Lyapunov-stable discrete-time model reference adaptive control

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SUMMARY

Discrete-time model reference adaptive control (MRAC) is considered with both least squares and projection algorithm parameter identification. For both cases complete Lyapunov proofs are given for stability and convergence. The results extend the approach of Johansson (*Int. J. Control* 1989; **50**(3): 859–869) to include Lyapunov stability for MRAC when the normalized projection algorithm is used for parameter identification. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: model reference adaptive control; discrete-time; Lyapunov stable

1. INTRODUCTION

In model reference adaptive control (MRAC), the objective is to have the plant emulate the dynamics of a specified model in response to a family of command signals. Model reference adaptive control has been extensively developed for continuous-time systems [1] and discrete-time systems [2], where the boundedness of the controller parameters and the convergence of the tracking error are demonstrated using the Gronwall–Bellman lemma and the key technical lemma, respectively. The objective of the present paper is to unify and extend discrete-time MRAC by constructing Lyapunov functions for error convergence and Lyapunov stability.

Discrete-time MRAC algorithms have been based on a variety of parameter identification algorithms. In particular, the recursive least squares (RLS) algorithm and the projection algorithm are used in Reference [3], where convergence is based on the key technical lemma. This method of proof yields convergence but does not imply Lyapunov stability of the error system. MRAC is considered in the presence of additive noise in References [2, 4–6]. In these results, convergence of the tracking error and parameters is guaranteed almost surely, but stochastic Lyapunov stability is not demonstrated.

Lyapunov stability of discrete-time MRAC and convergence of the error to a finite set is demonstrated in Reference [7], where the RLS algorithm is used for parameter identification and

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a Lyapunov candidate is applied to the time-varying error system. Stochastic Lyapunov stability of MRAC is addressed [8].

The novel Lyapunov construction of [7–10] is of independent interest since it involves the logarithm of a quadratic form. A similar construction is used in Reference [11] for full-state-feedback adaptive stabilization and in Reference [12] to a more general class of direct adaptive stabilization algorithms with normalized adaptive laws.

In view of these developments, the present paper has the following objectives. First, we refine the proof of Lyapunov stability and convergence for RLS parameter identification, where we account for the time-varying dynamics of the error system and clarify the assumptions under which the construction is valid. Next, we construct a Lyapunov proof of MRAC for the projection algorithm. These constructs remove the need for the key technical lemma used in Reference [3]. Finally, we simulate both MRAC algorithms to demonstrate their performance.

The contents of the paper are as follows. In Section 2 we present the solution to the model matching control problem in the case of a known plant. In Section 3 the model matching error dynamics in the case of an unknown plant are formulated. An adaptive control law with RLS parameter identification is derived in Section 4. Stability of RLS adaptive control is demonstrated in Section 5. An adaptive control law with the projection algorithm for parameter identification is presented in Section 6, and a proof of stability is given in Section 7. Section 8 presents simulation results. Finally, some concluding remarks on future extensions are made in Section 9.

2. MODEL REFERENCE CONTROL FOR A KNOWN PLANT

Consider a SISO process described by the DARMA model

$$y(k) = -\sum_{i=1}^{n} a_i y(k-i) + \sum_{j=0}^{m} b_j u(k-j), \quad k \geqslant 0 \tag{1}$$

The model (1) can be written in terms of the forward shift operator $\mathbf{q}$ as

$$\mathbf{A}(\mathbf{q})y(k) = \mathbf{B}(\mathbf{q})u(k) \tag{2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are polynomials of degree n and m, respectively, defined by

$$\mathbf{A}(\mathbf{q}) \triangleq \mathbf{q}^n + a_1 \mathbf{q}^{n-1} + \cdots + a_n \tag{3}$$

and

$$\mathbf{B}(\mathbf{q}) \triangleq b_0 \mathbf{q}^m + b_1 \mathbf{q}^{m-1} + \cdots + b_m \tag{4}$$

where $b_0 \neq 0$. We define the delay $d \triangleq n - m$ and make the following assumptions about the plant.

Assumption 2.1
The realization (2) is minimal, i.e. $\mathbf{A}$ and $\mathbf{B}$ are coprime.

Assumption 2.2
All roots of $\mathbf{B}(\mathbf{q})$ are inside the unit circle.

Assumption 2.3
n and m are known, and $m < n$.

Assumption 2.4
b_0 is known.

To modify the dynamics (2) we consider the 2-DOF model matching control law

$$u(k) = \frac{\mathbf{T(q)}}{\mathbf{R(q)}} u_{c}(k) - \frac{\mathbf{S(q)}}{\mathbf{R(q)}} y(k) \tag{5}$$

where u_c is the command signal. The closed-loop system (2)–(5) (see Figure 1) is then given by

$$y(k) = \frac{\mathbf{B(q)T(q)}}{\mathbf{A(q)R(q)} + \mathbf{B(q)S(q)}} u_{c}(k) \tag{6}$$

We want the response from the command signal u_c to the output y to be described by the reference model

$$y_{m}(k) = \frac{\mathbf{B_m(q)}}{\mathbf{A_m(q)}} u_{c}(k) \tag{7}$$

We make the following assumptions about the reference model.

Assumption 2.5
$\mathbf{A_m(q)}$ is monic and stable.

Assumption 2.6
$\deg \mathbf{A_m(q)} - \deg \mathbf{B_m(q)} = d$, i.e. the reference model has the same delay as the plant.

The closed-loop system (6) and the reference model (7) have the same forced response if

$$\frac{\mathbf{B(q)T(q)}}{\mathbf{A(q)R(q)} + \mathbf{B(q)S(q)}} = \frac{\mathbf{B_m(q)}}{\mathbf{A_m(q)}} \tag{8}$$
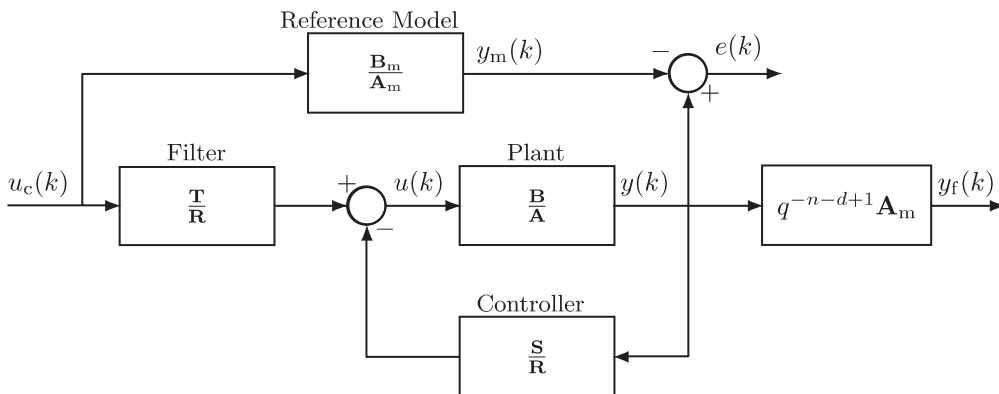
Figure 1. Model reference control.

which is equivalent to

$$\frac{\mathbf{T(q)}}{\mathbf{A(q)R(q)} + \mathbf{B(q)S(q)}} = \frac{\mathbf{B_m(q)}}{\mathbf{A_m(q)B(q)}} \tag{9}$$

The roots of the closed-loop characteristic polynomial $\mathbf{A_m(q)B(q)}$ consist of the roots of $\mathbf{A_m(q)}$ as well as the roots of $\mathbf{B(q)}$, all of which are stable by assumption. Let $n_m \overset{\triangle}{=} \deg \mathbf{A_m(q)}$ and define

$$\mathbf{P(q)} \overset{\triangle}{=} \mathbf{A_m(q)B(q)} = b_0 \mathbf{q}^{(n_m+m)} + p_1 \mathbf{q}^{(n_m+m)-1} + \cdots + p_{n_m+m} \tag{10}$$

To satisfy (9) it suffices to choose

$$\mathbf{T(q)} = \mathbf{B_m(q)} \tag{11}$$

and require that $\mathbf{R(q)}$ and $\mathbf{S(q)}$ satisfy

$$\mathbf{A(q)R(q)} + \mathbf{B(q)S(q)} = \mathbf{P(q)} \tag{12}$$

Defining $n_{\mathbf{R}} \overset{\triangle}{=} \deg \mathbf{R(q)}$, $n_{\mathbf{S}} \overset{\triangle}{=} \deg \mathbf{S(q)}$, $n_e \overset{\triangle}{=} n + n_{\mathbf{R}} + 1$, and $n_u \overset{\triangle}{=} n_{\mathbf{R}} + n_{\mathbf{S}} + 2$, (12) can be written as

$$M \begin{bmatrix} \mathscr{C}(\mathbf{R}) \\ \mathscr{C}(\mathbf{S}) \end{bmatrix} = \mathscr{C}(\mathbf{P}) \tag{13}$$

where $M \in \mathbb{R}^{n_e \times n_u}$ is the Sylvester matrix

$$M \overset{\triangle}{=} \begin{bmatrix} 1 & 0 & 0_{(n-1)\times 1} & 0_{d\times 1} & 0_{(d+1)\times 1} & 0_{(n+d-1)\times 1} \\ a_1 & 1 & 1 & b_0 & b_0 & b_0 \\ a_2 & a_1 & a_1 & b_1 & b_1 & b_1 \\ \vdots & \vdots & a_2 & \vdots & \vdots & \ddots & b_2 \\ a_n & a_n & \ddots & \vdots & b_m & b_m & \vdots \\ 0_{(n-1)\times 1} & 0_{(n-2)\times 1} & a_n & 0_{(n-1)\times 1} & 0_{(n-2)\times 1} & b_m \end{bmatrix}$$

Furthermore, $\mathscr{C}(\mathbf{R}) = [\, r_0 \;\; r_1 \;\; \cdots \;\; r_{n_{\mathbf{R}}} \,]^{\mathrm{T}}$, $\mathscr{C}(\mathbf{S}) = [\, s_0 \;\; s_1 \;\; \cdots \;\; s_{n_{\mathbf{S}}} \,]^{\mathrm{T}}$, and $\mathscr{C}(\mathbf{P}) = [\, b_0 \;\; p_1 \;\; \cdots \;\; p_{n_m+m} \,]^{\mathrm{T}}$ are vectors containing the coefficients of $\mathbf{R(q)}$, $\mathbf{S(q)}$, and $\mathbf{P(q)}$, respectively. In the remainder of the paper, we omit the explicit dependence of polynomial operators on $\mathbf{q}$.

Proposition 2.1
Assume that $n_m = 2n - m - 1$ and $n_{\mathbf{S}} = n - 1$. Then, for each $n_{\mathbf{R}} \geqslant 0$ there exist unique polynomials $\mathbf{R}$ and $\mathbf{S}$ satisfying (13), or equivalently (12). Furthermore, if $n_{\mathbf{R}} \geqslant n_{\mathbf{S}}$ then the control law (5) is causal.

Proof
Since $n_{\mathbf{S}} = n - 1$ it follows that $M \in \mathbb{R}^{(n+n_{\mathbf{R}}+1)\times(n+n_{\mathbf{R}}+1)}$ is square. Also, since $\mathbf{A}$ and $\mathbf{B}$ are relatively coprime, it follows that M is non-singular and the solution $[\, \mathscr{C}(\mathbf{R}) \;\; \mathscr{C}(\mathbf{S}) \,]^{\mathrm{T}}$ to (13) is unique. From (11) we have

$$\deg \mathbf{T} = \deg \mathbf{B_m} = \deg \mathbf{A_m} - d = 2n - m - 1 - n + m = n - 1 \tag{14}$$

Finally if $n_{\mathbf{R}} \geqslant n_{\mathbf{S}}$, then $\deg \mathbf{R} \geqslant \deg \mathbf{S} = \deg \mathbf{T}$, and thus the model matching controller (5) is causal. $\qquad\square$

Henceforth in accordance with Proposition 2.1 we assume that $\deg \mathbf{S} = n - 1$ and $\deg \mathbf{A}_{\mathrm{m}} = 2n - m - 1$ so that $\deg \mathbf{P} = 2n - 1$. Also, to obtain a minimum degree causal controller we assume that $\deg \mathbf{R} = n - 1$ so that $M \in \mathbb{R}^{2n \times 2n}$. Hence we write

$$\mathbf{R} \triangleq r_0 \mathbf{q}^{n-1} + r_1 \mathbf{q}^{n-2} + \cdots + r_{n-1} \tag{15}$$

and

$$\mathbf{S} \triangleq s_0 \mathbf{q}^{n-1} + s_1 \mathbf{q}^{n-2} + \cdots + s_{n-1} \tag{16}$$

where r_0 and s_0 are non-zero. In fact, it follows from (12) and (15) that $r_0 = b_0$.

Next to obtain a linear estimation model in terms of the controller we define the filtered output signal (see Figure 1)

$$y_{\mathrm{f}}(k) \triangleq \mathbf{q}^{-n-d+1} \mathbf{A}_{\mathrm{m}} y(k) \tag{17}$$

Then it follows from (2) that

$$y_{\mathrm{f}}(k) = \frac{\mathbf{q}^{-n-d+1} \mathbf{A}_{\mathrm{m}} \mathbf{B}}{\mathbf{A}} u(k) \tag{18}$$

With the model matching condition (12), y_{f} satisfies

$$\begin{aligned}
y_{\mathrm{f}}(k+d) &= \frac{\mathbf{q}^{-n+1}(\mathbf{AR} + \mathbf{BS})}{\mathbf{A}} u(k) \\
&= \left(\mathbf{R} + \mathbf{S} \frac{\mathbf{B}}{\mathbf{A}} \right) u(k-n+1) \\
&= \mathbf{R} u(k-n+1) + \mathbf{S} y(k-n+1)
\end{aligned} \tag{19}$$

Since $r_0 = b_0$ (19) can be written as the linear identification model

$$y_{\mathrm{f}}(k+d) = b_0 u(k) + \varphi^{\mathrm{T}}(k) \theta \tag{20}$$

where the parameter vector θ and the regressor $\varphi(k)$ are defined by

$$\theta \triangleq [\, r_1 \quad \cdots \quad r_{n-1} \quad s_0 \quad \cdots \quad s_{n-1} \,]^{\mathrm{T}} \in \mathbb{R}^{2n-1} \tag{21}$$

and

$$\varphi(k) \triangleq [\, u(k-1) \quad \cdots \quad u(k-n+1) \quad y(k) \quad \cdots \quad y(k-n+1) \,]^{\mathrm{T}} \in \mathbb{R}^{2n-1} \tag{22}$$

Using (19) and (20) the model matching control law (5) can be written as

$$u(k) = -\frac{1}{b_0} [\varphi^{\mathrm{T}}(k) \theta - \mathbf{q}^{-n+1} \mathbf{B}_{\mathrm{m}} u_{\mathrm{c}}(k)] \tag{23}$$

The filtered plant model (20) and the control law (23) are now in a form suitable for direct adaptive control.

3. MODEL MATCHING ERROR DYNAMICS

When the plant (2) is unknown we cannot solve (13) for the controller parameters $\mathbf{R}$ and $\mathbf{S}$. Hence, let $\hat{\mathbf{R}}(k)$ and $\hat{\mathbf{S}}(k)$ be polynomials in $\mathbf{q}$ that are estimates of $\mathbf{R}$ and $\mathbf{S}$ at time k. Then in place of (5), the estimated model matching controller is

$$u(k) = \frac{\mathbf{B}_{\mathrm{m}}}{\hat{\mathbf{R}}(k)} u_{\mathrm{c}}(k) - \frac{\hat{\mathbf{S}}(k)}{\hat{\mathbf{R}}(k)} y(k) \tag{24}$$

With (24) the closed-loop system has the form

$$y(k) = \frac{\mathbf{B}\mathbf{B}_{\mathrm{m}}}{\mathbf{A}\hat{\mathbf{R}}(k) + \mathbf{B}\hat{\mathbf{S}}(k)} u_{\mathrm{c}}(k) \tag{25}$$

Next let $\hat{\theta}(k)$ denote an estimate of θ at time k and define the parameter error

$$\tilde{\theta}(k) \triangleq \hat{\theta}(k) - \theta \tag{26}$$

and the filtered output error signal (see Figure 2)

$$e_{\mathrm{f}}(k) \triangleq y_{\mathrm{f}}(k) - \mathbf{q}^{-n-d+1}\mathbf{B}_{\mathrm{m}}u_{\mathrm{c}}(k) \tag{27}$$

To express $e_{\mathrm{f}}(k)$ in terms of $\tilde{\theta}$, note that

$$y_{\mathrm{f}}(k+d) = \mathbf{q}^{-n+1}\frac{\mathbf{A}_{\mathrm{m}}\mathbf{B}\mathbf{B}_{\mathrm{m}}}{\mathbf{A}\hat{\mathbf{R}}(k) + \mathbf{B}\hat{\mathbf{S}}(k)} u_{\mathrm{c}}(k) = \mathbf{q}^{-n+1}\frac{\mathbf{A}\mathbf{R} + \mathbf{B}\mathbf{S}}{\mathbf{A}\hat{\mathbf{R}}(k) + \mathbf{B}\hat{\mathbf{S}}(k)}\mathbf{B}_{\mathrm{m}}u_{\mathrm{c}}(k)$$

$$= \mathbf{q}^{-n+1}\frac{\mathbf{R} + \dfrac{\mathbf{B}\mathbf{S}}{\mathbf{A}}}{\hat{\mathbf{R}}(k) + \dfrac{\mathbf{B}\hat{\mathbf{S}}(k)}{\mathbf{A}}}\mathbf{B}_{\mathrm{m}}u_{\mathrm{c}}(k) = \mathbf{q}^{-n+1}\frac{\mathbf{R}u(k) + \mathbf{S}y(k)}{\hat{\mathbf{R}}(k)u(k) + \hat{\mathbf{S}}(k)y(k)}\mathbf{B}_{\mathrm{m}}u_{\mathrm{c}}(k)$$

$$= \frac{b_0 u(k) + \varphi^{\mathrm{T}}(k)\theta}{\mathbf{q}^{n-1}[b_0 u(k) + \varphi^{\mathrm{T}}(k)\hat{\theta}(k)]}\mathbf{B}_{\mathrm{m}}u_{\mathrm{c}}(k) \tag{28}$$
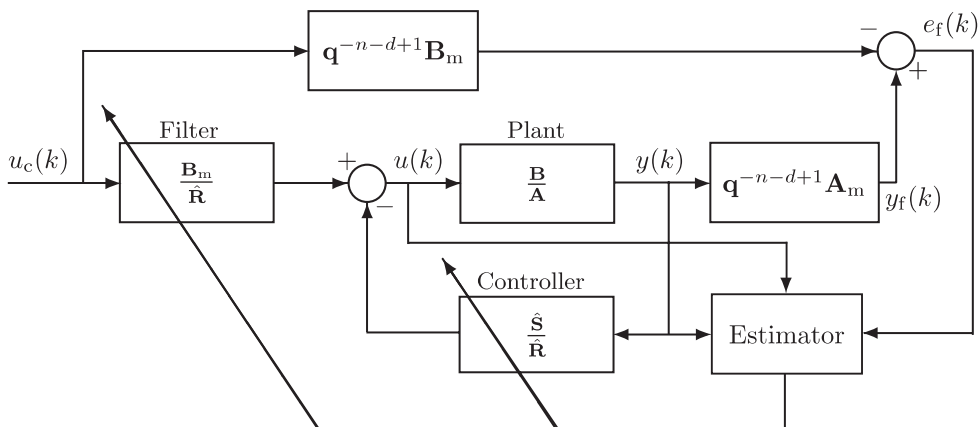

Figure 2. Model reference adaptive control block diagram.

Combining (20) and (28) yields

$$b_0 u(k) + \varphi^{\mathrm{T}}(k)\hat{\theta}(k) = \mathbf{q}^{-n+1}\mathbf{B}_{\mathrm{m}} u_{\mathrm{c}}(k) \tag{29}$$

From (20), (27) and (29) it follows that

$$\begin{aligned}
e_{\mathrm{f}}(k+d) &= y_{\mathrm{f}}(k+d) - \mathbf{q}^{-n+1}\mathbf{B}_{\mathrm{m}} u_{\mathrm{c}}(k) \\
&= b_0 u(k) + \varphi^{\mathrm{T}}(k)\theta - b_0 u(k) - \varphi^{\mathrm{T}}(k)\hat{\theta}(k) \\
&= -\varphi^{\mathrm{T}}(k)\tilde{\theta}(k) \tag{30}
\end{aligned}$$

To formulate the model matching error dynamics we note that the plant (2) can be written in the nth order fraction form as [13] (see Figure 3(a))

$$\mathbf{A}\xi(k-n) = u(k) \tag{31}$$

$$y(k) = \mathbf{B}\xi(k-n) \tag{32}$$

From (17) and (32) it follows that

$$y_{\mathrm{f}}(k+d) = \mathbf{q}^{-n+1}\mathbf{A}_{\mathrm{m}} y(k) = \mathbf{q}^{-n+1}\mathbf{A}_{\mathrm{m}}\mathbf{B}\xi(k-n) = \mathbf{q}^{-2n+1}\mathbf{P}\xi(k) \tag{33}$$

In a similar manner the reference model can be written in the $2n-1$th order non-minimal fraction form (see Figure 3(b))

$$\mathbf{A}_{\mathrm{m}}\mathbf{B}\xi_{\mathrm{m}}(k-n) = \mathbf{B}_{\mathrm{m}} u_{\mathrm{c}}(k) \tag{34}$$

$$y_{\mathrm{m}}(k) = \mathbf{B}\xi_{\mathrm{m}}(k-n) \tag{35}$$

From (34) it follows that

$$\mathbf{q}^{-n+1}\mathbf{B}_{\mathrm{m}} u_{\mathrm{c}}(k) = \mathbf{q}^{-n+1}\mathbf{A}_{\mathrm{m}}\mathbf{B}\xi_{\mathrm{m}}(k-n) = \mathbf{q}^{-2n+1}\mathbf{P}\xi_{\mathrm{m}}(k) \tag{36}$$

Using (33) and (36), the d-step ahead filtered output error can now be written as

$$\begin{aligned}
e_{\mathrm{f}}(k+d) &= y_{\mathrm{f}}(k+d) - \mathbf{q}^{-n+1}\mathbf{B}_{\mathrm{m}} u_{\mathrm{c}}(k) \\
&= \mathbf{q}^{-2n+1}\mathbf{P}\,[\xi(k) - \xi_{\mathrm{m}}(k)] \\
&= \mathbf{q}^{-2n+1}\mathbf{P}\xi_{\mathrm{e}}(k) \tag{37}
\end{aligned}$$
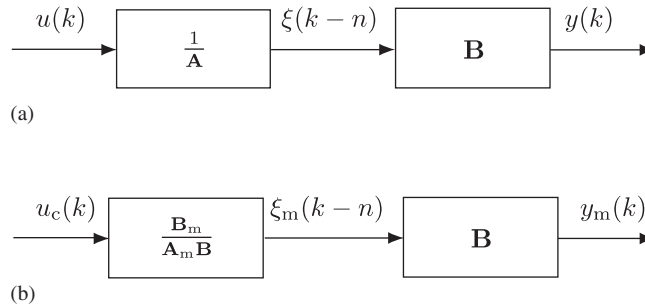
Figure 3. Fraction forms of the plant and the reference model.

where

$$\xi_e(k) \stackrel{\triangle}{=} \xi(k) - \xi_m(k) \tag{38}$$

Next define plant, reference model, and model-matching error states by

$$x(k) \stackrel{\triangle}{=} [\, \xi(k-1) \quad \cdots \quad \xi(k-2n+1)\,]^T \in \mathbb{R}^{2n-1} \tag{39}$$

$$x_m(k) \stackrel{\triangle}{=} [\, \xi_m(k-1) \quad \cdots \quad \xi_m(k-2n+1)\,]^T \in \mathbb{R}^{2n-1} \tag{40}$$

and

$$x_e(k) \stackrel{\triangle}{=} x(k) - x_m(k) \tag{41}$$

Then

$$x_e(k) = [\, \xi_e(k-1) \quad \cdots \quad \xi_e(k-2n+1)\,]^T \in \mathbb{R}^{2n-1} \tag{42}$$

Since

$$
\begin{aligned}
\xi_e(k) &= \mathbf{q}\xi_e(k-1) \\
&= -\mathbf{q}\left[\frac{1}{b_0} q^{-2n+1}\mathbf{P} - 1\right]\xi_e(k-1) + \frac{1}{b_0} q^{-2n+1}\mathbf{P}\xi_e(k) \\
&= -\mathbf{q}\left[\frac{1}{b_0} q^{-2n+1}\mathbf{P} - 1\right]\xi_e(k-1) + \frac{1}{b_0} e_f(k+d)
\end{aligned}
\tag{43}
$$

a state equation for x_e in controllable canonical form is given by

$$x_e(k+1) = A x_e(k) + \frac{1}{b_0} B e_f(k+d), \quad k \geqslant 0 \tag{44}$$

where

$$A \stackrel{\triangle}{=} \begin{bmatrix} -p_1/b_0 & \cdots & -p_{2n-1}/b_0 \\ & & & 0 \\ I_{(2n-2)\times(2n-2)} & & \vdots \\ & & & 0 \end{bmatrix}, \quad B \stackrel{\triangle}{=} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{45}$$

Note that A is asymptotically stable. Alternatively, using (30) the model matching dynamics (44) can be written as

$$x_e(k+1) = A x_e(k) - \frac{1}{b_0} B \varphi^T(k)\tilde{\theta}(k), \quad k \geqslant 0 \tag{46}$$

Next we show that the state x defined in (39) is related to φ through a non-singular transformation.

Lemma 3.1

The plant state x defined by (39) and the regressor (22) are related by

$$\varphi(k) = M_0 x(k) \tag{47}$$

where the non-singular matrix $M_0 \in \mathbb{R}^{(2n-1)\times(2n-1)}$ is defined by

$$M_0 \overset{\triangle}{=} \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_n & & & & & 0_{1\times(n-2)} \\ & & \ddots & & & \ddots & & & \ddots & \\ & 0_{1\times(n-2)} & & & & 1 & a_1 & a_2 & \cdots & a_n \\ 0_{1\times(d-1)}\, b_0 & b_1 & \cdots & b_m & & & & 0_{1\times(n-1)} & & \\ & & \ddots & & & \ddots & & & \ddots & \\ & 0_{1\times(n+d-2)} & & & & & b_0 & b_1 & \cdots & b_m \end{bmatrix} \tag{48}$$

Proof

It follows from (31) and (32) that

$$\varphi(k) = \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n+1) \\ y(k) \\ \vdots \\ y(k-n+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{q})\xi(k-n-1) \\ \vdots \\ \mathbf{A}(\mathbf{q})\xi(k-2n+1) \\ \mathbf{B}(\mathbf{q})\xi(k-n) \\ \vdots \\ \mathbf{B}(\mathbf{q})\xi(k-2n+1) \end{bmatrix} \tag{49}$$

Equations in (49) can be written as

$$u(k-1) = \mathbf{A}(\mathbf{q})\xi(k-n-1)$$

$$= [\mathbf{q}^n + a_1\mathbf{q}^{n-1} + \cdots + a_n]\xi(k-n-1)$$

$$= \xi(k-1) + a_1\xi(k-2) + \cdots + a_n\xi(k-n-1)$$

$$= [\,1 \quad a_1 \quad \cdots \quad a_n\,][\,\xi(k-1) \quad \cdots \quad \xi(k-n-1)\,]^{\mathrm{T}} \tag{50}$$

$$\vdots$$

$$u(k-n+1) = \mathbf{A}(\mathbf{q})\xi(k-2n+1)$$

$$= [\mathbf{q}^n + a_1\mathbf{q}^{n-1} + \cdots + a_n]\xi(k-2n+1)$$

$$= \xi(k-n+1) + \cdots + a_n\xi(k-2n+1)$$

$$= [\,1 \quad a_1 \quad \cdots \quad a_n\,][\,\xi(k-n+1) \quad \cdots \quad \xi(k-2n+1)\,]^{\mathrm{T}} \tag{51}$$

$$y(k) = \mathbf{B(q)}\xi(k - n)$$

$$= [b_0\mathbf{q}^m + b_1\mathbf{q}^{m-1} + \cdots + b_m]\xi(k - n)$$

$$= b_0\xi(k - d) + \cdots + b_m\xi(k - n)$$

$$= [\,b_0 \quad b_1 \quad \cdots \quad b_m\,][\,\xi(k - d) \quad \cdots \quad \xi(k - n)\,]^{\mathrm{T}} \tag{52}$$

$$\vdots$$

$$y(k - n + 1) = \mathbf{B(q)}\xi(k - 2n + 1)$$

$$= [b_0\mathbf{q}^m + b_1\mathbf{q}^{m-1} + \cdots + b_m]\xi(k - 2n + 1)$$

$$= b_0\xi(k - n - d + 1) + \cdots + b_m\xi(k - 2n + 1)$$

$$= [\,b_0 \quad b_1 \quad \cdots \quad b_m\,][\,\xi(k - n - d + 1) \quad \cdots \quad \xi(k - 2n + 1)\,]^{\mathrm{T}} \tag{53}$$

From (50)–(53) and (39) it follows that $\varphi(k) = M_0 x(k)$, where M_0 is given by (48). It can be seen that M_0 is the $(2n - 1) \times (2n - 1)$ submatrix of M^{T}, formed by omitting the first row and first column of M^{T}. Note that $\det M_0 = \det M$. Since $\mathbf{A}$ and $\mathbf{B}$ are relatively co-prime by assumption, it follows that M is non-singular, and thus M_0 is non-singular. $\qquad\square$

4. ADAPTIVE CONTROL USING RLS IDENTIFICATION

In this section we derive a recursive least squares update law for the parameter vector $\hat{\theta}(k)$ of a controller that drives $e_{\mathrm{f}}(k + d)$ to zero asymptotically. To do this we minimize the *retrospective cost function*

$$J(\hat{\theta}(k), k) \triangleq \sum_{i=d}^{k} E^2(\hat{\theta}(k), i) \tag{54}$$

where $k \geqslant d$ and where the *retrospective error* is defined by

$$E(\hat{\theta}(k), i) \triangleq y_{\mathrm{f}}(i) - b_0 u(i - d) - \varphi^{\mathrm{T}}(i - d)\hat{\theta}(k), \quad k \geqslant d \tag{55}$$

The retrospective cost function (54) defines the performance of $\hat{\theta}$ by evaluating the present value $\hat{\theta}(k)$ in terms of the past behaviour of the linear identification model (20) over the interval $d \leqslant i \leqslant k$.

Next define

$$\mathscr{E}(\hat{\theta}(k), k) \triangleq [\,E(\hat{\theta}(k), d) \quad \cdots \quad E(\hat{\theta}(k), k)\,]^{\mathrm{T}} \in \mathbb{R}^{k-d+1} \tag{56}$$

$$Y(k) \triangleq [\,y_{\mathrm{f}}(d) - b_0 u(0) \quad \ldots \quad y_{\mathrm{f}}(k) - b_0 u(k - d)\,]^{\mathrm{T}} \in \mathbb{R}^{k-d+1} \tag{57}$$

and

$$\Phi(k) \triangleq [\, \varphi(0) \quad \cdots \quad \varphi(k-d)\,]^{\mathrm{T}} \in \mathbb{R}^{(k-d+1) \times (2n-1)} \tag{58}$$

so that

$$\mathscr{E}(\hat{\theta}(k), k) = Y(k) - \Phi(k)\hat{\theta}(k) \in \mathbb{R}^{k-d+1} \tag{59}$$

With this notation the cost function (54) can be expressed as

$$J(\hat{\theta}(k), k) = \mathscr{E}^{\mathrm{T}}(\hat{\theta}(k), k)\mathscr{E}(\hat{\theta}(k), k)$$

$$= [Y(k) - \Phi(k)\hat{\theta}(k)]^{\mathrm{T}}[Y(k) - \Phi(k)\hat{\theta}(k)]$$

Next, we use a recursive least squares estimate of $\hat{\theta}(k)$ that minimizes $J(k, \hat{\theta})$, for details see, for example, Reference [14]. The RLS estimate for $\hat{\theta}(k)$ is given by

$$\mathscr{P}(k) = \mathscr{P}(k-1) - \frac{\mathscr{P}(k-1)\varphi(k-d)\varphi^{\mathrm{T}}(k-d)\mathscr{P}(k-1)}{1 + \varphi^{\mathrm{T}}(k-d)\mathscr{P}(k-1)\varphi(k-d)}, \quad \mathscr{P}(0) > 0 \tag{60}$$

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \mathscr{P}(k)\varphi(k-d)[y_{\mathrm{f}}(k) - b_0 u(k-d) - \varphi^{\mathrm{T}}(k-d)\hat{\theta}(k-1)] \tag{61}$$

where the model reference adaptive control law is

$$u(k) = -\frac{1}{b_0}[\varphi^{\mathrm{T}}(k)\hat{\theta}(k) - q^{-n+1}\mathbf{B}_{\mathrm{m}}u_{\mathrm{c}}(k)] \tag{62}$$

5. STABILITY OF MRAC WITH RLS IDENTIFICATION

Define

$$\tilde{\Theta}(k) \triangleq \begin{bmatrix} \tilde{\theta}(k) \\ \vdots \\ \tilde{\theta}(k+d-1) \end{bmatrix} \tag{63}$$

Then the error state vector consisting of the model matching error states and the parameter identification error states is defined by

$$X(k) \triangleq \begin{bmatrix} x_{\mathrm{e}}(k) \\ \tilde{\Theta}(k) \\ \mathrm{vec}(\mathscr{P}(k)) \\ \vdots \\ \mathrm{vec}(\mathscr{P}(k+d-1)) \end{bmatrix} \tag{64}$$

Using (26), (30), (60), (61) and (46), the closed-loop error dynamics with RLS identification can be represented as the $(d+1)(2n-1) + d(2n-1)^2$ dimensional system

$$x_e(k+1) = Ax_e(k) - \frac{1}{b_0} B\varphi^T(k)\tilde{\theta}(k) \tag{65}$$

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \mathscr{P}(k+1)\varphi(k-d+1)\varphi^T(k-d+1)\tilde{\theta}(k) \tag{66}$$

$$\vdots$$

$$\tilde{\theta}(k+d) = \tilde{\theta}(k+d-1) - \mathscr{P}(k+d)\varphi(k)\varphi^T(k)\tilde{\theta}(k+d-1) \tag{67}$$

$$\text{vec}[\mathscr{P}(k+1)] = \text{vec}\left[\mathscr{P}(k) - \frac{\mathscr{P}(k)\varphi(k-d+1)\varphi^T(k-d+1)\mathscr{P}(k)}{1 + \varphi^T(k-d+1)\mathscr{P}(k)\varphi(k-d+1)}\right] \tag{68}$$

$$\vdots$$

$$\text{vec}[\mathscr{P}(k+d)] = \text{vec}\left[\mathscr{P}(k+d-1) - \frac{\mathscr{P}(k+d-1)\varphi(k)\varphi^T(k)\mathscr{P}(k+d-1)}{1 + \varphi^T(k)\mathscr{P}(k+d-1)\varphi(k)}\right] \tag{69}$$

Note that the error system (65)–(69) is time varying since the regressor $\varphi(k)$ is a function of the exogenous signal $u_c(k)$. Furthermore, every equilibrium of the error system (65)–(69) is of the form $(0, \tilde{\theta}_q, \mathscr{P}_q)$, where $\mathscr{P}_q \geqslant 0$.

Remark 5.1
Although the future parameter errors $\tilde{\theta}(k+1)$ to $\tilde{\theta}(k+d-1)$ and the future adaptation gain matrices $\mathscr{P}(k+1)$ to $\mathscr{P}(k+d-1)$ are not computed by the algorithm at time k, they are included in the state vector $X(k)$ to facilitate the stability analysis.

To demonstrate that every equilibrium of the system (65)–(69) is Lyapunov stable, the following lemmas are required.

Lemma 5.1
Define

$$V_{\mathscr{P}}(\mathscr{P}) \triangleq \text{tr}\,\mathscr{P}^2 \tag{70}$$

$$\triangle V_{\mathscr{P}}(k) \triangleq \text{tr}\,[\mathscr{P}^2(k+1) - \mathscr{P}^2(k)] \tag{71}$$

$$V_{\tilde{\theta}}(\tilde{\theta}, \mathscr{P}) \triangleq \tilde{\theta}^T \mathscr{P}^{-1}\tilde{\theta} \tag{72}$$

and

$$\triangle V_{\tilde{\theta}}(k) \triangleq \tilde{\theta}^{\mathrm{T}}(k+1)\mathscr{P}^{-1}(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^{\mathrm{T}}(k)\mathscr{P}^{-1}(k)\tilde{\theta}(k) \tag{73}$$

Then, for all $k \geqslant 0$,

$$\triangle V_{\mathscr{P}}(k) \leqslant 0 \tag{74}$$

$$\triangle V_{\tilde{\theta}}(k) = \frac{-[\varphi(k-d+1)\tilde{\theta}(k)]^2}{1 + \varphi^{\mathrm{T}}(k-d+1)\mathscr{P}(k)\varphi(k-d+1)} \leqslant 0 \tag{75}$$

and $\lim_{k\to\infty} \mathscr{P}(k)$ and $\lim_{k\to\infty} \tilde{\theta}(k)$ exist.

Proof
The result follows from standard properties of RLS. See References [2, p. 60, 15, p. 202, 16, p. 22, 17, p. 58].

Lemma 5.2
For all $k > 0$

$$\frac{[\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)]^2}{1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)} \leqslant \sum_{i=k}^{k+d-1} \frac{[\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)]^2}{1 + \varphi^{\mathrm{T}}(i-d+1)\mathscr{P}(i)\varphi(i-d+1)} \tag{76}$$

Proof
From repeated self substitutions of (66) it follows that

$$\tilde{\theta}(k+d-1) = \tilde{\theta}(k) - \sum_{i=1}^{d-1} \mathscr{P}(k+d-i)\varphi(k-i)\varphi^{\mathrm{T}}(k-i)\tilde{\theta}(k+d-i-1)$$

$$= \tilde{\theta}(k) - \sum_{i=k}^{k+d-2} \mathscr{P}(i+1)\varphi(i-d+1)\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i) \tag{77}$$

Multiply both sides of (77) by $\varphi^{\mathrm{T}}(k)/\sqrt{1+\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)}$ to get

$$\frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[1+\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}} = \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k+d-1)}{[1+\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}}$$
$$+ \sum_{i=k}^{k+d-2} \frac{\varphi^{\mathrm{T}}(k)\mathscr{P}(i+1)\varphi(i-d+1)\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[1+\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}}$$

Now using (69) yields

$$\frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[1+\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}} = \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k+d-1)}{[1+\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}}$$
$$+ \sum_{i=1}^{k+d-2} \frac{\varphi^{\mathrm{T}}(k)\mathscr{P}(i)\varphi(i-d+1)\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[1+\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}[1+\varphi^{\mathrm{T}}(i-d+1)\mathscr{P}(i)\varphi(i-d+1)]}$$

Using the triangle inequality and the fact that $\mathscr{P}(i) \leqslant \mathscr{P}(k)$ for all $i \geqslant k$ it follows that

$$\left| \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}} \right| \leqslant \left| \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k + d - 1)}{[1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}} \right|$$

$$+ \sum_{i=k}^{k+d-2} \left| \frac{\varphi^{\mathrm{T}}(k)\mathscr{P}(i)\varphi(i - d + 1)}{[1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(i)\varphi(k)]^{1/2}[1 + \varphi^{\mathrm{T}}(i - d + 1)\mathscr{P}(i)\varphi(i - d + 1)]^{1/2}} \right|$$

$$\times \left| \frac{\varphi^{\mathrm{T}}(i - d + 1)\tilde{\theta}(i)}{[1 + \varphi^{\mathrm{T}}(i - d + 1)\mathscr{P}(i)\varphi(i - d + 1)]^{1/2}} \right|$$

Now using the Cauchy–Schwarz inequality we have

$$\left| \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}} \right| \leqslant \left| \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k + d - 1)}{[1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)]^{1/2}} \right|$$

$$+ \sum_{i=k}^{k+d-2} \left| \frac{\varphi^{\mathrm{T}}(i - d + 1)\tilde{\theta}(i)}{[1 + \varphi^{\mathrm{T}}(i - d + 1)\mathscr{P}(i)\varphi(i - d + 1)]^{1/2}} \right|$$

$$= \sum_{i=k}^{k+d-1} \left| \frac{\varphi^{\mathrm{T}}(i - d + 1)\tilde{\theta}(i)}{[1 + \varphi^{\mathrm{T}}(i - d + 1)\mathscr{P}(i)\varphi(i - d + 1)]^{1/2}} \right| \qquad \square$$

Lemma 5.3
Define

$$\Pi(k) \triangleq \operatorname{diag}[\,\mathscr{P}^{-1}(k) \quad \cdots \quad \mathscr{P}^{-1}(k + d - 1)\,] \tag{78}$$

$$V_{\tilde{\Theta}}(\tilde{\Theta}, \Pi) \triangleq \tilde{\Theta}^{\mathrm{T}}\Pi^{-1}\tilde{\Theta} \tag{79}$$

and

$$\triangle V_{\tilde{\Theta}}(k) \triangleq \tilde{\Theta}^{\mathrm{T}}(k + 1)\Pi^{-1}(k + 1)\tilde{\Theta}(k + 1) - \tilde{\Theta}^{\mathrm{T}}(k)\Pi^{-1}(k)\tilde{\Theta}(k) \tag{80}$$

Then

$$\triangle V_{\tilde{\Theta}}(k) \leqslant \frac{-[\varphi(k)\tilde{\theta}(k)]^2}{1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)} \tag{81}$$

Proof
From (63), (78) and (80) it follows that

$$\triangle V_{\tilde{\Theta}}(k) = \sum_{i=k+1}^{k+d} \tilde{\theta}^{\mathrm{T}}(i)\mathscr{P}^{-1}(i)\tilde{\theta}(i) - \sum_{i=k}^{k+d-1} \tilde{\theta}^{\mathrm{T}}(i)\mathscr{P}^{-1}(i)\tilde{\theta}(i) \tag{82}$$

Now using (75) and (76) we have

$$\triangle V_{\tilde{\Theta}} = - \sum_{i=k}^{k+d-1} \frac{[\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)]^2}{1 + \varphi^{\mathrm{T}}(i-d+1)\mathscr{P}(i)\varphi(i-d+1)} \leqslant - \frac{[\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)]^2}{1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)} \qquad \square$$

Lemma 5.4
Define

$$V_{\Pi}(\Pi) \stackrel{\triangle}{=} \mathrm{tr}\,[\Pi^{\mathrm{T}}\Pi] \tag{83}$$

and

$$\triangle V_{\Pi}(k) \stackrel{\triangle}{=} \mathrm{tr}\,[\Pi^{\mathrm{T}}(k+1)\Pi(k+1)] - \mathrm{tr}\,[\Pi^{\mathrm{T}}(k)\Pi(k)] \tag{84}$$

Then

$$\triangle V_{\Pi}(k) \leqslant 0, \quad k \geqslant 0$$

Proof
From (74) and (78) it follows that

$$\triangle V_{\Pi}(k) = \sum_{i=k-d+1}^{k+1} \mathrm{tr}\,\mathscr{P}^2(i) - \sum_{i=k-d}^{k} \mathrm{tr}\,\mathscr{P}^2(i)$$

$$= - \sum_{i=k-d}^{k} \triangle V_{\mathscr{P}}(k) \leqslant 0 \qquad \square$$

Lemma 5.5
Recall that A (45) is asymptotically stable. Let $P, R \in \mathbb{R}^{n \times n}$ be positive-definite matrices that satisfy

$$P = A^{\mathrm{T}}PA + R + I \tag{85}$$

and define

$$\sigma \stackrel{\triangle}{=} \sqrt{\lambda_{\max}(A^{\mathrm{T}}PA)} \tag{86}$$

Furthermore, let $\mu > 0$ and define

$$V_{x_e}(x_e) \stackrel{\triangle}{=} \ln(1 + \mu x_e^{\mathrm{T}} P x_e) \tag{87}$$

and

$$\triangle V_{x_e}(k) \stackrel{\triangle}{=} V_{x_e}(x_e(k+1)) - V_{x_e}(x_e(k))$$

Then

$$\triangle V_{x_e}(k) \leqslant \mu \frac{-x_e^{\mathrm{T}}(k)Rx_e(k) + b_0^{-2}(\sigma^2 + 1)B^{\mathrm{T}}PB[\varphi(k)\tilde{\theta}(k)]^2}{1 + \mu x_e^{\mathrm{T}}(k)Px_e(k)}, \quad k \geqslant 0$$

Proof
Define

$$F \triangleq \frac{1}{\sigma} P^{1/2}A, \quad G \triangleq \sigma P^{1/2}B, \quad \mathscr{J}(x_e) \triangleq x_e^T P x_e$$

Then,

$$\triangle \mathscr{J}_{x_e}(k) \triangleq x_e^T(k+1)Px_e(k+1) - x_e^T(k)Px_e(k)$$

$$= \left[Ax_e(k) - B\frac{\varphi(k)\tilde{\theta}(k)}{b_0} \right]^T P \left[Ax_e(k) - B\frac{\varphi(k)\tilde{\theta}(k)}{b_0} \right] - x_e^T(k)Px_e(k)$$

Omitting the explicit dependence on k we have

$$\triangle \mathscr{J}_{x_e}(k) = x_e^T A^T PA x_e - x_e^T A^T PB b_0^{-1}\varphi^T\tilde{\theta} - b_0^{-1}\varphi^T\tilde{\theta}B^T PAx_e + b_0^{-1}\varphi^T\tilde{\theta}B^T PB b_0^{-1}\varphi^T\tilde{\theta} - x_e^T Px_e$$

$$= x_e^T(A^T PA - P)x_e - x_e^T A^T PB b_0^{-1}\varphi^T\tilde{\theta} - b_0^{-1}\varphi^T\tilde{\theta}B^T PAx_e + b_0^{-2}(\varphi^T\tilde{\theta})^2 B^T PB$$

$$= x_e^T(A^T PA - P)x_e - x_e^T F^T G b_0^{-1}\varphi^T\tilde{\theta} - b_0^{-1}\varphi^T\tilde{\theta}G^T Fx_e + b_0^{-2}(\varphi^T\tilde{\theta})^2 B^T PB$$

$$= x_e^T(A^T PA - P + F^T F)x_e - x_e^T F^T G b_0^{-1}\varphi^T\tilde{\theta} - b_0^{-1}\varphi^T\tilde{\theta}G^T Fx_e + b_0^{-2}(\varphi^T\tilde{\theta})^2 B^T PB$$

$$\quad - x_e^T F^T Fx_e + b_0^{-2}(\varphi^T\tilde{\theta})^2 G^T G - b_0^{-2}(\varphi^T\tilde{\theta})^2 G^T G$$

$$= x_e^T(A^T PA - P + F^T F)x_e - [\, x_e^T \quad b_0^{-1}\varphi^T\tilde{\theta} \,] \begin{bmatrix} F^T F & F^T G \\ G^T F & G^T G \end{bmatrix} \begin{bmatrix} x_e^T \\ b_0^{-1}\varphi^T\tilde{\theta} \end{bmatrix}$$

$$\quad + (B^T PB + G^T G)b_0^{-2}(\varphi^T\tilde{\theta})^2$$

$$\leqslant x_e^T(A^T PA - P + F^T F)x_e + (B^T PB + G^T G)b_0^{-2}(\varphi^T\tilde{\theta})^2$$

Noting that

$$F^T F = \frac{A^T PA}{\sigma^2} = \frac{A^T PA}{\lambda_{\max}(A^T PA)} \leqslant \frac{\lambda_{\max}(A^T PA)I_n}{\lambda_{\max}(A^T PA)} = I_n$$

it follows from (85) that

$$A^T PA - P + F^T F \leqslant A^T PA - P + I = -R$$

Therefore,

$$x_e^T(k)(A^T PA - P + F^T F)x_e(k) \leqslant -x_e^T(k)Rx_e(k)$$

which implies that

$$\triangle \mathscr{J}_{x_e}(k) \leqslant -x_e^T(k)Rx_e(k) + (B^T PB + G^T G)b_0^{-2}[\varphi(k)\tilde{\theta}(k)]^2 \qquad (88)$$

Since $G^T G = \sigma^2 B^T PB$, it follows from (88) that

$$\triangle \mathscr{J}_{x_e}(k) \leqslant -x_e^T(k)Rx_e(k) + (\sigma^2 + 1)B^T PB b_0^{-2}[\varphi(k)\tilde{\theta}(k)]^2$$

Now, since $\ln x \leqslant x - 1$ for all $x > 0$,

$$\triangle V_{x_{\mathrm{e}}}(k) = \ln\left(1 + \mu \frac{\triangle \mathscr{I}_{x_{\mathrm{e}}}(k)}{1 + \mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)}\right)$$

$$\leqslant \mu \frac{x_{\mathrm{e}}^{\mathrm{T}}(k+1)Px_{\mathrm{e}}(k+1) - x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)}{1 + \mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)}$$

$$\leqslant \mu \frac{-x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_{\mathrm{e}}(k) + b_0^{-2}(\sigma^2 + 1)B^{\mathrm{T}}PB[\varphi(k)\tilde{\theta}(k)]^2}{1 + \mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)} \qquad \square$$

We now present the main stability result for MRAC with RLS identification.

Theorem 5.1
Assume that the reference signal $u_{\mathrm{c}}(k)$ is bounded. Then every equilibrium of the error system dynamics (65)–(69) is Lyapunov stable, $\tilde{\theta}(k)$ and $\mathscr{P}(k)$ converge, and $y(k) - y_{\mathrm{m}}(k) \to 0$ as $k \to \infty$.

Proof
Consider the Lyapunov function candidate

$$V(X) \overset{\triangle}{=} aV_{x_{\mathrm{e}}}(x_{\mathrm{e}}) + V_{\tilde{\Theta}}(\tilde{\Theta}, \Pi) + V_{\Pi}(\Pi) \tag{89}$$

Let $P, R \in \mathbb{R}^{n \times n}$ be positive definite and satisfy (85), and let $a > 0$. Then using Lemmas 5.5, 5.3 and 5.4 it follows that

$$\triangle V(k) \overset{\triangle}{=} V(X(k+1)) - V(X(k))$$

$$\leqslant a\mu \frac{-x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_{\mathrm{e}}(k) + (\sigma^2 + 1)B^{\mathrm{T}}PB[\varphi(k)\tilde{\theta}(k)]^2/b_0^2}{1 + \mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)} - \frac{[\varphi(k)\tilde{\theta}(k)]^2}{1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)}$$

Now from (47) it follows that

$$\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k) \leqslant \varphi^{\mathrm{T}}(k)\mathscr{P}(0)\varphi(k)$$

$$= x^{\mathrm{T}}M_0^{\mathrm{T}}\mathscr{P}(0)Mx$$

$$= (x_{\mathrm{e}} + x_{\mathrm{m}})^{\mathrm{T}}M_0^{\mathrm{T}}\mathscr{P}(0)M_0(x_{\mathrm{e}} + x_{\mathrm{m}})$$

$$\leqslant x_{\mathrm{e}}^{\mathrm{T}}M_0^{\mathrm{T}}\mathscr{P}(0)M_0x_{\mathrm{e}} + x_{\mathrm{m}}^{\mathrm{T}}M_0^{\mathrm{T}}\mathscr{P}(0)M_0x_{\mathrm{m}} \tag{90}$$

Let $\mu_1 > 0$ satisfy

$$\mu_1 P > M_0^{\mathrm{T}}\mathscr{P}(0)M_0 \tag{91}$$

Since, by assumption, the command signal $u_{\mathrm{c}}(k)$ is bounded and $\mathbf{A}_{\mathrm{m}}$ is stable, there exists $\beta > 0$ such that

$$x_m^{\mathrm{T}}(k)x_{\mathrm{m}}(k) \leqslant \beta \tag{92}$$

Using (90)–(92) it follows that

$$\varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k) \leqslant \mu_1 x_{\mathrm{e}}^{\mathrm{T}}Px_{\mathrm{e}} + \mu_1 x_{\mathrm{m}}^{\mathrm{T}}Px_{\mathrm{m}}$$

$$\leqslant \mu_1 x_{\mathrm{e}}^{\mathrm{T}}Px_{\mathrm{e}} + \beta\mu_1\lambda_{\max}(P) \tag{93}$$

Defining

$$\mu \stackrel{\triangle}{=} \frac{\mu_1}{1 + \beta\mu_1\lambda_{\max}(P)} \qquad (94)$$

and

$$a \stackrel{\triangle}{=} \frac{b_0^2}{1 + \mu_1(\sigma^2 + 1)B^{\mathrm{T}}PB} \qquad (95)$$

it follows that

$$\triangle V \leqslant \frac{-\mu_1 b_0^2 x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_{\mathrm{e}}(k) + \mu_1(\sigma^2 + 1)B^{\mathrm{T}}PB[\varphi(k)\tilde{\theta}(k)]^2}{(1 + \beta\mu_1\lambda_{\max}(P))(1 + \mu_1(\sigma^2 + 1)B^{\mathrm{T}}PB)(1 + \mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k))} - \frac{[\varphi(k)\tilde{\theta}(k)]^2}{1 + \varphi^{\mathrm{T}}(k)\mathscr{P}(k)\varphi(k)}$$

$$\leqslant \frac{-\mu_1 b_0^2 x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_{\mathrm{e}}(k)}{(1 + \beta\mu_1\lambda_{\max}(P))(1 + \mu_1(\sigma^2 + 1)B^{\mathrm{T}}PB)(1 + \mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k))}$$

$$= -\mu a \frac{x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_{\mathrm{e}}(k)}{1 + \mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)} \qquad (96)$$

Since $V(X)$ is positive definite and radially unbounded it follows from (96) that the origin of the error system (65)–(69) is Lyapunov stable. Furthermore, using Theorem A.1 it follows that $x_{\mathrm{e}}(k) \to 0$ as $k \to \infty$. Then using (32) and (35) we have that $y(k) - y_{\mathrm{m}}(k) \to 0$ as $k \to \infty$. The convergence of $\tilde{\theta}(k)$ and $\mathscr{P}(k)$ follows from Lemma 5.1.   □

6. PROJECTION ADAPTIVE CONTROL ALGORITHM

In Sections 4 and 5 we developed and proved the stability of MRAC in which the controller parameters are updated using RLS. In Sections 6 and 7 we consider the case in which the controller parameters are updated using the projection algorithm.

Consider the cost function

$$J_\lambda(\hat{\theta}(k), \hat{\theta}(k-1), k) \stackrel{\triangle}{=} \tfrac{1}{2}\|\hat{\theta}(k) - \hat{\theta}(k-1)\|_2^2 + \lambda[y_{\mathrm{f}}(k) - b_0 u(k-d) - \varphi^{\mathrm{T}}(k-d)\hat{\theta}(k)] \qquad (97)$$

where λ is the Lagrange multiplier. A recursive expression for $\hat{\theta}(k)$ that minimizes (97) is given by [2, p. 51]

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \frac{\varphi(k-d)[y_{\mathrm{f}}(k) - b_0 u(k-d) - \varphi^{\mathrm{T}}(k-d)\hat{\theta}(k-1)]}{\varphi^{\mathrm{T}}(k-d)\varphi(k-d)} \qquad (98)$$

Therefore, the adaptive control law is

$$u(k) \stackrel{\triangle}{=} -\frac{1}{b_0}[\varphi^{\mathrm{T}}(k)\hat{\theta}(k) - q^{-n+1}\mathbf{B}_{\mathrm{m}}u_{\mathrm{c}}(k)] \qquad (99)$$

with the parameter update

$$\hat{\theta}(k) = \begin{cases} \hat{\theta}(k-1) + \dfrac{\varphi(k-d)[y_{\mathrm{f}}(k) - b_0 u(k-d) - \varphi^{\mathrm{T}}(k-d)\hat{\theta}(k-1)]}{\varphi^{\mathrm{T}}(k-d)\varphi(k-d)}, & \varphi(k-d) \neq 0 \\ \hat{\theta}(k-1), & \varphi(k-d) = 0 \end{cases} \qquad (100)$$

7. STABILITY OF MRAC WITH PROJECTION IDENTIFICATION

The error state vector consisting of the model matching error states and the parameter identification error states is defined by

$$X(k) \triangleq \begin{bmatrix} x_e(k) \\ \tilde{\Theta}(k) \end{bmatrix}^T \tag{101}$$

Using (26), (30), (46) and (100), the closed-loop error dynamics with projection identification can be represented in the state space form, for $k \geqslant 0$, by

$$x_e(k+1) = Ax_e(k) - \frac{1}{b_0} B\varphi^T(k)\tilde{\theta}(k) \tag{102}$$

$$\tilde{\theta}(k+1) = \left[I - \frac{\varphi(k-d+1)\varphi^T(k-d+1)}{\varphi^T(k-d+1)\varphi(k-d+1)} \right] \tilde{\theta}(k) \tag{103}$$

$$\vdots$$

$$\tilde{\theta}(k+d) = \left[I - \frac{\varphi(k)\varphi^T(k)}{\varphi^T(k)\varphi(k)} \right] \tilde{\theta}(k+d-1) \tag{104}$$

Before presenting the main stability results we state the following lemmas.

Lemma 7.1
Define

$$V_{\tilde{\theta}}(\tilde{\theta}) \triangleq \tilde{\theta}^T\tilde{\theta} \tag{105}$$

and

$$\triangle V_{\tilde{\theta}}(k) \triangleq \tilde{\theta}^T(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)\tilde{\theta}(k)$$

Then

$$\triangle V_{\tilde{\theta}}(k) \leqslant \frac{-[\varphi^T(k-d+1)\tilde{\theta}(k)]^2}{\varphi^T(k-d+1)\varphi(k-d+1)} \leqslant 0, \quad k \geqslant 0 \tag{106}$$

and, for all $t \geqslant 0$, $\{\|\tilde{\theta}(k+t) - \tilde{\theta}(k)\|\}_{k=0}^{\infty} \in l_2$.

Proof
See References [2, p. 51, 16, p. 78].

Lemma 7.2
For all $k > 0$

$$\frac{[\varphi^T(k)\tilde{\theta}(k)]^2}{\varphi^T(k)\varphi(k)} \leqslant \sum_{i=k}^{k+d-1} \frac{[\varphi^T(i-d+1)\tilde{\theta}(i)]^2}{\varphi^T(i-d+1)\varphi(i-d+1)}$$

Proof

From repeated self substitutions of (103) it follows that

$$\tilde{\theta}(k+d-1) = \tilde{\theta}(k) - \sum_{i=1}^{d-1} \frac{\varphi(k-i)\varphi^{\mathrm{T}}(k-i)\tilde{\theta}(k+d-i-1)}{\varphi^{\mathrm{T}}(k-i)\varphi(k-i)}$$

$$= \tilde{\theta}(k) - \sum_{i=k}^{k+d-2} \frac{\varphi(i-d+1)\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]} \tag{107}$$

Multiplying both sides of (107) by $\varphi^{\mathrm{T}}(k)/\sqrt{\varphi^{\mathrm{T}}(k)\varphi(k)}$ yields

$$\frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}} = \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k+d-1)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}} + \sum_{i=k}^{k+d-2} \frac{\varphi^{\mathrm{T}}(k)\varphi(i-d+1)\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]}$$

The triangle inequality implies

$$\left| \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}} \right| \leqslant \left| \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k+d-1)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}} \right| + \sum_{i=k}^{k+d-2} \left| \frac{\varphi^{\mathrm{T}}(k)\varphi(i-d+1)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]^{1/2}} \right|$$

$$\times \left| \frac{\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]^{1/2}} \right|$$

Now using the Cauchy–Schwarz inequality we have

$$\left| \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}} \right| \leqslant \left| \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k+d-1)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}} \right| + \sum_{i=k}^{k+d-2} \left| \frac{\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]^{1/2}} \right|$$

$$= \sum_{i=k}^{k+d-1} \left| \frac{\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]^{1/2}} \right| \qquad \square$$

Lemma 7.3

Define

$$V_{\tilde{\Theta}}(\tilde{\Theta}) \triangleq \tilde{\Theta}^{\mathrm{T}}\tilde{\Theta} \tag{108}$$

and

$$\triangle V_{\tilde{\Theta}}(k) \triangleq \tilde{\Theta}^{\mathrm{T}}(k+1)\tilde{\Theta}(k+1) - \tilde{\Theta}^{\mathrm{T}}(k)\tilde{\Theta}(k) \tag{109}$$

Then

$$\triangle V_{\tilde{\Theta}}(k) \leqslant - \frac{[\varphi(k)\tilde{\theta}(k)]^2}{\varphi^{\mathrm{T}}(k)\varphi(k)}, \quad k \geqslant 0 \tag{110}$$

Proof
From (63) and (109) it follows that

$$\triangle V_{\tilde{\Theta}}(k) = \sum_{i=k+1}^{k+d} \tilde{\theta}^{T}(i)\tilde{\theta}(i) - \sum_{i=k}^{k+d-1} \tilde{\theta}^{T}(i)\tilde{\theta}(i) \tag{111}$$

Using Lemmas 7.1 and 7.2 yields

$$\triangle V_{\tilde{\Theta}}(k) = - \sum_{i=k}^{k+d-1} \frac{[\varphi^{T}(i-d+1)\tilde{\theta}(i)]^2}{\varphi^{T}(i-d+1)\varphi(i-d+1)} \leqslant - \frac{[\varphi^{T}(k)\tilde{\theta}(k)]^2}{\varphi^{T}(k)\varphi(k)} \qquad \square$$

We now present the main stability result for MRAC with projection identification.

Theorem 7.1
Assume that the reference signal $u_c(k)$ is bounded . Then every equilibrium of the error system dynamics (102)–(104) is Lyapunov stable, and $y(k) - y_m(k) \to 0$ as $k \to \infty$.

Proof
The proof follows along the same lines as Theorem 5.1 except that the Lyapunov candidate is defined by

$$V(X) \stackrel{\triangle}{=} aV_{x_e}(x_e) + V_{\tilde{\Theta}}(\tilde{\Theta}) \tag{112}$$

and μ_1 is chosen to satisfy

$$\mu_1 P > M_0^T M_0$$

where P is the positive-definite matrix satisfying (85). $\qquad \square$

8. EXAMPLE

Example 8.1
   Consider the unstable minimum phase SISO plant with relative degree $d = 2$ given by

$$\frac{y(\mathbf{q})}{u(\mathbf{q})} = \frac{\mathbf{q} + 0.5}{\mathbf{q}^3 + \mathbf{q}^2 + \mathbf{q} + 1.5} \tag{113}$$

To track reference signals we choose an FIR filter as the reference model. We require $\deg \mathbf{A}_m = 2n - m - 1 = 4$ and $\deg \mathbf{B}_m = \deg \mathbf{A}_m - d = 2$. Let the reference model be

$$\frac{y_m(\mathbf{q})}{u_c(\mathbf{q})} = \frac{\mathbf{q}^2}{\mathbf{q}^4} \tag{114}$$

and let $u_c(k)$ be a square wave with period of 100 samples. The plant (113) with the control law (62) and the parameter updates (60)–(61) and (100) is simulated in MATLAB. The simulation results are shown in Figures 4 and 5.
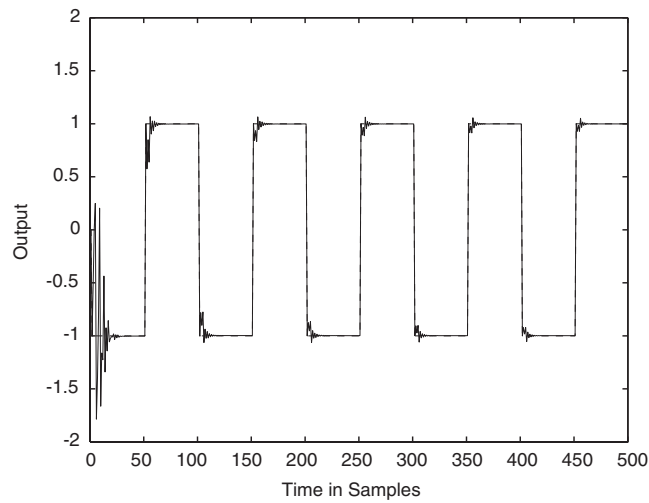
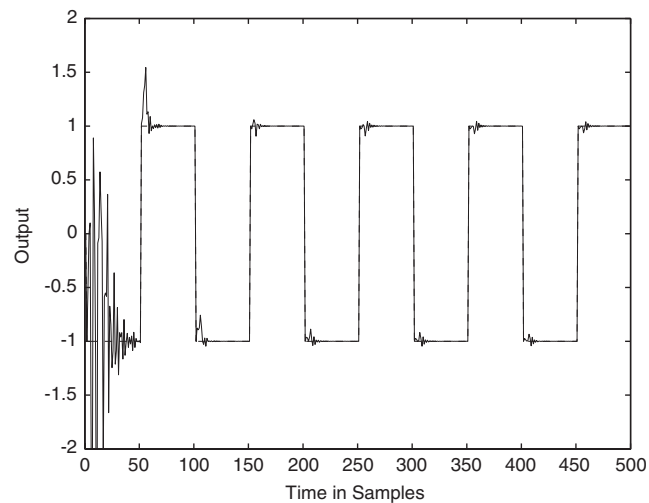Figure 4. Tracking performance with RLS identification.

Figure 5. Tracking performance with projection identification.

9. CONCLUDING REMARKS

In this paper we considered model reference adaptive control using both recursive least squares and projection identification. In both cases Lyapunov stability is established. A future extension of the present work will include the supermartingale analysis of Reference [8] with alternative identification algorithms including projection, stochastic approximation, and least mean squares.

APPENDIX A

The following result is a discrete-time version of Theorem 4.4 of Reference [18].

Theorem A.1
Let $\mathbb{N}$ denote the set of non-negative integers, let $\mathscr{D} \subset \mathbb{R}^n$ be an open neighbourhood of the origin, $W_1 : \mathscr{D} \to (0, \infty)$, $W_2 : \mathscr{D} \to (0, \infty)$, $W : \mathscr{D} \to [0, \infty)$, $f : \mathbb{N} \times \mathscr{D} \to \mathbb{R}^n$ and $V : \mathbb{N} \times \mathscr{D} \to \mathbb{R}$ be such that $W_1(x) \leqslant V(k, x) \leqslant W_2(x)$ for all $k \in \mathbb{N}$ and $x \in \mathscr{D}$ and $\triangle V \triangleq V(k + 1, f(k, x)) - V(k, x) \leqslant - W(x)$ for all $k \in \mathbb{N}$ and $x \in \mathscr{D}$. Furthermore, define $B_r \triangleq \{x \in \mathscr{D} : \|x\| < r\}$ and let $\tau_r < \min_{\|x\|=r} W_1(x)$. Then all solutions of $x_{k+1} = f(k, x(k))$ with $x_0 \in \{x \in B_r : W_2(x) \leqslant \tau_r\}$ are bounded and satisfy $W(x(k)) \to 0$ as $k \to \infty$.

Proof
Choose $\tau_r > 0$ such that $\tau_r < \min_{\|x\|=r} W_1(x)$. Then $\{x \in B_r : W_2(x) \leqslant \tau_r\} \subset \{x \in B_r : V(k, x) \leqslant \tau_r\} \subset \{x \in B_r : W_1(x) \leqslant \tau_r\}$. Let $x_0 \in \{x \in B_r : W_2(x) \leqslant \tau_r\}$. Then $x \in \{x \in B_r : V(k, x) \leqslant \tau_r\}$ for all $k \geqslant 0$. Hence $\|x(k)\| < r$ for all $k \geqslant 0$. Since $V(k, x)$ is bounded from below, the limit

$$\lim_{k \to \infty} \sum_{\zeta=0}^{k} W(x(\zeta)) \leqslant - \lim_{k \to \infty} \sum_{\zeta=0}^{k} \triangle V(\zeta, x(\zeta)) = V(0, x_0) - \lim_{k \to \infty} V(k, x(k))$$

exists. This implies that $W(x(k)) \to 0$ as $k \to \infty$. $\square$

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