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On Cardy's formula for the critical crossing probability in 2D percolation

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Abstract. Cardy's formula for the probability $\pi_v(r)$ of crossing a rectangular critical percolation system in the vertical direction, with free boundaries on the two sides, is written explicitly in terms of the aspect ratio r = height/width. The first three terms are given by $\pi_v(r) = 2^{4/3}3\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})^2[e^{-\pi r/3} - \frac{4}{7}e^{-7\pi r/3} + \frac{2}{13}e^{-13\pi r/3}\ldots]$. which, with $\pi_v(r) = 1 - \pi_v(1/r)$, are sufficient to determine $\pi_v(r)$ to high accuracy for all r.

1. Introduction

Cardy [1] has recently shown that the crossing probability in the vertical direction $\pi_v(r)$ for a percolation system with rectangular boundaries of aspect ratio r at the critical threshold p_c is given parameterically by

$$\pi_v(r) = c\eta^{1/3}F(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; \eta) \tag{1a}$$

$$= 1 - c(1-\eta)^{1/3}F(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; 1-\eta) \tag{1b}$$

where

$$\eta = \left(\frac{1-k}{1+k}\right)^2 \tag{2}$$

$$r = \frac{2K(k^2)}{K(1-k^2)} \tag{3}$$

$c \equiv 3\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})^2 \approx 0.566\,046\,680$, $F(a, b; c; z) = {}_2F_1(a, b, c; z)$ is Gauss' hypergeometric function (in the notation of Abramowitz and Stegun [2] (AS) ch 15), and $K(m)$ is the elliptic integral as in AS ch 17. Here, r is the ratio of height h to width w of the rectangle, and vertical crossing is defined as a continuous connected path from any site on the top with any site on the bottom, with free boundaries on the two sides. For convenience, r is defined here as the reciprocal of that in [1]—equivalent to choosing vertical rather than horizontal crossing. With this definition, $\pi_v(r)$ decreases mononically from the value of 1 to 0 as r goes from 0 to ∞.

The predictions of (1) agree with numerical studies carried out by Langlands *et al* [3, 4], and the point $\pi_v(1) = 1/2$ for a square system was also confirmed by Grassberger [5] and by Ziff [6]. Note that (1) is valid for any kind of percolation (site, bond etc) and on any lattice (square, triangular, random etc) at the critical threshold, for large h and w with

$r = h/w$. For finite h and w, finite-size corrections also enter [6, 7]. Studies of percolation in rectangular systems have also been made by Monetti and Albano [8], and numerical studies of crossing in three-dimensional percolation have been carried out recently [9, 10]. Note $\pi_v(r)$ is also designated by $p'(p_c)$, $\Pi(p_c)$ [11], and $R(p_c)$ [12].

The crossing probability $\pi_v(r)$ satisfies

$$\pi_v(r) + \pi_v(1/r) = 1 \tag{4}$$

because a system that does not percolate vertically must percolate horizontally on its dual, which is equivalent to the original system in boundary condition and shape, but of reciprocal aspect ratio [3]. When $r \to 1/r$, then $\eta \to 1 - \eta$, and (4) is satisfied by (1*a*) and (1*b*). The equivalence of (1*a*) and (1*b*) follows from AS 15.3.6.

Besides its definition as a crossing probability for a finite system, $\pi_v(r)$ may also be interpreted as the probability that the maximum height of the clusters connected to the bottom of an infinitely high rectangle of unit width is greater than r (again with free boundaries on the sides). It follows that the probability density that the maximum height of those clusters just equals r is given by (minus) the derivative of π_v with respect to r:

$$P_{max}(r) = -\pi_v'(r) \tag{5}$$

and the average maximum height is therefore given by

$$\langle r \rangle = -\int_0^\infty r\pi_v'(r)\,\mathrm{d}r = \int_0^\infty \pi_v(r)\,\mathrm{d}r. \tag{6}$$

In this paper, I report on some formal simplifications of Cardy's results (1)–(3). These were found with the help of the computer math programs MATHEMATICA 2.2 [13] and MAPLE V Rel. 2 [14]. For some of these calculations, MAPLE proved to be less useful because of its (apparently) limited ability with hypergeometric functions, but MATHEMATICA also missed some important identities. The series $h(s)$ in (13) below was identified with the help of a sequence database [15]. Some of the relations were found by an empirical extrapolation of series results rather than by deductive proof, but there is little doubt that they are correct as they can be verified numerically to arbitrarily high precision. The results are given in the next section, and discussed further in section 3.

2. Results

2.1. Elimination of *k*

First of all, note that k can be eliminated from (2) and (3) to yield

$$r = \frac{K(1-\eta)}{K(\eta)}. \tag{7}$$

To prove this, use Landen's transformation (AS 17.3.29) with $m = 1 - k^2$ and $m = 4k/(1+k)^2 = 1 - \eta$, to find $K(1-k^2) = 2K(\eta)/(1+k)$ and $K(1-\eta) = (1+k)K(k^2)$, respectively. Equation (7) implies that $\mathrm{e}^{-\pi r}$ is the Jacobi nome q for parameter η (AS 17.3.17).

2.2. Expansion of $\pi_v(r)$ about $r = 1$

The point $r = 1$ corresponds to $\eta = 1/2$. Writing $K(\eta)$ as a hypergeometric function (AS 17.3.9) and expanding about this point, one finds

$$K(\tfrac{1}{2}+x) = \frac{\pi^{1/2}}{\Gamma(\frac{3}{4})^2}\left(1+\frac{1}{2}x^2+\frac{25}{24}x^4+\frac{45}{16}x^6+\cdots\right)$$
$$+\frac{4\pi^{1/2}}{\Gamma(\frac{1}{4})^2}\left(x+x^3+\frac{32}{15}x^5+\frac{17}{3}x^7+\frac{1054}{63}x^9+\cdots\right)$$
$$= \frac{\pi^{1/2}}{\Gamma(\frac{3}{4})^2}F(\tfrac{1}{4},\tfrac{1}{4};\tfrac{1}{2};4x^2)+\frac{4\pi^{1/2}}{\Gamma(\frac{1}{4})^2}xF(\tfrac{3}{4},\tfrac{3}{4};\tfrac{3}{2};4x^2) \tag{8a}$$

where $x \equiv \eta - \frac{1}{2}$. Because there are two separate series in this expansion, a somewhat complicated expression results for $r = K(\frac{1}{2}-x)/K(\frac{1}{2}+x)$. It is thus convenient to introduce the quantity $(1-r)/(1+r)$, which is odd under the interchange $r \to 1/r$ and odd in x. (In [1] and [3], the function $\ln r$, which is also an odd function of x, is used, but it does not lead to simple expressions in this development.) It follows from (8*a*) that

$$a\left(\frac{1-r}{1+r}\right) = xF(\tfrac{3}{4},\tfrac{3}{4};\tfrac{3}{2};4x^2)/F(\tfrac{1}{4},\tfrac{1}{4};\tfrac{1}{2};4x^2)$$
$$= x+x^3+\tfrac{32}{15}x^5+\tfrac{17}{3}x^7+\tfrac{1054}{63}x^9+\tfrac{368}{7}x^{11}+\cdots \tag{8b}$$

where $a = \Gamma(\frac{1}{4})^2/4\Gamma(\frac{3}{4})^2 \approx 2.188\,439\,615$. This equation can be inverted to find x as a function of $y = a(1-r)/(1+r)$:

$$x = y - y^3 + \tfrac{13}{15}y^5 - \tfrac{3}{5}y^7 + \tfrac{113}{315}y^9 - \tfrac{103}{525}y^{11} + \cdots. \tag{9}$$

Expanding (1) about $x = 0$ yields

$$\pi_v(r) = \frac{1}{2} + \frac{2^{4/3}c}{3}\left(x+\frac{8}{9}x^3+\frac{16}{9}x^5+\frac{2560}{567}x^7+\frac{28\,160}{2187}x^9\ldots\right). \tag{10}$$

(Note that $\pi_v(r) - 1/2$ must be an odd function of x to satisfy (4).) Substituting (9) into (10), one finds

$$\frac{\pi_v(r)-\frac{1}{2}}{2^{4/3}c/3} = y - \frac{y^3}{9} - \frac{y^5}{45} + \frac{11}{2835}y^7 + \frac{19}{76\,545}y^9\ldots \tag{11}$$

which is the desired expansion. It implies that $-\pi_v'(1) = \pi_v''(1) = 2^{1/3}ac/3 \approx 0.520\,246\,171$. While (11) is useful to find the behaviour of $\pi_v(r)$ for r near 1, it converges too slowly to be accurate when r is much greater than 2 or less than 1/2. Consequently, another approach was also pursued.

2.3. Closed-form expression for $\pi_v(r)$

I found that $\pi_v(r)$ can also be written

$$\pi_v(r) = 2^{4/3}c\int_0^{e^{-\pi r/3}} h(s)\,ds \tag{12a}$$

$$= 1 - 2^{4/3}c\int_0^{e^{-\pi/3r}} h(s)\,ds \tag{12b}$$

where

$$h(s) = \prod_{n=1}^{\infty}(1-s^{6n})^4 \tag{13a}$$

$$= \left[\sum_{n=-\infty}^{\infty}(-1)^n s^{3(3n^2+n)}\right]^4 \tag{13b}$$

$$= 1 - 4s^6 + 2s^{12} + 8s^{18} - 5s^{24} - 4s^{30} - 10s^{36} - \cdots. \tag{13c}$$

That is, explicitly,

$$\pi_v(r) = 2^{4/3}c(e^{-\pi r/3} - \tfrac{4}{7}e^{-7\pi r/3} + \tfrac{2}{13}e^{-13\pi r/3}\ldots) \tag{14a}$$

$$= 1 - 2^{4/3}c(e^{-\pi/3r} - \tfrac{4}{7}e^{-7\pi/3r} + \tfrac{2}{13}e^{-13\pi/3r}\ldots). \tag{14b}$$

Additional coefficients that appear in (13c) and (14) are $h_n = 1, -4, 2, 8, -5, -4, -10, 8, 9, 0, 14, -16, -10, -4, 0, -8, 14, 20, 2, 0, -11, 20, -32, -16, 0, -4, 14, 8, -9, 20, 26, 0$ for $n = 0, \ldots 31$.

Because (14a) converges rapidly for $r > 1$ and (14b) converges rapidly for $r < 1$, these simple formulae allow $\pi_v(r)$ to be calculated to high accuracy for all r. At the worst point, $r = 1$, the three terms in (14a) or (14b) give $\pi_v(1)$ to nine significant figures. That just a few terms in such expansions are sufficient was also found by Cardy in the context of other models [16–18]. The leading exponential behaviour of $\pi_v(r)$ in (14a) was previously given by Cardy [18] and Langlands *et al* [3].

I arrived at these results as follows. First, using AS 17.3.21, (7) can be written

$$q \equiv e^{-\pi r} = \frac{\eta}{16} + 8\left(\frac{\eta}{16}\right)^2 + 84\left(\frac{\eta}{16}\right)^3 + 992\left(\frac{\eta}{16}\right)^4 + \cdots \tag{15}$$

which can be inverted for small η or large r to yield:

$$\eta = 16(q - 8q^2 + 44q^3 - 192q^4\ldots). \tag{16}$$

Next, expand (1) in powers of η,

$$\pi_v(r) = c\eta^{1/3}[1 + \tfrac{1}{6}\eta + \tfrac{5}{63}\eta^2 + \tfrac{4}{81}\eta^3\ldots]. \tag{17}$$

The series for $\eta^{1/3}, \eta^{4/3}, \ldots$ in powers of q follow from (16), and when substituted into (17), they yield the result

$$\pi_v(r) = 2^{4/3}c[q^{1/3} - \tfrac{4}{7}q^{7/3}\ldots] \tag{18}$$

where a term of order $q^{4/3}$ has cancelled out. At this point, I extended the series empirically by using the requirement that $\pi_v(1) = 1/2$. At $r = 1$, $q = e^{-\pi}$ and the above two terms already give $\pi_v(1) \approx 0.499\,999\,730\,085$. Hypothesizing that the next non-zero term is of order $q^{13/3}$ and that its coefficient is an integer divided by 13 (analogous to the previous terms), I guessed that that coefficient was 2 by calculating $13(0.5 - 0.499\,999\,730\,085)/(2^{4/3}q^{13/3}c) = 2.01021\ldots$. Repeating this procedure for the higher powers $q^{19/3}, q^{25/3}, \ldots$, using multiple-precision arithmetic, I found all the coefficients h_n following (14). I then found that the fourth root of $h(s)$ is simply $1 - s - s^2 + s^5 + s^7 - s^{12} - s^{15} + s^{22} + s^{26} \ldots$, and the database [15] suggested that the reciprocal of this series is the generating function of the number of unrestricted partitions of n elements $p(n)$ (AS 24.2.1), which finally led me to the closed-form expressions (13*a*) and (13*b*). I have since found that [15] also contains the expansion of the nome (15) to 17th order, and using this in the calculation above leads to the same first nine values of h_n.

While (12) was thus found numerically, there seems to be little doubt that it is equivalent to (1), since it can be checked to arbitrary precision using MATHEMATICA or MAPLE. For example, $r = 2$ corresponds to $k = 2^{-1/2}$ and $\eta = [(2^{1/2} - 1)/(2^{1/2} + 1)]^2 = 0.029\,437\,252\ldots$, and (1*a*) or (1*b*) gives $\pi_v(2) = 0.175\,646\,893\,800\,655\,239\,129\ldots$. This number is reproduced to 12 digits by the first two terms of (14), and 5–6 additional digits are correctly given with each addtional term of (14).

Note that the five terms in (11) give $\pi_v(2)$ to only 7 digits of accuracy, and each additional term adds only one additional digit.

The point $r = 4$ corresponds to $k = 2^{5/4}/(2^{1/2} + 1)) \approx 0.029\,437$, $\eta = [(2^{1/4} - 1)/(2^{1/4} + 1)] \approx 5.579\,59 \times 10^{-5}$, and $\pi_v(4) \approx 0.021\,630$. These exact values of k and η for $r = 2$ and 4 were found by using the transformations $k' = 2k^{1/2}/(1 + k)$ and $\eta' = [(1 - (1 - \eta)^{1/2})/(1 + (1 - \eta))^{1/2}]^2$ for $r' = 2r$, which follow from Landen's transformation.

Equation (12*b*) was derived from (12*a*) by assuming (4), but the equivalence of these two can also be shown directly. It is convenient to consider the derivative of $\pi_v(r)$, for which (12*a*) gives

$$\pi_v'(r) = -\frac{2^{4/3}\pi c}{3}\left[\sum_{n=-\infty}^{\infty}(-1)^n e^{-3\pi r(n+1/6)^2}\right]^4 \tag{19}$$

while (12*b*) gives a similar expression, but with r replaced by $1/r$ and a coefficient of r^{-2} added in front. The equivalence of these two expressions is implied by the identity

$$\sum_{n=-\infty}^{\infty}(-1)^n e^{-3\pi r(n+1/6)^2} = r^{-1/2}\sum_{n=-\infty}^{\infty}(-1)^n e^{-(3\pi/r)(n+1/6)^2} \tag{20}$$

which follows from the Poisson summation formula ([19], p 124; also [16]).

Considering the function $\pi_v'(r)$ may also be useful in finding a direct proof that (12) follows from (1). From (1), one has

$$\pi_v'(r) = \frac{c}{3[\eta(1-\eta)]^{2/3}}\frac{d\eta}{dr}. \tag{21}$$

Equating this to (21) yields an expression for $dr/d\eta$, which by (7) is equal to $-[K(\eta)K'(1-\eta) + K(1-\eta)K'(\eta)]/K(\eta)^2$. However, a direct proof of the resulting identity is not immediately obvious. Note that functions of the type of (13*a*) and (13*b*) appear quite frequently in conformal field theories (e.g. [17]), although not usually as a fourth power. This result clearly reflects some underlying identities in these Jacobi-type functions.

3. Discussion

Equation (14a) shows that $\pi_v(r)$ drops essentially by a single exponential $e^{-\pi r/3}$ for $r \gtrsim 1$, as noted previously [3, 18]. This behaviour suggests a simple interpretation of $\pi_v(r)$ as a product of independent probabilities. Increasing r by unity, which corresponds to adding an additional square to the rectangular system, reduces the probability of percolating by a factor $e^{-\pi/3} \approx 0.350\,919\,807$ when r is large. This factor is significantly lower than the probability that the first square percolates (1/2), evidently because for the first square the bottom of the system is completely occupied, while for squares added to a high rectangular system, only a fraction of their bottom rows are occupied. This simple exponential behaviour of $\pi_v(r)$ (for $h, w \to \infty$) presumably occurs only at the critical point p_c.

For a finite system, the renormalization-group (RG) fixed point p^* is determined by $p^* = \pi_v(h, w, p^*)$, where $\pi_v(h, w, p)$ gives the spanning or crossing probability at occupation probability p in a finite system of dimensions $h \times w$ [12]. When the system is made infinite with $h/w = r$ fixed, p^* limits to p_c because $\pi_v(h, w, p)$ becomes a step function in p. It follows, however, that if $p_c \neq \pi_v(r)$, then $\pi_v(h, w, p^*)$ will not limit to $\pi_v(r)$, and in [6] it is shown that this leads to a slower convergence of p^* to p_c as $w \to \infty$. For a square boundary, this requirement is met by bond percolation on the square lattice, where, due to a number of symmetries, $p^* = 1/2 = \pi_v(1)$ for systems of all size, but for site percolation, $\pi_v(w, w, p^*)$ limits to $p_c \approx 0.592\,746$ as $w \to \infty$ rather than 1/2 [6,20]. However, if the aspect ratio is adjusted appropriately, the RG fixed point for site percolation can be made to coincide with $\pi_v(r)$; for such a system, p^* will tend to p_c much faster than for systems of other shape, as $w \to \infty$. From (1) or (12) I find that $\pi_v(r) = 0.592\,746$ corresponds to $r = 0.835\,668\,5$, $k = 0.092\,991\,07$ and $\eta = 0.688\,636\,28$. Unfortunately, this value of r does not correspond to any obvious number (presumably reflecting some underlying symmetry in the system) which would imply a closed expression for p_c.

For $\langle r \rangle$, (12) implies

$$\langle r \rangle = 2^{4/3} \sum_0^{\infty} \frac{h_n e^{(6n+1)/3\pi}}{(6n+1)^2} \approx 1.280\,356\,234\,428\,962\,546. \qquad (22)$$

(This numerical value was found by integrating (14a) from 1 to ∞ and (14b) from 0 to 1, requiring the use of just a few terms of those expansions.) Thus, for an infinitely high rectangular system, the average maximum height of the clusters connected to the bottom is about 1.28. Because $\pi_v(1) = 1/2$, it follows that exactly half of the times the maximum height is greater than 1, and half the times it is less than 1; the average maximum height is greater than 1 because of weighting by r. The average of the quantity y is, however, identically zero, reflecting its symmetry about $r = 1$.

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