Solution of the Problem of Homing in a Vacuum with a Point Mass

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I. INTRODUCTION AND SUMMARY

It is the main purpose of this report to give a simple, general solution to the problem of homing with a point mass in a vacuum.

This report is associated closely with Report No. UMM 18 on the subject of homing with minimum fuel consumption. The background of the problem is given there in some detail and we shall not go into it here. By <u>homing</u> is meant the application of <u>thrust</u>[1] to the first particle, essentially a rocket, called the <u>craft</u>, in such a way that its position will at a later time coincide with that of the second particle, the <u>target</u>. The target is assumed to be in free flight or, in the later sections, following any prescribed course.

The method of this report also gives the method for determining the paths of minimum fuel consumption and their existence. It has important applications for the designer in

[1] The <u>thrust</u> is a force gained by the emission of part of the mass of the craft. The average velocity of these particles over any time interval is assumed to be constant, designated by c, the <u>effective gas velocity</u>.

determining craft specifications for homing. The method also gives the engineer an approximate answer to many other problems that arise in rocket work. Some of these approximations are given and the sources of errors due to the approximations are pointed out.

A. Points Particularly Important to the Homing Problem

The method can be used whenever the thrust is a specified function of time and when the target course can be predicted. This includes the case in which the acceleration due to thrust is a specified function of time.

The key to the method is this: a single grid can be drawn up which, except for parameters of the particular rocket chosen, completely describes the motion of all rockets which have a similar thrust[1]. As an example, all rockets whose thrust is constant have similar thrust. Because of its importance the work is carried through for this case with supplementary explanations of the direct generalizations. The grid is drawn up in dimensionless form. The initial conditions, that is, the initial relative velocity and position of the target with respect to the craft, are expressed in terms of the parameters of the rocket. This determines a curve which is superimposed on the grid.

[1]Explained in detail in the text.

The following quantities can be read directly from the graphs.

1. The times when homing can be effected.
2. The burning time (the duration of thrust) and fuel consumption corresponding to any chosen homing time.
3. The existence (or non-existence) of a minimum path[1].
4. The homing time, burning time, and fuel consumption corresponding to the minimum path.

The method appears to be well adapted to use in the field. The important reason is this: the characteristics of the craft do not need to be specified beforehand; they need only to be specified at the instant homing starts. This seems important if we consider that homing will take place in a succession of steps as the information becomes more and more accurate. Each step will begin with an amount of fuel which cannot be predicted beforehand but which can be computed, since the initial weight (at the beginning of the first step) and the amount of fuel burned in each step will be known as soon as the step is over.

[1] Let us refer to paths of minimum fuel consumption as <u>minimum paths</u>.

As shown in UMM-18 the problem of homing is entirely a problem in relative motion. In the early stages of the homing problem it may be necessary to use information from earth-based equipment. In this case there are at least fifteen parameters in the problem: the initial position of the craft (three coordinates), its initial velocity (three more parameters), the position and velocity of the target, the thrust of the craft, the effective velocity of the jet, and the initial weight of the craft.[1]

The second important feature is this. It is shown that the fifteen parameters above can be combined to yield three significant parameters; these three parameters completely specify the problem.[2]

This simplifies the over-all problem greatly. Of course any solution in terms of initial values must be reduced to this form eventually, either explicitly or implicitly in that the computations carried out are entirely equivalent. Immediate reduction to this form removes the mysticism from the solution.

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[1]These can be expressed in numerous other equivalent ways.

[2]There is a fourth parameter (a sixteenth parameter in the first set) which corresponds to the maximum allowable burning time, or to the total amount of fuel which the rocket has.

B. Points Particularly Important in the Design of a Homing Craft

The method can be used to find the following information of particular interest to a person designing a homing craft:

For given target conditions and known jet velocity one can determine the following:

1. The times when homing can be effected by burning all the way.[1]
2. For a chosen homing time, the fuel consumption corresponding to burning all the way (if it is possible).

By virtue of properties (1) and (2) we can draw a graph which will show the lowest burning rate which can effect homing (and the homing time corresponding to this burning rate). There may not be a lowest burning rate; the burning rate may decrease steadily as homing time increases.

3. For a chosen homing time, the fuel consumption corresponding to thrust applied as an impulse.

[1] We will see that if the average velocity during homing exceeds $\underline{c}$, the jet velocity, then homing cannot be effected by burning all the way at a constant rate.

Familiarity with the method will reveal other important properties to those who use it. Many of the properties of minimum paths can be shown directly from the graphs.

In UMM-18 certain principles of minimizing fuel consumption were given. These could equally well have been called principles for maximum performance; that is, how to achieve maximum performance with a given craft and a given amount of fuel.

The two most important principles were that

1. thrust must be fixed in direction (the first fundamental principle for minimizing fuel consumption), and
2. thrust must be high during the early moments of the homing, then it must be cut off (the second fundamental principle for minimizing fuel consumption).

The homing paths determined here are found under the assumption that these principles are observed. Fortunately these principles, particularly the second, seem to be quite natural ones to follow when the initial velocity and initial position are known.

For a detailed analysis of a particular craft it is often simpler not to change to the dimensionless form, that is, to leave distances expressed in feet, times in seconds, etc.

C. Obvious Generalizations and Limitations of the Method

The method can be used any time that the vector of relative position can be broken down into two components, one independent of the thrust and the other due entirely to thrust. Simple extensions of the methods are given to solve the following problems.

1. The problem where the target follows a known dodging course.
2. The problem of sending a rocket to a fixed point in space.

One can read directly from the graph all the quantities mentioned previously such as the time when the craft can home (or arrive), fuel consumption, the fuel consumption corresponding to burning all the way and corresponding to an impulse, the determination of paths of minimum fuel consumption, if they exist, etc.

The engineer and the designer are frequently called upon to make quick estimates and approximations. One aim of this report is to provide them with a method for getting these. Therefore we point out the limitations and the places where approximations are made, so that they can evaluate their results.

In the first place we treat the point mass problem. This is a good approximation so long as the times required to rotate the craft can be neglected and the displacements resulting from the rotations can be neglected in comparison to the other times and displacements.

In the second place it is assumed that the action takes place in a vacuum. This approximation is good as long as the acceleration due to interaction with the air can be neglected. In many cases the method gives a first approximation. Using this, we can compute the aerodynamic forces and superimpose them as a perturbation upon the first approximation to yield two results, one above the true value for distance and velocity and the second below the true value.

Part of the performance analysis in Project WIZARD Phase One Report was carried out in a similar way. Some detailed computation should be carried out to give a measure of the validity of the answers.

The third approximation is this. It is assumed that the target and the craft are close enough together that the acceleration due to gravity is the same on each. This introduces an error in the acceleration of approximately one foot per sec. per sec. for each three hundred thousand feet distance between them when they are near the earth's surface. When we have accelerations from thrust of the order of three to ten times the acceleration of gravity, this seems negligible. For a long period of free flight this may introduce an appreciable error.

II. FORMULAS AND EQUATIONS

A. General Formulas

The equation of motion of a rocket in linear motion in a field-free space can be written

(2.1) $$\dot{w} = \frac{c\dot{r}}{1 - r}$$

where

w is the velocity,

c is the effective velocity of the emitted gas, usually assumed constant,

r is the burnt fuel ratio $= \frac{m}{M_o}$,

m is the mass of the fuel consumed at any time,

M_o is the initial mass of the rocket complete with fuel,

M is the mass at any time. ($M_o = M + m$)

Let us use dots over variables to indicate the derivative with respect to the time t. the thrust force is given (in magnitude) by

(2.2) $$T = \dot{m}c = -\dot{M}c .$$

We can integrate equation (2.1) to get the well-known formula for velocity,

$$w = - c \ln (1 - r) \tag{2.3}$$

if the rocket starts at rest. If thrust is constant we can integrate equation (2.3) to get distance

$$s(t) = ct \left[1 + \left(\frac{1}{r} - 1\right) \ln (1 - r)\right] \tag{2.4}$$

during burning if the craft starts at rest.

We define the burning time $\underline{t_1}$ by the relation

$$\begin{aligned} \dot{r} &= \dot{r}_0 , \quad \text{a constant, for } t < t_1 \\ \dot{r} &= 0 \quad \text{for } t > t_1. \end{aligned} \tag{2.5}$$

We have the second formula for distance

$$s(t,t_1) = - ct \ln (1 - r_1) + ct_1 \left[1 + \frac{1}{r_1} \ln (1 - r_1)\right] \tag{2.6}$$

for $t \geqq t_1$. This equation includes equation (2.4) if we define $t_1 = t$ during burning.

B. Dimensionless Form of the Equations of Motion

Let us define the dimensionless quantities

$$\begin{aligned} t^* &= \dot{r}_0 t \\ w^* &= \frac{w}{c} \\ s^* &= \frac{s}{c/\dot{r}_0} ; \end{aligned} \tag{2.7}$$

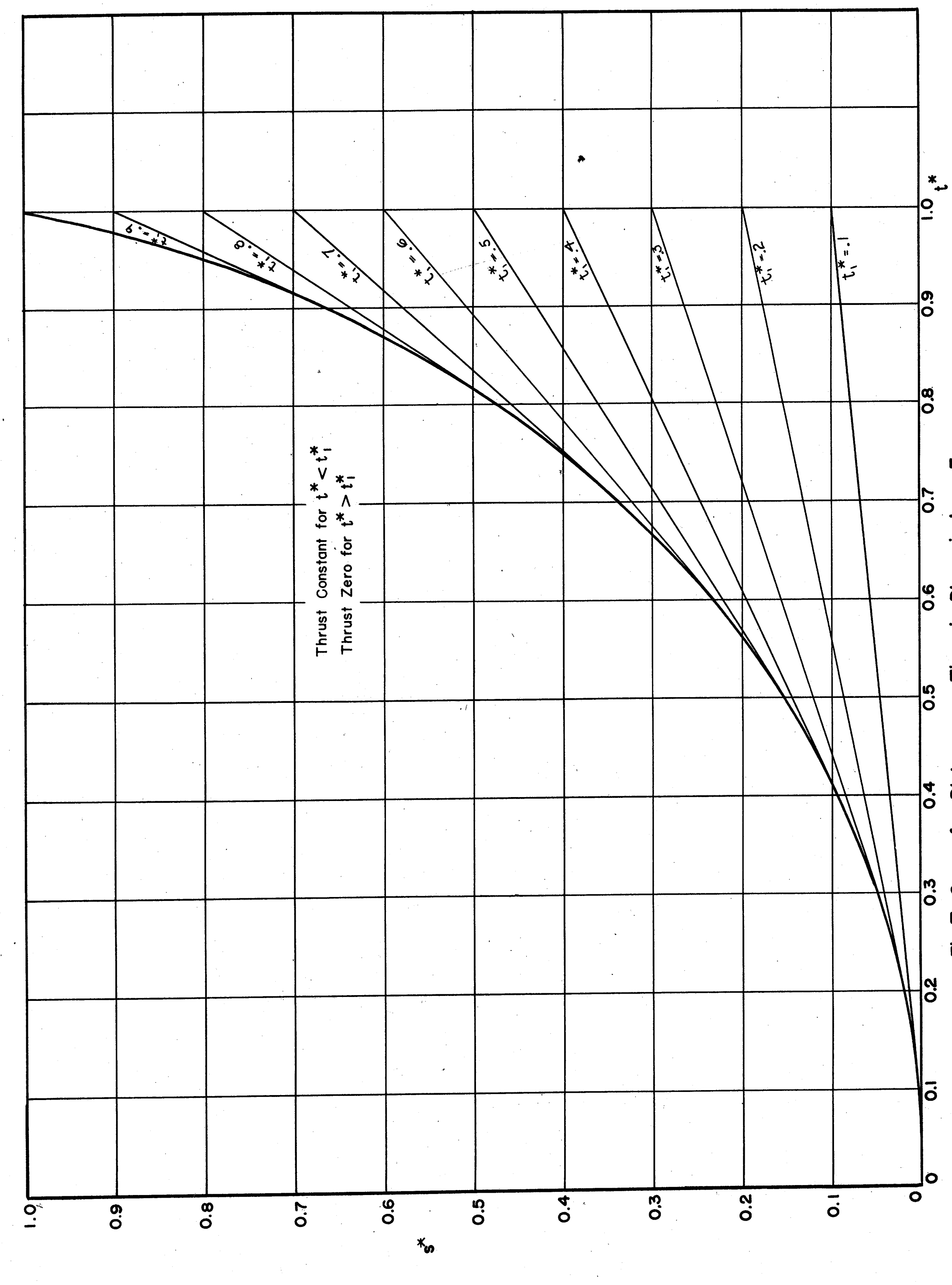

Fig. I Curve for Distance vs. Time in Dimensionless Form the equations above become

(2.8) $w^* = -\ln(1 - t^*)$,

(2.9) $s^*(t^*) = t^* + (1 - t^*)\ln(1 - t^*)$,

and

(2.10) $s^*(t^*, t_1^*) = -t^* \ln(1 - t_1^*) + t_1^* + \ln(1 - t_1^*)$.

Note that these formulas, which are the same for all rockets whose thrust is constant, completely describe the motion of the craft except for the parameters $\underline{c}$ and $\underline{\dot{r}_0}$.

Figure I is a graph of $\underline{s^*(t^*, t_1^*)}$ versus $\underline{t^*}$ with $\underline{t_1^*}$ as a parameter. We see the following interesting property: if it were possible to burn the craft up with the last particles delivering the same momentum, the craft would, in its last moments, reach an infinite velocity, but it would travel only a finite distance.

It has a second interesting and useful property. By direct substitution we get the relation

(2.11) $s^*(1, t_1^*) = t_1^*$.

This is used in the following way. We shall want to find the point of tangency to the curve for $\underline{s^*(t^*)}$. By noting the intercept of the tangent with the line $t^* = 1$ we get the value of $\underline{t^*}$ at the point of tangency much more accurately than we could by examining the intersection of the tangent with the curve.

Intuitively we define $1/\dot{r}_0$ as the burnup time, that is, the time it would take the craft to burn up entirely if it continued at the initial rate. We define $c/\dot{r}_0$ as the burnup distance, the distance the craft would travel if the same equation of motion (2.1) held until the craft burned up. The dimensionless terms above then express times in terms of the burnup time, velocities in terms of the jet velocity, and distances in terms of the burnup distance.

If thrust is constant then $r = t^*$ during burning and $r_1 = t_1^*$.

C. Similarity Considerations

The previous work is a simple application of similarity considerations. These are frequently used to reduce many particular problems to a small number of general problems[1].

Let us consider the motion of two craft. Let us denote quantities associated with the second craft by Greek letters. Let the burned fuel ratios be r and ρ respectively.

Definition. If the unit fuel consumptions r and ρ satisfy the relation

$$\rho(t) = r(kt) ; \tag{2.12}$$

[1]These are particularly important in problems in compressible flow. See, for example, Dodge and Thompson, Fluid Mechanics, New York: McGraw Hill, 1937, pp 420 ff.

then the thrust of the two rockets is similar; the two rockets are said to burn in a similar manner.

The curves for r and ρ vs. t are similar in the usual sense of the definition of similar curves in dynamics.

For linear motion we see that the velocities w and ω satisfy the relation

$$(2.13) \qquad \omega(t) = -\gamma \ln(1 - \rho(t)) = -\gamma \ln(1 - r(kt)) = \frac{\gamma}{c} w(kt)$$

where c and γ are the respective effective gas velocities. If $\gamma = c$ then

$$(2.13') \qquad \omega(t) = w(kt) .$$

If we differentiate in equation (2.12) we see that

$$(2.14) \qquad \dot{\rho}(t) = k\,\dot{r}(kt) ,$$

and that the accelerations a and α satisfy the relation

$$(2.15) \qquad \alpha(t) = \frac{\gamma\ \dot{\rho}(t)}{1 - \rho(t)} = \frac{\gamma\ k\dot{r}\ (kt)}{1 - r(kt)} = \frac{\gamma k}{c} a(kt) .$$

If $c = \gamma$, then

(2.15') $$\alpha(t) = k\, a(kt) \,.$$

If we integrate in equation (2.13) we find that

(2.16) $$\sigma(t) = \frac{\gamma}{ck}\, s(kt)$$

where $\underline{s}$ and $\underline{\sigma}$ are the distances for the two craft. The craft are assumed to start from rest at the origin. If $\gamma = c$, then

(2.16') $$\sigma(t) = \frac{1}{k}\, s(kt) \,.$$

It is clear that the curves for acceleration, velocity, distance and fuel consumption as functions of time for all rockets that have similar thrust differ only in the scale used along the axes and that points on one of these curves for any rocket of the set can be obtained from the corresponding curve for any other rocket of the set by the proper use of the factors $\underline{k}$ and $\underline{\gamma/c}$. Hence a single curve can be drawn up in dimensionless form obtained by dividing $\underline{t}$ by some parameter corresponding to $\underline{1/\dot{r}_0}$ of paragraph B and dividing $\underline{s}$ by the corresponding quantity involving $\underline{c}$. A curve of the dimensionless quantities $\underline{s^*}$ as a function of $\underline{t^*}$ with parameter $\underline{t_1^*}$ is obtained corresponding to the curves of Figure I. In general a second curve is needed to convert $\underline{t_1^*}$ to $\underline{r}$.

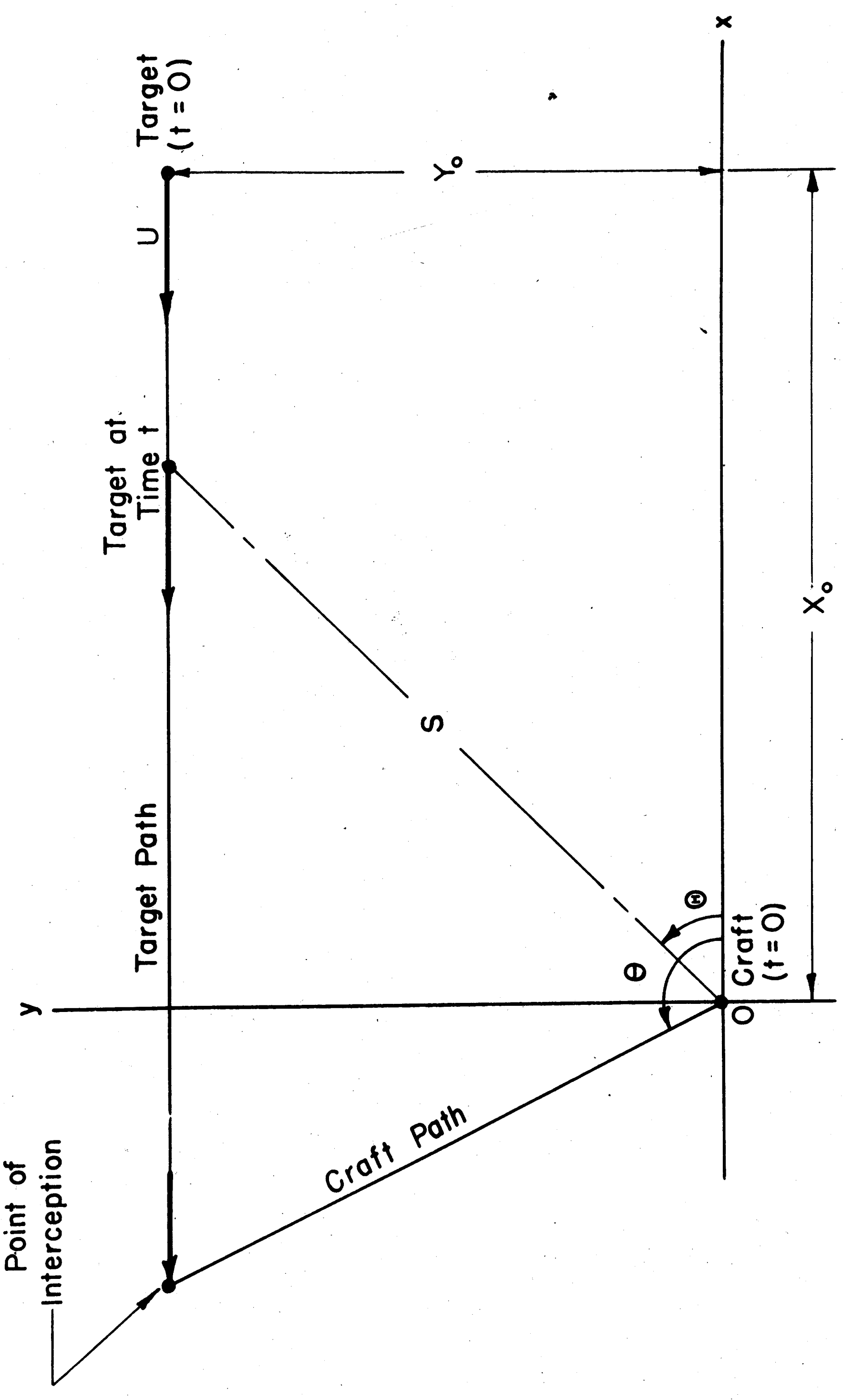

Fig II Coordinate Set and Target Path

III. THE METHOD OF SOLUTION OF THE HOMING PROBLEM

We shall choose our coordinate set so that the craft is originally at the origin with zero velocity and its motion is due entirely to the thrust applied. We choose the coordinate set so that the target is initially at ($\underline{X_0}$,$\underline{Y_0}$) with its velocity ($\underline{U}$,$\underline{0}$) parallel to the $\underline{x}$-axis as in Figure II. This can always be done[1].

The coordinates of the target position in this system are

$$X = X_0 + Ut \equiv U(t - t') ,$$

(3.1)

$$Y = Y_0 ,$$

where $t' = -\frac{X_0}{U}$; $\underline{t'}$ is the time when the target is nearest to the origin. In polar form the target position becomes

$$S = \sqrt{X^2 + Y^2}$$

(3.2)

$$\Theta = \arctan \frac{Y_0}{X_0 + Ut} .$$

[1]See UMM-18, the appendix.

We can write the first of these as

(3.3). $$S^2 - U^2(t - t')^2 = Y_o^2 ,$$

the equation of a hyperbola in the tS-plane.

The craft position can be expressed in polar form (s,θ) and we note that s is given by the formulas of Section II. We can change s to its dimensionless form s* and then the grid of Figure I is the graph of $s^*(t^*,t_1^*)$. Only a limited amount of fuel will be available and this will place a limit on t_1 and t_1^*. This determines the heavy tangent curve of Figure III, called the curve of maximum performance, which s* cannot exceed. In Figure III this maximum value of t_1^* was taken arbitrarily as .5 corresponding to a fuel weight of one half the initial gross weight.

Now let us change the quantities S, X, Y, U, and t to a dimensionless form in the same manner as in the preceding section, by multiplying times by $\dot{r}_o$, dividing velocities by c and distances by $c/\dot{r}_o$. We denote the corresponding dimensionless terms by * as before.

Equation (3.3) becomes

(3.4) $$S^{*2} - U^{*2}(t^* - t'^*)^2 = Y_o^{*2} .$$

In the t*S*-plane this is the equation of a hyperbola with center at $(t'^*,0)$ with asymptotes of slope $\pm U^*$ and semi-transverse

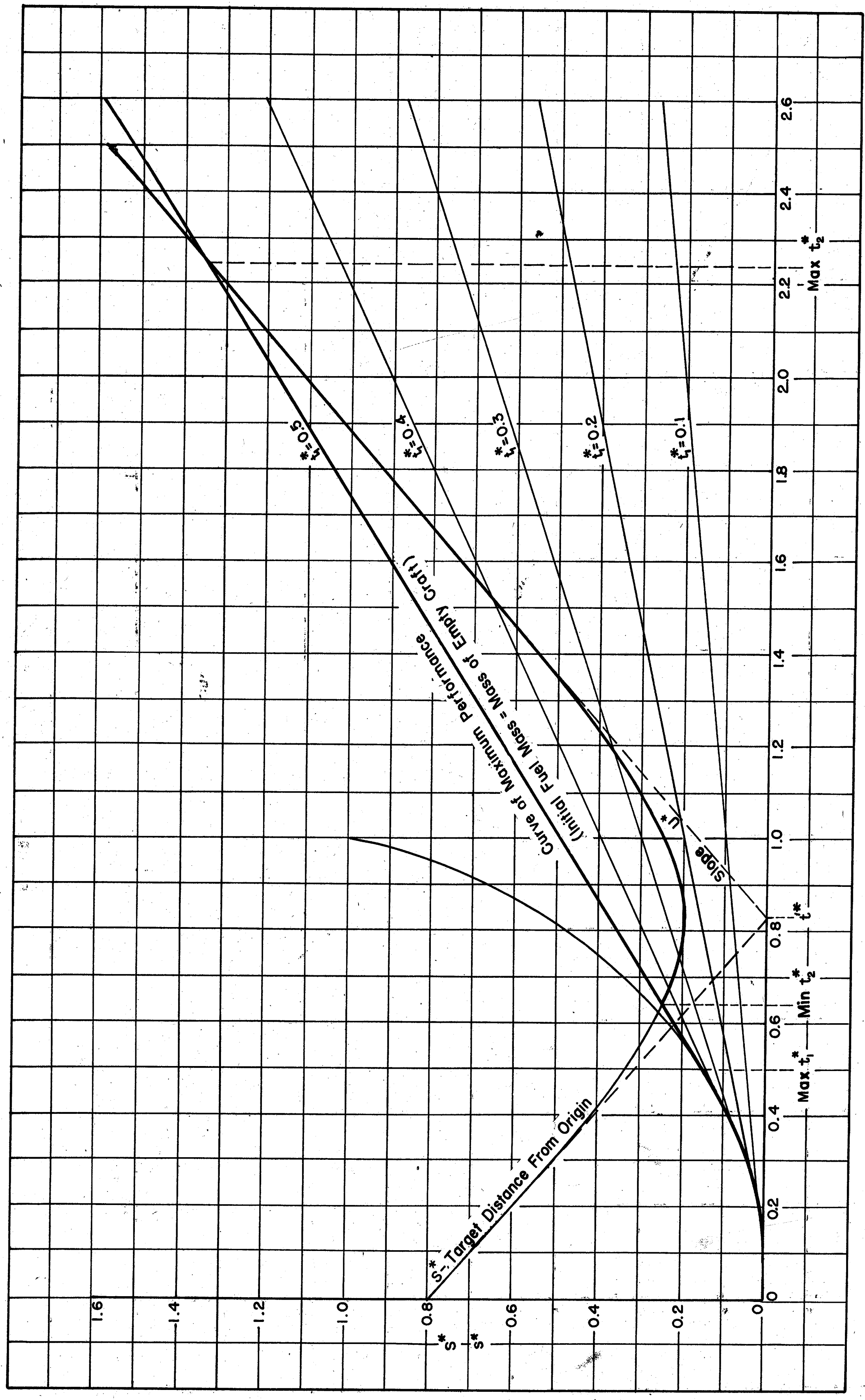

Fig. III. Use of Curve in Solving a Problem

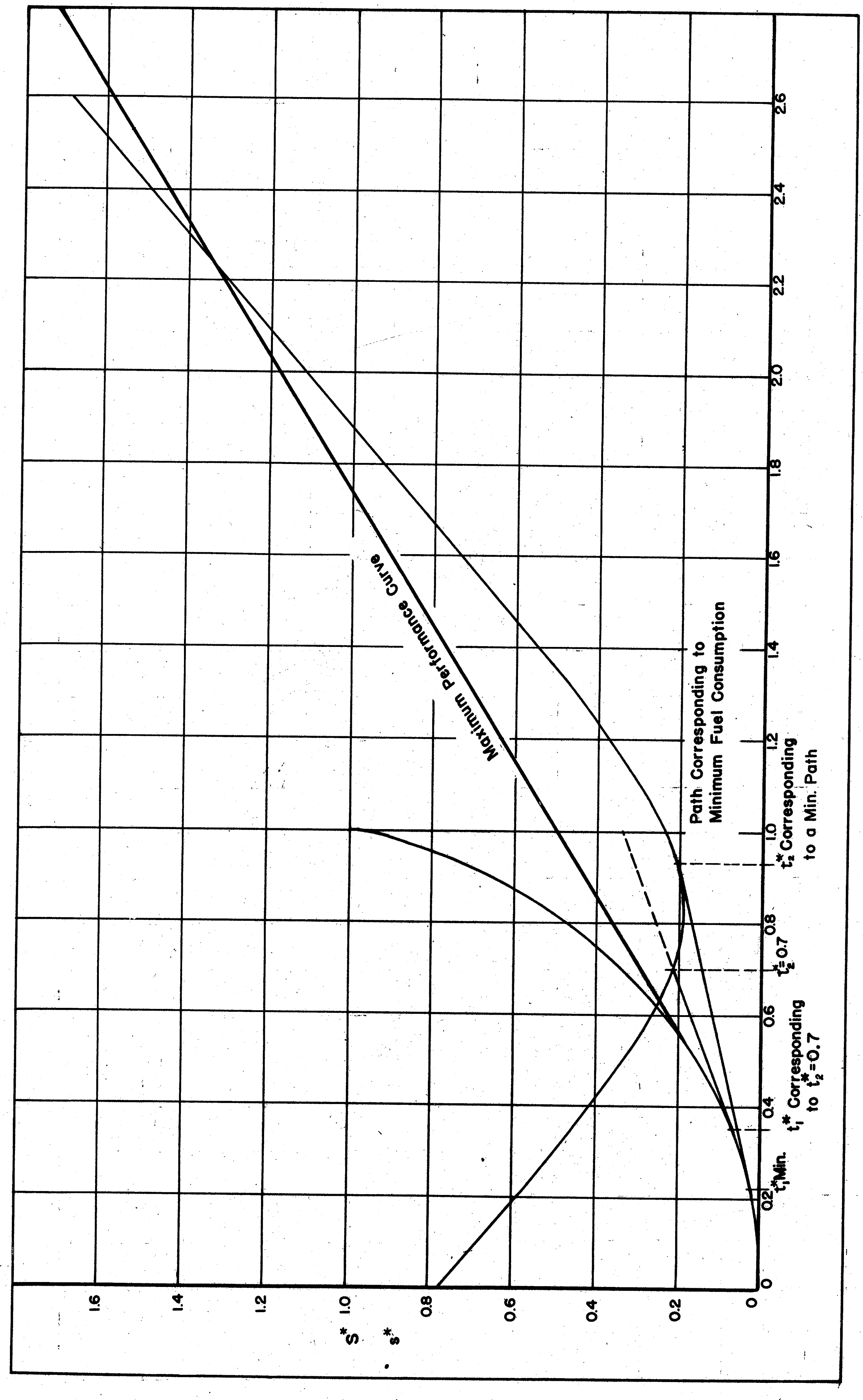

Fig. IV. Use of Curve in Solving a Problem axis Y_0*. Let us draw the graph of equation (3.4) on the same set of coordinates as those for equations (2.9) and (2.10) (see Figure III). Since S* is essentially a distance we shall consider only the upper branch of the curve.

Now all points of the S*-curve which lie below the curve of maximum performance represent possible homing times and homing points.

For a chosen t_2* in this range we find t_1* as follows (see Figure IV where we chose t_2* arbitrarily as 0.7). Draw the tangent to the s*-curve from the point on the S*-curve. The point of tangency determines t_1*. As remarked in Section II, it is difficult to determine the exact point of tangency directly but we can use the relation

$$(2.11) \qquad s^*(1, t_1^*) = t_1^*$$

to find t_1* quite accurately. For $t_2^* = .7$ we see from Figure III that $t_1^* = .335$.

We see that there is a path of minimum fuel consumption. The corresponding values of t_2* and t_1* are .93 and .225 respectively.

We see that for this example homing is possible for $.65 < t_2^* < 2.25$. For the extreme values, $t_1^* = .5$, and it is less for the points in between.

We have satisfied the first condition of homing, that $s = S$. The second condition is

$$(3.7) \qquad \theta = \Theta(t_2)$$

$$= \arctan \frac{Y_0}{X_0 + Ut_2} \qquad \left(= \arctan \frac{Y_0^*}{X_0^* + U^* t_2^*}\right) .$$

If it is simpler to apply thrust at some particular angle than at any other, this condition can be satisfied, the corresponding homing times be determined (from equation 3.7) and the burning time be determined from the graph as in Figure IV.

If we choose our axes differently, equation (3.3) will have the form

$$S^2 - W^2(t - t')^2 = S^2_{min} ,$$

where $\underline{W}$ is the magnitude of the initial relative velocity, S_{min} is the distance from the origin to the target when the target is nearest, and $\underline{t'}$ is the time corresponding to $S = S_{min}$. For our choice of axes we see that $U = W$ and $Y_0 = S_{min}$.

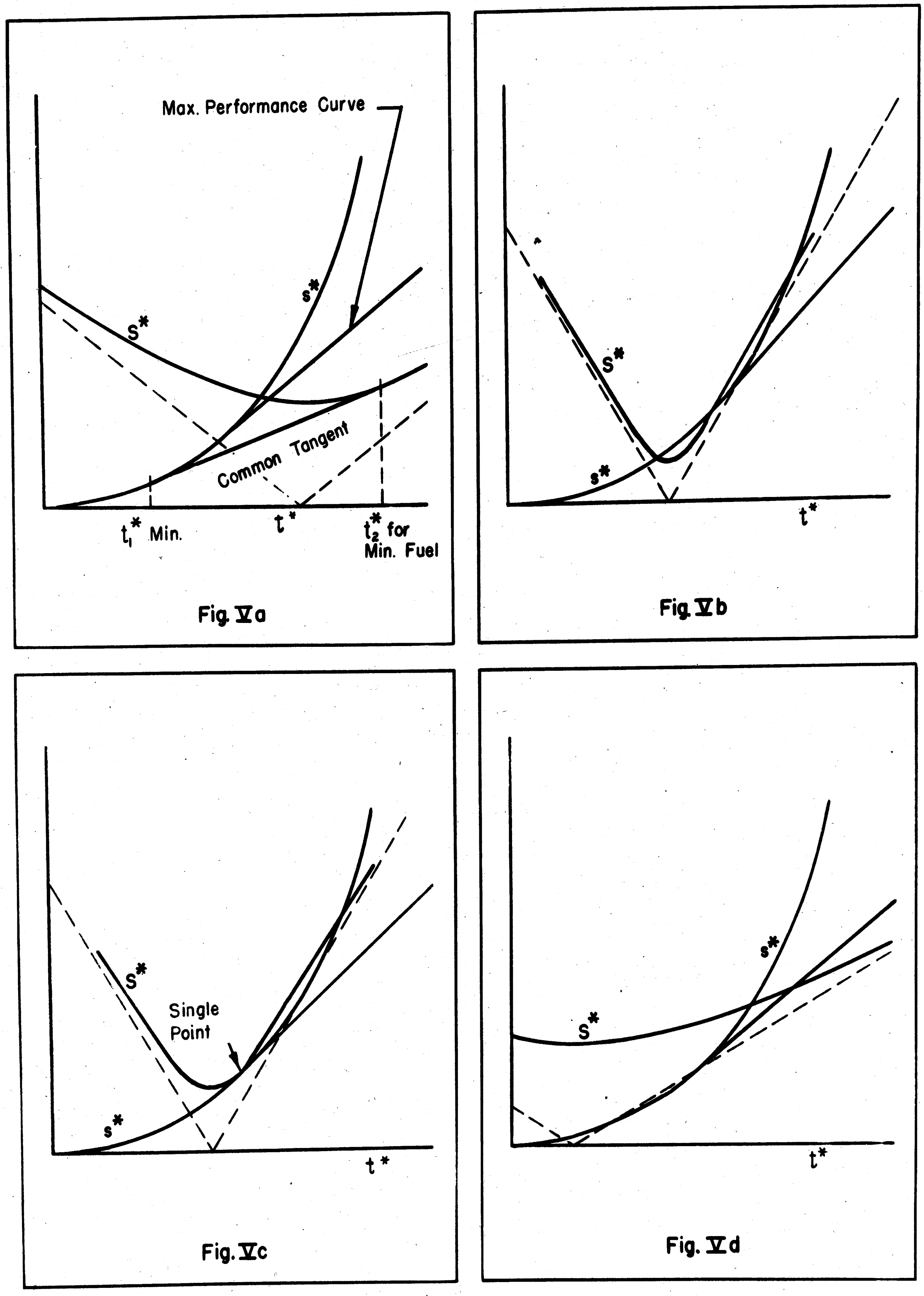

Fig. V Some Possible Intersections of Curves for S^* and s^*

IV. DISCUSSION OF POSSIBLE TIMES OF INTERCEPTION AND OF THE PATHS OF MINIMUM FUEL CONSUMPTION.

The curve of Figure I is the same for all rockets which have similar thrust, except for the maximum fuel consumption (or burning time). The conditions of homing then depend only upon the hyperbola.

We have just seen that we could determine at a glance whether homing could be effected or not: homing was possible at all times when the curve of maximum performance was above or touching the hyperbola.

It is equally easy to determine whether or not a path of minimum fuel consumption exists: the necessary and sufficient condition is that the curve of $\underline{s^*(t^*)}$ and the hyperbola have a common tangent such that $t_1^* < t_2^*$. The burning time $\underline{t_1}$ corresponds to the point of tangency to the curve of $\underline{s^*(t^*)}$, and in this case the homing time $\underline{t_2}$ corresponds to the point of tangency to the hyperbola (see Figure V.a).

For example, if the asymptote of positive slope does not cut the curve of $s^*(t^*)$, then there is a path of minimum fuel consumption. This condition is not necessary for a path of minimum fuel consumption, as is shown in Figure V.b.

There may also be isolated points of interception such that $\underline{t_1*} = \underline{t_2*}$ and such that the two curves have a common tangent as in Figure V.c. These are not paths of minimum fuel consumption since homing is not possible for any neighboring points (times). In this case the two curves for $\underline{s*}$ and $\underline{S*}$ may or may not have a common tangent later, corresponding to a relative minimum. If they have, the relative minimum requires more fuel than the path for the isolated homing time. It can also happen that the curve for $\underline{S*}$ may cross the curve for $\underline{s*}$ (during burning) three times and that the asymptote does not cut the $\underline{s*}$-curve. In this case there are two relative minima, the lower one being also an absolute minimum to fuel consumption. These are the cases referred to in UMM-18, p 12.

If there is no common tangent to the two curves, then there is no path of minimum fuel consumption. There is a critical fuel consumption

$$r'' = 1 - e^{-\frac{W}{c}}$$

which can be approached arbitrarily close. In this case fuel consumption goes up as homing time decreases and, vice versa, as homing time becomes infinite the unit fuel consumption $\underline{r}$ approaches $\underline{r''}$. On the graph this is represented by the curves of $\underline{s*}$ and of

S* being asymptotically parallel. That is, the curve for s* is parallel to and lies on or under the asymptote of the curve for S*.

It is not difficult to show that the paths found above are paths of minimum fuel consumption, for the curves describe all paths for which thrust is fixed in direction and for which thrust is high at first, then zero[1]. Hence we need only consider these paths and find the one for which $r \equiv t_1{}^*$ is lowest.

In the case where it is not possible to throttle the burning, the curve of maximum performance is the only curve allowed. This is the case for the solid fuel rockets available today.

Several other properties come of considering hyperbolas. For example, as pointed out above, there may be no intersection of the two curves or there may be as many as three, all representing possible homing situations.

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[1]These principles were expressed and proved in UMM-18 to be necessary if fuel consumption was to be minimized.

V. EXAMPLE OF A HOMING PROBLEM

As a specific example, consider a rocket with an initial weight of 1500 pounds, thrust 15,000 pounds, specific impulse 250, (and -dW/dt = 60 pounds/sec). Then

$$c = Ig = 8000 \text{ (ft/sec)}$$

$$\dot{r}_o = \frac{-dW}{dt}/W_o = \frac{60}{1500} = .04$$

$$\frac{c}{\dot{r}_o} = \frac{8000}{.04} = 200,000 \text{ (ft)} .$$

Consider an incoming target initially 100,000 feet distant, relative velocity 10,000 ft/sec, the velocity vector 10° (170°) from the position vector. Change to dimensionless form. Then the initial range becomes, in the dimensionless form,

$$S_o^* = \frac{100,000}{200,000} = .5 ,$$

velocity becomes

$$U^* = \frac{10,000}{8,000} = 1.25 ;$$

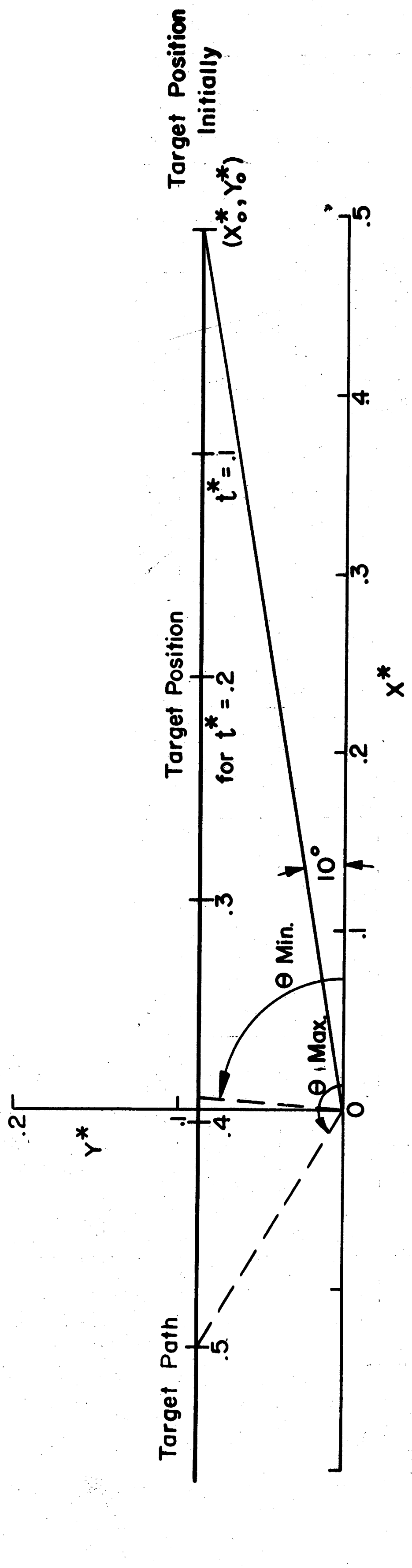

Fig. VII Target Position, Indicating Possible Homing Points

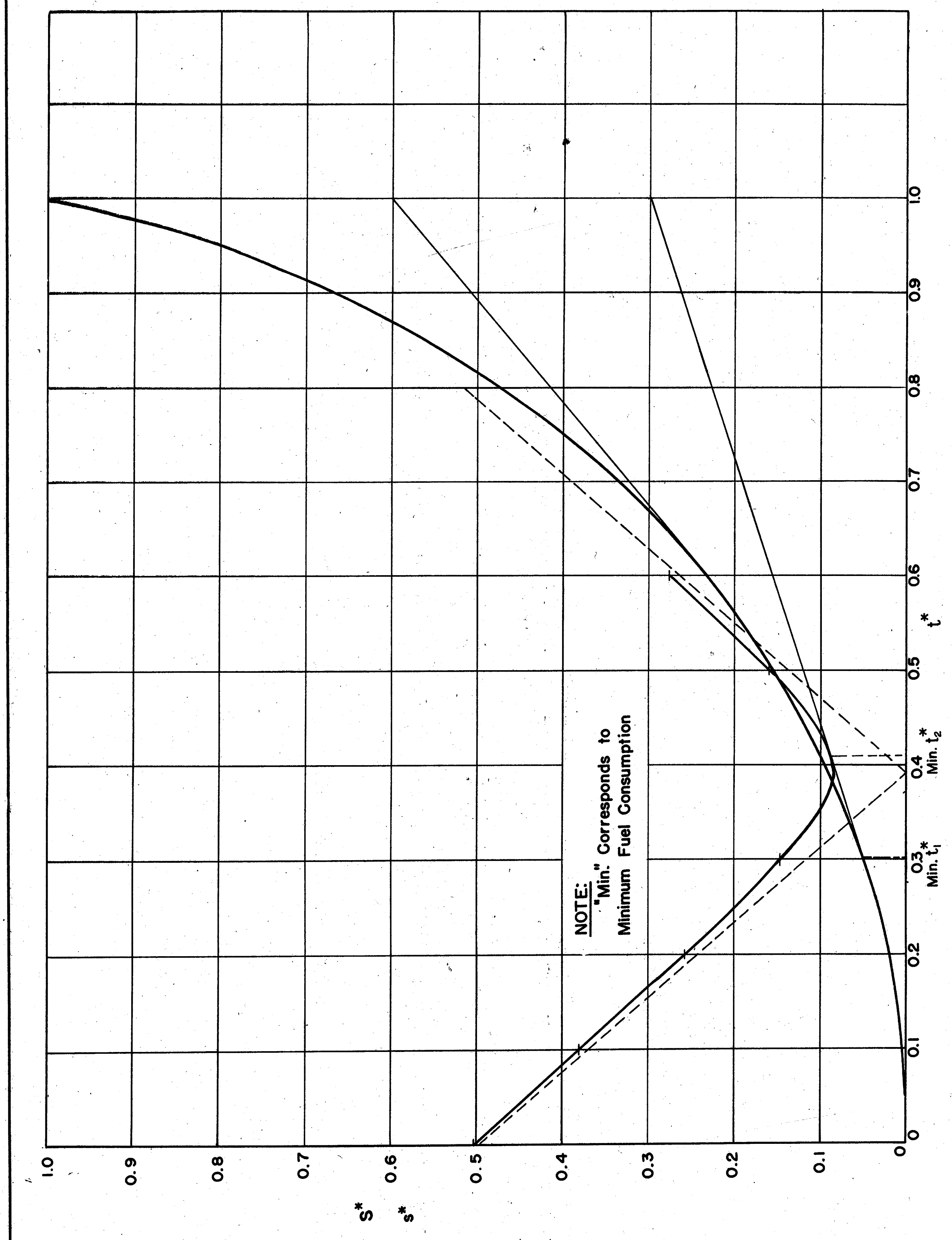

Fig. VII Solution of Homing Problem

and

$$t^* = \frac{t}{25} .$$

Now one way to solve the problem would be to obtain the equation of the hyperbola and plot it. However, a simpler way is to take the graph of the target position versus "time" in the dimensionless form, Figure VI and lay off the target position as a function of t*, say in intervals of .1 for t*. Then the distance from the origin to the point is the ordinate of the hyperbola as a function of "time". Transfer these points onto the curve of rocket performance with a pair of dividers, Figure VII. This is a ruler and compass construction of the hyperbola that can be done quickly. The asymptotes are also known from the time when the target is nearest to the origin and its velocity. Figure VII shows the hyperbola sketched in. As many points as are wanted may be found.

In this particular case homing could take place for $.39 \leq t_2^* \leq .5$, and a burning time $.3 \leq t_1^* < .5$. Since t_1^* corresponds to r, the minimum fuel consumption would be 450 pounds to cause interception. The corresponding angles may be found from Figure VI as $\theta_{min} = 86^\circ \leq \theta \leq 147^\circ = \theta_{max}$ since homing time would be known. The method may be made as accurate as desired.

It seems probable that the parameters could all be fed into a machine that would construct the hyperbola at once and solve the equation. Either this machine would have to be in the craft (this seems unlikely) or the information would have to be converted to information for the craft in space. That is, a ground-based computer would have to obtain position and velocity from some source, and convert this to angle and burning time.

VI. THE SOLUTION OF A HOMING PROBLEM IN SPACE

The solution of the homing problem in Section V was given for the problem after it had been reduced to the two-dimensional form. For practical purposes, the solution can be carried out almost as simply for the three-dimensional form. Let us work out a second example to show the method. The computations involved are simple. The hyperbola is drawn, a homing time is selected from the graph, the burning time is then read from the graph, and then three equations give the direction cosines for the thrust.

Consider the problem in the three-dimensional form. The relative position of the target is given by

$$\begin{aligned} X &= X_o + Ut \\ Y &= Y_o + Vt \qquad (6.1) \\ Z &= Z_o + Wt \end{aligned}$$

in our coordinate system.

Its distance from the origin is

$$(6.2) \qquad S = \sqrt{X^2 + Y^2 + Z^2},$$

$$= \sqrt{(X_0 + Ut)^2 + (Y_0 + Vt)^2 + (Z_0 + Wt)^2}.$$

We shall consider only thrust fixed in direction. Hence the position of the craft is given by

$$x = \lambda s$$

$$(6.3) \qquad y = \mu s$$

$$z = \nu s$$

where λ, μ, ν are the direction cosines of the thrust, and

$$s = \iint^t a\, dt^2 .$$

For the case of constant thrust applied for a time t_1, with a specific fuel consumption r_1, s is given by

$$(2.6) \qquad s(t,t_1) = -ct \ln(1 - r_1) + ct_1 \left[1 + \frac{1}{r_1} \ln (1 - r_1)\right]$$

$$\text{for } t \geqq t_1 .$$

The condition of homing is that there be a time $t_2 \geqq t_1$ such that

$$s(t_2,t_1) = S(t_2) \text{ , and}$$

(6.4)
$$\frac{\lambda}{X_0 + Ut_2} = \frac{\mu}{Y_0 + Vt_2} = \frac{\nu}{Z_0 + Wt_2} .$$

Consider now a specific problem. Let the origin be on the earth with the z-axis vertical initially, and the x- and y-axes any direction to form an orthogonal set. Let the axes be fixed in orientation like the axis of a free gyroscope.

Let the craft be originally at the point x_0, y_0, z_0 with components

400,000, 100,000, 400,000, (feet),

with velocity components, u_0, v_0, w_0

3,000, -1,000, 2,000 (ft/sec).

Let the target be initially at the point X_0', Y_0', Z_0', with components

464,000, 160,000, 448,000, (feet),

and with its velocity components U_0, V_0, W_0 equal respectively to

-3,000, -7,200, -3,056, (ft/sec).

Then the relative position has components X_0, Y_0, Z_0,

64,000, 60,000, 48,000, (feet),

and the relative velocity has components $\underline{U}$, $\underline{V}$, $\underline{W}$,

$$-6{,}000, \qquad -6{,}200, \qquad -5{,}056, \quad \text{(ft/sec)}.$$

(These particular numbers were chosen to make the distance between the two craft 100,000 feet and the relative velocity 10,000 ft/sec.)

Let the craft weigh 1500 pounds initially, let it develop 15,000 pounds thrust, let it carry 400 pounds of fuel, let the specific impulse $\underline{I}$ be 200. Then the effective gas velocity is

$$c = 6{,}400 \qquad \text{(ft/sec)}.$$

Its burning rate $\underline{\dot{r}_o}$ is

$$\dot{r}_o = \frac{15{,}000}{1500\ I} = .05 \ ,$$

and the maximum burned fuel ratio is

$$r_{max} = \frac{400}{1500} = .2666 \ .$$

The maximum burning time is

$$t_{1max} = \frac{.2666}{.05} = 5.33 \text{ sec.}$$

Let us reduce to dimensionless form by dividing distances by

$$c/\dot{r}_o = 128{,}000$$

and times by $1/\dot{r}_o$ (= 20).

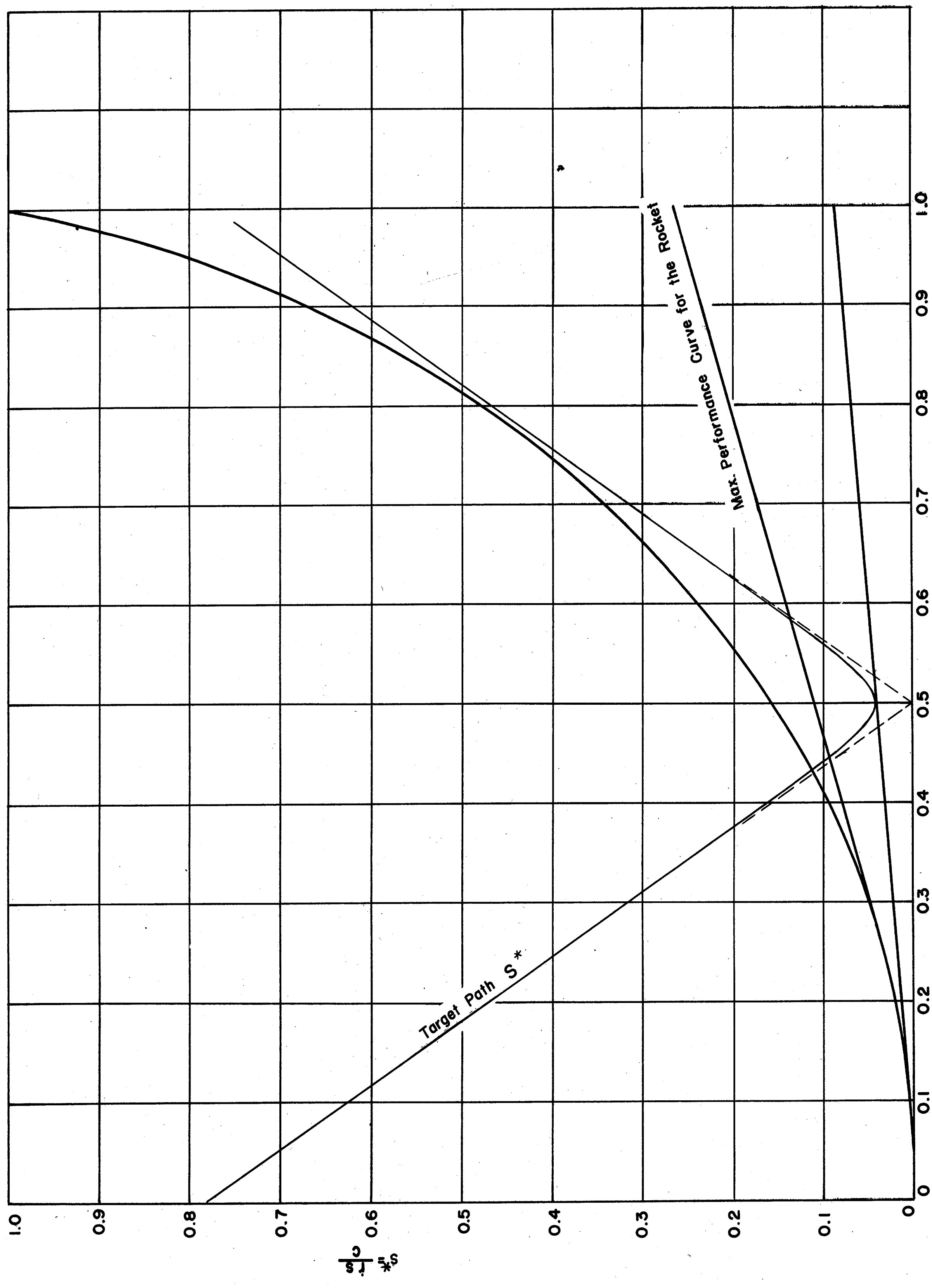

Fig. VIII Solution of Homing Problem

The curve of rocket performance is the graph of Figure I. We plot

$$S^* = \frac{\sqrt{(X_0 + Ut)^2 + (Y_0 + Vt)^2 + (Z_0 + Wt)^2}}{128,000}$$

as a function of <u>t*</u> ($= \dot{r}_0 t$) on the same graph (Figure VIII). We see from the graph that homing is possible for

$$.445 \leqq t^* \leqq .585 ,$$

that is for

$$8.9 \leqq t \leqq 11.7 \qquad \text{(sec)} .$$

The minimum fuel consumption is given by

$$t_1^* = r = .085$$

with a homing time of

$$t = 10 \qquad \text{(sec)} .$$

The direction cosines of thrust, corresponding to these homing times are given by equation (11.4)

t_2	λ	μ	ν
8.9	.882	.401	.249
10	.776	-.388	-.497
11.7	-.348	-.704	-.626

The point of homing may be found by the target position at the time of homing. If the homing time chosen is 10 sec., the point of homing will be

$$X = X_0' + U_0 t_2 = 434,000$$

$$Y = Y_0' + V_0 t_2 = 88,000$$

$$Z = Z_0' + W_0 t_2 - \frac{1}{2} g t_2^2 = 415,890$$

The gravitational acceleration is $30.81 \pm .10$ ft/sec.^{2} on both target and craft[1]. The error by assuming it to be the same on each is less than 10 feet displacement during homing. These distances would be measured in a non-rotating reference frame to avoid the complications of centrifugal forces and Coriolis forces.

The angular error in the above problem (the supplement of the angle between the vectors of relative position and relative velocity) can be found by the methods of analytic geometry, using the formula

$$\cos \beta = - (\lambda_1 \lambda_2 + \mu_1 \mu_2 + \nu_1 \nu_2)$$

where λ_1, μ_1, ν_1, are the direction cosines of the first vector, etc. The angle $\underline{\beta}$ for this problem turns out to be 2.95°.

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[1]These values are based on the value 32.16 for the gravitational constant at sea level.

VII. BASIC THEORY UNDERLYING THE METHOD OF SOLVING THE HOMING PROBLEM

In this section the essentials of this method of solving the homing problem will be pointed out.

In general we will have a rocket craft which we wish to move from one point to another by means of its thrust. This point to which we wish to move it may be a point fixed with respect to the earth, it may be a point fixed in some inertial space or it may be a point which moves according to an arbitrary physical law.

The rocket will usually be moving initially and will be subject to the acceleration of gravity and possibly other specified accelerations. Then the displacement of the rocket with respect to the point is given by

$$(7.1) \qquad \bar{\xi} = \bar{X} - \bar{x}$$

where $\underline{X}$ is the position vector of the point in an arbitrarily chosen rectangular coordinate system whose axes are not rotating and $\underline{x}$ is the position vector of the craft. Now we can write this as

$$(7.2) \qquad \bar{\xi} = -\iint^t \bar{a}\, dt^2 + \bar{\xi}_o + \dot{\bar{\xi}}_o t + \iint^t \bar{G}\, dt^2 - \iint^t \bar{g}\, dt^2 ,$$

where $\bar{a}$ is the acceleration due to thrust, $\bar{\xi}_o$ is the vector of initial relative position, $\dot{\bar{\xi}}_o$ is the vector of initial relative velocity, $\bar{g}(t)$ is the acceleration to which the craft is subject and $\bar{G}(t)$ is the acceleration to which the target point is subject.

The basic assumption of this method is that the vector of displacements due to extraneous forces

$$(7.3) \qquad \iint^t \bar{G}\, dt^2 - \iint^t \bar{g}\, dt^2$$

does not depend upon the thrust applied to the craft. We shall review this assumption in the next section.

Definition. We shall say that homing is effected if there is a time $t_2 > 0$ such that

$$(7.4) \qquad \bar{\xi}(t_2) = \bar{0} .$$

If we denote by $\bar{s}$ and $\bar{S}$ the vectors

$$\bar{s} = \iint^t \bar{a}\, dt^2$$

$$\bar{S} = \bar{\xi}_o + \dot{\bar{\xi}}_o t + \iint^t (\bar{G} - \bar{g})\, dt^2 ,$$

then we can express

$$\bar{\xi} = \bar{S} - \bar{s} \quad (7.6)$$

We have expressed $\bar{\xi}$ in terms of two vectors, one of which depends only upon conditions specified by the target point and the other depends only upon thrust applied to the rocket. The definition of homing can be written

$$\bar{s}(t_2) = \bar{S}(t_2) \quad (7.7)$$

Now for two vectors to be equal the first condition is that their magnitudes be equal:

$$s = S , \quad (7.8)$$

where

$$s = |\bar{s}|$$
$$S = |\bar{S}| .$$

But $\bar{S}$ is a vector known as a function of time. Hence, we can plot S as a function of time. We can also reduce it to the dimensionless form S^* as a function of t^* where[1] $S^* = S/(c/\dot{r}_0)$, $t^* = \dot{r}_0 t$, as in Section III.

Now consider thrust fixed in direction. Then s is a given function of the burning time t_1 and of time t. We can

[1]We assume that $\dot{r}_0$ is not zero; if it is zero, then another parameter needs to be chosen.

draw a graph of s as a function of t_1 and t or of s^* as a function of t_1^* and t^*. Consider the latter.

We will have a set of curves exactly as in Figure III except that in general $S^*(t^*)$ is not a hyperbola. All points such that $S^*(t^*)$ lies below the curve of maximum performance for the rocket represent possible homing times. For any chosen homing time we read the burning time, represented by t_1^*, the abscissa at the point of tangency to the curve for $s^*(t^*)$.

The remaining problem is to determine the direction to apply thrust. The direction cosines are the direction cosines of $\overline{S}(t_2)$ where t_2 is the homing time chosen, namely

$$(7.9) \qquad \lambda(t_2) = \frac{S_x(t_2)}{S_2}, \quad \mu(t_2) = \frac{S_{y_2}}{S_2}, \quad \nu(t_2) = \frac{S_{z_2}}{S_2}$$

where S_x, S_y and S_z are the components of $\overline{S}$.

Paths of minimum fuel consumption can be determined in the same manner as before. If the S-curve has an oscillatory character there may be several relative minima.

VIII. DISCUSSION AND LIMITATIONS OF THE METHOD

Let us consider some cases where the method applies.

First there is the problem of homing against a target in free flight. We assumed that $\bar{G} = \bar{g}$. So long as $\underline{\bar{G} - \bar{g}}$ is an acceleration vector negligible with respect to other accelerations this is a valid assumption.

Second, there is the problem of sending the craft to a specified point in space. For this case $\underline{\bar{G} - \bar{g}}$ is the gravitational acceleration on the rocket. Within the limits that we can estimate this as a function of time, this method is exact, ignoring aerodynamic forces.

Third, the problem of homing against a target which follows a dodging course that can be predicted. Again $\underline{\bar{S}}$ becomes a known vector function of time.

Let us consider some cases where it does not apply.

First, there is the dodging target whose maneuvers are based on intelligence about the pursuing craft. The vector $\underline{\bar{S}}$ is no longer a function of time only since it depends upon the maneuvers of the craft.

Second, if the homing time is large, the displacement

$\iint^t (\bar{G} - \bar{g})\, dt^2$ may not permit accurate evaluation.

Third, the time required for rotation may be appreciable and require consideration.

The condition that $\iint^t (\bar{G} - \bar{g})\, dt^2$ can be computed as an explicit function of time is important for three reasons.

First, if it is satisfied, the paths determined as above are the best paths that can be found from the point of view of obtaining maximum correction for given fuel consumption, and conversely, for obtaining a definite correction with minimum fuel consumption.

Second, if it is satisfied, the problem of homing is reduced to a problem in algebra and the determination of paths of minimum fuel consumption, if they exist, is reduced to a problem in calculus with a direct graphic solution.

Third, if it is not satisfied, the homing problem is very difficult to solve.

One feature which can hardly be overemphasized is the gain in reducing to dimensionless terms. In that way, a single grid covers all cases and the number of significant variables is reduced to a minimum.

Of course, if a detailed analysis of a single craft is to be made, it may be more satisfactory to work with $\underline{s}$, $\underline{S}$, $\underline{t}$, etc. The graph for $\underline{s}$ vs. $\underline{t}$ can be drawn up exactly as the curve for $\underline{s^*}$ vs. $\underline{t^*}$ was. If the curve for $\underline{s^*}$ is available, the curve for $\underline{s}$ can be obtained directly from it; indeed the same curve can be used by altering the scales on the axes. All operations are performed as before except that times, distances, etc., are read in seconds, feet, etc., and require no conversion.

IX. SOME APPLICATIONS OF THE METHOD TO PROBLEMS OF DESIGN

The method can be used to get the answer to several design problems.

As an example, assume that a set of target conditions is chosen and we wish to determine some craft specifications for homing. Assume that the specific impulse is given[1].

Now choose, as a first approximation a value of $\underline{\dot{r}_0}$ that seems reasonable. Draw the graph of $\underline{S^*}$ on the grid of Figure I. There are three possibilities. If the value of $\underline{\dot{r}_0}$ is well chosen, a figure is obtained similar to Figures IX.a, or IX.b. If $\underline{\dot{r}_0}$ is chosen too small, the resulting figure will look like Figure IX.c. If $\underline{\dot{r}_0}$ is chosen too large, the figure will look like Figure IX.d; the grid will appear dwarfed beside the hyperbola (this corresponds to thrust too large). It is best to choose a new value of $\underline{\dot{r}_0}$ in the latter cases so that the resultant figure is similar to IX.a. Otherwise, one of the

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[1]The specific impulse depends principally upon the fuel for a well-designed craft.

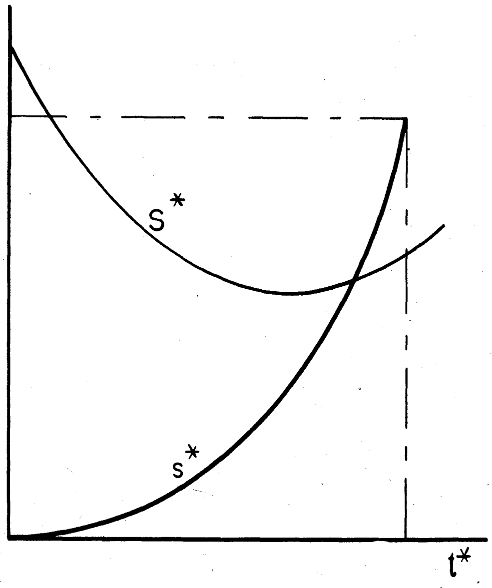

Fig. IX a

$\dot{r}_0$ in the proper range

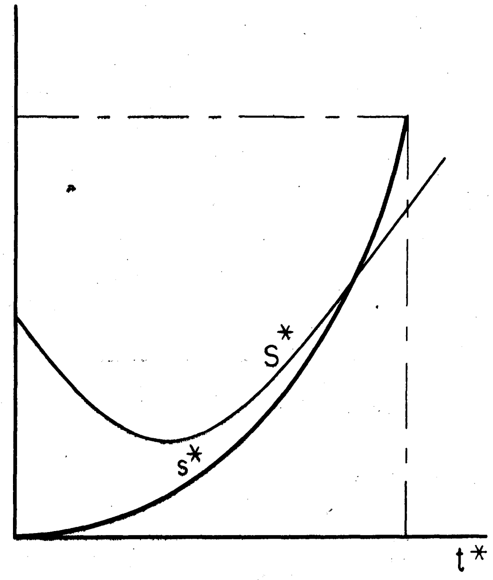

Fig. IX b

$\dot{r}_0$ in the proper range

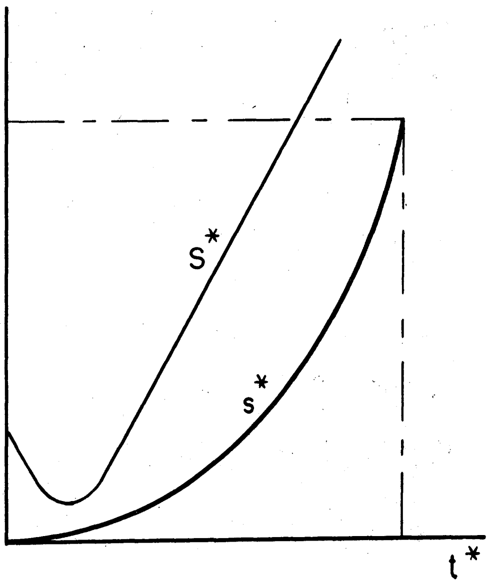

Fig. IX c

$\dot{r}_0$ too small

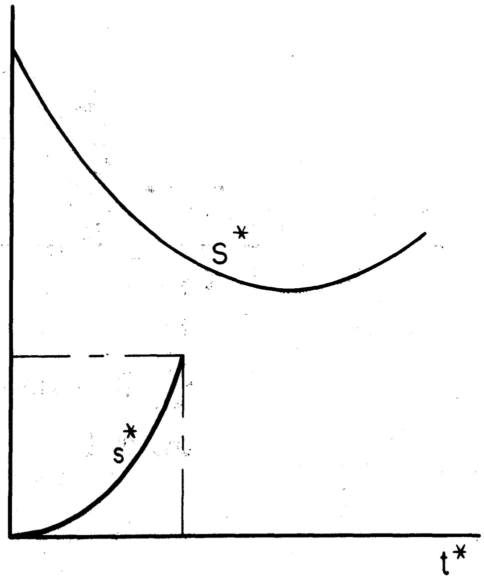

Fig. IX d

$\dot{r}_0$ too large

Fig. IX Grids for Various Values of $\dot{r}_0$ and Fixed Target Conditions

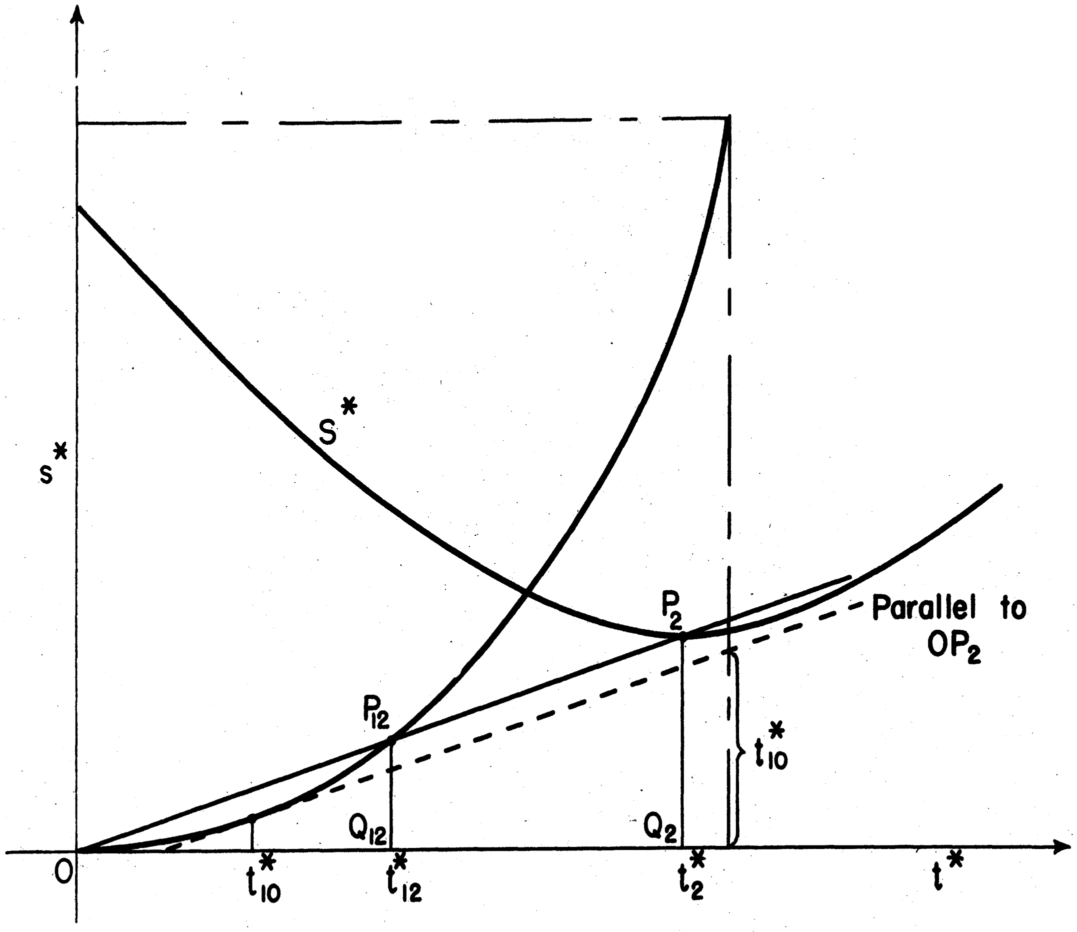

Fig. X Determination of $\dot{t}_0$ for Interception and of Lower Bound to Fuel Consumption.

two figures, the grid or the hyperbola, is so small compared to the other that interpretations are difficult.

Two or three trials will give a reasonable value for $\dot{r}_0$. Then one may refine the choice of $\dot{r}_0$ in various ways.

First, let us show that for a chosen homing time, we can pick the minimum value of $\dot{r}_0$ that will cause homing at a selected homing time. This value is the value for burning all the way. Assume that we have the graph in a form like Figure IX.a, see Figure X.

Choose any value of t_2^* (or of t_2). Designate the point $(t_2^*, S^*(t_2^*))$ as P_2. Draw the line OP_2. Designate the point where it cuts the curve of $s^*(t^*)$ by P_{12}. Designate the abscissa of P_{12} by t_{12}^*.

Then the burning rate for homing at time t_2^* by burning all the way is

$$(9.1) \qquad \dot{\rho}_0 = \frac{t_{12}^* \, \dot{r}_0}{t_2^*}$$

where $\dot{r}_0$ is the value for which the curve was originally drawn.

It is a matter of direct substitution into the equations of Section II, Article C to show this. We have from Figure X

$$(9.2) \qquad S^*(t_2^*) = s^*(t_{12}^*) \, \frac{t_2^*}{t_{12}^*}$$

since $OQ_{12}P_{12}$ and OQ_2P_2 are similar triangles.

Now $S(t_2) = \frac{c}{\dot{r}_0} S^*(t_2^*)$, $s(t_{12}) = \frac{c}{\dot{r}_0} s^*(t_{12}^*)$, and $\frac{t_2^*}{t_{12}^*} = \frac{t_2}{t_{12}}$.

So equation (9.2) becomes

(9.3) $$S(t_2) = \frac{t_2}{t_{12}} s(t_{12}) \, .$$

In Article II.C. we saw that the burned fuel ratio for two craft satisfied the relation

(2.12) $$\rho(t) = r(kt) \, .$$

Hence, letting $k = t_{12}/t_2$ we obtain the desired result,

(9.4) $$\sigma(t_2) = \frac{t_2}{t_{12}} s(t_{12})$$

$$= S(t_2)$$

if

(9.5) $$\dot{\rho}_0 = \frac{t_{12}^*}{t_2^*} \dot{r}_0 \, .$$

A rate of burning less than $\dot{\rho}_0$ cannot effect interception at the selected time and any rate of burning greater than $\dot{\rho}_0$ will effect homing at the desired time if t_1 is properly chosen.

From equation (2.12) above we obtain the relation

$$\rho(t_2) = r(kt_2)$$

$$= r(t_{12}) \, ,$$

or

(9.6) $$\rho(t_2) = t_{12}^* \, .$$

Hence $\underline{t_{12}*}$ is the unit fuel consumption for burning all the way.

In the illustration $\underline{t_2*}$ was greater than $\underline{t_{12}*}$. This was irrelevant to the method and the result.

However it is seen that certain points on the hyperbola, those for which $S*/t* > 1$, have no corresponding point $\underline{P_{12}}$ on the $\underline{s*}$-curve. For these points it is simply not possible to home by burning all the way. We see that for $\underline{\dot{r}}$ a constant

$$s*(t*) < t* \qquad \text{for } t* < 1 ,$$

(9.7)
$$\lim_{t*\to 1} s* = \lim_{t*\to 1} \left[t* + (1 - t*) \ln (1 - t*)\right]$$

$$= 1 ,$$

as indicated on the graph. This gives the result

(9.8)
$$\frac{s}{t} < c \qquad \text{for } r < 1 ;$$

in other words, the average velocity during burning is less than $\underline{c}$, the effective gas velocity.

Another quantity which may be read directly from the graph is the fuel consumption corresponding to burning in an impulse. In Figure X, draw the line (shown dotted) parallel to $\underline{OP_2}$ and tangent to the $\underline{s*}$-curve. The abscissa $\underline{t_{10}*}$ of the point of tangency gives the value of $\underline{r}$ corresponding to an impulse. This can be shown by considerations on velocity similar to those carried out on distance in the previous paragraphs, using the similarity considerations given in Section II.

This value $\underline{t_{10}*}$ is the greatest lower bound to fuel consumption[1] for the homing time $\underline{t_2}$. Fuel consumption is monotonic increasing with burning time for similar burning. Hence we have bounds to the fuel consumption for homing at the selected time t_2

$$t_{10}^* \leq r \leq t_{12}^* .$$

If the amount of fuel is designated, this will show whether or not homing can be effected.

The method of solution is also applicable to the problem of homing with minimum acceleration. A memorandum is being prepared on this.

[1] This was shown in detail in UMM-18.

X. TWO THEOREMS ON HOMING

Some interesting results follow from simple consideration of the hyperbolas and the grids.

THEOREM I[1]. If the burnt fuel ratio r can exceed $r'' = 1 - e^{-\frac{W}{c}}$, then homing can always be effected.

Proof. The asymptote to the hyperbola will have slope $W^* = \frac{w}{c}$ and the hyperbola approaches it arbitrarily closely as t* becomes infinite. Since $r > r''$, the curve of maximum performance has a slope greater than W*. Hence for t* sufficiently great there are points on the hyperbola below those on the curve of maximum performance and these represent possible homing paths.

THEOREM II. For an outgoing target the lower bound to the burnt fuel ratio is $r'' = 1 - e^{-\frac{W}{c}}$.

Proof. For an outgoing target, the center of the hyperbola is on the t*-axis either at or to the left of the

[1]These theorems were stated in UMM-18 as Corollaries VI.2 and VI.3 and the reference given is to this proof.

origin. If $r < r''$ then the curve of maximum performance cannot touch the hyperbola, hence homing is not effected. By Theorem I, if $r > r''$ homing can be effected.

It is interesting that this limit does not depend upon the manner of burning; that is, upon the particular thrust function. We saw that the existence of paths of minimum fuel consumption depended upon the rate of burning.

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