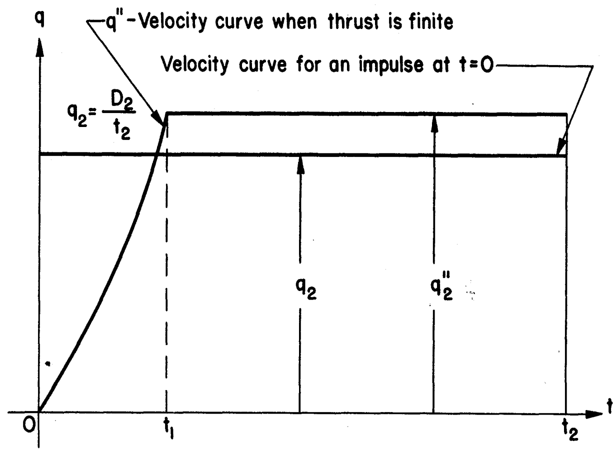

Figure III. Velocity curves

We define the impulse as the limiting case when the burning time approaches zero[1]. This implies that the acceleration becomes infinite. We characterize the other cases by finite acceleration[2], that is, by finite thrust.

---

[1]See footnote, p. 7.

[2]These characterizations seem to emphasize the important properties of the two cases. The two categories are not all-inclusive, however; for example, if the acceleration a be defined by

Let us expand equation (5.1)

$$(5.1') \qquad q_2 t_2 - \int_0^{t_1} t\, a\, dt = D_2 .$$

From the results of the preceding chapter we consider only $a \geqq 0$. We have then

$$(5.4) \qquad q_2 \geqq D_2/t_2 .$$

Now

$$0 \leqq \int_0^{t_1} t\, a\, dt \qquad (= s_1')$$

$$\leqq t_1 \int_0^{t_1} a\, dt ,$$

or

$$(5.5) \qquad s_1' \leqq t_1 q_2 .$$

$$a = \begin{cases} a_1, & a_1 \text{ constant, for } t < \dfrac{D_2}{a_1 t_2} \\ \dfrac{D_2^2}{a_1 t_2^2} \dfrac{1}{(t_2 - \frac{D_2}{a t_2})^2}, & \text{for } t > \dfrac{D_2}{a_1 t_2} , \end{cases}$$

then substitution into equation (5.1) shows that a satisfies the equation of homing for all values of $a_1 > D_2/t_2$, including the case when a_1 becomes infinite; yet the burning time t_1 is always equal to the homing time t_2. Such systems do not seem practical and we shall not consider them further.

Hence

$$q_2 \leqq \frac{D_2}{t_2} + \frac{t_1 q_2}{t_2} . \qquad (5.6)$$

If we combine (5.4) and (5.6) to get

$$\frac{D_2}{t_2} \leqq q_2 \leqq \frac{D_2}{t_2 - t_1} , \qquad (5.7)$$

it follows that

$$\lim_{t_1 \to 0} q_2 = \frac{D_2}{t_2} \qquad (5.8)$$

The first part of the theorem is proved. We see from relations (5.5) and (5.8) that

$$\lim_{t_1 \to 0} s_1' = 0 . \qquad (5.9)$$

On the other hand, consider the case when the acceleration <u>a</u> is finite. Let <u>a"</u> be an upper bound to <u>a</u>. Let <u>q</u> be $\int_0^t a \, dt$, and define <u>t_1''</u> and <u>q_1''</u> by the relations

(5.10)

$$\int_0^{t_1''} (t_2 - t) \, a'' \, dt = D_2$$

$$q'' = a''t , \qquad \text{for } t < t_1''$$

$$q_1'' = a_1''t_1'' \qquad (\text{not } a_1''t_1) .$$

Now

$$\int_0^{t_2} q \, dt = \int_0^{t_2} q'' \, dt \quad (= D_2) .$$

But

$$q \leqq q'' \qquad \text{for } t < t_1''$$

so that

$$\int_0^{t_1''} q \, dt \leqq \int_0^{t_1''} q'' \, dt$$

and

$$\int_{t_1''}^{t_2} q \, dt \geqq \int_{t_1''}^{t_2} q'' \, dt .$$

Since the maximum value $\underline{q_2}$ of the integrand is greater than or equal to its average value, it follows that

$$q_2 \geqq q_2''$$

$$= \frac{D_2}{t_2 - \frac{t_1''}{2}}$$

or

$$q_2 \geqq \frac{D_2}{t_2 - \frac{D_2}{2a''t_2}} > \frac{D_2}{t_2} .$$

The last relations come directly from the definitions in relations (5.10). By Theorem II, the case of the impulse requires less fuel.

The proof is complete.

There now remains but one problem in the determination of the lower bound to fuel consumption, the determination of the best homing time. That is taken up in the next section.

VI. THE LOWER BOUND TO FUEL CONSUMPTION

We have just seen that if the homing time is chosen, the best way to apply thrust is to make it as large as possible during the early moments of homing. This leads to the impulse as a limit as the burning time $\underline{t_1}$ approaches zero. The only remaining problem is to determine the proper angle of thrust[1] $\underline{\theta}$ for the impulse, or to choose the best homing time since the two are connected by the formula

$$\tan\theta = \frac{Vt_2}{X_0 + Ut_2} . \tag{6.1}$$

We state the results as two theorems, one for incoming targets ($U < 0$) and the other for outgoing targets ($U > 0$).

THEOREM VI A: For an incoming target the lower bound of fuel is obtained if thrust is applied as an impulse normal to the initial line of sight.

---

[1]Since we are now working in two dimensions, a single angle $\underline{\theta}$ will specify the direction of thrust: we call $\underline{\theta}$ the angle of thrust.

Proof. Assume the target is initially at the point (X_o,0) with velocity components (U,V) where U < 0 (see Figure IV). Consider an impulse such that it gives

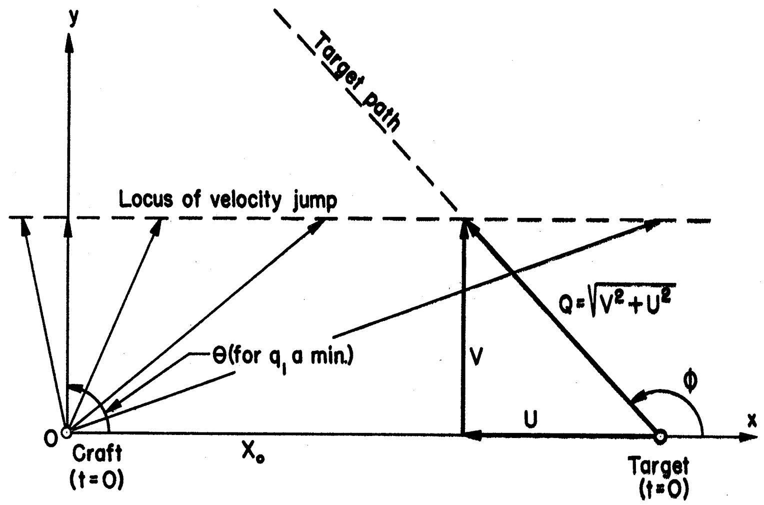

Figure IV. Family of velocities due to impulse which will cause interception.

a velocity jump of magnitude q making an angle θ with the x-axis. The equation of homing

(3.18) $$X_oV + s_1'\,V\cos\theta - q_1X_o\sin\theta - Us_1'\sin\theta = 0$$

becomes

(6.2) $$V - q_1\sin\theta = 0$$

since $s_1' = 0$ for an impulse and $X_o \neq 0$.

We wish to minimize

$$q_1 = \frac{V}{\sin \theta} \tag{6.2'}$$

as a function of $\underline{\theta}$. This has the obvious minimum for $\theta = 90^\circ$:

$$q_{1min} = V . \tag{6.3}$$

The proof is complete.

The graphic proof is so simple and enlightening that we include it, see Figure IV: consider an impulse such that the velocity jump $\underline{q_1}$ satisfies the relation

$$q_1 \sin \theta - V = 0 . \tag{6.2}$$

From the graph we see that q_1 is smaller for $\theta = 90^\circ$ (and that all conditions of homing are satisfied since $\dot{Y}_1 = \dot{y}_1$, $Y_1 = y_1$, $\dot{X}_1 - \dot{x}_1 < 0$, $X_1 - x_1 > 0$).

The corresponding fuel consumption is

$$r' = 1 - e^{-\frac{V}{c}} ; \tag{6.4}$$

this is the <u>lower bound to fuel consumption</u>.

It seems worthwhile at this point to review the chain of reasoning for this important result. We have shown:

1. If a homing path is given for which angle of thrust is varied, one can determine directly a better path for

which angle of thrust is constant. Consequently a minimum path must have thrust fixed in direction.

2. If the angle of thrust is fixed, the best way to apply thrust is to apply thrust as high as possible early, then coast; the impulse is the ideal.

3. For an incoming target, the best direction for an impulse is at right angles to the initial line of sight.

Thus, there is no possible way to home which would require less fuel.

For the outgoing target, we have

THEOREM VI B: Against an outgoing target, the lower bound to fuel consumption is

$$r'' = 1 - e^{-\frac{Q}{c}} \tag{6.5}$$

where $Q^2 = U^2 + V^2$.

Proof. Let us combine the inequality

$$\cos \varphi < \cos \theta \tag{6.6}$$

(see Figure IV) with the equation of homing for an impulse (6.2). Now $\varphi \leqq 90^\circ$, hence

$$\cos \varphi \geqq 0 .$$

We can rewrite equation (6.2) as

$$(6.7) \qquad q_1 = \frac{V}{\sqrt{(1 - \cos^2 \theta)}} .$$

With relation (6.6) this gives the greatest lower bound

$$(6.8) \qquad \text{glb.}\ (q_1) = \frac{V}{\sqrt{(1 - \cos^2 \varphi)}} = Q$$

and we can come arbitrarily close to this limit. The corresponding lower bound to fuel consumption for an outgoing target is

$$(6.9) \qquad r'' = 1 - e^{-\frac{Q}{c}} .$$

There are several interesting consequences.

LEMMA. As the angle of thrust $\underline{\theta}$ approaches the angle $\underline{\varphi}$ between the initial line of sight and the initial velocity vector, the homing time becomes infinite.

We can solve equations (6.1) and (6.2) for $\underline{t_2}$ when thrust is fixed in direction

$$(6.10) \qquad t_2 = \frac{X_0 \sin \theta}{Q \sin (\varphi - \theta)} .$$

The denominator goes to zero as θ approaches $\underline{\phi}$ and the numerator remains finite. From this we get an important corollary to the theorem.

COROLLARY VI.1. For practical purposes, more fuel than the lower bound $r'' = 1 - e^{-\frac{Q}{c}}$ is required for homing against an outgoing target.

Proof. The limit is approached only as $\underline{\theta}$ approaches $\underline{\phi}$. In this case the homing time becomes infinite (equation 6.10): the target would have completed its mission before homing occurred.

Two other corollaries can be proved. However, they involve considerable manipulation and they follow from other considerations as obvious conclusions. Hence we state them here and indicate a proof. A different proof is given later[1].

COROLLARY VI.2. If thrust is not infinitesimal, then the lower bound to fuel consumption, $r'' = 1 - e^{-\frac{Q}{c}}$ for homing against an outgoing target is independent of the thrust function.

That is, if the thrust is finite, the lower bound approached as $\underline{t_2}$ becomes infinite does not de end upon the thrust function.

[1] UMM-19, Section X.

That can be proved by developing the relations

$$\frac{D_2}{t_2} \leq q_2 \leq \frac{D_2}{t_2 - t_1} \leq \frac{X_0 + Wt_2}{t_2 - t_1}$$

and using the relation that as $\underline{t_2}$ becomes infinite, $\underline{t_1}$ remains bounded if $\underline{T}$ is not infinitesimal.

COROLLARY VI.3. If the fuel consumption exceeds $r'' = 1 - e^{-\frac{Q}{c}}$, then homing is always possible.

Inspection will show that we have not used the relation $\varphi \leq 90^\circ$ in the previous proof and that the results hold independent of φ. The corollary follows[1].

So we see that it is always possible to home if $r > r''$. It is an important property of minimum paths that only for paths near them is it possible to home with less.

[1]These two corollaries can also be proved directly from considerations on the equation of homing

$$X_0V + s_1' V \cos \theta - q_1 X_0 \sin \theta - s_1' U \sin \theta = 0 \tag{3.18}$$

but the proof involves many details to make it complete.

VII. THE NECESSARY AND SUFFICIENT CONDITIONS FOR A PATH OF MINIMUM FUEL CONSUMPTION

In this section are developed the necessary and sufficient conditions which a path must satisfy to be a path of minimum fuel consumption.

The proof is long and detailed. Hence we state the principal result at the beginning as a theorem and put other direct results in the next section.

*THEOREM VII. The necessary and sufficient conditions for a path to be a path of minimum fuel consumption are:

(1) The first two fundamental principles of homing with minimum fuel consumption (of Sections IV and V) must be satisfied; that is, (a) thrust must be fixed in direction and (b) thrust must be as large as possible during the early stage and then as small as possible.

(2) It must satisfy the conditions of homing with $t_1 < t_2$.

(3) The vector of relative position at end of burning must be perpendicular to the thrust vector (see Figure V).

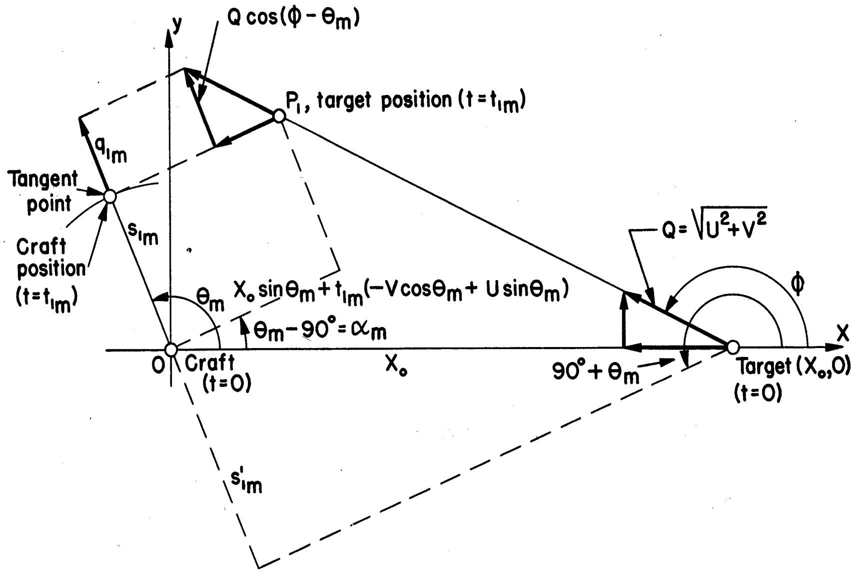

Figure V. Conditions for a minimum path

Proof. We have established that for a chosen homing time, conditions (1) must be satisfied to minimize fuel consumption. The remaining problem is to establish conditions on

t_2, or what is equivalent, on θ since they are connected by the relation (4.4)

$$\theta = \arctan \frac{Vt_2}{X_0 + Ut_2} . \tag{7.1}$$

By the above principles, the unit burning rate $\dot{r}$ and the unit fuel consumption r are given functions of time for $t < t_1$. Hence the acceleration

$$a = \frac{c\dot{r}}{1 - r} \tag{7.2}$$

is also a known function of time.

We must determine the angle of thrust θ to minimize fuel consumption. Consider the equation of homing (3.18),

$$X_0 V + s_1' V \cos\theta - q_1 X_0 \sin\theta - U s_1' \sin\theta = 0 , \tag{7.3}$$

This is a function of the variables s_1', q_1, and θ (and of the known parameters and functions $\dot{r}(t)$, c, U, V and X_0). Hence we can write it as

$$F(s_1', q_1, \theta) = 0 . \tag{7.4}$$

Assume that $\dot{r}$ (or that the thrust) is not zero for any time $t < t_1$. Under the above assumptions (that $\dot{r}(t)$ is given and that thrust is fixed in direction) it is obvious from the definitions $\left(a = \frac{c\dot{r}}{1 - r}\right)$,

$$(7.5) \qquad q_1 = \int_0^{t_1} a\, dt , \qquad s_1 = \int_0^{t_1} (t_1 - t)\, a\, dt ,$$

$$s_1' = \int_0^{t_1} t\, a\, dt , \qquad r_1 = \int_0^{t_1} \dot{r}\, dt ,$$

that q_1, s_1' and r_1 are all strictly increasing functions of t_1. Hence, implicit function relations are satisfied; all are determined as soon as one is chosen; and to minimize one of them is to minimize them all. We find it simplest to minimize q_1.

If homing is to be effected, equation (7.3) must be satisfied. Define the total partial derivative,

$$(7.6) \qquad \frac{\delta F}{\delta q_1} = \frac{\partial F}{\partial q_1} + \frac{\partial F}{\partial s_1'} \frac{ds_1'(q_1)}{dq_1}$$

Then so long as $\delta F/\delta q_1 \neq 0$, equation (7.3) or (7.4) then defines q_1 as an implicit function of θ; we can write

$$\frac{DF}{D\theta} = \frac{\partial F}{\partial \theta} + \frac{\delta F}{\delta q_1} \frac{dq_1}{d\theta} = 0 ,$$

since equation (7.4) must be treated as an identity in θ, $q_1(\theta)$; hence

$$(7.7) \qquad \frac{dq_1}{d\theta} = - \frac{\partial F}{\partial \theta} \div \frac{\delta F}{\delta q_1} .$$

For our case we can (and will later) show that $\partial F/\partial q_1$ is bounded; hence, a first condition for $\underline{q_1}$ to have a minimum is

(7.8) $$\frac{\partial F}{\partial \theta} = 0 .$$

Substituting from (7.3), this becomes

(7.9) $$s_1' V \sin\theta + q_1 X_0 \cos\theta + U s_1' \cos\theta = 0 .$$

Now define $\underline{\alpha}$ by the relation

(7.10) $$\tan\alpha = \frac{s_1' V}{q_1 X_0 + s_1' U} .$$

We can use the conditions of homing to show that

(7.11) $$\frac{s_1' V}{q_1 X_0 + s_1' U} = \frac{V t_1 - s_1 \sin\theta}{X_0 + U t_1 - s_1 \cos\theta}$$

$$= \frac{Y_1 - y_1}{X_1 - x_1} .$$

Hence $\underline{\alpha}$ is the angle made by the vector of relative position at the end of burning with the $\underline{x}$-axis (see Figure V).

From equation (7.9) we get

(7.12) $$\tan\theta = - \frac{q_1 X_0 + U s_1'}{s_1' V} .$$

Comparing this with equation (7.10), we see that for a minimum path it is necessary that the vector of relative position at end of burning be perpendicular to the angle of thrust. Conclusion (3) is proved.

We must prove that the conditions of the theorem are sufficient; that is, that equation (7.9) and the conditions of homing lead to a minimum, not to a maximum nor an inflection point when $t_1 < t_2$.

We shall need the relation

$$\frac{ds_1'}{dq_1} = t_1 \,. \tag{7.13}$$

We substitute this relation into equation (7.7) to get

$$\frac{dq_1}{d\theta} = - \frac{q_1 X_o \cos\theta + s_1' V \sin\theta + s_1' U \cos\theta}{X_o \sin\theta + t_1(-V\cos\theta + U\sin\theta)} \,. \tag{7.14}$$

Let us denote the values associated with a minimum path by subscript m. If we expand the numerator of the right hand side of equation (7.14) in powers of $\theta - \theta_m$ we find that θ_m is a simple zero, and the numerator is positive for $\theta < \theta_m$ (θ near θ_m). Hence, taking into account the minus sign in front, we see that the denominator must be positive for a minimum; we must have

$$X_o \sin\theta_m + t_1(-V\cos\theta_m + U\sin\theta_m) > 0 \,. \tag{7.15}$$

This we can write as

$$\frac{X_o + Ut_1}{Vt_1} > \cot\theta_m \,. \tag{7.16}$$

Now if we express the position of the target in polar form

$$S = \sqrt{X^2 + Y^2} \tag{7.17}$$

$$\Theta = \arctan \frac{X}{Y} = \arctan \frac{Vt}{X_o + Ut} ,$$

we see that Θ is an increasing function of time (since V and X_o are positive). We can write (7.16) as

$$\cot \Theta (t_1) > \cot \Theta (t_2) . \tag{7.18}$$

Hence

$$t_{1m} < t_{2m} .$$

Inspection of the conditions of homing with thrust fixed in direction will show that the inequality, which is the condition that the vector of relative position and relative velocity be opposite in sense, can be expressed as

$$\theta \geqq \Theta_1 .$$

Hence the condition (7.16) is satisfied for all homing paths such that $t_1 < t_2$. Hence we have proved that conclusion (2) is necessary and that conlcusions (2) and (3) are also sufficient.

Let us summarize the proof: the equations of motion when thrust is a specified function of time[1], fixed in

[1]This specified function is the upper bound.

direction and not zero for $t < t_1$ define $\underline{q_1}$, $\underline{s_1}$, $\underline{s_1}'$, $\underline{r}$ and $\underline{t_1}$ as strictly increasing functions of one another; hence, there is an implicit function relation between any two and if conditions are chosen to minimize one, all are minimized. The equation of homing in turn defines $\underline{q_1}$ as a function of $\underline{\theta}$. We find $dq_1/d\theta$. For $\underline{q_1}$ a minimum $dq_1/d\theta$ must vanish. This is shown to be the condition that thrust be perpendicular to the vector of relative position at end of burning. The numerator, of $dq_1/d\theta$ has a simple zero so the second condition for a minimum is that the denominator be positive. This condition is satisfied by the conditions of homing whenever the burning time is less than the homing time.

No real trouble is experienced if we do not assume $\underline{a}$ to be non-zero, since $\underline{q_1}$ and $\underline{s_1}'$ are still strictly increasing functions of one another.

The proof is complete.

VIII. FURTHER PROPERTIES OF PATHS OF MINIMUM FUEL CONSUMPTION

The following properties of minimum paths can be deduced at once from the results of the last section and are essentially corollaries to Theorem VII. Because of their importance we include them as theorems.

THEOREM VIII.A. The angle of thrust for a minimum path is greater than or equal to 90° (from the initial line of sight). It is greater than 90° if thrust is finite and equal to 90° for an impulse.

Proof. Let us eliminate q_1 between equations (7.3) and (7.9). We get

(8.1) $$\cos \theta_m = - \frac{s_1'}{X_0} ;$$

θ_m is the angle of thrust for a minimum path. Now

$$s_1' = \int_0^{t_1} t \; a \; dt.$$

$$> 0$$

if thrust is finite[1]. For the case of an impulse we saw that

(5.9) $$\lim_{t_1 \to 0} s_1' = 0 .$$

Hence

$$\theta_m > 90^\circ \qquad \text{if thrust is finite}$$

(8.2)

$$\theta_m = 90^\circ \qquad \text{for homing with an impulse.}$$

COROLLARY VIII.1: There can be no minimum path against an outgoing target.

One condition of homing is

$$\theta < \varphi .$$

The condition for a target to be outgoing is $\varphi \leqq 90^\circ$. These two conditions are incompatible with equation (8.1).

We defined $r'' = 1 - e^{-\frac{Q}{c}}$ and we saw (Corollary VI.3) that if the allowed unit fuel consumption r exceeded r'' we could effect homing.

THEOREM VIII.B: For a minimum path the unit fuel consumption r_m is less than r''.

[1]See analytic proof of Theorem V for details.

Proof. We can eliminate θ between equations (7.3) and (7.9) to get

$$q_{1m} = V \sin \theta_m + U \cos \theta_m$$
$$= Q \cos (\varphi - \theta_m)$$

or

(8.3) $$q_{1m} < Q$$

since

$$\varphi > \theta_m .$$

The unit fuel consumption is then

$$r_m = 1 - e^{-\frac{q_1 m}{c}}$$
$$< 1 - e^{-\frac{Q}{c}}$$

whence

(8.4) $$r_m < r'' .$$

We had a lower bound to unit fuel consumption $r' = 1 - e^{-\frac{V}{c}}$ for incoming targets. This was the least unit fuel consumption that could possibly effect homing (equation (6.3) following Theorem VI).

COROLLARY VIII.2. For a minimum path the unit fuel consumption r_m is greater than or equal to r'. If thrust is finite, then $r_m > r'$.

Proof. Consider equation (8.3)

$$q_{1m} = Q \cos(\varphi - \theta_m)$$
$$\geqq Q \cos(\varphi - 90^\circ)$$

since $\varphi > \theta_m \geqq 90^\circ$. We saw that if thrust was finite the strict inequality held:

(8.5) $$q_{1m} > Q \sin \varphi$$

for acceleration bounded.

Hence we have bounds for r_m when thrust is finite.

(8.6) $$r' < r_m < r'' .$$

IX. SPECIAL MINIMUM PATHS

In the previous sections, the work on minimum fuel consumption was done under the assumption that the second fundamental principle of homing with minimum fuel consumption (that thrust be large early and then cut off, see Theorem V) was observed. This included the case where thrust is a specified function of time for $t < t_1$, and the case where throttling could take place with either thrust or acceleration having a known maximum value as a function of time.

We consider now a special class of paths of minimum fuel consumption. We consider two particular cases, then a general class of problems including both of them.

A) Consider the following problem: let it be decided to home with acceleration constant all during homing, the constant being properly chosen to effect homing.

The first principle of homing, that thrust must be fixed in direction, must be satisfied. Hence, we can write the position of the craft

(9.1)
$$x = 1/2\ a(\theta)t^2 \cos\theta ,$$
$$y = 1/2\ a(\theta)t^2 \sin\theta ,$$

and the condition of homing

(9.2)
$$1/2\ a\ t_2^2 \cos\theta - Ut_2 - X_0 = 0 ,$$
$$1/2\ a\ t_2^2 \sin\theta - Vt_2 = 0 .$$

The integral

(9.3)
$$\int_0^{t_2} a(\theta)\, dt = t_2\, a(\theta)$$

is to be minimized. We write $a(\theta)$ to emphasize that the value of the required acceleration depends upon the angle of thrust.

(9.4)
$$t_2\, a(\theta) = \frac{2V}{\sin\theta} ,$$

and now we wish to minimize the left side. Since V is known, the right side attains its minimum for

(9.5)
$$\theta = 90^\circ ;$$

that is, for minimum fuel consumption, the thrust must be applied along the vertical to the initial line of sight.

B) Consider the following case: let thrust T be constant during homing, the constant chosen to effect homing. Using the equations of Section II, we can write the condition of homing

$$(9.5)\quad \begin{aligned} ct_2 \left[1 + \left(\frac{1}{r} - 1\right) \ln (1 - r)\right] \cos \theta &= X_0 + Ut_2 \,, \\ ct_2 \left[1 + \left(\frac{1}{r} - 1\right) \ln (1 - r)\right] \sin \theta &= Vt_2 \,. \end{aligned}$$

We can rewrite the second of these as

$$(9.6)\quad \left[1 + \left(\frac{1}{r} - 1\right) \ln (1 - r)\right] = \frac{V}{c \sin \theta} = \frac{\text{constant}}{\sin \theta} \,.$$

Now the function

$$1 + \left(\frac{1}{r} - 1\right) \ln (1 - r) = \frac{r}{2} + \frac{r^2}{6} + \frac{r^3}{12} + \cdots \frac{r^n}{n(n+1)} + \cdots$$

is clearly monotonic increasing in r. Hence to minimize r we need to minimize the right side of equation (9.6). As before, it attains its minimum for $\theta = 90^\circ$.

C) General Case: these are all examples of the following case.

THEOREM IX. Whenever the velocity required for homing can be expressed in the form

(9.7) $$v = f(A, t/t_2) ,$$

where *f* is an increasing function of *A* for any fixed value of *t/t_2*, minimum fuel consumption is achieved for thrust applied at 90° from the initial line of sight.

We consider only incoming targets; against outgoing targets there is no minimum.

Proof: From the results of section IV we must minimize

(9.8) $$v_2 = f(A,1) ,$$

a monotonic function of *A*; that is we need to select the angle of thrust *θ* to minimize *A*.

For homing to occur we must have

(9.9) $$s_2 = S_2 .$$

Now,

(9.10) $$S_2 = \frac{Vt_2}{\sin\theta}$$

and

$$s(A,t_2) = \int_0^{t_2} v \, dt$$

$$s(A,t_2) = \int_0^{t_2} f(A,t/t_2)\,dt$$

$$= t_2 \int_0^1 f(A,\tau)\,d\tau$$

or

$$(9.11) \qquad s(A,t_2) = t_2 F(A) ,$$

defining $\underline{F(A)}$, a non-decreasing function of $\underline{A}$.

Hence, by relations (9.9), (9.10) and (9.11) it is required for homing that

$$F(A) = \frac{V}{\sin \theta} .$$

The right side of this equation has its minimum when $\theta = 90^\circ$. Hence $\underline{A}$ attains its minimum when $\theta = 90^\circ$, $\underline{v_2}$ has its smallest value for $\theta = 90^\circ$, and minimum fuel consumption is attained for $\theta = 90^\circ$.

Corollary IX.1. The conditions (2) can be replaced by the condition that $v_2 = f(A,1)$ be a monotonic increasing function of $F(A) = \int_0^1 f(A,\tau)\,d\tau$.

This was the only property of $\underline{f}$ used in the proof.

Corollary IX.2. If the acceleration has the form $a = \frac{1}{t_2} g(A, t/t_2)$ where g is an increasing function of A, thrust must be applied at 90° from the initial line of sight to achieve minimum fuel consumption.

If the acceleration has the form

$$a = \frac{1}{t_2} g(A, t/t_2) ,$$

$$v = \frac{1}{t_2} \int_0^t g(A, t/t_2)\, dt$$

$$= \int_0^{t/t_2} g(A, \tau)\, d\tau .$$

This we can rewrite as

$$v = f(A, t/t_2)$$

and this velocity function satisfies the hypotheses of the theorem.

Corresponding to this first corollary we could specify weaker conditions on the acceleration function. It is required only that $\int_0^1 g(A, \tau)\, d\tau$ be a monotonic increasing function of $\int_0^1 (1 - \tau)\, g(A, \tau)\, d\tau$.

In paragraph A of this section we considered the case when the acceleration $\underline{a}$ was constant during homing. We wrote this as $\underline{a(\theta)}$ to indicate that the value of this constant depended upon the direction in which thrust was applied. We could equally well have written $\underline{a(t_2)}$ since $\underline{\theta}$ and $\underline{t_2}$ are connected by the formula

$$\tan\theta = \frac{Vt_2}{X_0 + Ut_2} .$$

If we set $A(t_2) = t_2\ a(t_2)$ we see that

$$v = A\ t/t_2 \qquad \left[= f(A,t/t_2)\right]$$

$$v_2 = A \qquad \left[= f(A,1)\right]$$

$$s_2 = t_2 A \qquad \left[= t_2\ F(A)\right] \quad ;$$

so that $f(A,1)$ is a strictly increasing function of $F(A)$, the hypothesis for corollary IX.1.

This, the example of constant acceleration, is the second most simple example of the theorem, leading to the case when $\underline{v}$ is linear in $\underline{A}$ and linear in $\underline{t/t_2}$. The case of the impulse is the simplest; $\underline{v}$ is linear in $\underline{A}$ and constant in $\underline{t/t_2}$.

Corollary IX.3. If the unit burning rate $\underline{\dot{r}}$ has the form $\dot{r} = \frac{1}{t_2}\ h(A,t/t_2)$ with $\underline{h}$ an increasing function of $\underline{A}$ the conclusion of the theorem holds.

We see that

$$r = \int_0^t \dot{r}\, dt$$

$$= \frac{1}{t_2} \int_0^t h(A, t/t_2)\, dt$$

$$= \int_0^{t/t_2} h(A, \tau)\, d\tau$$

$$= H(A, t/t_2) ,$$

a strictly increasing function of <u>A</u>. Now

$$v = - c \ln (1 - r)$$

$$= - c \ln \left[1 - H(A, t/t_2)\right] .$$

Since an increasing function of an increasing function is an increasing function, the hypotheses of the theorem are satisfied.

In paragraph B we considered the case when thrust was constant during homing, this constant being properly chosen to effect homing. In any given situation $\dot{r} = \dot{r}(t_2)$ since the value depends only upon the homing time chosen.

In this case

$$r(t,t_2) = \int_0^t \dot{r}(t_2)\, dt$$

$$= t\, \dot{r}(t_2) \;;$$

this we can rewrite

$$r(t,t_2) = t_2\, \dot{r}(t_2) \cdot t/t_2 \, .$$

Now, if we set $t_2\, \dot{r}(t_2) = A$, this assumes the form of the function for fuel consumption in corollary IX.3,

$$r = A\, t/t_2 \qquad [= f(A,t/t_2)] \, .$$

This is the case when fuel consumption can be expressed in a particularly simple form, linear in each variable $\underline{A}$ and $\underline{t/t_2}$. Velocity then has the form

$$v = -c \ln (1 - At/t_2) \, .$$

APPENDIX[1]

DISCUSSION OF COORDINATE SYSTEMS

The first problem in a study is to choose a proper coordinate set in which to work. We shall see that any cartesian coordinate set whose axes are not rotating in space may be used equally well. We shall see that the fundamental equations depend only upon the initial relative position and the initial relative velocity of the target with respect to the craft and upon the thrust applied to the craft. They are independent of the actual velocity of either the craft or the target, of body forces, and of arbitrary displacements of the origin of coordinates. This allows a great deal of freedom in the choice of a coordinate system.

Since Newton's laws furnish the basis for our study, we shall start with a Newtonian reference frame or coordinate

[1]The material of this section is essentially the same as in many books on mechanics and theoretical physics. See, for example, Synge and Griffith, Principles of Mechanics, Chapter XII, or G. Joos, Theoretical Physics.

system. Let $\underline{\bar{x}}$ be the position vector of the craft and $\underline{\bar{X}}$ the position vector of the target in this reference frame. The equation of motion of the craft is then

$$M \frac{d^2\bar{x}}{dt^2} = \bar{T} + M\bar{G}$$

where $\underline{M\bar{G}}$ is the body force. We can rewrite this as

(A.1) $$\frac{d^2\bar{x}}{dt^2} = \frac{c\,\dot{r}\,\bar{\tau}}{1 - r} + \bar{G} ,$$

as in Section II; $\underline{\bar{\tau}}$ is the unit vector parallel to $\underline{\bar{T}}$. Let

$$\bar{a} = \frac{c\,\dot{r}\,\bar{\tau}}{1 - r} ;$$

then $\underline{\bar{a}}$ is the vector of acceleration due to thrust. The integration of equation (A.1) leads to

(A.2) $$\frac{d\bar{x}}{dt} = \left(\frac{d\bar{x}}{dt}\right)_0 + \int_0^t \bar{a}\, dt + \int_0^t \bar{G}\, dt .$$

This can be integrated once more to yield

(A.3) $$\bar{x} = \bar{x}_0 + t\left(\frac{d\bar{x}}{dt}\right)_0 + \int_0^t \int_0^\tau \bar{a}\, d\sigma\, d\tau + \int_0^t \int_0^\tau \bar{G}\, d\sigma\, d\tau .$$

Consider a new coordinate system, the position of the origin of which is given by the vector $\underline{\bar{q}}(t)$. Then the position vector $\underline{\bar{x}}'$ of the craft in this new system is

(A.4) $\quad \bar{x}' = \bar{x} - \bar{q}$

$$= \bar{x}_0 + t\left(\frac{d\bar{x}}{dt}\right)_0 + \int_0^t \int_0^\tau \bar{a}\, d\sigma\, d\tau + \int_0^t \int_0^\tau \bar{G}\, d\sigma\, d\tau - \bar{q}(t).$$

If $\bar{X}'$ denotes the position vector of the target in this new system, then we have

(A.5) $$\bar{X}' = \bar{X}_0 + t\left(\frac{d\bar{X}}{dt}\right)_0 + \int_0^t \int_0^\tau \bar{G}\, d\sigma\, d\tau - \bar{q}(t)\ .$$

The coordinates of interest are those that give the position of the target with respect to the craft. We see that

(A.6) $\quad \bar{X}' - \bar{x}' = \bar{X} - \bar{x}$

$$= (\bar{X} - \bar{x})_0 + t\left(\frac{d\bar{X}}{dt} - \frac{d\bar{x}}{dt}\right)_0 - \int_0^t \int_0^\tau \bar{a}\, d\sigma\, d\tau\ .$$

The only terms that enter in these equations are those giving the relative position at the time t = 0, the relative velocity at that time, and the thrust applied to the craft. Hence for the two-body problem we can relax the restriction that the system be Newtonian, so long as the axes are not rotating. The importance of this is that the origin can be moving. For example, a point on the earth's surface may be taken as origin without influencing the fundamental equations[1].

---

[1]Account must be taken of the earth's rotation.

It will be helpful later if we choose $\bar{q}$ in the following manner

$$\bar{q} = \bar{x}_o + t(\frac{d\bar{x}}{dt})_o + \int_0^t \int_0^\tau \bar{G} \, d\sigma \, d\tau .$$

In this system

$$\text{(A.7)} \qquad \bar{x}' = \int_0^t \int_0^\tau \bar{a} \, d\sigma \, d\tau ,$$

$$\text{(A.8)} \qquad \bar{X}' = (\bar{X} - \bar{x})_o + t(\frac{d\bar{X}}{dt} - \frac{d\bar{x}}{dt})_o .$$

We see that $\bar{x}'$ depends only upon the applied thrust and $\bar{X}'$ depends only upon the initial relative motion.

Let us consider a rectangular coordinate system in which x, y, z are the coordinates of the craft position, and X, Y, Z are those of the target. Equation (A.7) is equivalent to the three scalar equations

$$x = \int_0^t \int_0^\tau a_x(\sigma) \, d\sigma \, d\tau$$

$$\text{(A.9)} \qquad y = \int_0^t \int_0^\tau a_y(\sigma) \, d\sigma \, d\tau$$

$$z = \int_0^t \int_0^\tau a_z(\sigma) \, d\sigma \, d\tau$$

where $\bar{a}_x = \frac{\bar{T}_x}{M}$ is the x-component of the acceleration due to thrust, etc.

Equation (A.8) can also be expressed

$$
\begin{aligned}
X &= X_o + Ut \\
Y &= Y_o + Vt \qquad \text{(A.10)} \\
Z &= Z_o + Wt\ .
\end{aligned}
$$

If we let

$$
\begin{aligned}
\xi &= X - x \\
\eta &= Y - y \qquad \text{(A.11)} \\
\zeta &= Z - z
\end{aligned}
$$

expression (A.6) has the form

$$
\begin{aligned}
\xi &= X_o + Ut - \int_0^t \int_0^\tau a_x \, d\sigma \, d\tau \\
\eta &= Y_o + Vt - \int_0^t \int_0^\tau a_y \, d\sigma \, d\tau \qquad \text{(A.12)} \\
\zeta &= Z_o + Wt - \int_0^t \int_0^\tau a_z \, d\sigma \, d\tau\ .
\end{aligned}
$$

In general we shall choose the orientation of our axes so that $Y_o = Z_o = W = 0$. This can always be in the following manner. Set up the square array[1] representing a rotation of coordinates

[1]For example, see Snyder and Sisam, Analytic Geometry of Space, Article 37.

		x	y	z
	x'	ℓ_{11}	ℓ_{12}	ℓ_{13}
(A.13)	y'	ℓ_{21}	ℓ_{22}	ℓ_{23}
	z'	ℓ_{31}	ℓ_{32}	ℓ_{33}

where the ℓ's represent the cosine of the angle between the various axes; ℓ_{12}, for example, being the cosine of the angle between the old y and the new x' axes. Then the transformation is represented by

$$x' = \ell_{11} x + \ell_{12} y + \ell_{13} z ,$$

etc. The quantities X_o, Y_o, Z_o, U, V, W, being components of vectors, are transformed in a similar manner. If we choose ℓ_{31}, ℓ_{32}, ℓ_{33}, to satisfy the relations

(A.14)
$$Z_o' = \ell_{31} X_o + \ell_{32} Y_o + \ell_{33} Z_o = 0$$
$$W' = \ell_{31} U + \ell_{32} V + \ell_{33} W = 0$$

and ℓ_{21}, ℓ_{22}, ℓ_{23}, to satisfy the relation

(A.15)
$$Y_o' = \ell_{21} X_o + \ell_{22} Y_o + \ell_{23} Z_o = 0 ;$$

then these conditions with the conditions of orthogonality and normality

$$\text{(A.16)} \qquad \sum_{i=1}^{3} \ell_{ij} \ell_{ik} = \sum_{i=1}^{3} \ell_{ji} \ell_{ki} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j = k \; (j,k = 1,2,3) \end{cases}$$

which the cosines satisfy, will determine the ℓ's except for three signs. We shall choose X_o' to be positive, V' to be positive and the orientation of the z-axis to form a right hand act of coordinates. Then equations (A.10) are in the desired form, we can drop the primes,

$$\text{(A.17)} \qquad \begin{aligned} X &= X_o + Ut \\ Y &= Vt \\ Z &= 0 . \end{aligned}$$

Since a_x, a_y, a_z are components of a vector, equations (A.9) for the rocket motion are unchanged in form by the rotation. Equations (A.12) now have the form

$$\text{(A.18)} \qquad \begin{aligned} \xi &= X_o + Ut - \int_0^t \int_0^\tau a_x \, d\sigma \, d\tau \\ \eta &= Vt - \int_0^t \int_0^\tau a_y \, d\sigma \, d\tau \\ \zeta &= - \int_0^t \int_0^\tau a_z \, d\sigma \, d\tau . \end{aligned}$$

It follows from the nature of vectors that it is not essential whether the subtractions (A.6) or the rotations are performed first. The integral

$$\int_0^t \sqrt{a_x^2 + a_y^2 + a_z^2}\, dt$$

in the fundamental equation (2.5) connecting fuel consumption to thrust is the integral of the magnitude of the vector $\bar{a} = \frac{\bar{T}}{M}$, hence is not affected by a rotation of coordinates.

The condition that interception occur without further thrust is that the vector of relative position (X_0, Y_0, Z_0) is parallel to the vector of relative velocity (U, V, W) and opposite in sense, that

(A.19) $$\frac{X_0}{U} = \frac{Y_0}{V} = \frac{Z_0}{W} \quad (= -t_2) \leq -t_1 \; .$$

This can be seen by substitution into equation (A.12). Terms in expression (A.19) with numerator and denominator both zero must be omitted. A relation corresponding to equation (A.19) must be satisfied at the end of burning if homing is to be achieved without further corrections. This problem is discussed in Section III.

We have just seen that we can express the problem as a plane one. If we solve the plane problem, we can solve the problem in space, taking into account the forces of gravity, etc.

The real importance of the transformation is this: to solve the problem of homing it is necessary to go to the plane problem; this may be done explicitly, or it may be done implicitly. It is not necessary to go to the trouble of carrying out the above computations but only to determine $\underline{X}_0$, $\underline{U}$ and $\underline{V}$[1].

A second significance of this transformation is this: we can study the motion in the plane above and know that we have not omitted anything of significance to the space problem, subject to the assumptions in the introduction.

[1]This is done in UMM-19, Section VI.

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