THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

DIFFUSION LENGTHS IN HETEROGENEOUS MEDIA

Joel H. Ferziger
George S.C. Wang
Paul F. Zweifel

Department of Nuclear Engineering
The University of Michigan
Ann Arbor, Michigan

September, 1960

IP-468

Engn
UMR
1921

ACKNOWLEDGMENTS

We wish to thank Professor R. K. Osborn for several illuminating discussions and the National Science Foundation and the Babcock and Wilcox Company, Atomic Energy Division, for their support.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

An adaptation of Behrens' Method[7] to the calculation of diffusion lengths in heterogeneous media is given. In all cases, the diffusion length in a medium containing absorbing lumps can be related to the self-shielding factor of the lumps. Calculations are presented only for the simplest case given but the results display considerable disagreement with a frequently used formula. On grounds which are mainly intuitive, it is believed that this method is more accurate, particularly for large moderator to absorber ratio. Final conclusions cannot be drawn, however, until more experimental data becomes available. Calculations using some of the corrections and evaluation of some of the integrals shown here will be given in a future paper.

DIFFUSION LENGTHS IN HETEOGENEOUS MEDIA

I Introduction

A knowledge of thermal diffusion lengths for non-homogeheous media is of particular importance in survey type criticality and shielding studies for which an accurate estimate of the thermal leakage may be required. In cases in which inhomogeneities are not very large compared to a neutron near-free path the use of the homogeneous reactor theory with appropriate self-shielded cross sections probably gives reasonable results. However, in a typical low enrichment reactor, the inhomogeneities may be large, and some other approach must be sought.

Weinberg and Wigner(1) and Russell(2) have defined the diffusion length, L, as that value which makes the well known formula

$$\frac{1}{1+L^2B_g^2} \qquad (1)$$

yield the correct thermal leakage, B_g^2 being the geometric buckling of the system. As the leakage is the quantity in which one is directly interested, this is a very convenient means of defining the diffusion length. By applying diffusion theory to alternating slabs of moderator and fuel, Weinberg and Wigner arrive at the result:

$$L^2 = L_F^2 f + L_M^2 (1-f) \qquad (2)$$

where L_F^2 and L_M^2 are the diffusion areas for the pure fuel and moderator respectively and f is the thermal utilization, defined conventionally.

In slightly enriched uranium-water systems, the two terms on the right hand side of Equation (2) are of comparable magnitude, whereas is a similar homogeneous medium, the second term alone represents the correct diffusion area. Thus, this method gives L^2 incorrectly in the homogeneous limit. The difficutly apparently lies in the application of diffusion theory to thin regions.

By using a variational principle, Russell arrives at the result

$$L^2 = \overline{D}/\overline{\Sigma}_a \tag{3}$$

where $\overline{D}$ and $\overline{\Sigma}_a$ are the flux-volume averaged diffusion coefficient and macroscopic absorption cross section, respectively. This method does not appear to be completely satisfactory either. First, the details of the energy-space distribution of the flux are required for computing the required averages and second, directional effects are neglected. In rodded media, one expects a large diffusion length parallel to the rods than perpendicular to them. This effect has been found to be significant and is taken into account by the French (3) in the design of the their graphite reactors. By retaining diffusion theory, with a tensor rather than a scalar diffusion coefficient, one finds the leakage to be

$$\frac{1}{1 + \sum_i L_i^2 B_i^2} \tag{4}$$

Here L_i^2 and B_i^2 are the diffusion area and the buckling in the i direction, the i being the principal axes of the diffusion tensor.

Spinrad[4] and Shevelev[5] are able to predict different L^2 in various directions when anisotropies exist. They define L^2 in terms of the asymptotic decay of the neutron flux from a source, and apply diffusion theory to thin regions. There is some question as to the validity of this procedure. In fact, Spinrad's method has the wrong homogeneous limit.

Triflaj[6] applied a variational principle to the transport equation and arrived at an expression for the diffusion length, which, while apparently accurate is rather cumbersome. A result amenable to hand calculation would appear to be more desirable.

In searching for a new method for calculating diffusion lengths, one is struck by the remarkable agreement of Behrens' theory[7] with experimental results[8] for the case of a homogeneous medium containing empty holes. The method employed here is in essence an adaptation of Behrens' method to the more general case of a heterogeneous medium.

II CALCULATION OF THE DIFFUSION LENGTH

As a starting point we take the definition of the diffusion length as one-sixth of the mean square crow flight distance that a neutron travels from the time it is thermalized to the time it is absorbed. For convenience, we shall deal only with infinite media in which the structure is periodic, so that a unit cell may be studied.

In such a medium the thermal flux shape is also periodic and it is assumed that this shape can be calculated. First, we seek ℓ^2, the mean square distance travelled by neutrons in the stationary distribution. (It is assumed here that the neutrons have the same spatial distribution after each fight, and that the effect of sources and sinks balance so as to maintain this distribution.) Then, P_A, the probability that a neutron is absorbed in a single flight, is to be calculated. Under the assumptions made above ℓ^2 and P_A are identical for each flight made by a neutron, and one can write

$$L^2 = \frac{\ell^2}{6P_A} \tag{5}$$

provided that scattering is isotropic, i.e., provided that the direction of a neutron's travel does not depend on its past history. Equation (5) results from the fact that under the assumption of a stationary distribution P_A is constant for each flight, and thus $1/P_A$ is the average number of flights made by thermal neutron before being absorbed. The assumption of a stationary distribution is very nearly fulfilled when a neutron makes many collisions before being absorbed.

In calculating ℓ^2 and P_A, it will be further assumed that the lumps are widely separated and thus a neutron has a negligibly small probability of entering more than one lump of a given flight. (This restriction may be relaxed in a subsequent paper.) Then contributions to ℓ^2 and P_A come from five sources:

1) Neutrons suffering their ith collisions in the moderator and their (i+1)st collisions in a lump;

2) Neutrons suffering their ith collisions in the moderator which pass through a lump and suffer their (i+1)st collisions in the moderator;

3) Neutrons which do not enter a lump at all or a given flight;

4) Neutrons which suffer their ith collisions in a lump, leave the lump and suffer their (i+1)st collisions in the moderator; and

5) Neutrons which do not leave the lump at all on a given flight.

The probabilities of each type of event will be calculated, and the mean square flight distance and absorption probability will be averaged over the events to yield the required value of ℓ^2 and P_A.

A. Isotropic Scattering, Flat Flux

As a first example, the following highly idealized case will be considered (some of the previous assumptions repeated for clarity):

a) The medium is composed of only two materials, one of which is highly absorbing;

b) The absorber is in the form of widely separated lumps, distributed periodically in the moderator;

c) Scattering is isotropic (Equation (5) applies);

d) The neutron flux is uniform in the moderator;

e) Neutrons scattered in the absorber have the same ℓ^2 and P_A as those scattered in the moderator. (Contributions (4) and (5) above can be neglected, it can be assumed that all neutrons start in the moderator.)

Some of these restrictions will be relaxed later. Macroscopic cross sections will be denoted by Σ for the heavily absorbing material and σ for the moderating material, both with the conventional subscripts a, s, and tr for absorption, scattering, and transport respectively. Lack of a subscript will denote total cross section.

We define the directional chord length distribution $\psi(R,\underline{\Omega})$, such that $\psi(R,\underline{\Omega})dR$ is the number of chords in the lump with lengths between R and R+dR in direction $\underline{\Omega}$, (see Figure 1). It is related to the conventional (normalized) chord length distribution $\varphi(R)$ defined by Case, Placzek and DeHoffmann[9] by:

$$\varphi(R) = \frac{S}{4} \int \psi(R,\underline{\Omega})d\underline{\Omega} \qquad (6)$$

where S is the surface area of a lump; $\varphi(R)dR$ is the relative number of chords with lengths between R and R+dR in a lump. A volume v of moderator is associated with each lump. From Figure 1, one sees that the neutrons which travel a distance between z and z+dz in direction $\underline{\Omega}$ before entering a lump, and then enter the lump such that they travel along a chord of length between R and R+dR, must have started their

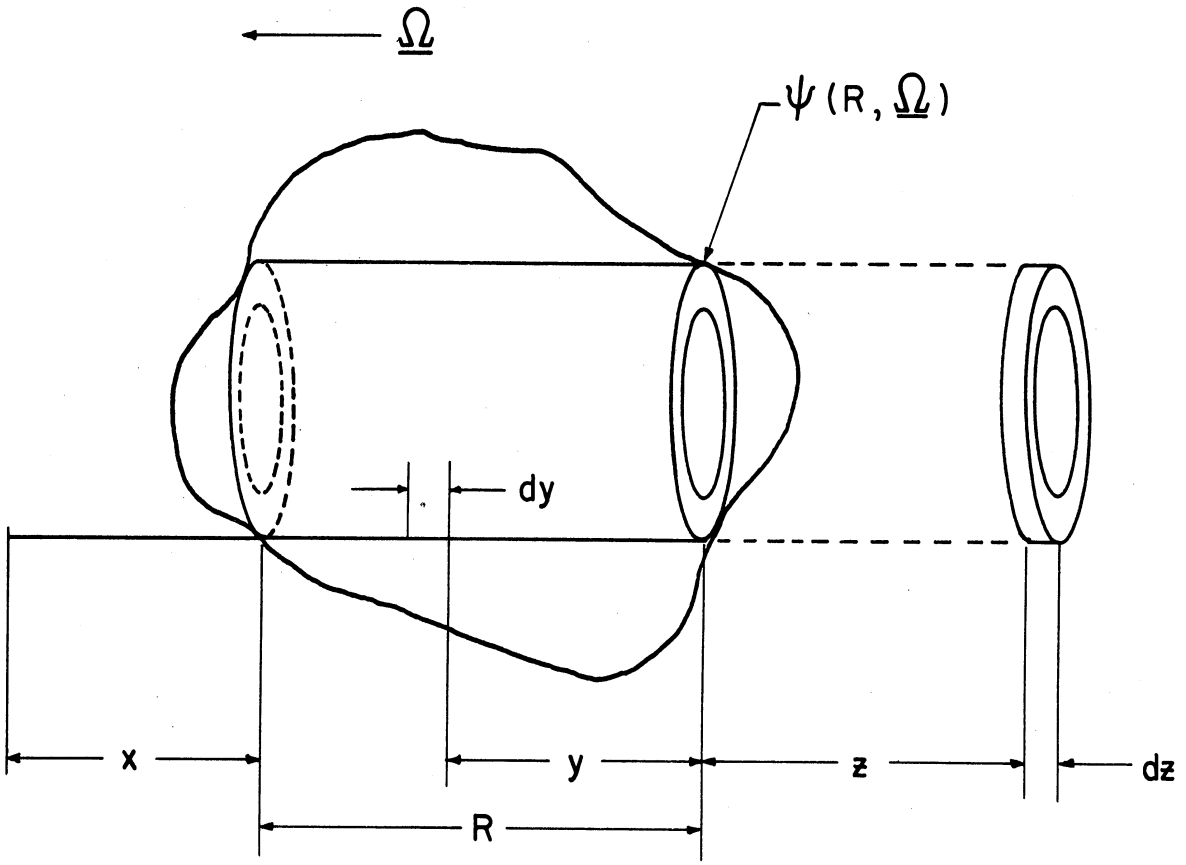


Figure 1. Geometry for Calculation of ℓ_o and P_{A_o}

flights in the element of volume

$$dv = \psi(R,\underline{\Omega})dRdz$$

The fraction of neutrons in volume elements of this type is dv/v, under the assumption of a flat flux distribution. But these neutrons are attenuated by a factor $e^{-\sigma z}$ in travelling to the lump. Therefore, the probability that a neutron travels a distance between z and z+dz, in direction $\underline{\Omega}$, and then enters the lump along a chord with length between R and R+dR is:

$$q(z,R,\Omega)dRdz = \frac{e^{-\sigma z}\psi(R,\Omega)}{v}\, dRdz \qquad (7)$$

Since the probability of travelling a distance between y and y+dy(y<R) in the lump and then colliding is $\Sigma\, e^{-\Sigma y}dy$, the square distance travelled by those neutrons which make their next collisions within a lump is:

$$\int_0^\infty dR \int \frac{d\Omega}{4\Pi} \int_0^\infty dz \int_0^R dy\,(y+z)^2 q(z,R,\Omega)\Sigma\, e^{-\Sigma y} \qquad (8)$$

The z integral is allowed to go to infinity as a result of the assumed wide separation of the lumps; the factor 4Π is for normalization purposes.

On the other hand, the neutron has probability $e^{-\Sigma R}$ of going through the lump and it may then travel a distance between x and x+dx, with probability $\sigma e^{-\sigma x}dx$, in the moderator after leaving the lump. These neutrons contribute

$$\int \frac{d\Omega}{4\Pi} \int_0^\infty dR \int_0^\infty dz \int_0^\infty dx\,(z+R+x)^2 q(z,R,\Omega)\sigma e^{-\sigma x} \qquad (9)$$

to the square distance travelled (ℓ^2).

Lastly, some of the neutrons may not enter the lump at all on a given flight. For these neutrons, it does not matter what is in the lump. Thus, it may thus be assumed to be a vacuum, and we may follow Behrens' argument to say that the probability of entering the hole for a neutron which travels <u>total</u> distance between w and w+dw in the <u>moderator</u> (w = z+x as defined above) in direction $\underline{\Omega}$ on a given flight is given by:

$$P(w,\underline{\Omega})dw = dw \int_0^\infty dR \int_0^w dz \int_0 dx\ q(z,R,\underline{\Omega})e^{-\sigma z}\sigma e^{-\sigma x}\ (z+x-w) \tag{10}$$

$$= \sigma w e^{-\sigma w}\, dw \int_0^\infty \psi(R,\underline{\Omega})dR$$

a result derived by Behrens'. Then the contribution to ℓ^2 of neutrons which do not enter the lump is:

$$\int \frac{d\Omega}{4\pi} \int_0^\infty w^2\ [1-P(w,\underline{\Omega})]\ \sigma e^{-\sigma w} dw \tag{11}$$

It may be noted that the probabilities of the three processes described above add to unity.

The integrals may be carried out and expressed in terms of the escape probability (self-shielding factor) of the lumps. The result, obtained by adding expressions (8), (9), and (11), is:

$$\ell_o^2 = \frac{S}{v}(1-g)\ [\frac{1}{2\sigma^2\Sigma} + \frac{1}{2\sigma\Sigma^2} - \frac{1}{\sigma^3}] + \frac{S}{v}(\frac{dg}{d\Sigma})[\frac{1}{2\sigma\Sigma} - \frac{1}{2\sigma^2}] + \frac{2}{\sigma^2} \tag{12}$$

where the function g is defined by:

$$g = \frac{4}{S} \int \frac{d\Omega}{4\pi} \int_0^\infty e^{-\Sigma R} \psi(R, \underline{\Omega}) dR = 1 - \langle R \rangle \Sigma P_o \qquad (13)$$

Here, P_o is the usual escape probability and is tabulated for spheres, slabs and cylinders in Reference [9]. The average chord length $\langle R \rangle$ is simply the surface area of a lump, S, divided by four times the volume, V, of the lump.

To obtain the diffusion length, one needs only to calculate the probability that a neutron is absorbed on a given flight, P_A. From previous arguments, the probability that a neutron in the assumed flat distribution outside the lump will suffer its next collision in a lump is given by Expression (8) with the $(y+z)^2$ term deleted from the integrand:

$$P_c = \int \frac{d\Omega}{4\pi} \int_0^\infty dR \int_0^\infty dz \int_0^R dy \; q(z, R, \Omega) \Sigma \, e^{-\Sigma y} = \frac{S}{4v\sigma} (1-g) \qquad (14)$$

It would be unreasonable to assume that all of these neutrons are absorbed, especially since the absorption and scattering cross sections are approximately equal for low enrichment uranium. A better approximation is to assume that only Σ_a/Σ of them are absorbed. This explains why assumption (e) above was invoked. Neutrons do scatter from the lumps, but if they have the same ℓ^2 and P_A as neutrons scattered from the moderator, their effect on L^2 can be neglected. When the ratio of moderator to absorber is high, relatively few neutrons are scattered from the moderator and these neutrons can be safely ignored. This is clearly a poor approximation for small moderator to absorber ratios but

but is an excellant approximation for very heavily absorbing materials, eg. highly enriched uranium.

Neutrons which collide in the moderator have probability σ_a/σ of being absorbed, so the absorption probability becomes:

$$P_{A_o} = \left(\Sigma a/\Sigma\right) P_c + \left(\sigma_a/\sigma\right)(1-P_c)$$
$$= \frac{\Sigma a}{\Sigma}\frac{S}{4v\sigma}(1-g) + \frac{\sigma a}{\sigma}\left[1-\frac{S}{4v\sigma}(1-g)\right] \qquad (15)$$

By inserting Equations (12) and (15) into Equation (5), one obtains a formula for the diffusion length.

In addition to the neglect of scattering from the lump, there is another major flaw in the argument. All of the integrals within the moderator have been extended to infinity as a consequence of the assumed wide separation of the lumps. Again, this breaks down if the moderator to absorber ratio (M/A) is small. Means of avoiding these difficulties will be given in section C.

In the limit of small lump dimensions, for constant moderator to fuel ratio, the result should go to the homogeneous diffusion length. One can show that this is so, provided that (M/A) is sufficiently large. For small lumps, the simplified case $\Sigma_s = 0$, $\sigma_a = 0$ yields:

$$L_o^2 \longrightarrow \frac{\Sigma a/\sigma + M/A}{3\Sigma a\sigma} \qquad (16)$$

which, for large (M/A), goes to the result $(3\Sigma_s\Sigma_a)^{-1}$, which is correct for isotropic scattering in a weakly absorbing homogeneous medium. The validity, of course, depends on the cross sections to some extent through the ratio $(\Sigma a/\sigma)$.

B. Anisotropic Scattering

If anisotropic scattering is present Equation (5) is invalid since the direction of the (n+1)st flight depends on the direction of the nth flight. The method employed in the previous section is essentially a random walk technique. Generalization of this problem to anisotropic scattering if very difficult if a solution is to be obtained in closed form. Instead we have done the random walk problem for a homogeneous medium and obtained the not unexpected result:

$$L^2 = \frac{1}{3\Sigma_a(\Sigma_t - \Sigma_s\bar{\mu})} = \frac{1}{3\Sigma_a\Sigma_{tr}} \qquad (17)$$

where $\bar{\mu}$ is the average cosine of the laboratory scattering angle. For the same problem with isotropic scattering using the notation of section A, we have:

$$\ell^2 = 2/\Sigma_t^2$$

$$P_A = \Sigma a/\Sigma_t$$

From which one obtains the well known result

$$L^2 = (3\Sigma_a\Sigma_t)^{-1} \qquad (18)$$

Comparing (17) and (18), one sees that (17) is derived from (18) by replacing the total cross section by the transport cross section, $\Sigma_{tr} = \Sigma_t - \Sigma_s\bar{\mu}$. From this it was assumed that for anisotropic scattering one need only replace the total cross section in the moderator by the transport cross section.

The derivation of Equation (17) is of some interest. Let $P(\underline{r}_i)$ be the probability that the ith neutron flight be represented by the vector $\underline{r}_i$. Then the mean square distance travelled by a neutron in exactly n flights is

$$\langle r^2 \rangle_n = \int_{\underline{r}_1} \int_{\underline{r}_2} \cdots \int_{\underline{r}_n} (\underline{r}_1 + \underline{r}_2 + \cdots \underline{r}_n)^2 P(\underline{r}_1) \cdots P(\underline{r}_n) d\underline{r}_1 \cdots d\underline{r}_n \quad (19)$$

Now, the $P(\underline{r}_i)$ can be decomposed into the product of the probability $P(r_i)$ that $\underline{r}_i$ have length r_i and the probability $P(\underline{\Omega}_i)$ that it have direction $\underline{\Omega}_i$. For integration purposes, it is convenient to use $\underline{\Omega}_{i-1}$ as the polar axis of the coordinates. Terms of the following type arise from Equation 19.

$$\int_0^\infty dr_1 \cdots \int_0^\infty dr_n \int d\underline{\Omega}_1 \cdots \int d\underline{\Omega}_n (\underline{r}_i \cdot \underline{r}_j) P(r_1) P(\underline{\Omega}_1) \cdots P(r_i) P(\underline{\Omega}_i) \cdots P(r_j) P(\underline{\Omega}_j) \cdots P(r_n) P(\underline{\Omega}_n) \quad (20)$$

Now $\underline{r}_i \cdot \underline{r}_j$ depends only on the lengths of $\underline{r}_i$ and $\underline{r}_j$ and the scattering angles for collisions i + 1, i + 2 .. j. When the various probabilities $P(r_k)$ and $P(\underline{\Omega}_k)$ are properly normalized, all of the integrals over r_k for $k \neq i, j$ and all of integrals over $\underline{\Omega}_k$ for $k \leq i$ and $k > j$ may be carried out to yield unity. Then Equation (20) reduces to:

$$\int_0^\infty r_i P(r_i) dr_i \int_0^\infty r_j P(r_j) dr_j \int d\underline{\Omega}_{i+1} \cdots \int d\underline{\Omega}_j (\underline{\Omega}_i \cdot \underline{\Omega}_j) P(\underline{\Omega}_{i+1}) \cdots P(\underline{\Omega}_j) \qquad i \neq j \quad (21)$$

Honeck[10] has shown that

$$\int d\underline{\Omega}_{i+1} \cdots \int d\underline{\Omega}_j (\underline{\Omega}_i \cdot \underline{\Omega}_j) P(\underline{\Omega}_{i+1}) \cdots P(\underline{\Omega}_j) = (\bar{\mu})^{j-1} \quad (22)$$

a result that has been verified by using the properties of rotation groups. Finally, since

$$P(r_i) = \Sigma e^{-\Sigma r_i}$$

expression (21) becomes (including the case i = j):

$$\frac{1+\delta_{ji}}{\Sigma^2} \langle\mu\rangle^{j-i} \tag{23}$$

Now, collecting terms and noting that there all n terms of the type $\underline{r}_i \cdot \underline{r}_i$, n-1 of the type $\underline{r}_i \cdot \underline{r}_{i+1}$, etc. we have:

$$\langle r^2\rangle_n = [n+(n-1)\bar{\mu} + (n-2)\bar{\mu^2} + \ldots + \bar{\mu}^{n-1}] \frac{2}{\Sigma^2} \tag{24}$$

The probability that a neutron will make exactly n collisions is:

$$P_n = \left(\frac{\Sigma_s}{\Sigma}\right)^{n-1} \left(\frac{\Sigma_a}{\Sigma}\right) \tag{25}$$

and, finally, L^2 is given by

$$\langle r^2\rangle = 6L^2 = \sum_{n=1}^{\infty} P_n \langle r^2\rangle_n = \left(\frac{2}{\Sigma^2}\right)\left(\frac{\Sigma_a}{\Sigma}\right)\sum_{n=1}^{\infty} \left(\frac{\Sigma_s}{\Sigma}\right)^{n-1} \sum_{m=1}^{n} m\bar{\mu}^{n-m} \tag{26}$$

The double sum may be carried out in closed form by rearranging the series and yields[17].

C. Small Moderator to Absorber Ratio

It was shown that the method of section A is best for large (M/A) and it was pointed out that the method has at least two difficulties:

1) That integrals in the moderator extend to infinity and

2) that scattering from the lump is treated in a very approximate manner.

The first of these difficulties is overcome by cutting the integrals off at some limit point Q. The determination of Q is rather difficult and it would be best to obtain it, or at least verify any guess at it, by comparison with experiment. As experimental data were lacking at the writing of this paper, no authoritiative value can be given. A reasonable just guess might be $S_o/4v$ where S_o and v are the surface area and volume of a cell. By allowing the integrals in the moderator to range only from zero to Q one obtains the correction:

$$\delta \ell_o^2 = \frac{S}{4v}(1-g)e^{-\sigma Q}\left[Q^3 + \frac{2Q^2}{\sigma} + Q\left[\frac{4}{\sigma^2} - \frac{2}{\sigma\Sigma}\right] - \frac{2}{\sigma\Sigma^2} - \frac{2}{\sigma^2\Sigma}\right]$$

$$+ \frac{S}{4v}\frac{dg}{d\Sigma}e^{-\sigma Q}\left[2Q^2 + \frac{2Q}{\sigma} + \frac{2}{\sigma^2} - \frac{2}{\sigma\Sigma}\right] - \frac{SQ}{4v}e^{-\sigma Q}\frac{d^2g}{d\Sigma^2} \qquad (27)$$

$$- e^{-\sigma Q}\left[Q^2 + \frac{2Q}{\sigma} + \frac{2}{\sigma^2}\right]$$

which is to be added to Equation (12) and give a corrected ℓ_o^2. The probability of collision in the lump now becomes

$$P_c = \frac{S}{4v\sigma}(1-g)(1-e^{-\sigma Q}) \qquad (28)$$

which should be used in place of (14) in computing P_{A_o}.

The second difficulty mentioned, that of treating scattering in the lumps, may be handled as follows: on each flight a fraction P_c of the neutrons which started outside the lump suffer their next

collisions within the lump. Of these, $(\Sigma_s/\Sigma_t)P_C = t_1$ are scattered and we assume that they have probability $(1-P_o)$ of having their next collision within the lump. This is equivalent to assuming that the entering neutrons have a uniform collision density in the lump. Then $(\Sigma_s/\Sigma_t)(1-P_o)t_1 = t_2t_1$ of the neutrons in the moderator are twice scattered in the lump. Continuing in this manner, always assuming a flat distribution, we find:

$$t_1(1 + t_2 + t_2^2 + \ldots) = \frac{t_1}{1 - t_2} \qquad (28)$$

as the ratio of the number of neutrons scattering in the lump to those scattering outside. The mean square distance travelled by a neutron starting in the lump may then be calculated by arguments similar to those used in section A.

Consider the neutrons originating in the volume element $\psi(R,\underline{\Omega})\,dR_s$, in the lump (see Figure 2), travelling in $d\underline{\Omega}$ about $\underline{\Omega}$. They are $\frac{\psi(R,\Omega)dR_s}{V}\frac{d\Omega}{4\Pi}$ of the total neutrons scattering in the lump, V being the volume of the lump, again assuming uniform collision density and isotropic scattering. Those which do not leave the lump on a single flight contribute

$$\int \frac{d\underline{\Omega}}{4\Pi} \int_0^\infty dR \int_0^R dR_s \int_0^{Rs} dx\; x^2\, \Sigma e^{-\Sigma x} \frac{\psi(R,\Omega)}{V} \qquad (29)$$

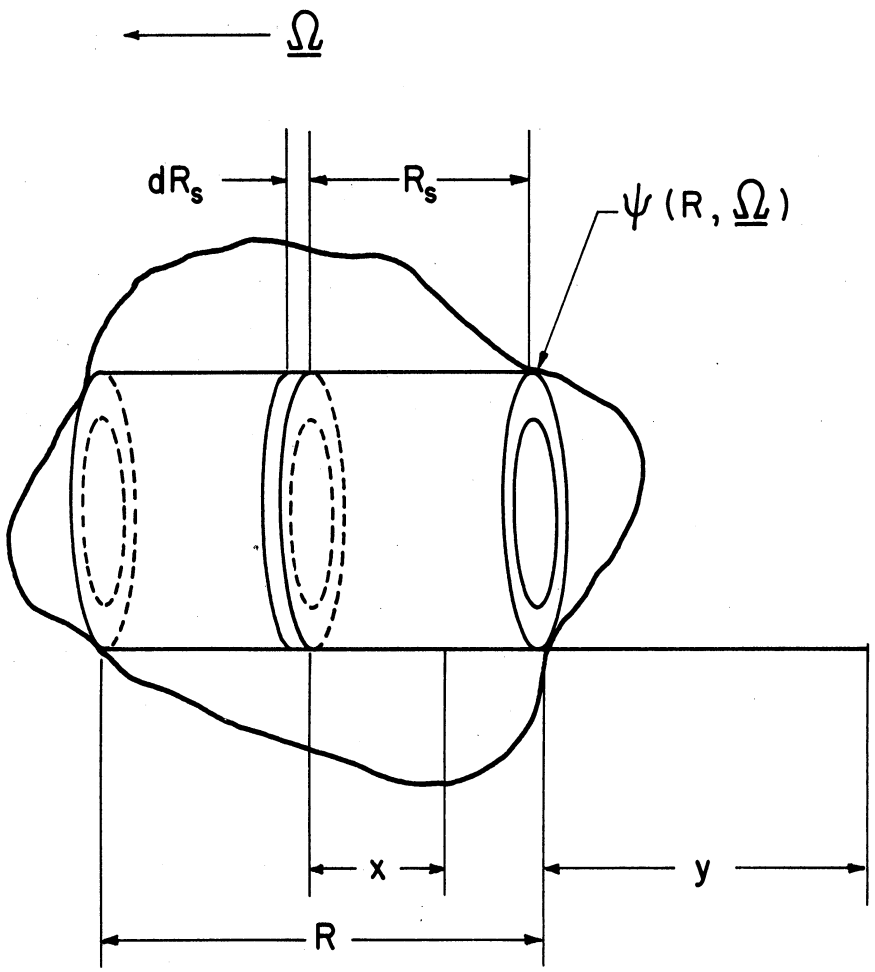


Figure 2. Geometry for Calculation of ℓ_1 and P_{A_1}

to the mean square length travelled. Those that do leave contribute:

$$\int \frac{d\Omega}{4\Pi} \int_0^\infty dR \int_0^R dR_s \int_0^Q dy \; (R_s + y)^2 \sigma e^{-\sigma y} \frac{\psi(R,\Omega)}{v} e^{-\Sigma R_s} \qquad (30)$$

Adding these up we have:

$$\ell_1^2 = \frac{2}{\Sigma^2} + \frac{S(1-g)}{4V\Sigma} \left\{ \frac{4}{\Sigma^2} - \frac{1}{\sigma\Sigma} + \frac{2}{\sigma^2} + e^{-\sigma Q} \left[-Q^2 + \frac{2Q}{\sigma} - \frac{2}{\sigma^2} + \frac{Q}{\Sigma} - \frac{1}{\sigma\Sigma} + \frac{2}{\Sigma^2} \right] \right\}$$
$$+ \frac{S}{4V\Sigma} \frac{dg}{d\Sigma} \left\{ \frac{6}{\Sigma} + \frac{1}{\sigma} + e^{-\sigma Q} \left[Q - \frac{1}{\sigma} - \frac{2}{\Sigma} \right] \right\} \qquad (31)$$
$$+ \frac{S}{4V\Sigma} \left[e^{-\sigma Q} - 2 \right] \frac{dg^2}{d\Sigma^2}$$

As the square distance travelled in one flight by neutrons originating in the lump. Lastly, P_A for neutrons originating in the lump is needed.

Since the collision density in the lump is assumed uniform, the probability of the next collision being in the lump is just $(1-P_o)$ and the probability of absorption for neutrons scattered from the lump is:

$$P_{A_1} = \frac{\Sigma_a}{\Sigma}(1-P_o) + \frac{\sigma_a}{\sigma} P_o \qquad (32)$$

To obtain an L^2 for all neutrons, it seems more reasonable to average ℓ^2 and P_A individually rather than averaging L^2. This results from an argument that a neutron generally spends part of its life in the moderator and part of its life in the full and is subject to mean square flights of ℓ_o^2 and ℓ_1^2 and absorption probabilities P_{A_o} and P_{A_1}, and

not L^2, on each flight. The fraction of neutrons scattering in the fuel is:

$$\frac{t_1/1-t_2}{1+t_1/1-t_2} = \frac{t_1}{1+t_1-t_2} \tag{33}$$

while the fraction of neutrons scattering in the moderator is:

$$1 - \frac{t_1}{1+t_1-t_2} = \frac{1-t_2}{1+t_1-t_2} \tag{34}$$

Thus, the proper values of ℓ^2 and P_A should be:

$$\ell^2 = \ell_o^2 \frac{1-t_2}{1+t_1-t_2} + \ell_1^2 \frac{t_1}{1+t_1-t_2} \tag{35}$$

$$P_A = P_{A_o} \frac{1-t_2}{1+t_1-t_2} + P_{A_1} \frac{t_1}{1+t_1-t_2} \tag{36}$$

where the value of ℓ_o^2 is to be obtained from (12) and (27), P_{A_o} is given by (28) and (15), ℓ_1^2 may be taken from (31) and P_{A_1} from (32).

D. Anisotropy Factors:

Earlier in the paper it was stated that neutrons generally travel further in some directions than others when the diffusing medium is anisotropic. A tensor diffusion coefficient is then required and Equation (4) is used to describe the leakage from such a system. The method proposed here may be used to obtain the diagonal elements of this tensor, which, if the coordinate system is aligned with the principal axes, are the only necessary elements. The L_i^2 of Equation (4) are

the projections of L^2 onto the three principal axes of the diffusion tensor.

Then, all one need do is to calculate the projections of L^2. This is readily accomplished by multiplying the integrands used to obtain ℓ^2 by $(\underline{\Omega}\cdot\underline{i})^2$, where $\underline{i}$ is a unit vector in the direction in which the diffusion length is required. Thus as an example, (9) would read:

$$\int \frac{d\Omega}{4\pi} \int_0^\infty dR \int_0^\infty dZ \int_0^\infty dx \; (\underline{\Omega}\cdot\underline{i})^2 (Z+R+x)^2 q(Z,R,\underline{\Omega}) \sigma e^{-\sigma x} \qquad (37)$$

as one contribution to ℓ_o^2 projected on the $\underline{i}$ direction. Of course, P_A will also vary with direction, but one may follow a suggestion of Beeler[11] and determine a projected P_A by multiplying the integrands of the integrals for P_A by $\underline{\Omega}\cdot\underline{i}$.

The integrals obtained in this manner are related to g but have not been tabulated. In a subsequent paper these integrals will be developed and a comparison of the results with those of Beeler will be given.

E. Flux Depression

In all of the previous discussion, the effect of flux shape has been ignored. This is particularly important in the case of widely separated black lumps; the neutrons tend to distribute themselves away from black lumps, resulting in a decrease in the probability of entering the lumps, and hence decreasing P_A. A smaller effect on ℓ^2 is also introduced.

In general, the previous calculations become very difficult for a neutron density which is not uniform. As a first attempt, one might use a diffusion theory flux and obtain values of ℓ^2 and P_A. Even this appears to be a rather formidable task. It is possible to carry out the integrals in slab geometry. This has been done, assuming alternate layers of absorber and moderator, using a flux calcualted from an albedo boundary condition:

$$\frac{J+}{J-} = \beta \tag{38}$$

at the moderator-fuel interfaces. The method follows the method previously used except that the probability that a neutron is found in the volume element $\psi(R,\Omega)dRdz$ is:

$$\frac{\varphi(\underline{r})\psi(R,\Omega)dRdz}{\int \varphi(\underline{r})d\underline{r}} \tag{39}$$

where the integral extends over the moderator associated with one lump.

It is difficult to apply these calculations to more complicated geometry. It does not appear that a reasonable attempt could be made until more data are available.

III RESULTS

The results obtained from the method of section II-A are plotted as L^2 vs M/A, and L^2 vs rod radius for UO_2 and uranium metal containing natural and enriched uranium in both H_2O and D_2O moderator (see Figures 3-14). Extension of the L^2 vs rod radius plots to zero rod radius should give the L^2 values for a homogeneous mixture. The comparison of the extrapolated homogeneous diffusion length (L^2_{ex}) against the calculated homogeneous diffusion length (L^2_{cal}) provides a check on the validity of the results. For example, for the case of 1.0 percent enriched uranium in pure D_2O, L^2_{ex} is, on the average, about 3.5 percent lower than L^2_{cal}, but for light water the difference is somewhat larger.

At low M/A, as discussed in section II-C the L^2 obtained by the methods of section II-A are incorrect, and are, in fact, below the homogeneous values. Physically, one expects self-shielding effects to increase L^2 in a heterogeneous media. Corrections could be made using the methods of section II-C and would be in the right direction. These calculations have not been made as yet but will be reported on in a subsequent paper.

For purposes of comparison, L^2 was calculated using Equation (2). To obtain f, the thermal utilization, we used the following method.

The disadvantage factor G, the ratio of flux at the surface of the element in the mean flux in the element, was calculated using the method by P. Lehamann, et al.[12] and Amouyal, Benoist, and

Horowitz [13]:

$$G = 1 + \frac{\Sigma a}{\Sigma t} A \left[1 + \alpha \frac{\Sigma s}{\Sigma t} + \beta \left(\frac{\Sigma s}{\Sigma t} \right)^2 \right] \qquad (40)$$

The coefficients A, α, and β are functions of the parameter $a\Sigma_t$ (where a is the rod radius) In turn, the thermal utilization is given by [13]:

$$\frac{1}{f} = 1 + \frac{\sigma a}{\Sigma a} \left(\frac{M}{A} \right) G \quad \ldots \qquad (41)$$

Use of these formulas in Equation (2) yields the values of L^2 shown in Figure 15. As M/A increases, L^2 drops below the value for a homogeneous mixture.

Sher and Kouts [14] used Equation (2) with measured values of f and obtained the values shown in Figure 15. These results appear more reasonable but the above mentioned discrepancy is observed if the graphs are extrapolated. The difference between our results and those of Kouts and Sher may be due to a different choice of nuclear constants.

IV CONCLUSIONS

The results given by the method proposed here appears to be more satisfactory than those obtained by the older method, Equation (2) (see Figure 15). The major effect of the heterogeneity of the system on L^2 is due to a decrease in P_A which is, in turn, due to self-shielding and flux depression. Thus, one expects that for a heterogeneous medium, the diffusion length will be greater than for a homogeneous medium with the same (M/A). As can be seen in Figure 15, Equation (2) predicts L^2 which are sometimes lower than the homogeneous L^2. For this reason it is believed that the present calculation represents a better approximation of L^2.

No calculations have been made using the corrections derived in sections II-B through II-E. As experimental data are not as yet available, it was impossible to determine which of these corrections is necessary and we have therefore decided to delay further calculations until such time as it will be possible to compare results with experiment.

From Figure 15, it is seen that the results of section II-A are too small for low M/A. This is as anticipated but detailed calculations have not been carried out for this case. Sample calculations indicate that the corrections are in the right direction and yield reasonable results.

V ACKNOWLEDGEMENTS

We wish to thank Professor R. K. Osborn for several illuminating discussions and the National Science Foundation and the Babcock and Wilcox Company, Atomic Energy Division, for their support.

NUCLEAR CONSTANTS USED

Enrichment	N^{UO_2}	N^{25}	N^{28}
Natural (10g/cc)	2.23×10^{22}		
1.0 %		4.971×10^{20}	4.732×10^{22}
1.15%		5.540×10^{20}	4.731×10^{22}
1.3 %		6.288×10^{20}	4.719×10^{22}

Element	Microscopic Cross Sections (barns)		
	Total	Absorption	Scattering
U^{235}	475	465	10
U^{238}	10.74	2.44	8.3
U(Natural)	14.05	6.76	8.3
O		~0	4.2
D_2O	13.55	~0	13.55
D	4.91	0	4.91
H_2O	90.5		
	22.08	6.66	15.42
H	31	.292	30.71

			10g/cc Natural	Enriched Uranium		
	H_2O	D_2O	UO_2	1.0%	1.15%	1.3%
L_o^2	8.31	104	5.40	1.300	1.165	1.020

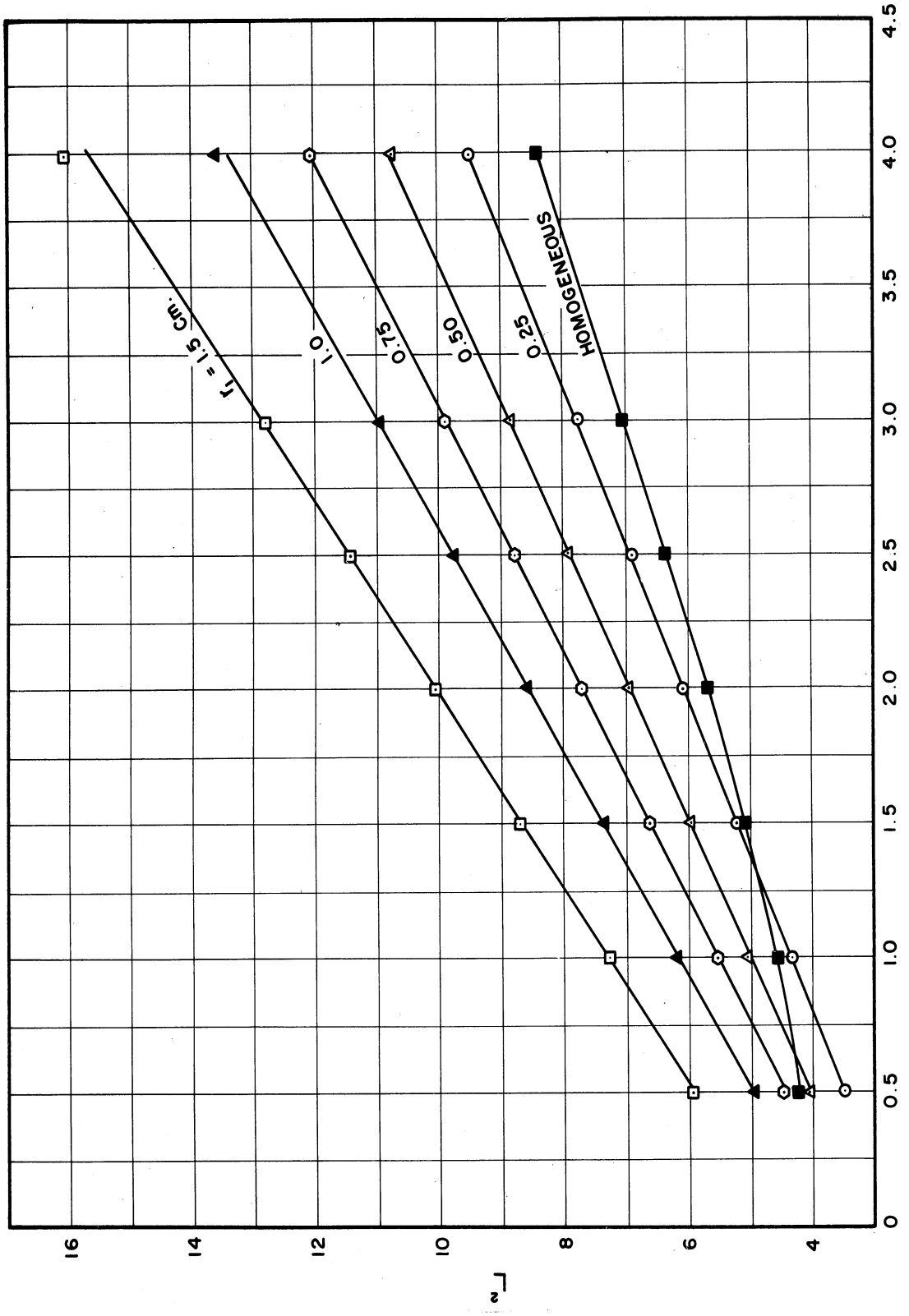
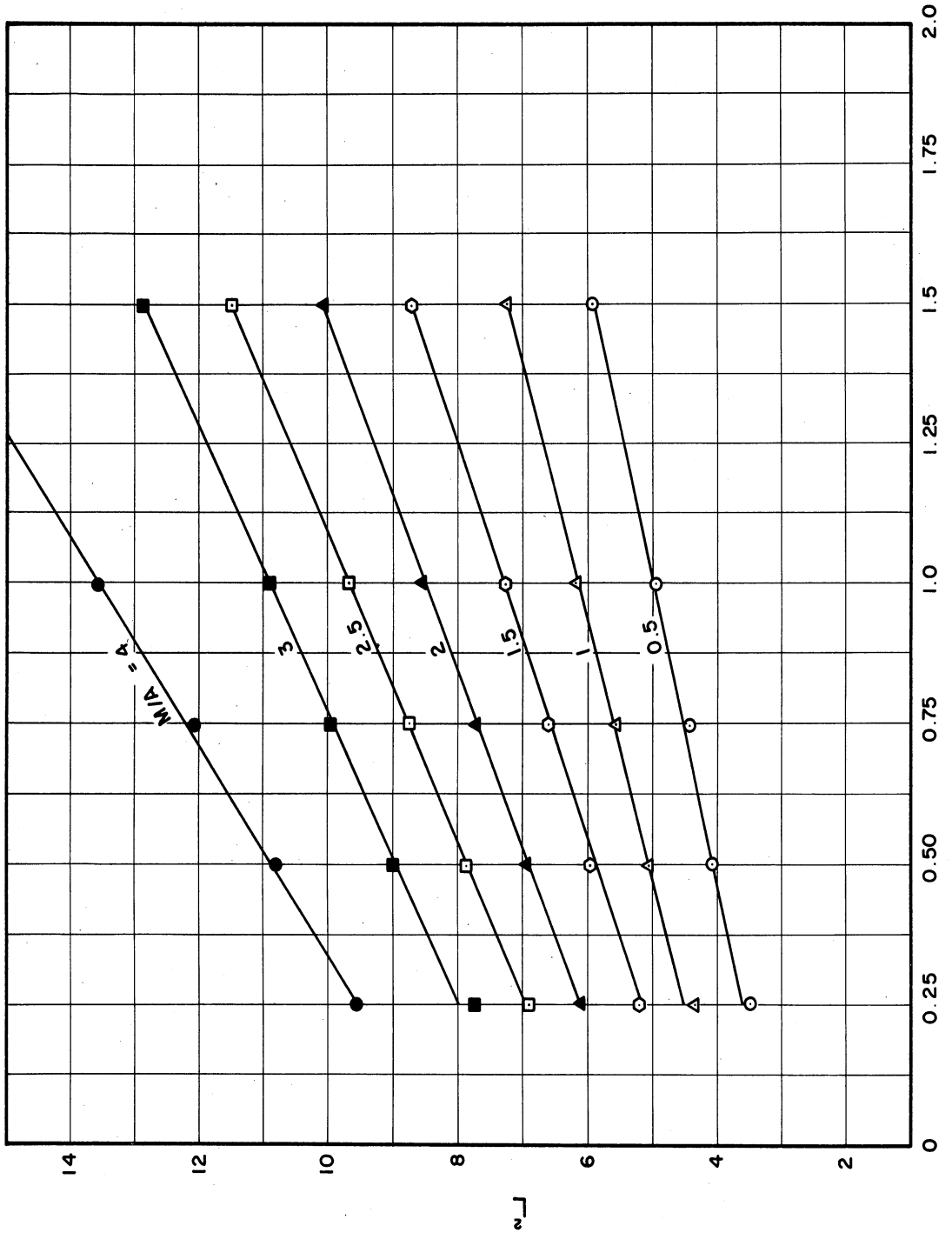
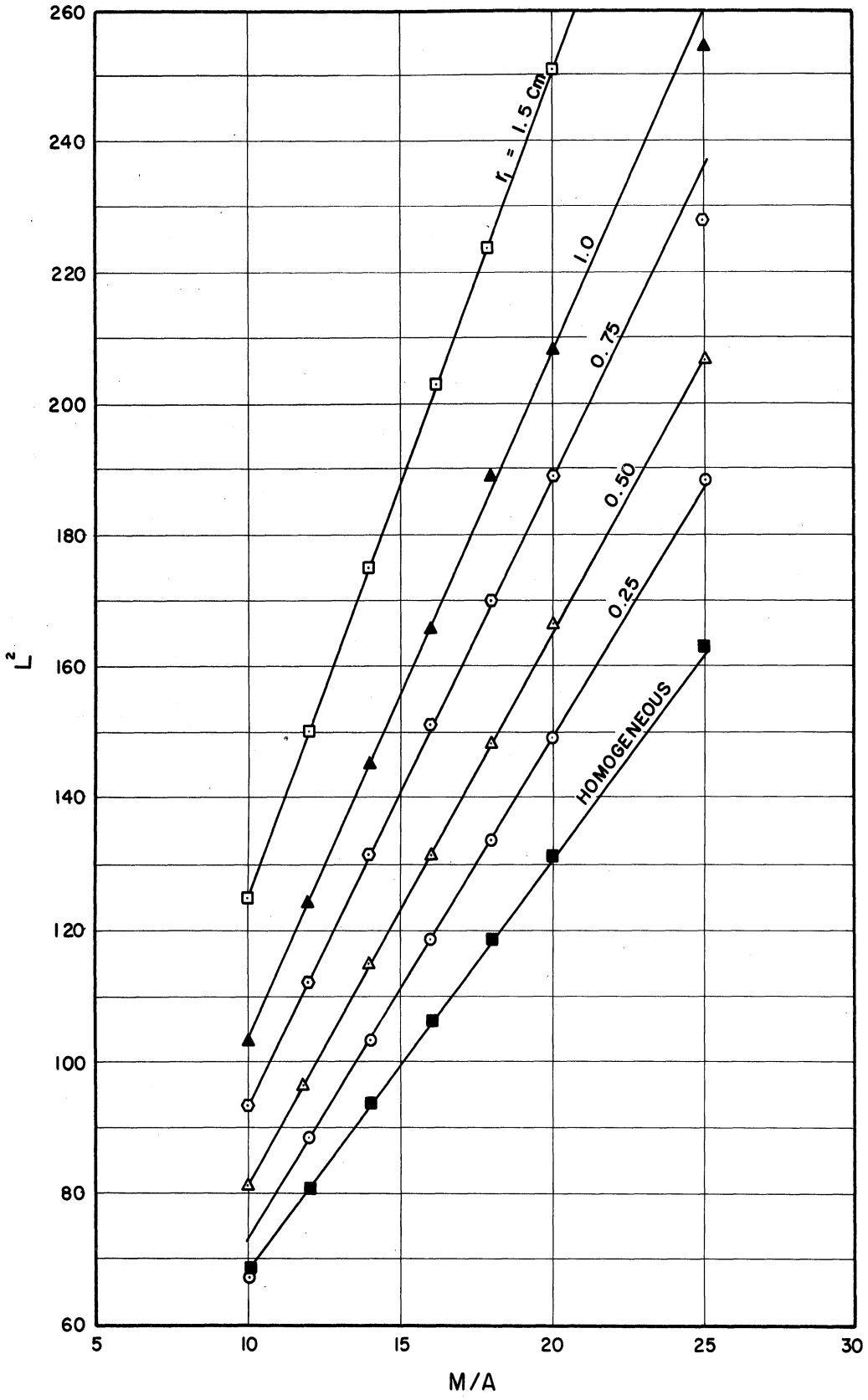


Figure 3. Natural UO_2(10g/cc) in Pure H_2O
L^2 vs M/A
r_1 = Rod Radius



L^2 vs Rod Radius

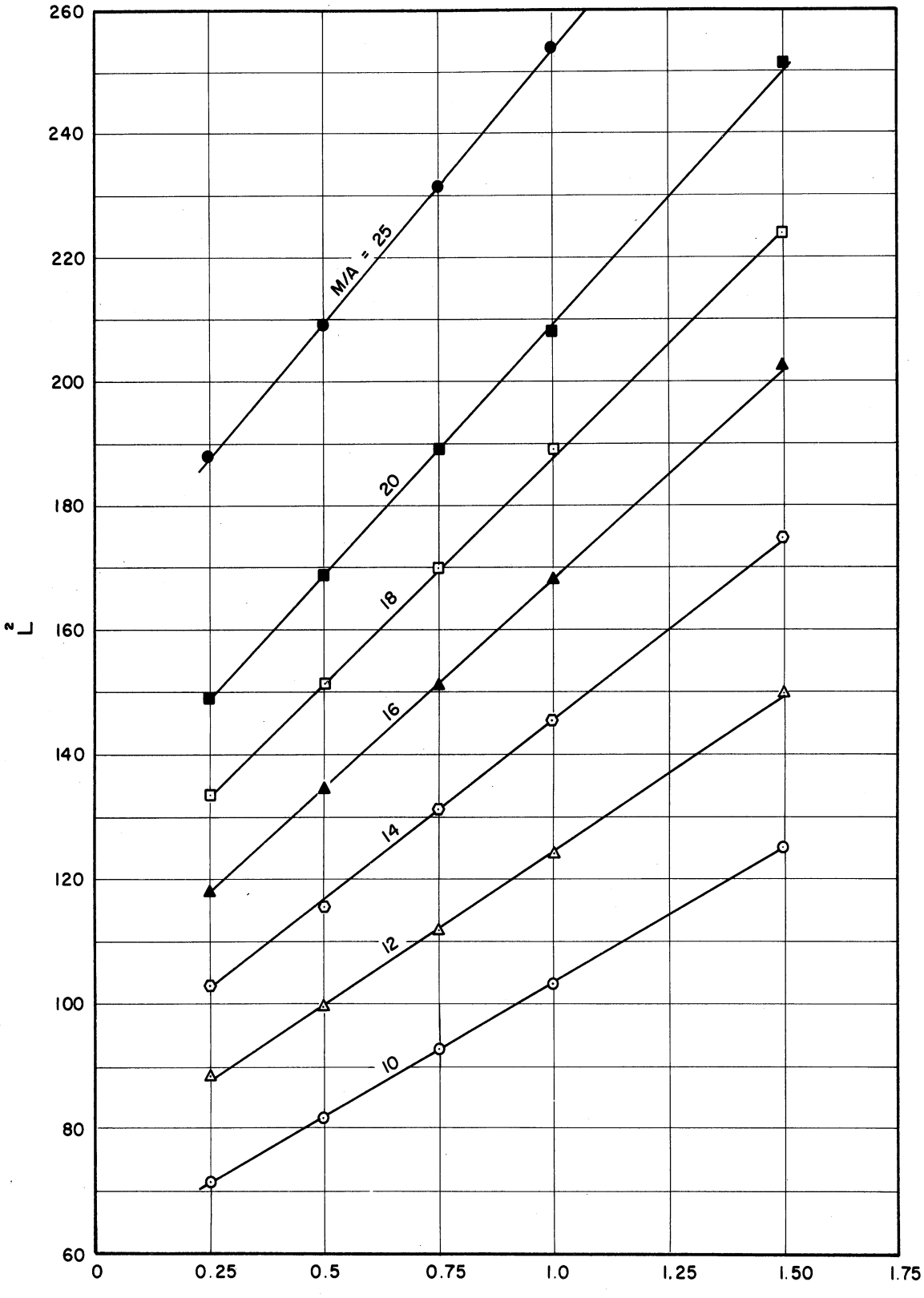
Figure 4. Natural UO_2 (10g/cc) in Pure H_2O



L^2 vs M/A

Figure 5. Natural UO_2 (10g/cc) in Pure D_2O

r_1 = Rod Radius



L^2 vs Rod Radius

Figure 6. Natural UO_2 (10g/cc) in Pure D_2O

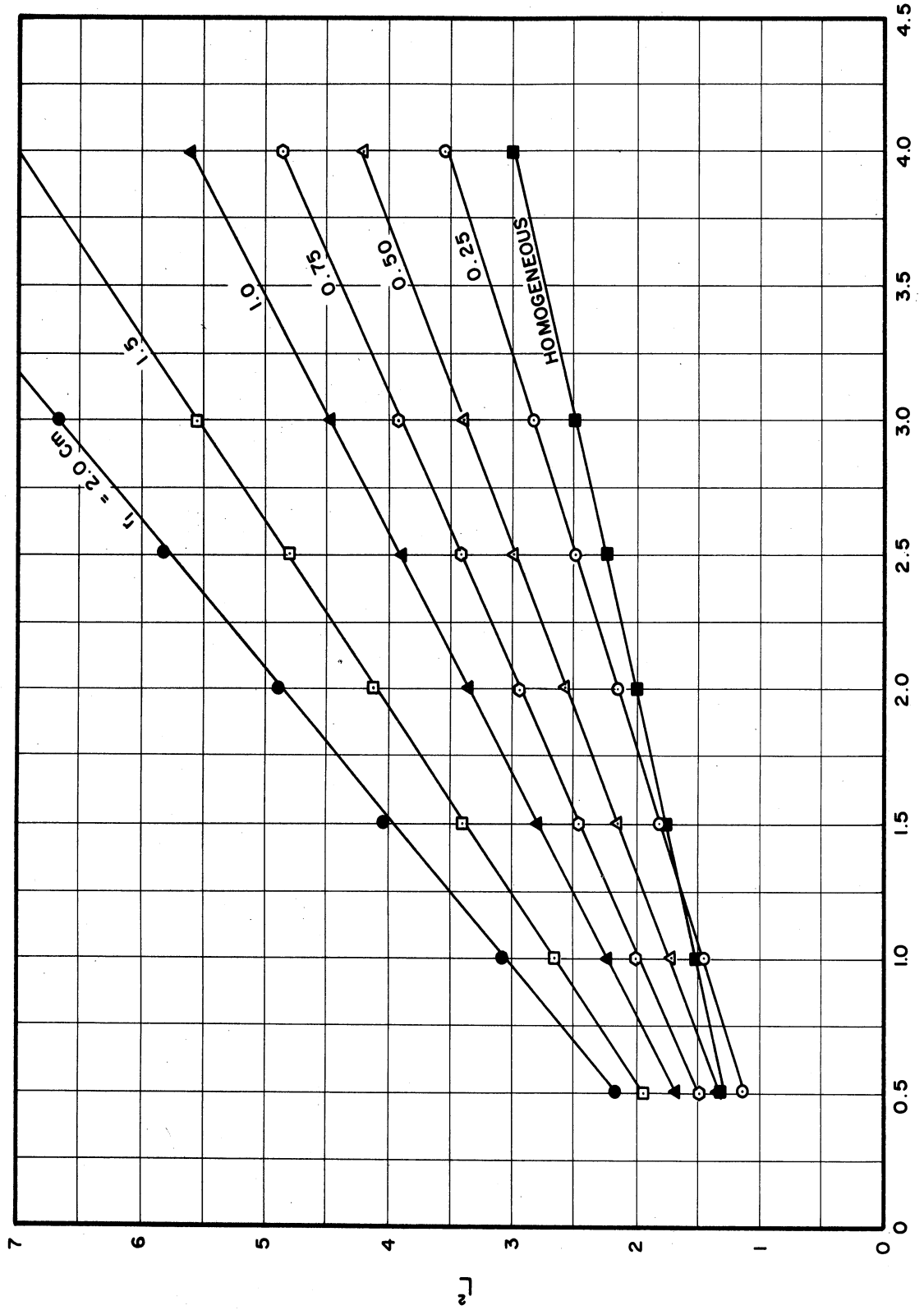


Figure 7. 1.00% Enriched Uranium in Pure H_2O

L^2 vs M/A

r_1 = Rod Radius

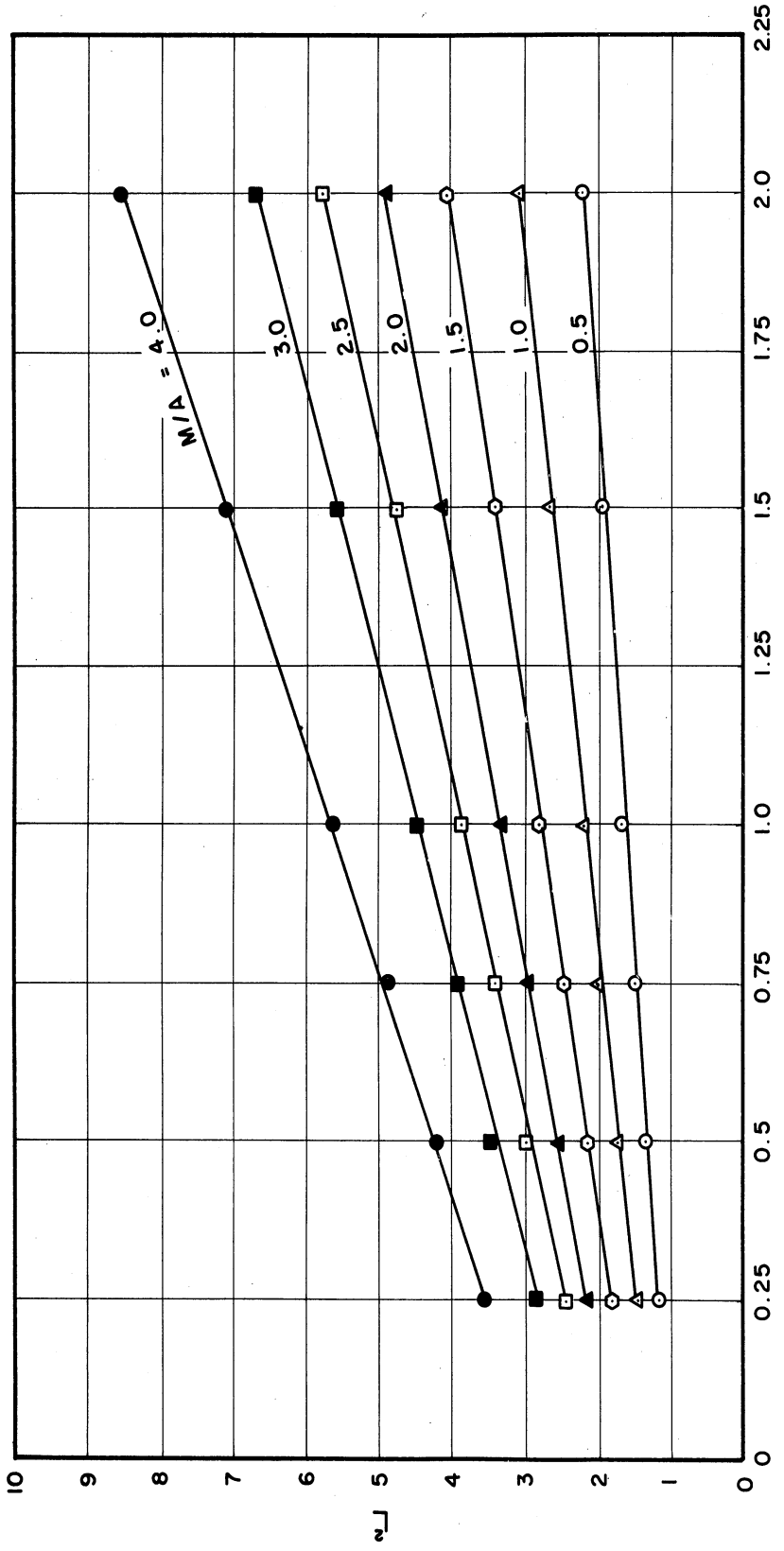


Figure 8. 1.00% Enriched Uranium in Pure H_2O
L^{ϵ} vs Rod Radius
M/A = Water to Metal Ratio

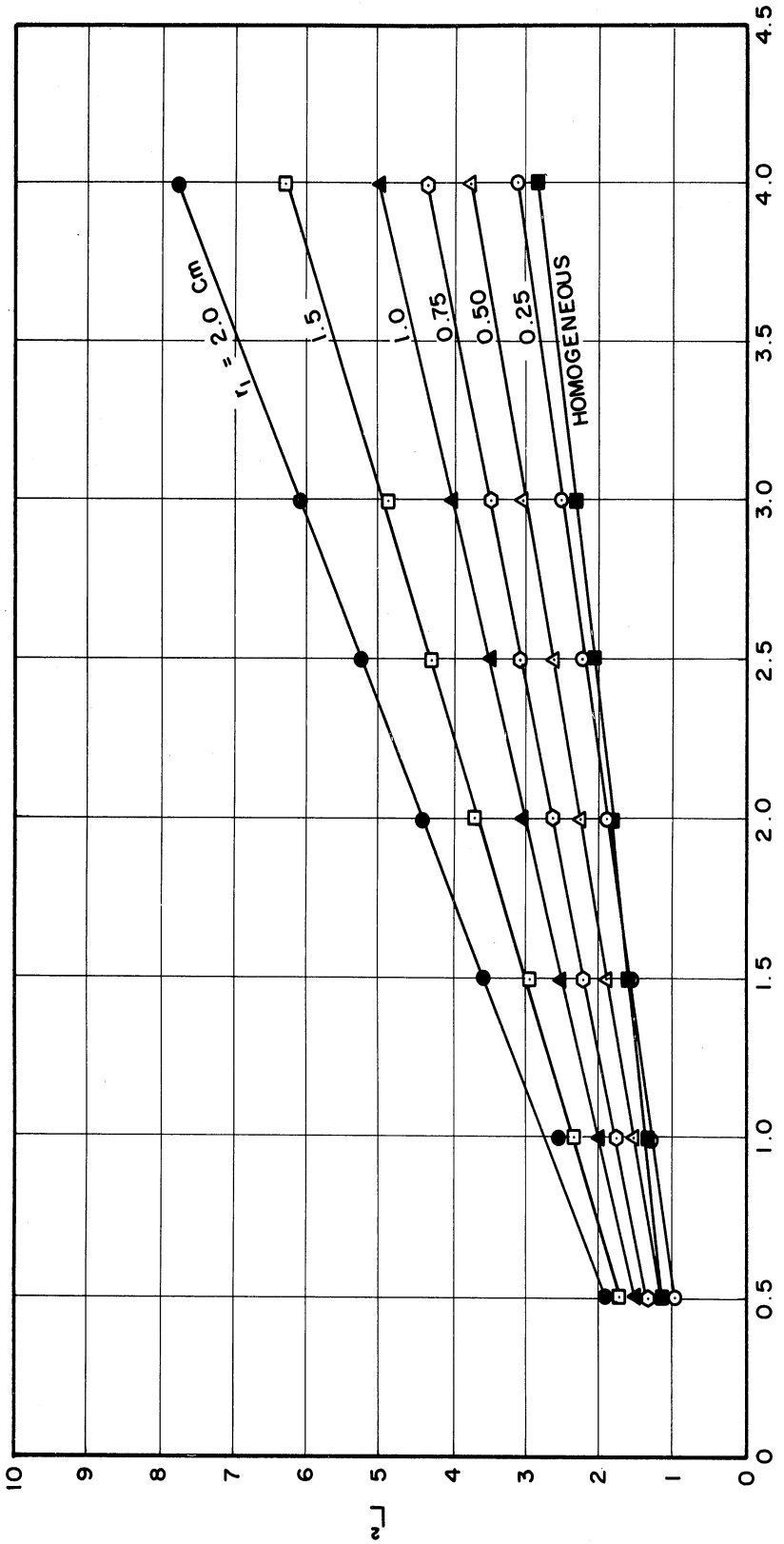


Figure 9. 1.15% Enriched Uranium in Pure H_2O
L^2 vs M/A
r_1 = Rod Radius

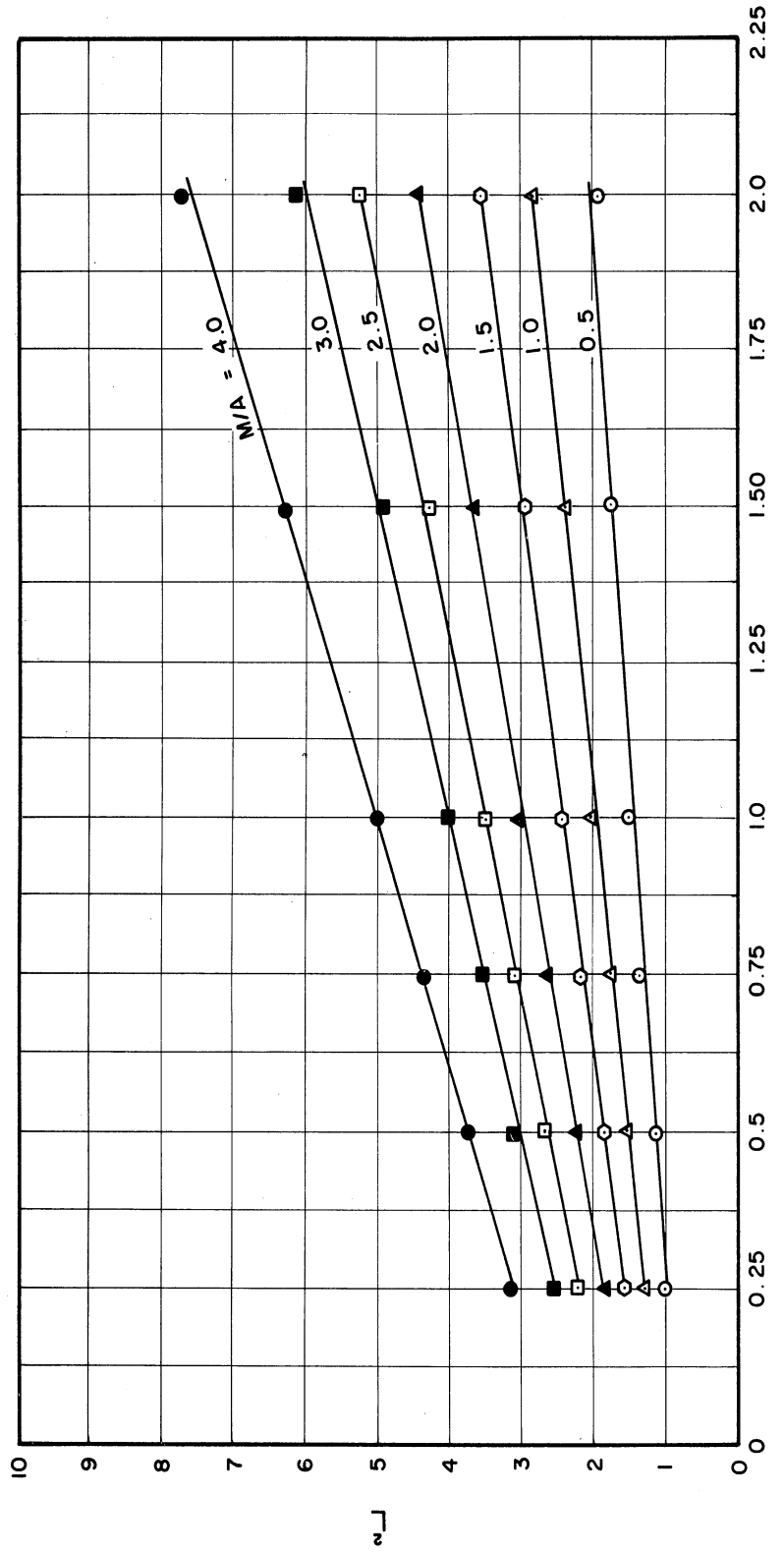


Figure 10. 1.15% Enriched Uranium in Pure H_2O

L^2 vs Rod Radius

M/A = Water to Metal Ratio

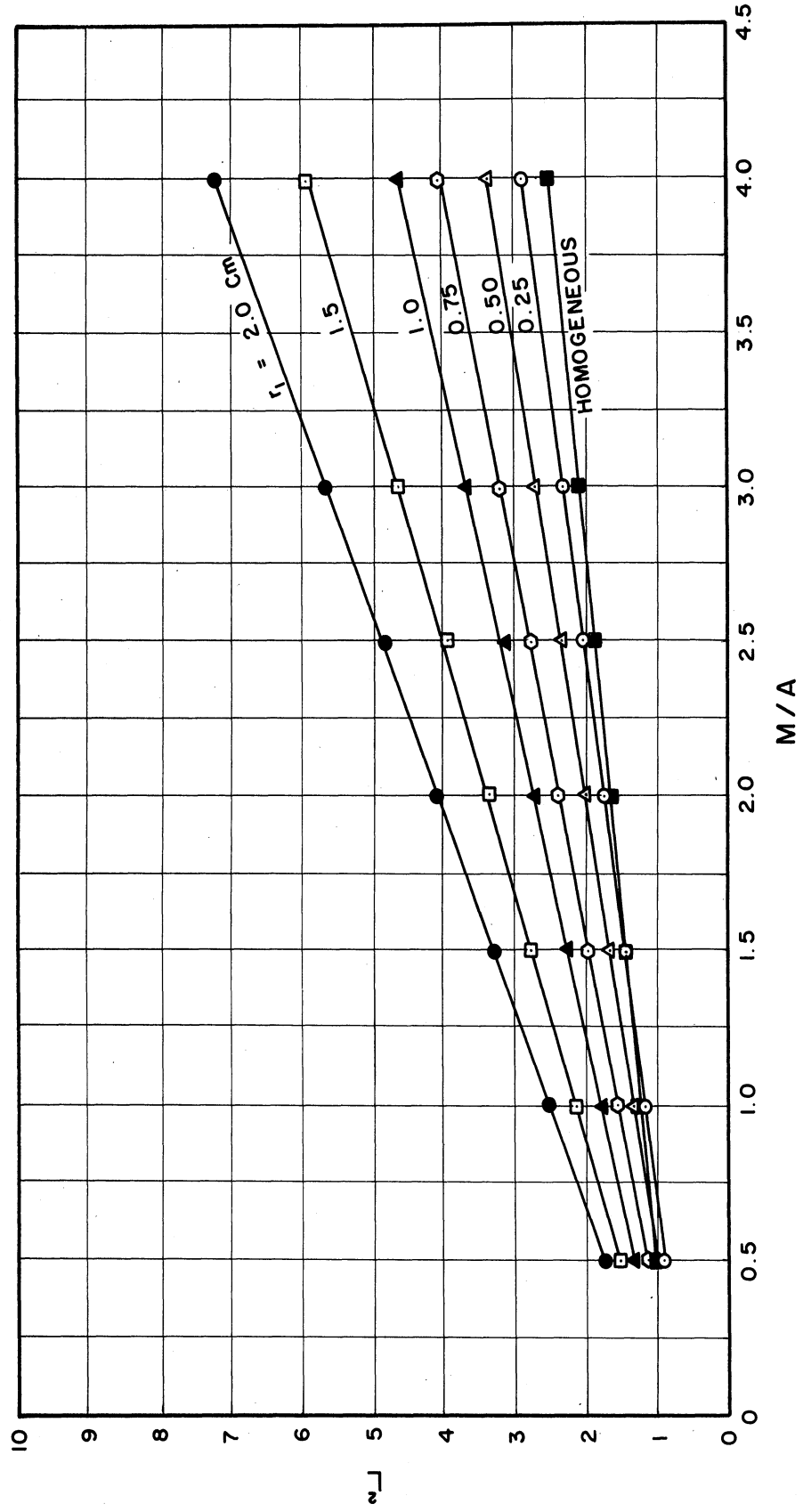
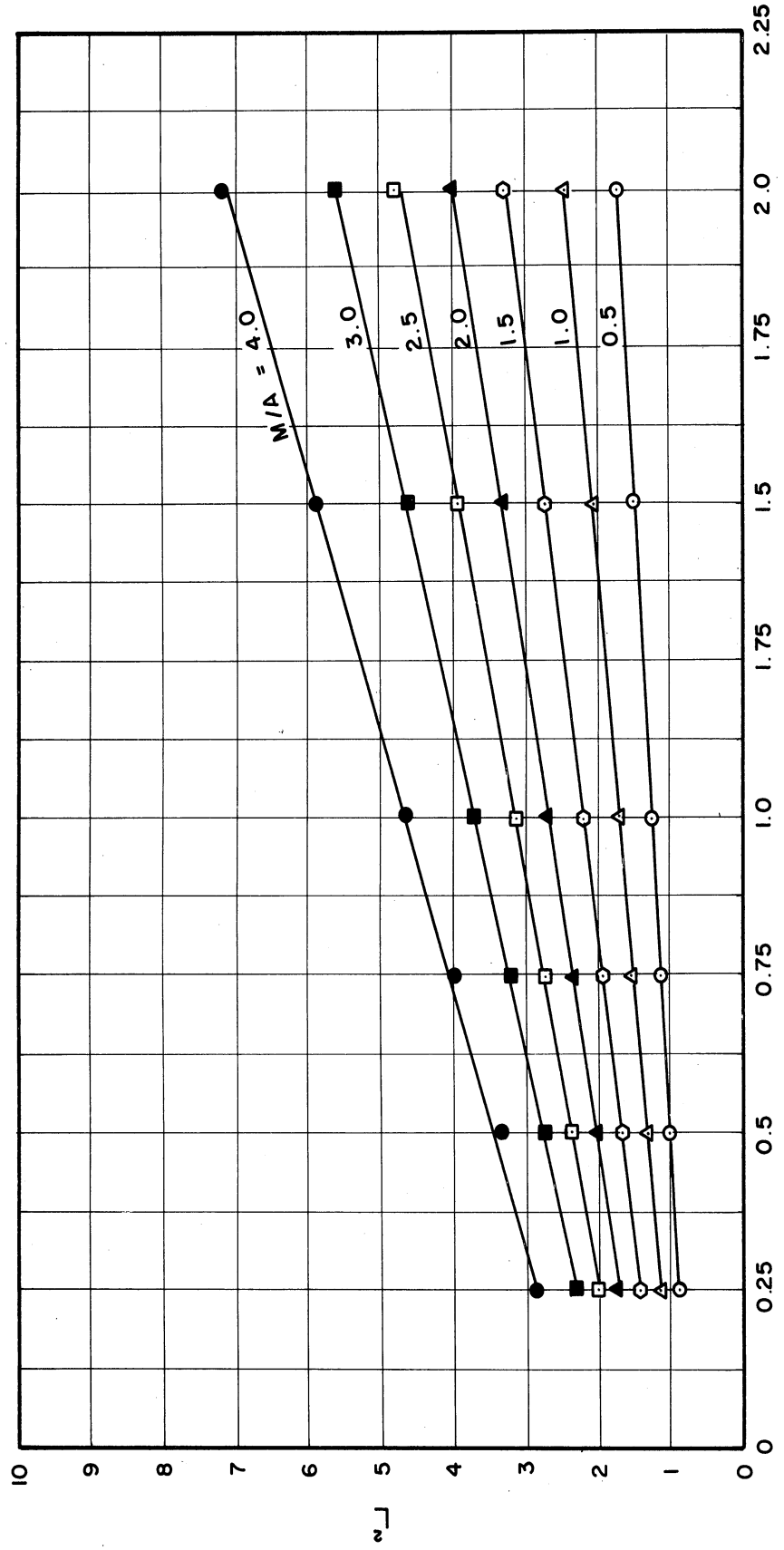


Figure 11. 1.30% Enriched Uranium in Pure H_2O

L^2 vs M/A

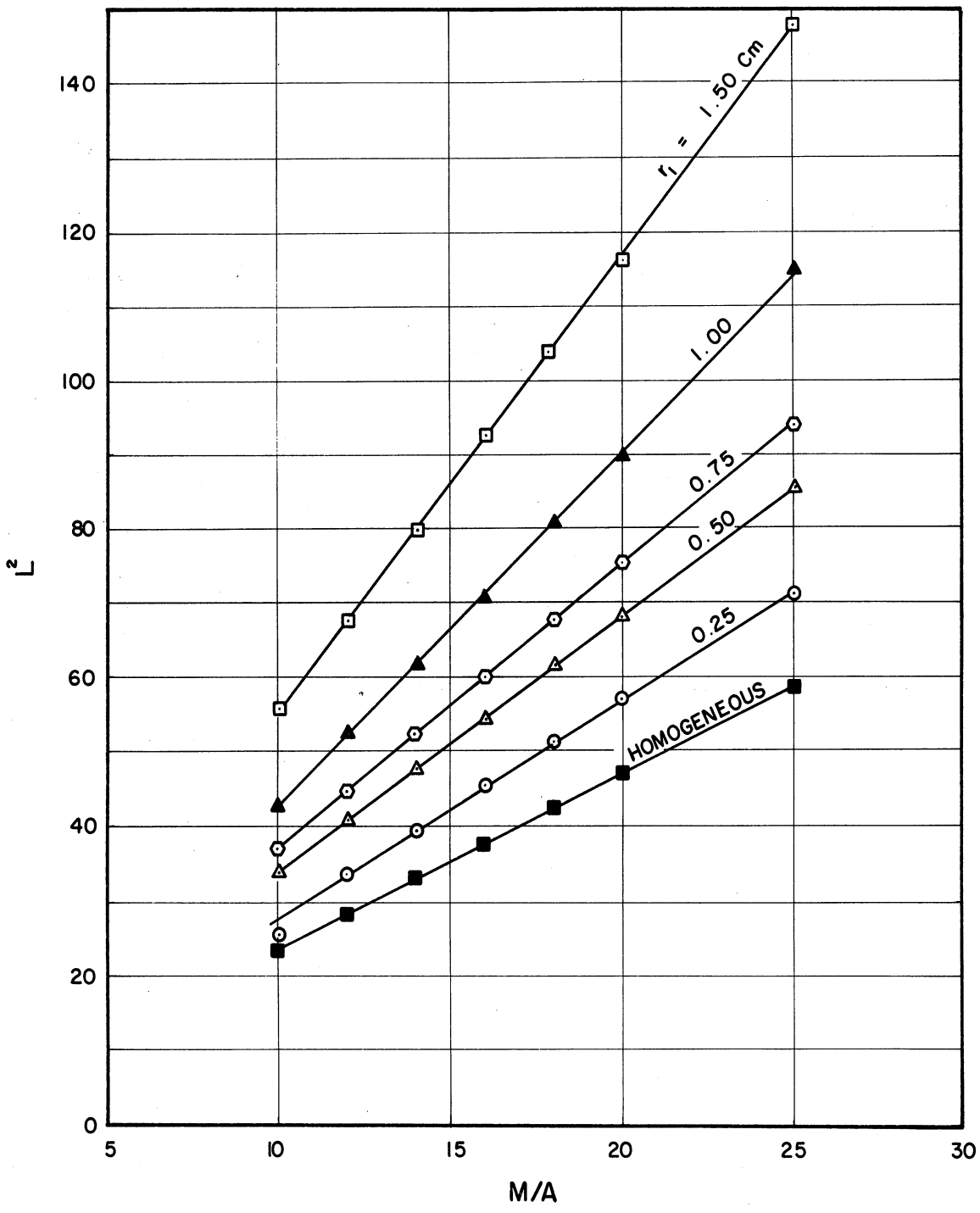
r_1 = Rod Radius



L^2 vs Rod Radius

Figure 12. 1.3% Enriched Uranium in Pure H_2O

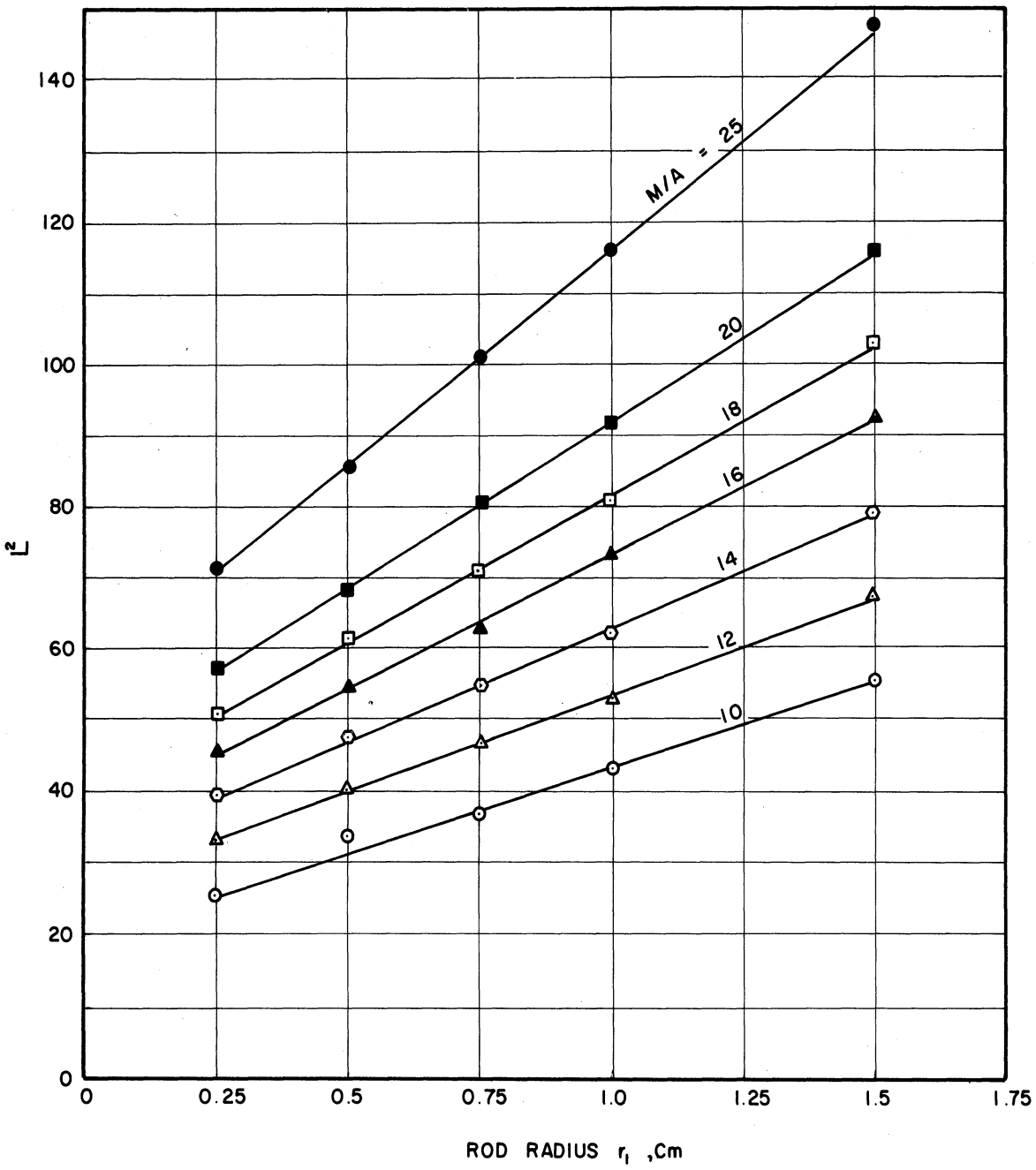
W/M = Water to Metel Ratio



L^2 vs M/A

Figure 13. 1.00% Enriched Uranium in Pure D_2O

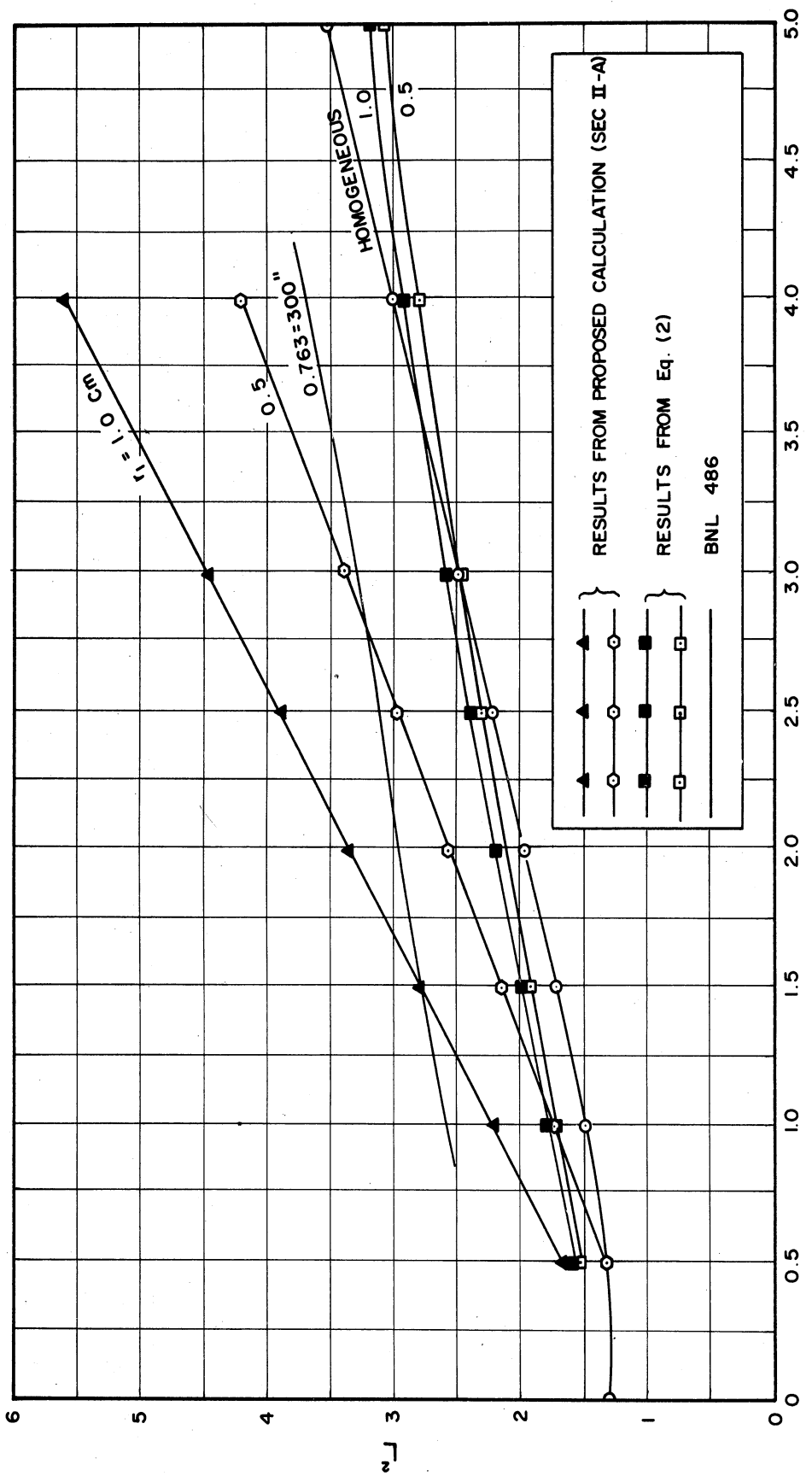
r_1 = Rod Radius



L^2 vs Rod Radius

Figure 14. 1.00% Enriched Uranium in Pure D_2O

M/A = Water to Metal Ratio



L^2 vs M/A
Comparison of Results r_1 = Rod Radius

Figure 15. 1.00% Enriched Uranium in Pure H_2O

BIBLIOGRAPHY

1. Weinberg, A. M. and E. P. Wigner, "The Physical Theory of Neutron Chain Reactors," University of Chicago Press (1958).

2. Russell, J. L., Trans. Amer. Nuc. Soc, 2, (Nov. 1959), Paper 17-8.

3. Schmitt, A. P., et al. Proc. Int'l. Conf., Geneva, (1958) P. 1191.

4. Spinrad, B. I., J. Appl. Phys., 26, 548, (1955).

5. Shevelev, Y. V., J. Nuc. Energy, 6, 132, (1957).

6. Triflaj, Y., J. Nuc. Energy, 6, 126, (1957).

7. Behrens, D. J., Proc. Phys. Soc., 52, 607, (1949).

8. Seren, L., Trans. Amer. Nuc. Soc., 1, 2, Paper 3-6, (Nov. 1959), also Trans. Amer. Nuc. Soc., (June 1960).

9. Case, K. M., F. DeHoffmann, G. Placzek, "Information to the Theory of Neutron Diffusion," U. S. Government Printing Office, (1953)

10. Honeck, H., "Monte Carlo Calculation of Fermi Age," ORNL Report CF 55-8-193, (1955).

11. Beeler, J. R. "Anisotropy of Neutron Diffusion in Slab Systems with Simple Discontinuities", Report XDC-60-6-176, (1960).

12. Lehmann, P., R. W. Meier and J. P. Schneeberger, "The Physical Properties of Some Composite Fuel Elements in U-D_2O Type Reactors," Proc. Int. Conf., Geneva, P. 245 (1958).

13. Amouyal, Benoist, and Horowitz, J. Nuc. Energy, 6, 79 (1957).

14. Kouts, H. and R. Sher, "Experimental Studies of Slightly Enriched Uranium, Water Moderated Lattices", BNL 486.

THE UNIVERSITY OF MICHIGAN

DATE DUE

2/14 15:43 4/2 23:06	