

LETTER TO THE EDITOR

Anomalous fluctuations in surface growth

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Abstract. We have studied fluctuations in the steady state of a modified ballistic deposition model. The ensemble fluctuations of the surface width averaged over a time sequence, which involves temporal correlations, exhibit a peak around the previously proposed phase transition point in both 2+1 and 3+1 dimensions. We show that the time series of a finite system is self-averaging and the anomaly is in the fluctuations of the temporal correlations. We discuss the implications of our results in 2+1 dimensions for the structure of the renormalization group flows.

Non-equilibrium surface (interface) growth problems have attracted great interest recently. In particular, much attention has been focused on a roughening phase transition between two phases with different scaling [1-6]. These problems are associated with models which are described in the continuum limit by a nonlinear stochastic equation

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x, t) \quad (1)$$

proposed by Kardar, Parisi and Zhang (κPZ) [7]. Here $h(x, t)$ is the height of the interface, whose roughness we study, at position x and time t . The first term on the right-hand side of (1) represents surface relaxation driven by a surface tension ν , while the noise $\eta(x, t)$ in the last term is an uncorrelated white noise with Gaussian distribution and zero mean. The nonlinear term accounts for the dependence of the growth velocity on the slope of the interface.

The studies are done in $d+1$ dimensions with d the dimension of the substrate which is perpendicular to the additional growth direction. Renormalization group analyses [7-9] show that, while in 1+1 dimensions the nonlinear term always dominates, i.e. strong-coupling scaling prevails as long as $\lambda \neq 0$. In higher dimensions with $d > 2$, it is possible to have a phase transition at some finite λ between the strong-coupling (rough) phase and the weak-coupling ('smooth') phase. In the critical dimension $d = 2$, the situation is not clear. Numerical simulations have confirmed unambiguously that in 3+1 dimensions there is a phase transition at some $\lambda \neq 0$ [3, 5, 6, 10]. In 2+1 dimensions, however, controversy exists over whether there is a genuine phase transition [3, 4, 6, 10] or just a viciously slow crossover [5, 11]. Tang *et al* [11] propose an exponentially slow logarithmic-to-power-law crossover scaling from a one-loop renormalization group analysis of the continuum κPZ equation in

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2+1 dimensions. The latest work of Hwa *et al* [12], on the other hand, suggests a new and unusual fixed point controlling the crossover between rough and faceted growth, if both lattice effects and nonlinearities are included.

As in equilibrium critical phenomena, the fluctuations of physical quantities should play significant roles near a phase transition point. It is important to examine the fluctuations in the non-equilibrium growth models, especially in critical regions. In principle, there are two different kinds of fluctuations present, ensemble fluctuations and fluctuations in a steady-state time series, both due to the noise inherent in the process, as represented by η in the KPZ equation (1). In this letter, we will study both of these in a modified ballistic deposition model. We find that it is the temporal fluctuations that have interesting properties. We will describe a conjecture about the phase transition in 2+1 dimensions that has emerged from our study.

The dynamics of deposition in the model we study obeys:

$$\begin{aligned} h(\mathbf{x}, t+1) &= h(\mathbf{x}, t) + 1 \\ h(\mathbf{x}', t+1) &= \max[h(\mathbf{x}, t), h(\mathbf{x}', t)] \end{aligned} \quad (2a)$$

with probability p , and

$$h(\mathbf{x}, t+1) = \frac{1}{n_\alpha + 1} \left[h(\mathbf{x}, t) + 1 + \sum_{nn} h(\mathbf{x}', t) \right] \quad (2b)$$

with probability $1-p$. Here \mathbf{x} is picked at random, and the sum runs over the n_α nearest neighbours, \mathbf{x}' . If $p=1$, we recover the ballistic deposition rules, where h represents the height of the active zone [13, 14], while $p=0$ corresponds to the Edwards-Wilkinson model [15]. In a previous study of this model [3, 16]†, we showed the existence of a phase transition between $p=0.2$ and $p=0.4$ in 3+1 dimensions. In 2+1 dimensions, simulation results suggest a phase transition between $p=0.2$ and $p=0.4$ with complex scaling exhibited for $p \leq 0.2$. In this study, we focus on the steady state and investigate the fluctuations in surface width w .

In the steady state, the width w_t saturates and fluctuates significantly. We consider the ensemble fluctuations of $\xi = (1/N_T) \sum w_t^2$ where N_T is the number of time steps we measure in the steady state (each time step represents a sweep over the lattice). The ensemble fluctuation is defined by

$$\begin{aligned} \langle \xi^2 \rangle - \langle \xi \rangle^2 &= \frac{1}{N_T^2} \left(\left\langle \left(\sum_{ij} w_i^2 w_j^2 \right)^2 \right\rangle - \left\langle \sum_i w_i^2 \right\rangle^2 \right) \\ &= \frac{1}{N_T^2} \sum_{ij} \left(\langle w_i^2 w_j^2 \rangle - \langle w_i^2 \rangle \langle w_j^2 \rangle \right) \end{aligned} \quad (3)$$

where $\langle \rangle$ denotes an ensemble average. From (3) we can see that we actually measure the ensemble fluctuation of an averaged correlation in the time series. We have carried out our calculations on long time series to get reliable values for ξ and over 10 samples for the ensemble average. In figure 1, we plot the relative standard deviation of ξ , i.e. $\Delta = \sqrt{\langle \xi^2 \rangle - \langle \xi \rangle^2} / \langle \xi \rangle$ in 1+1, 2+1 and 3+1 dimensions, respectively. For 1+1 dimensions, in figure 1(a), barring the obvious statistical errors in the data, Δ monotonically increases with decreasing p , showing no singularity on the curves, as we expect

† We note that in [3] we presented the results for the model described here, rather than the one inadvertently defined there. However, these two models are in the same universality class, and both have been shown to exhibit the roughening phase transition and the anomalous scaling.

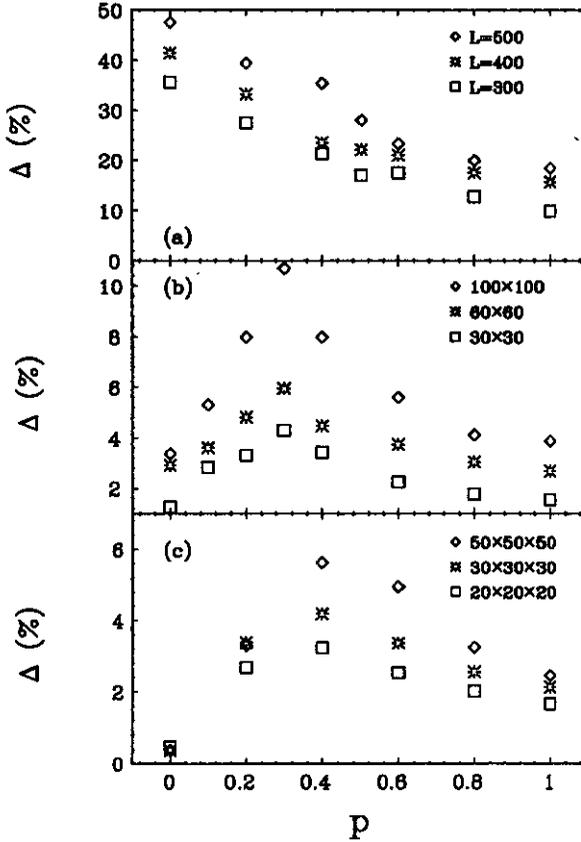


Figure 1. Δ against p in (a) 1+1 dimensions, (b) 2+1 dimensions, (c) 3+1 dimensions.

no phase transition except at $p=0$. In 3+1 dimensions, where we know there is a phase transition, we observe, in figure 1(c), a peak around $p=0.4$, in the vicinity of the transition point. Due to the numerical constraints and since the results are consistent with theoretical expectations, we have measured at only enough values to demonstrate the effect. For the critical dimension $d=2$, there is also a peak around $p=0.3$, as shown in figure 1(b). With increase in lattice size, the peak grows higher and sharper. This is consistent with our previous studies [3, 16], as well as other work [4, 6, 10], where there is evidence to suggest a phase transition between $p=0.2$ and $p=0.4$. The numerical difficulties due to the intrinsic fluctuations prevent us from getting better statistics. However, the data in figure 1 unambiguously demonstrate an anomaly in the collective fluctuations in both 2+1 and 3+1 dimensions and the lack thereof in 1+1 dimensions.

To better understand these results, let us first examine the time series of the surface width. In figure 2, we plot the distribution of the squared width w^2 in 2+1 dimensions for lattice size 100×100 and 4000 time steps after saturation, averaged over 40 samples. The inset in figure 2 shows the distribution obtained from one of those 40 samples which looks very erratic and noisy. However, if we take one sample and measure the distribution of w^2 over 60 000 time steps after saturation, we get exactly the same one as in figure 2. Normally, we expect the longest correlation length in the steady state time series to be of order L^z , which is about 1500 time steps for $L=100$ and $z=1.6$

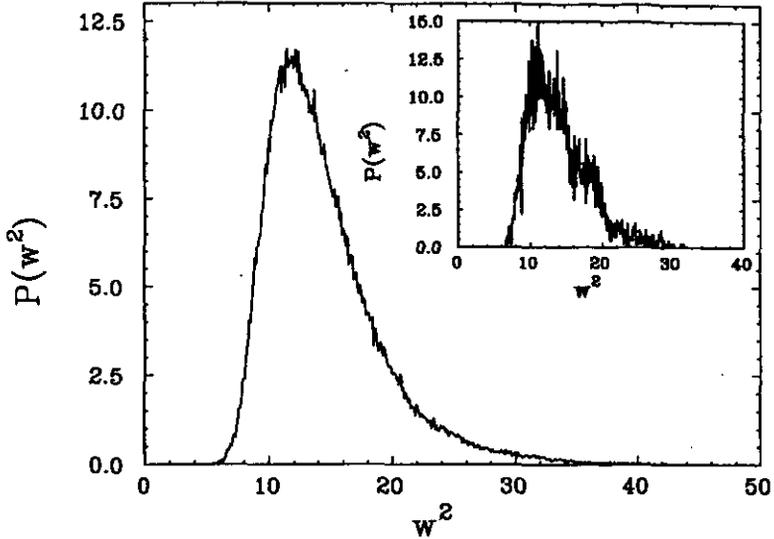


Figure 2. Distribution of squared width w^2 in the steady state time series of 4000 time steps for $p = 1$ and lattice size 100×100 , averaged over 40 samples. The inset shows the distribution obtained from only one sample.

in 2+1 dimensions, so that in a sequence of 60 000 time steps we actually have about 40 segments of length L^2 to average over. Thus we conclude that the time series of the surface width w^2 for a finite system is self-averaging in this model†. This result holds for all values of p and lattice sizes we have studied.

In figure 3, we plot the distributions for different p 's with lattice size 100×100 and in the inset those for different lattice sizes with $p = 1$. If we scale w^2 by $L^{2\alpha}$, the distributions with different lattice sizes in the inset collapse onto a single curve. We have also examined the tails in figure 3 by fitting them to both power law and exponential forms. Generally, the exponential form gives a better fit, especially for small p 's. An analytical analysis of the Edwards-Wilkinson model ($p = 0$) yields an exponential form $P(w^2) \sim e^{-aw^2}$ for the distribution. Thus, it appears that the anomaly in figure 1(b) is due neither to a failure of self-averaging nor to a dangerously long tail in the distribution.

In addition, we have calculated the ensemble fluctuation of w_t^2 at time t in the steady state. Notice that the temporal correlation is missing here. What we have found is a fluctuation monotonically decreasing with p , as measured by the standard deviation, in all three dimensions studied. This implies that it is the temporal correlation that is essential in the dynamical behaviour of the systems. This notion is also supported by a recent study of temporal correlations in the χ KS model [18]. In that study, we found that, in 2+1 dimensions, the correlation of the growth velocity $\langle v(0)v(\tau) \rangle$ scales with τ as $\tau^{-\gamma} e^{-\tau/\tau_0}$ for a range of τ , with $\gamma = 0.4$ for $p \geq 0.4$ and $\gamma = 0.8$ for $p = 0.2$. The characteristic decay time τ_0 for $p = 0.4$ is much larger than that for $p = 0.2$ and for $p > 0.4$.

Figure 4 shows plots of ξ against p . A careful examination of the graph reveals that there are inflexion points in the plots of 2+1 and 3+1 dimensions. Moreover,

† Amar and Family [17] have recently also obtained similar distributions in another context and arrived at the same conclusion.

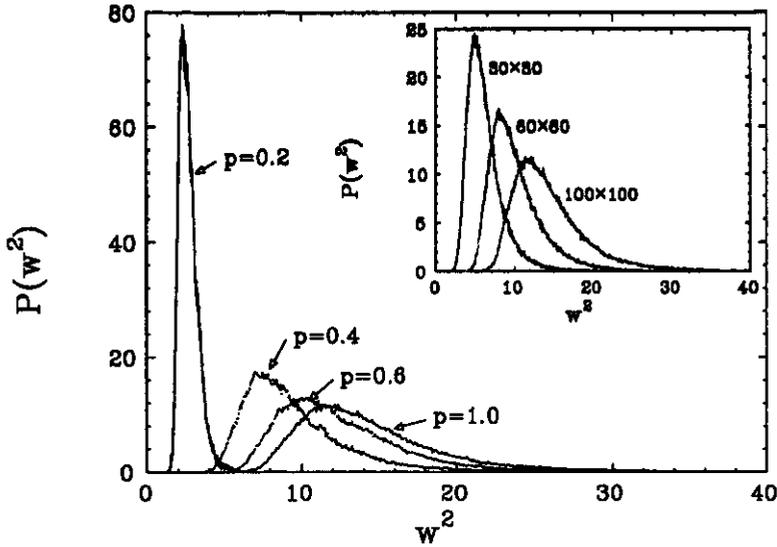


Figure 3. Distributions of squared width w^2 in the steady state time series of 4000 time steps, averaged over 40 samples, for different p 's with lattice size 100×100 . The inset shows the distributions for different lattice sizes with $p = 1$.

with increasing lattice sizes, the inflexion points move away from $p = 0$. This is again consistent with what we have found above.

While the results in $3+1$ dimensions are generally expected and understood, the ones in $2+1$ dimensions are still puzzling. Tang *et al* [11] suggest that for the continuum KPZ equation there should be a slow crossover. This, however, cannot account for all the effects we have seen. The results presented above and those from previous simulations [3, 4, 6, 10, 16] seem to indicate that, at least for the lattice sizes we have studied in $2+1$ dimensions, there is a point p_c , such that for $p > p_c$, systems flow† to the strong-coupling fixed point, while for $p < p_c$, systems flow to the weak-coupling fixed point. To make the renormalization group analysis and the simulation results consistent, it is necessary that the renormalization group flow should be intrinsically two dimensional, so that the system with $p < p_c$ can flow, at intermediate length scales, towards the unstable weak-coupling fixed point and turn at some larger length scale to follow the crossover to the strong-coupling phase. It is, however, not clear to us at present what physical parameter characterizes this second dimension. Hwa *et al* [12] point out that the discreteness in heights can give rise to anomalous scaling. However, it seems unlikely to us that this is responsible for the transition at p_c , since in our modified ballistic deposition model [3] and in a direct simulation on the KPZ equation [20], where this same effect has been observed, the surface heights are not quantized. Also, an apparently analogous transition is seen in a directed polymer version of the problem [10], where again the variable equivalent to the height is continuous.

In this two-dimensional flow diagram, there could be another, doubly unstable, fixed point corresponding to p_c , which controls the flows of the system at intermediate length scales. Though the nature of this unstable fixed point is not clear at this point, this notion might explain why the system with small p exhibits complex scaling relations

† The word 'flow' here means the direction for the system to evolve when the system size is increased.

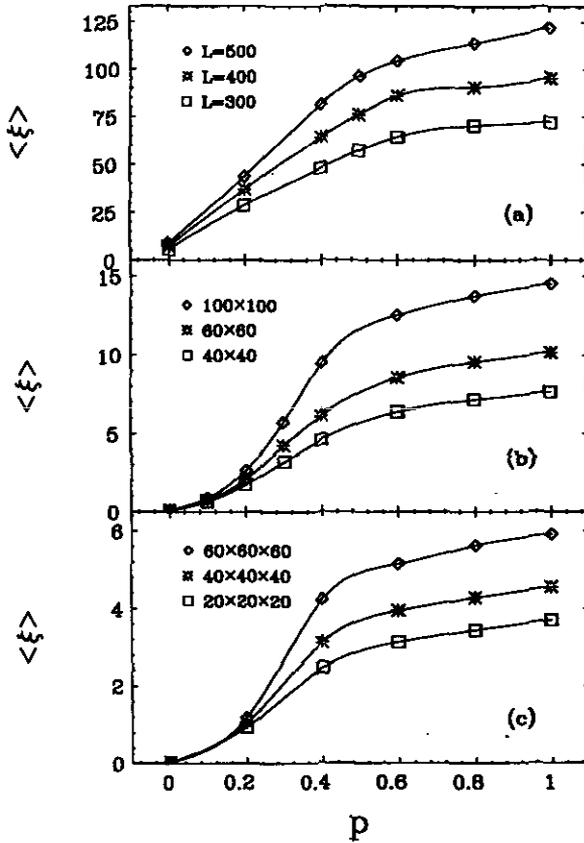


Figure 4. ξ against p in (a) 1+1 dimensions, (b) 2+1 dimensions, (c) 3+1 dimensions.

[3] for $p < p_c$, since the system is crossing over to the singly unstable weak-coupling fixed point. We conjecture that the point p_c is an unstable critical point which decides in which direction the system should flow, though the ultimate flow is presumably always towards strong-coupling for $p \neq p_c$. Closer to this point, the system would take a longer time to choose, in a manner similar to critical slowing down in equilibrium critical phenomena. We could define a time scale [1] $\tau \sim |p - p_c|^{-\mu}$ to characterize this situation. It is this scenario that may be responsible for the peaks observed both here and in [18] in the fluctuation of the temporal correlations for $p \sim p_c$. Although this conjecture of a 'phase transition' without a change of phase still needs to be examined with more numerical and analytical work, such as a real-space renormalization group analysis, it does seem to be reasonable and consistent with the available numerical and analytical results.

In summary, we have presented numerical results on fluctuations in the steady state of a modified ballistic-deposition model. We have studied the ensemble fluctuations of the averaged surface width, which involves temporal correlations, and found a peak in the vicinity of the previously proposed phase transition point in both 2+1 and 3+1 dimensions. We propose the existence of a phase transition of a special kind in 2+1 dimensions to account for the current results. More studies on this idea are certainly warranted to understand better the behaviour of the system in 2+1 dimensions.

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