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Spectral and dynamic properties of a 2D Luttinger liquid with current–current interactions

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Abstract. We calculate the Green function, particle–hole susceptibility and Cooper pair susceptibility of a 2D Luttinger liquid that results from an electron gas with long-range current–current interactions. The Haldane bosonization scheme allows this model to be written in terms of boson operators, from which the required fermion operators are constructed. We find weakly decaying power law tails in the spectral function and a Kohn anomaly in the $2k_F$ particle–hole susceptibility, but no divergence in the Cooper amplitude.

The physics of the normal state of high-T_c systems is considered in many quarters to be of a new kind of fermion state completely different from that of the normal Fermi liquid. One of the phenomenological approaches to this issue stresses the similarity of the normal state of high-T_c materials to that of the 1D Luttinger liquid [1, 2]. This state is characterized by, among other features, (i) the removal of the Fermi surface discontinuity in the momentum distribution and (ii) a power law decay in the single particle spectral function. This latter feature is regarded as a definitive property of normal high-T_c systems [3]. Several aspects of the Luttinger liquid phenomenology have been shown to hold rigorously for the 1D t–J model [4] and for the 2D t–J model; numerical evidence to this extent has been presented [5]. Until recently, however, it has not been understood what the necessary or sufficient conditions are for Luttinger liquid phenomenology to be valid above one dimension.

Most recently, this phenomenology has been shown to be valid for a 2D electron system with long-range current–current interactions, by Khveshchenko *et al* [6], using the bosonization scheme devised by Haldane [7]. This scheme replaces the 2D Fermi surfaces by patches (of size Λ) characterized by the vector normal to the Fermi surface. The normal, non-singular electron interaction, in principle, couples all such patches, but after renormalization and taking the limit of vanishing patch size, this type of interaction is found to be an irrelevant quantity. The special feature of the interaction considered by Khveshchenko *et al* [6] is the fact that the Ampere force responsible for the current–current interaction is (i) long ranged, (ii) wavevector dependent, and (iii) persists in the limit of vanishing patch size. In this limit, it allows a bosonization of the electronic degrees of freedom, which, taken together with a non-vanishing interaction strength, leads to the result that one of the above mentioned Luttinger results holds, namely the vanishing of the momentum discontinuity at the Fermi surface and its replacement by a weak power law dependence.

A natural question to ask is then to what extent the other properties of this 2D model *with current–current interactions follow the expected properties of a Luttinger liquid.* In this paper, we follow up on the bosonization procedure of Khveshchenko *et al* [6] by calculating various fermion correlation functions in 2D. We shall start by calculating the single particle propagator, and then consider two related two-particle Green functions that reflect the instability of the Luttinger liquid against two types of ordering. We shall consider the $2k_F$ particle–hole susceptibility and the Cooper pair susceptibility. For all cases, we shall need to develop the bosonic representation of fermion operators—we shall therefore follow quite closely the procedures adopted by Luther and Peschel [8] for the 1D problem, although a number of calculational details are different.

Our starting point is the model of Khveshchenko *et al* [6]written in terms of bosonized density operators $R_\alpha(q) = \sum_{p\in\Lambda_\alpha} c^+_\alpha(p+q)c_\alpha(p)$, where $c(p)$ denotes the (spinless) fermion operators, Λ_α denotes the region in momentum space assigned to the patch α and the wavevector sum is restricted to states such that $|p - p_F\,\widehat{n}_\alpha| < \Lambda$. The resulting boson density operator, when commuted with the kinetic energy $H_{kin} = \sum_p \varepsilon(p)c^+_p c_p$, satisfies $[H_{kin}, R_\alpha(q)] = v_\alpha \cdot q R_\alpha(q)$, where $v_\alpha = v_F\,\widehat{n}_\alpha$ is the Fermi velocity along the patch α, as long as $\varepsilon(p)$ is constant in the direction perpendicular to the ray $\widehat{n}_\alpha$. This allows one to write the kinetic energy as

$$H_{kin} = \frac{(2\pi)^2|v_F|}{\Omega\Lambda}\sum_q\sum_\alpha R_\alpha(q)R_\alpha(-q) \tag{1}$$

where Ω is the volume of the system, and the patch size Λ is taken independent of α. The central aspect of the model is the nature of the interaction vertex $\Gamma(p, p', q, \omega)$ (for scattering of incoming states p, p' through wavevector q), which expressed in bare form becomes

$$\Gamma(p, p', q, \omega) = \mathrm{Re}\left[\left(\frac{g}{m^2}\right)\frac{(p\cdot q)(p'\cdot q)/q^2 - p\cdot p'}{(q^2 + \mathrm{i}\gamma/q|\omega|)}\right] \tag{2}$$

for which the singular q dependent nature of the interaction is apparent. Here g denotes a coupling strength and m the electron mass, while γ represents Landau damping. The effective interaction Hamiltonian corresponding to the expression (2) can be written [6]

$$H_{int} = \frac{1}{2\Omega}\sum_q\sum_{\alpha,\beta}\left(\frac{-g}{m^2}\right)\frac{|p_F|^2 q^2\sin\alpha\sin\beta}{q^4 + \gamma^2|v_F|^2(\cos\alpha + \cos\beta)^2}R_\alpha(q)R_\beta(-q) \tag{3}$$

where α and β denote angles made by patches Λ_α, Λ_β with q. In the limit $\Lambda \to 0$, this takes a simple form

$$H_{int} = \frac{1}{4}\frac{(2\pi)^2|v_F|}{\Omega\Lambda}\sum_q\sum_\alpha \delta|\sin\alpha|(R_\alpha(q) - R_{-\alpha}(q))(R_{\alpha^*}(-q) - R_{-\alpha^*}(q)) \tag{4}$$

where $-\alpha$ denotes the mirror patch of α in q, and α^* denotes the opposite patch of the Fermi surface to α. The parameters in (2) and (3) are combined as a dimensionless coupling $\delta = g|p_F|/2\gamma$. The interaction term depends only on the antisymmetric combinations of density operators with respect to q, $A_\alpha(q) = (1/\sqrt{2})(R_\alpha(q) - R_{-\alpha}(q))$, while the kinetic part also depends on the symmetric part $S_\alpha(q) = (1/\sqrt{2})(R_\alpha(q) + R_{-\alpha}(q))$. In terms of these operators, the Hamiltonian takes the form

$$\begin{aligned} H = \frac{(2\pi)^2|v_F|}{4\Lambda\Omega}\sum_\alpha\sum_{\widehat{n}_\alpha\cdot q>0} & \big[\{S_\alpha(q)S_\alpha(-q) + S_{\alpha^*}(-q)S_{\alpha^*}(q)\} \\ & + \{A_\alpha(q)A_\alpha(-q) + A_{\alpha^*}(-q)A_{\alpha^*}(q) \\ & + 2\delta|\sin\alpha|A_\alpha(q)A_{\alpha^*}(-q) + A_\alpha(-q)A_{\alpha^*}(q)\}\big] = H_A + H_S \end{aligned} \tag{5}$$

where H_S and H_A refer to the symmetric and antisymmetric parts of the Hamiltonian respectively. The latter is simply diagonalized via the canonical transformation $e^{iS} H_A e^{-iS}$, where

$$S = \frac{i(2\pi)^2}{4\Lambda\Omega} \sum_\beta \sum_q \frac{\phi_\beta(q)}{\widehat{n}_\beta \cdot q} A_\beta(q) A_{\beta^*}(-q) \tag{6}$$

and $\tanh 2\phi_\beta(q) = -2\delta|\sin\beta|$. The diagonalized form of the Hamiltonian involves two types of Bose quasiparticle with energies $E_\alpha^A(q) = (v_\alpha \cdot q)[1 - (2\delta)^2(\sin\alpha)^2]^{1/2}$ and $E_\alpha^S(q) = v_\alpha \cdot q$. The crucial step in formulating correlation functions is to find a bosonic representation of the fermion operators. Such a representation (i) has to satisfy the same commutation relation with the density operator $R_\alpha(q)$ as the original fermion operators, (ii) has to obey the same equation of motion as the fermion state function $\psi_\alpha(x)$, and (iii) has to transform the same way as $\psi_\alpha(x)$ under the canonical transformation parametrized by (6). Such requirements are satisfied by a field operator

$$\psi_\alpha(x) = f(x) \exp\left[\frac{(2\pi)^2}{\Lambda\Omega} \sum_{n_\alpha \cdot q > 0} \frac{\{R_\alpha(-q)e^{iq\cdot x} - R_\alpha(q)e^{-iq\cdot x}\}}{\widehat{n}_\alpha \cdot q}\right] \tag{7}$$

where $f(x)$ is chosen so that the free fermion propagator takes on its correct value. Separating (7) into symmetric and antisymmetric parts yields

$$\psi_\alpha(x) = f(x)\Phi_\alpha^S(x)\Phi_\alpha^A(x) \tag{8}$$

where

$$\Phi_\alpha^{A,S}(x) = \exp\left[\frac{(2\pi)^2}{\sqrt{2}\Lambda\Omega} \sum_{n_\alpha \cdot q > 0} \frac{1}{\widehat{n}_\alpha \cdot q} \left\{ \begin{pmatrix} A_\alpha(-q) \\ S_\alpha(-q) \end{pmatrix} e^{iq\cdot x} - \begin{pmatrix} A_\alpha(q) \\ S_\alpha(q) \end{pmatrix} e^{-iq\cdot x} \right\}\right] \tag{9}$$

The transformation of $\Phi_\alpha^A(x)$is then $e^{iS}\Phi_\alpha^A(x)e^{-iS} = W_\alpha^A(x)R_\alpha^A(x)$, where

$$W_\alpha^A(x) = \exp\left[\frac{(2\pi)^2}{\sqrt{2}\Lambda\Omega} \sum_{n_\alpha \cdot q > 0} \frac{\cosh\phi_\alpha(q)}{\widehat{n}_\alpha \cdot q} \{A_\alpha(-q)e^{iq\cdot x} - A_\alpha(q)e^{-iq\cdot x}\}\right] \tag{10}$$

$$R_\alpha^A(x) = \exp\left[\frac{(2\pi)^2}{\sqrt{2}\Lambda\Omega} \sum_{n_\alpha \cdot q > 0} \frac{\sinh\phi_\alpha(q)}{\widehat{n}_\alpha \cdot q} \{A_{\alpha^*}(-q)e^{iq\cdot x} - A_{\alpha^*}(q)e^{-iq\cdot x}\}\right] \tag{11}$$

We note that $W_\alpha^A(x)$ is different from that given in [6], the difference being due to the explicit extra fermion operator retained in [6]. The Green function of interest is then given by

$$G_\alpha^>(x,t) = \langle G|\psi_\alpha(x,t)\psi_\alpha^+(0,0)|G\rangle = \langle 0|e^{iH_D t}e^{iS}\psi_\alpha(x)e^{-iS}e^{-iH_D t}e^{iS}\psi_\alpha^+e^{-iS}|0\rangle \tag{12}$$

where $H_D = H_D^A + H^S$, and H_D^A is the diagonalized antisymmetric component of the Hamiltonian. Here $|G\rangle$ denotes the ground state of H and $|0\rangle$ denotes the ground state of H_D, related to $|G\rangle$ through $|0\rangle = e^{iS}|G\rangle$. After some rearrangements, and after deployment of the relation $e^{iS}f(A)e^{-iS} = f(e^{iS}Ae^{-iS})$, we obtain

$$\begin{aligned}\langle G|\psi_\alpha(x,t)\psi_\alpha^+(0,0)|G\rangle &= f(x)f^*(0)\langle 0|e^{iH_S t}\Phi_\alpha^S(x)e^{-iH_S t}\widetilde{W}_\alpha^A(x,t)\widetilde{R}_\alpha^A(x,t) \\ &\quad \times \Phi_\alpha^S(0)R_\alpha^A(0,0)^+W_\alpha^A(0,0)^+|0\rangle\end{aligned} \tag{13}$$

where

$$\widetilde{W}_\alpha^{\rm A}(x,t)=\exp\left[\frac{(2\pi)^2}{\sqrt{2}\Lambda\Omega}\sum_{n_\alpha\cdot q}\frac{\cosh\phi_\alpha(q)}{\widehat{n}_\alpha\cdot q}\{A_\alpha(-q)\mathrm{e}^{\mathrm{i}q\cdot x-\mathrm{i}E_\alpha^{\rm A}(q)t}-A_\alpha(q)\mathrm{e}^{-\mathrm{i}q\cdot x+\mathrm{i}E_\alpha^{\rm A}(q)t}\}\right] \quad (14)$$

$$\widetilde{R}_\alpha^{\rm A}(x,t)=\exp\left[\frac{(2\pi)^2}{\sqrt{2}\Lambda\Omega}\sum_{n_\alpha\cdot q}\frac{\sinh\phi_\alpha(q)}{\widehat{n}_\alpha\cdot q}\{A_{\alpha^*}(-q)\mathrm{e}^{\mathrm{i}q\cdot x+\mathrm{i}E_\alpha^{\rm A}(q)t}-A_{\alpha^*}(q)\mathrm{e}^{\mathrm{i}q\cdot x-\mathrm{i}E_\alpha^{\rm A}(q)t}\}\right] \quad (15)$$

The above combination of exponentials is then rearranged, using repeatedly the relation $\mathrm{e}^{A}\mathrm{e}^{B}=\mathrm{e}^{A+B+(1/2)[A,B]}$, so that the exponential of destruction operators lies to the right. The result for the Green function $G_\alpha^>(x,t)$ is then

$$\begin{aligned}\langle G|\psi_\alpha(x,t)\psi_\alpha^+(0,0)|G\rangle &= f(x)f^*(0)\langle 0|\mathrm{e}^{\mathrm{i}H_{\rm S}^{\rm D}t}\Phi_\alpha^{\rm S}(x)\mathrm{e}^{-\mathrm{i}H_{\rm S}^{\rm D}t}\Phi_\alpha^{\rm S}(0)|0\rangle\\ &\quad\times\exp\left[\frac{2\pi^2}{\Lambda\Omega}\sum_{n_\alpha\cdot q>0}\frac{\cosh^2\phi_\alpha(q)}{\widehat{n}_\alpha\cdot q}\{\mathrm{e}^{\mathrm{i}q\cdot x-\mathrm{i}E_\alpha^{\rm A}(q)t}-1\}\right.\\ &\quad\left.+\frac{\sinh^2\phi_\alpha(q)}{\widehat{n}_\alpha\cdot q}\{\mathrm{e}^{-\mathrm{i}q\cdot x-\mathrm{i}E_\alpha^{\rm A}(q)t}-1\}\right] \end{aligned}\quad (16)$$

which can be written in terms of the free Green function $G_0^>(x,t)$, as $G_\alpha^>(x,t)=G_0^>(x,t)\exp X(x,t)$, where $G_0^>(x,t)=-\frac{i}{\Omega}\sum_k[1-n_k]\mathrm{e}^{-\mathrm{i}\varepsilon_k t+\mathrm{i}k\cdot x}$ and

$$X(x,t)=\frac{2\pi^2}{\Lambda\Omega}\sum_{n_\alpha\cdot q}\frac{\sinh^2\phi_\alpha(q)}{\widehat{n}_\alpha\cdot q}\{\mathrm{e}^{\mathrm{i}q\cdot x-\mathrm{i}E_\alpha^{\rm A}(q)t}+\mathrm{e}^{-\mathrm{i}q\cdot x-\mathrm{i}E_\alpha^{\rm A}(q)t}-2\} \quad (17)$$

From (16), we can check that setting $\phi=0$ does in fact yield the free Green function, and hence complete the derivation of $f(x)$. Following the argument of Khveshchenko *et al* [6] and making use of the facts that at zero temperature the $q_\perp$ integral in the exponent in (16) is restricted to a region $\Lambda\ll|p_{\rm F}|$ within the patch α, and, that ε_{k_α} is independent of $q_\perp$, the $q_\perp$ integral yields a factor $(2/x_\perp)\sin(\Lambda x_\perp/2)$, which is significant only for $x_\perp<1/\Lambda$. This allows the $x_\perp$ dependence of the exponents to be suppressed. We find in this way that both the symmetric component of the Green function, namely $\langle 0|\mathrm{e}^{\mathrm{i}H_{\rm D}^{\rm S}t}\Phi_\alpha^{\rm S}(x)\mathrm{e}^{-\mathrm{i}H_{\rm D}^{\rm S}t}\Phi_\alpha^{\rm S}(0)|0\rangle$, and the exponential term in (16), yield a factor $[\Lambda^2/4(x_\parallel-v_{\rm F\parallel}t)]^{-1/2}$. This is to be expected since in the non-interacting limit we have simply split the fermion operator into identical symmetric and anti-symmetric parts, and each of these bosonic contributions therefore yields a square root of the expected free electron denominator (see (19)). Hence, we determine $f(x)=\mathrm{e}^{\mathrm{i}k_{\rm F}\cdot x}(\Lambda e^{C}/2)^{1/2}$, where C is Eulers's constant. Our particular interest is in the Fourier transform of $G_\alpha^>(x,t)$

$$G_\alpha^>(p,\omega)=\int\mathrm{d}^2x\,\mathrm{e}^{-\mathrm{i}p\cdot x}\int\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\left(\frac{-\mathrm{i}}{\Omega}\right)\sum_k\mathrm{e}^{\mathrm{i}k\cdot x}[1-n_k]\mathrm{e}^{-\mathrm{i}\varepsilon_k t}\mathrm{e}^{X(x,t)} \quad (18)$$

which becomes,

$$G_\alpha^>(p,\omega)=\int_{-\infty}^{+\infty}\frac{\mathrm{d}x_\parallel}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t\frac{\exp[-\mathrm{i}(p_\parallel-k_{\rm F\parallel})x_\parallel+\mathrm{i}\omega t]}{(x_\parallel-v_{\rm F\parallel}t)}\exp[\widetilde{X}(x_\parallel,t)] \quad (19)$$

from which we can extract the correct form of the free Green function, by setting $\widetilde{X}\to 0$. To evaluate $\widetilde{X}(x_\parallel,t)$, we assume a small coupling parameter δ so that $\sinh^2\phi_\alpha(q)\simeq\delta^2\sin^2\alpha=\delta^2q_\perp^2/(q_\parallel^2+q_\perp^2)$ after which the $q_\perp$ integral can be evaluated, with the result that

$$\widetilde{X}(x_\parallel,t)=\frac{\delta^2}{2\Lambda}\int_0^{\Lambda/2}\frac{\mathrm{d}q_\parallel}{q_\parallel}[\mathrm{e}^{\mathrm{i}q_\parallel(x_\parallel-v_{\rm F\parallel}t)}+\mathrm{e}^{-\mathrm{i}q_\parallel(x_\parallel+v_{\rm F\parallel}t)}-2]\left[\Lambda-2q_\parallel\tan^{-1}\left(\frac{\Lambda}{2q_\parallel}\right)\right] \quad (20)$$

To be consistent with our small-δ assumption, we must work to leading order in $x_\parallel, t$ and Λ. It is straightforward to show that $\widetilde{X}(x_\parallel, 0, t) = -(\delta^2/2)\ln[(\Lambda^2/4)(x_\parallel^2 - v_{F\parallel}{}^2 t^2)]$, apart from constant terms of order δ. Substituting in (19), we obtain, after some manipulation

$$G_\alpha^>(p,\omega) = \frac{1}{4\pi v_{F\parallel}^{2+\delta^2}}\left(\frac{2}{\Lambda^2}\right)^{\delta^2/2} F(1+\delta^2/2, k_{F\parallel} - p_\parallel - \omega/v_{F\parallel}) \\ \times F(\delta^2/2, (k_{F\parallel} - p_\parallel + \omega/2v_{F\parallel})/2) \tag{21}$$

where

$$F(r,z) = \int_{-\infty}^{\infty} dy(e^{izy}/z^r).$$

We expand above prefactors to leading order in δ, which yields $F(\delta^2/2, z) \simeq 2\sin(\delta^2/4) \times(2/(\omega - \tilde{p}))^{1-\delta^2/2}$ and $F(1+\delta^2/2, z) \simeq -i\pi|\widetilde{p} + \widetilde{\omega}|^{\delta^2/2}$, where $\widetilde{p} \equiv p_\parallel - k_{F\parallel}$, $\omega/v_F \equiv \widetilde{\omega}$. The result is that

$$\mathrm{Im}\, G_\alpha^>(p,\omega) \simeq (-\pi\delta^2/4v_{F\parallel})[|\widetilde{\omega}^2 - \widetilde{p}^2|^{\delta^2/2}/(\widetilde{\omega} - \widetilde{p})]\Lambda^{-\delta^2}. \tag{22}$$

Thus, the spectral function shows a power law dependence on the forward and backward moving combinations of frequency and wavevector. For electrons at the Fermi surface, we obtain a sublinear power law decay in the spectral function

$$\mathrm{Im}\, G_\alpha^>(p_{F\parallel},\omega) = -(\pi\delta^2/4v_{F\parallel})\widetilde{\omega}^{\delta^2-1}\Lambda^{-\delta^2} \tag{23}$$

which has the same behaviour for both signs of the interaction parameter.

Turning to the particle–hole susceptibility, we note the susceptibility

$$\chi_\alpha(\boldsymbol{x},t) = -i\Theta(t)\langle[\rho_\alpha(\boldsymbol{x},t), \rho_\alpha^+(0,0)]\rangle \tag{24}$$

where $\rho_\alpha(\boldsymbol{x}) = \psi_\alpha^+(\boldsymbol{x})\psi_{\alpha^*}(\boldsymbol{x})$ is the particle–hole operator, the operator that scatters an electron across the Fermi surface. The Hamiltonian is not diagonal in these particle–hole operators so the same boson representation of the Fermi operators as in (8) has to be used. Following through as before requires that we calculate the quantity

$$\langle G|\rho_\alpha(\boldsymbol{x},t)\rho_\alpha^+(0,0)|G\rangle = \langle 0|e^{iH_{D}t}e^{iS}\rho_\alpha(\boldsymbol{x})e^{-iS}e^{-iH_{D}t}e^{iS}\rho_\alpha^+(0)e^{-iS}|0\rangle \tag{25}$$

which, on substituting (8) and carrying through the transformations under $e^{iS}$and e^{iH_Dt}, requires that we evaluate the combinations $\widetilde{W}_\alpha^A(\boldsymbol{x},t)^+\widetilde{R}_{\alpha^*}^A(\boldsymbol{x},t)R_{\alpha^*}(0,0)^+W_\alpha^A(0,0)$ and $\widetilde{R}_\alpha^A(\boldsymbol{x},t)^+\widetilde{W}_{\alpha^*}^A(\boldsymbol{x},t)W_{\alpha^*}^A(0,0)^+R_\alpha^A(0,0)$, which are slightly more involved combinations of exponentials of boson operators than occurred in the evaluation of the single-particle Green function. These are, as before, rearranged until the exponentials of destruction operators lie to the right. The difference, compared to the single particle Green function case, is that somewhat more involved phase functions result, with the upshot that we can write the interacting particle–hole propagator (25), in terms of the free particle–hole propagator $\langle 0_{NI}|\rho_\alpha(\boldsymbol{x},t)\rho_\alpha^+(0,0)|0_{NI}\rangle$ with $|0_{NI}\rangle$ denoting the non-interacting ground state, as

$$\langle G|\rho_\alpha(\boldsymbol{x},t)\rho_\alpha^+(0,0)|G\rangle = \langle 0_{NI}|\rho_\alpha(\boldsymbol{x},t)\rho_\alpha^+(0,0)|0_{NI}\rangle\exp[T(\boldsymbol{x},t)] \tag{26}$$

where

$$T(\boldsymbol{x},t)=\frac{2\pi^2}{\Lambda\Omega}\sum_{n_\alpha\cdot q>0}\frac{[\{\cosh\phi_\alpha(q)+\sinh\phi_{\alpha^*}(q)\}^2-1]}{\widehat{n}_\alpha\cdot q}\{\mathrm{e}^{\mathrm{i}q\cdot x-\mathrm{i}E_\alpha^\Lambda(q)t}+\mathrm{e}^{-\mathrm{i}q\cdot x-\mathrm{i}E_\alpha^\Lambda(q)t}-2\}. \tag{27}$$

Applying exactly the same procedure to the quantity $\langle 0|\rho_\alpha^+(0)\rho_\alpha(\boldsymbol{x},t)|0\rangle$, we obtain, for the Fourier transformed quantity $\chi_\alpha(q,\omega)$,

$$\chi_\alpha(q,\omega)=\int \mathrm{d}^2x\,\mathrm{e}^{-\mathrm{i}q\cdot x}\int_{-\infty}^{+\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}(-\mathrm{i}\Theta(t))\{\langle 0_{\mathrm{NI}}|\rho_\alpha(\boldsymbol{x},t)\rho_\alpha(0,0)|0_{\mathrm{NI}}\rangle\mathrm{e}^{T(\boldsymbol{x},t)}$$
$$-\langle 0_{\mathrm{NI}}|\rho_\alpha^+(0,0)\rho_\alpha(\boldsymbol{x},t)|0_{\mathrm{NI}}\rangle\mathrm{e}^{T(-\boldsymbol{x},-t)}\} \tag{28}$$

The simplification that may now be made concerns the non-interacting particle–hole propagator which simply factorizes into a product of free single-particle Green functions. In addition, the same arguments, that led to the neglect of the $x_\perp$ dependence of the single particle phase shift $X(\boldsymbol{x},t)$, apply to $T(\boldsymbol{x},t)$, with the result that, for q in the patch α,

$$\chi_\alpha(q,\omega)=\mathrm{i}\int \mathrm{d}x_\parallel\,\mathrm{e}^{-\mathrm{i}q_\parallel x_\parallel}\int_0^\infty \mathrm{d}t\frac{\mathrm{e}^{\mathrm{i}\omega t+2k_{\mathrm{F}\parallel}x_\parallel}}{(x_\parallel^2-v_{\mathrm{F}\parallel}^2t^2)}S(q_\perp)$$
$$\times[\exp(T(x_\parallel,0,t))-\exp(T(-x_\parallel,0,-t))] \tag{29}$$

where

$$S(q_\perp)=\frac{1}{(2\pi)^2}\int_{-\Lambda/2}^{\Lambda/2}\mathrm{d}k_\perp\int_{-\Lambda/2}^{\Lambda/2}\mathrm{d}q_\perp\,\delta(q_\perp+p_\perp-k_\perp)$$

is a phase space factor of order Λ, reflecting the fact mentioned earlier, that for these systems, two particle effects pick up extra factors of the patch size Λ. The evaluation of $T(x_\parallel,0,t)$ parallels that of $X(x_\parallel,0,t)$ with the result that

$$T(x_\parallel,0,t)=\delta\ln\left[\mathrm{i}\frac{\Lambda}{2}(x_\parallel-v_{\mathrm{F}\parallel}t)\right]+\delta\ln\left[-\mathrm{i}\frac{\Lambda}{2}(x_\parallel+v_{\mathrm{F}\parallel}t)\right] \tag{30}$$

apart from non-singular terms of order δ. The imaginary part of the susceptibility can be written, using the oddness of $\chi(q,\omega)$ with ω, and the evenness of $T(x_\parallel,t)$ with respect to $x_\parallel$, as (defining $Q=q_\parallel-2k_{\mathrm{F}\parallel}$)

$$\operatorname{Im}\chi_\alpha(q,\omega)=\frac{1}{2}\int \mathrm{d}x_\parallel\,\mathrm{e}^{-\mathrm{i}Qx_\parallel}S(q_\perp)\int \mathrm{d}t\frac{\mathrm{e}^{\mathrm{i}\omega t}}{(x_\parallel^2-v_{\mathrm{F}\parallel}^2t^2)}[\mathrm{e}^{T(x_\parallel,0,t)}-\mathrm{e}^{T(-x_\parallel,0,-t)}] \tag{31}$$

which, on inserting (30) and making various substitutions, yields

$$\operatorname{Im}\chi_\alpha(Q,\omega)=\frac{S(q_\perp)(\Lambda^2/4)^\delta}{2^{3-\delta}\,v_{\mathrm{F}\parallel}^{1-2\delta}}\{F(\omega-v_{\mathrm{F}\parallel}Q)F(\omega+Qv_{\mathrm{F}\parallel})-F(-\omega-v_{\mathrm{F}\parallel}Q)F(-\omega+v_{\mathrm{F}\parallel}Q)\} \tag{32}$$

where

$$F(z)=\int_{-\infty}^{\infty}\mathrm{d}s\exp(\mathrm{i}sz/2)(-\mathrm{i}s)^{\delta-1}.$$

It follows that the combination in brackets above depends crucially on the relative phases of $\omega - v_{F\parallel} Q$ and $\omega + v_{F\parallel} Q$.

If $\omega^2 - {v_{F\parallel}}^2 Q^2 < 0$, the two terms cancel, whereas for $\omega^2 > {v_{F\parallel}}^2 Q^2$ the phase factors combine to yield, to lowest order in δ,

$$\mathrm{Im}\,\chi_\alpha(Q, \omega) = \frac{\pi^2\delta^2}{2v_{F\parallel}} S(q_\perp)\left(\frac{\omega^2 - v_{F\parallel}^2 Q^2}{\Lambda^2}\right)^{-\delta} \tag{33}$$

showing, for positive δ, a weak power law divergence at the edges of the particle–hole continuum $\omega = \pm v_{F\parallel} Q$. The Kramers–Kronig relation

$$\mathrm{Re}\,\chi_\alpha(Q, 0) = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}\omega}{\omega}\mathrm{Im}\,\chi_\alpha(Q, \omega)$$

yields, for small δ,

$$\mathrm{Re}\,\chi_\alpha(Q, 0) = \frac{\pi\delta^2}{v_{F\parallel}} A_1 S(q_\perp)(Q\Lambda)^{-2\delta} \tag{34}$$

where

$$A_1 = \int_0^\infty \mathrm{d}x \frac{x^{-\delta}(x+2)^{-\delta}}{(x+1)}.$$

Clearly the real part of the static particle–hole susceptibility shows the same weak power law divergence as $|q| \to 2|k_F|$ for positive δ. The essential point here is the severity of this power law divergence compared with the logarithmic dependence on $|q - 2k_F|$ (in 1D) and certainly when compared with the slope discontinuity (in 2D), that are usually associated with the Kohn anomaly.

A simple modification of the particle–hole susceptibility analysis allows us to compute the Cooper pair propagator

$$P_\alpha(\boldsymbol{x}, t) = -\mathrm{i}\Theta(t)\langle G|[\psi_\alpha(\boldsymbol{x}, t)\psi_{\alpha^*}(\boldsymbol{x}, t), \psi_\alpha^+(0, 0)\psi_{\alpha^*}^+(0, 0)]|G\rangle \tag{35}$$

where, for example, the calculation of $\langle G|\psi_\alpha(\boldsymbol{x}, t)\psi_{\alpha^*}(\boldsymbol{x}, t)\psi_{\alpha^*}^+(0, 0)\psi_{\alpha^*}^+(0, 0)|G\rangle$ $\widetilde{W}_\alpha^A(\boldsymbol{x}, t)\widetilde{R}_{\alpha^*}^A(\boldsymbol{x}, t)W_\alpha^A(0, 0)^+ R_{\alpha^*}^A(0, 0)^+$ and $\widetilde{R}_\alpha^A(\boldsymbol{x}, t)\widetilde{W}_{\alpha^*}^A(\boldsymbol{x}, t)R_\alpha^A(0, 0)^+ W_{\alpha^*}^A(0, 0)^+$. The only modifications from the particle–hole susceptibility lie in the relative signs of the $\cosh\phi_\alpha(q)$ and $\sinh\phi_\alpha(q)$ phase factors, as well as the fact the wavevector dependence is on $\boldsymbol{q}$, not on $\boldsymbol{q} - 2\boldsymbol{k}_F$. For example, one component of the pair propagator takes the form

$$\langle G|\psi_\alpha(\boldsymbol{x}, t)\psi_{\alpha^*}(\boldsymbol{x}, t)\psi_{\alpha^*}^+\psi_{\alpha^*}^+|G\rangle = \langle 0_{NI}|\psi_\alpha(\boldsymbol{x}, t)\psi_{\alpha^*}(\boldsymbol{x}, t)\psi_{\alpha^*}^+\psi_{\alpha^*}^+|0_{NI}\rangle \exp S(\boldsymbol{x}, t) \tag{36}$$

where

$$S(\boldsymbol{x}, t) = \frac{2\pi^2}{\Lambda\Omega}\sum_{n_\alpha\cdot q>0}\frac{\{(\cosh\phi_\alpha(q) - \sinh\phi_{\alpha^*}(q))^2 - 1\}}{\widehat{n}_\alpha\cdot\boldsymbol{q}} \times\{\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x} - \mathrm{i}E_\alpha^A(q)t) + \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x} - \mathrm{i}E_\alpha^A(q)t) - 2\} \tag{37}$$

The essential difference is in the sign of the $\sinh\phi$, which in the small δ limit reverses the sign of the power law found for the particle–hole susceptibility. Thus for positive δ, there will be no divergence in the Cooper pair susceptibility on the Fermi surface, so a superconducting instability is not favoured.

In conclusion, we have calculated the spectral function, particle–hole susceptibility and Cooper pair susceptibility, to our knowledge, for the first microscopically derived and exactly solvable model that demonstrates Luttinger liquid behavior in 2D. This model relies on the fact that the Ampere force underlying the current–current interaction is long ranged and cannot be totally screened, so that it survives the renormalization procedure that lies behind bosonization in higher dimensions. We find that the spectral function shows a power law tail with a power dependent on the square of the interaction strength, while the particle–hole susceptibility shows, for the conventional sign of the Ampere force, a power law divergence at $q = 2k_F$, with a power law given by the interaction strength. At the same time, the Cooper pair susceptibility remains finite. While these results essentially follow the known behaviour of 1D Luttinger liquids, the knowledge of these quantities is a prerequisite for the calculation of effects due to additional interactions or interlayer 3D couplings.

In particular, it has been suggested [9] that for systems where the spectral function shows the generic Luttinger liquid scaling behaviour, the additional of a weak attractive coupling will lead to superconductivity, provided that the attractive coupling exceeds some critical value. This conclusion is based on treating the additional interaction within the ladder approximation, in the absence of a detailed knowledge of the inherent superconducting properties of the underlying Luttinger liquid. However, the formulation presented here does allow for a detailed examination of this issue because of its treatment of both inherent and additional superconducting interactions on an equal footing. Work on including interlayer coupling and additional interactions is in progress.

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