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RESEARCH NOTE

Electron and ion escape over a potential barrier in a mirror field*

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THE PROBLEM of calculating the electron loss rate and the corresponding energy loss rate from a magnetic mirror machine having a positive potential has been examined and solved to a reasonable approximation by PASTUKHOV (1974). However, his results cannot be directly applied to heavy ions escaping over a potential barrier since his equations, which are based on a two-species plasma, contain the assumption that the escaping species has a much higher speed than the remaining species. In the case of heavy ions escaping over a potential barrier (as in the central section of a Tandem Mirror (FOWLER and LOGAN, 1977) this assumption must in fact be reversed. In this note we follow Pastukhov to produce particle and energy confinement times for ions which can be readily utilized in the study of plasma confinement in Tandem Mirrors. We will show that a single additional parameter, labeled C, can be incorporated into the appropriate equations to make them equally applicable to electrons and ions; $C=1$ for electrons and $C=\sqrt{2}$ for ions.

The derivation is based on the Fokker–Planck equation in a square well potential, which, when expressed in terms of the Rosenbluth potentials (ROSENBLUTH *et al.*, 1957) g and h, assumes the form

$$\frac{1}{\Gamma}\frac{\partial f}{\partial t}=-\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2 f\frac{\partial h}{\partial v}\right]+\frac{1}{2v^2}\left[v^2 f\frac{\partial^2 g}{\partial v^2}\right]-\frac{1}{v^2}\frac{\partial}{\partial v}\left[f\frac{\partial g}{\partial v}\right]+\frac{1}{2v^3}\frac{\partial^2}{\partial\mu^2}\left[(1-\mu^2)f\frac{\partial g}{\partial v}\right]+\frac{1}{v^3}\frac{\partial}{\partial\mu}\left[\mu f\frac{\partial g}{\partial v}\right]. \quad (1)$$

Where

$$\Gamma=\frac{4\pi Z^2 e^4}{m^2}\ln\Lambda$$

$$g=g_i+g_e$$

$$h=h_i+h_e$$

and $\ln\Lambda$ is the familiar Coulomb logarithm.

We make the common assumption that the Rosenbluth potentials are independent of the angle term μ, although the distribution function f may depend on μ. For the electron loss problem, since $v_e \gg v_i$ for virtually all electrons and ions involved, we find that the potential g_e can be expressed as

$$g_e=g_{ee}+g_{ei}\simeq g_{ee}+n_i v_e. \quad (2)$$

For the ion loss term, it is generally true even for the high energy tail of the ion distribution that $v_i<v_e$. For this calculation, we shall assume that v_i is in fact negligible compared to v_e, so that we can write

$$g_i=g_{ii}+g_{ie}\simeq g_{ii}+2n_e(\pi\alpha_e)^{-1/2} \quad (3)$$

where $\alpha_e^{-1/2}$ is the electron thermal velocity. Thus, we have

$$\frac{\partial g_e}{\partial v_e}=\frac{\partial g_{ee}}{\partial v_e}+n_i$$
$$\frac{\partial g_i}{\partial v_i}=\frac{\partial g_{ii}}{\partial v_i}. \quad (4)$$

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If we now introduce the dimensionless variables used by PASTUKHOV (1974), namely

$$v_0=\alpha_e^{-1/2}=\left(\frac{2T_e}{m_e}\right)^{1/2} \quad \text{or} \quad v_0=\alpha_i^{-1/2}=\left(\frac{2T_i}{m_i}\right)^{1/2}$$

$$\tau_0=\frac{v_0^3}{n_e\Gamma_e}=\frac{m_e^{1/2}T_e^{3/2}}{\sqrt{2}\pi n_e e^4 \ln\Lambda} \quad \text{or} \quad \tau_0=\frac{v_0^3}{n_i\Gamma_i}=\frac{m_i^{1/2}T_i^{3/2}}{\sqrt{2}\pi n_i e^4 \ln\Lambda}$$

$$x=\frac{v}{v_0}$$

$$y=\frac{g_{ee}}{n_e v_0} \quad \text{or} \quad y=\frac{g_{ii}}{n_i v_0}$$

$$F=\frac{v_0^3 f_e}{n_e} \quad \text{or} \quad F=\frac{v_0^3 f_i}{n_i}$$

we obtain the equation

$$\frac{v_0^3}{n}\tau_0\frac{\partial}{\partial t}\left(\frac{nF}{v_0^3}\right)=4\pi F^2+\frac{1}{2}\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 F}{\partial x^2}+\frac{1}{x^2}\frac{\partial y}{\partial x}\frac{\partial F}{\partial x}+\frac{1}{2x^3}\left[\frac{\partial y}{\partial x}+\left(\frac{2}{c^2}-1\right)\right]\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial F}{\partial \mu}\right] \tag{5}$$

where the quantity $\left(\frac{2}{c^2}-1\right)=1$ for the electron case, and $\left(\frac{2}{c^2}-1\right)=0$ for the ion case; or more explicitly $c_e=1$ and $c_i=\sqrt{2}$. This form of c appears somewhat awkward in equation (5). However, when we let g_{ee} and g_{ii} take their Maxwellian forms, and let x become very large, so that

$$\frac{\partial y}{\partial x}\to 1-\frac{1}{2x^2}$$

$$F\to 0$$

then we obtain an equation of the form

$$\hat{L}(F)=\frac{1}{2x^3}\frac{\partial^2 F}{\partial x^2}+\frac{1}{x^2}\frac{\partial F}{\partial x}+\frac{1}{c^2x^3}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial F}{\partial \mu}\right], \tag{6}$$

The solution F can be obtained by following the procedure outlined by Pastukhov. This method entails extending F into the loss cone region by requiring it to satisfy an equation of the form

$$\hat{L}(F)-Q(x,\mu)=0 \tag{7}$$

where $\hat{L}(F)$ is the operator given in equation (6) and $Q(x,\mu)$ is a source (or loss) term which is non-zero only for $x>x_0$ and $\mu_0^2<\mu^2\le 1$, where

$$x_0^2=-\frac{Ze\phi}{T}$$

$$\mu_0^2=\frac{R-1}{R}+\frac{x_0^2}{Rx^2}$$

with ϕ being the potential and R the mirror ratio.

Pastukhov proposes the function

$$Q(x,\mu)=-qe^{-x^2}\delta(1-\mu^2)\eta(x-a) \tag{8}$$

where $\eta(Z)$ is the step function, being zero for $Z<0$ and unity for $Z>0$. The constants q and a are determined from the boundary conditions:

$$F(x=x_0,\mu^2=1)=0$$

$$\left[\frac{\partial F}{\partial x}-\frac{1}{Rx_0}\frac{\partial F}{\partial \mu}\right]_{\substack{x=x_0\\ \mu^2=1}}=0 \tag{9}$$

Obtaining the solution F, in Pastukhov's method, involves two co-ordinate transformations. First, we let

$$r=e^{x^2}$$

and second, we define

$$\rho = cr\sqrt{1-\mu^2}$$
$$\eta = r\mu/\sqrt{\ln r^2}.$$

Pastukhov's paper defines ρ without the factor c, of course. With this definition, however, the transformed differential equation becomes identical to his, and, after much calculation, we obtain

$$a = x_0 \tag{10}$$

$$q = \frac{8}{\pi^{3/2}c^2x_0^3 \ln\left(\frac{4R}{c^2}+2\right)}. \tag{11}$$

The loss rates are obtained from

$$\begin{aligned} \frac{\tau_0}{n}\frac{dn}{dt} &= 4\pi \int_0^\infty \int_0^1 x^2 Q(x, \mu)\, d\mu\, dx \\ \frac{3}{2}\frac{\tau_0}{nv_0^2}\frac{d}{dt}(nv_0^2) &= 4\pi \int_0^\infty \int_0^1 x^4 Q(x, \mu)\, d\mu\, dx. \end{aligned} \tag{12}$$

Inserting the results, (10) and (11) into (8), and defining

$$t = x^2 - a^2$$

we obtain

$$\frac{dn}{dt} = -\frac{4n}{c^2\sqrt{\pi}\tau_0}\left(\frac{R}{R+c^2}\right)^{1/2} \frac{e^{-Ze\phi/T}}{\ln\left(\frac{4R}{c^2}+2\right)}\left(\frac{T}{Ze\phi}\right)\int_0^\infty \left[1+\frac{T}{Ze\phi}t\right]^{1/2} e^{-t}\, dt \tag{13}$$

$$\frac{d(nT)}{dt} = -\frac{4nT}{c^2\sqrt{\pi}\tau_0}\left(\frac{R}{R+c^2}\right)^{1/2} \frac{e^{-Ze\phi/T}}{\ln\left(\frac{4R}{c^2}+2\right)}\left\{\frac{2}{3}+\left(\frac{T}{Ze\phi}\right)\int_0^\infty \left[1+\frac{T}{Ze\phi}t\right]^{1/2} e^{-t}\, dt\right\}. \tag{14}$$

These equations are identical to those of Pastukhov when we set $c = 1$ (the electron case) except that his loss rates are only half as large as our results. The source of this factor of two discrepancy is somewhat obscure, since his published work must necessarily omit many of the intermediate steps required to obtain the results. The quantity q given by (11) is four times as large as Pastukhov's value. One factor of two appears to be related to the integral over μ, and thus disappears when the integrals of (12) are performed. The other factor of two remains unexplained. However, our results agree with those given (without derivation) in LOGAN (1977).

Finally, we note that when more than one heavy ion species is present, the loss rate for each species can be represented in the form given by (13) and (14), with the quantity c^2 now defined as follows:

for electrons,

$$c_e^2 = \frac{2\Gamma_{ee}n_e}{\Gamma_{ee}n_e + \sum_k \Gamma_{ek}n_k} \tag{15}$$

and for heavy ions,

$$c_j^2 = \frac{2\sum_k \frac{m_j}{m_k}\Gamma_{jk}n_k}{\sum_k \Gamma_{jk}n_k}. \tag{16}$$

In (15) and (16), the summations over k include the heavy (positive) ion species, but not the electrons, and Γ_{jk} is given by

$$\Gamma_{jk} = \frac{4\pi Z_j^2 Z_k^2 e^4 \ln \Lambda_{jk}}{m_j^2} \tag{17}$$

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