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CLASSICAL TRANSPORT COEFFICENTS IN A FIELD-REVERSED CONFIGURATION*

K. NGUYEN and T. KAMMASH
University of Michigan, Ann Arbor, MI 48109, U.S.A.

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Abstract—A Hill's Vortex equilibrium model is utilized to calculate the classical transport properties of a plasma confined in a field-reversed magnetic configuration. Consistent with zero dimensional steady state conservation of particles and energy we compute the corresponding fluxes at the separatrix using volume-averaged values for the density and conductivity. We find that the confinement times scale directly with the size of the Vortex and conductivity at that point, and inversely with a function of the profile parameter.

MUCH attention has recently been aimed at studying plasma confinement properties in field-reversed configurations as a result of the great interest generated in Compact Tori as potential fusion reactors. Several experiments (ARMSTRONG *et al.*, 1980) have been carried out where confinement times were measured, and several theoretical studies have been advanced to interpret and predict the data observed. For example, a model that takes into account a loss-cone-like mechanism in the field-reversed theta pinch confinement has been suggested (FANG and MILEY, 1980) and the results obtained appear to fall within a factor of 2–3 of the range of values reported from the experiments FRX-A and B (LINFORD *et al.*, 1978). Another investigation (HAMASAKI and KRALL, 1980*a*) examining both classical theory and the role of microinstabilities in the transport of plasma in these experiments has also shown that agreement between theory and experiment can be affected if the anomalous diffusion is singled out as the dominant mechanism. In this paper we employ the Hill's Vortex model (THOMPSON, 1962) to represent the plasma equilibrium in such field-reversed configurations, and on the basis of this model we compute the particle and energy fluxes across magnetic surfaces. We find that the resulting confinement times can be in good agreement with the FRX-B experimental results if certain choice is made for the plasma effective charge.

We begin with the steady-state generalized form of Ohm's law (BRAGINSKII 1965), i.e.

$$\frac{J_\parallel}{\sigma_\parallel}+\frac{J_\perp}{\sigma_\perp}=\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}-\frac{1}{en}\nabla p_i+\frac{0.7}{e}\nabla_\parallel T_e+\frac{3}{2}\frac{cn}{B^2\sigma_\perp}\mathbf{B}\times\nabla T_e \tag{1}$$

where J is the current density, **E** and **B** are the electric and magnetic fields

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respectively, p is the pressure, T is the temperature, and σ is the conductivity. The subscripts on J and σ denote parallel and perpendicular directions to the magnetic field, respectively, and in the case of a field-reversed configuration with only a poloidal magnetic field the current density is in the toroidal direction only. If we cross multiply equation (1) by $\mathbf{B}$ we obtain, assuming $E_\perp = E_\theta = 0$:

$$\mathbf{v}_\perp = -\frac{c}{\sigma_\perp}\frac{\mathbf{J}\times\mathbf{B}}{B^2} + \frac{3}{2}\frac{c^2 n}{\sigma_\perp B^2}\nabla_\perp T_e, \tag{2}$$

having used the fact that for the system under consideration $\mathbf{J}\cdot\mathbf{B} = 0$, and $J_\perp = J_\theta$ in cylindrical coordinates. If we further assume that the densities, temperatures, and conductivity are functions of the flux function, ψ, only then equation (2) can be put in the form

$$\mathbf{v}_\perp = \frac{c^2}{\sigma_\perp B^2}\nabla\psi\frac{\mathrm{d}p}{\mathrm{d}\psi}\left[1 - \frac{3}{2}n\frac{\mathrm{d}T_e}{\mathrm{d}p}\right]$$

$$\mathbf{v}_\perp = \left[\frac{c^2 r}{\sigma_\perp B}\frac{\mathrm{d}p}{\mathrm{d}\psi}\left(1 - \frac{3}{2}n\frac{\mathrm{d}T_e}{\mathrm{d}p}\right)\right]\hat{e}_\psi \tag{3}$$

where we recall that the second term represents the contribution to the particle flux arising from the electron temperature gradient.

We now specialize to the case of the Hill's Vortex equilibrium given by

$$\frac{\psi}{\psi_0} = \frac{r^2}{r_0^4}[2r_0^2 - r^2 - \alpha^2 z^2] \tag{4}$$

where ψ_0 is the value of the flux function ψ at the vortex or "0" point whose coordinates are given by $r = r_0$, $z = 0$, and α is the ellipticity parameter. Equation 4 is simply the solution to the Grad–Shafranov equation in a cylindrically symmetric system for which the pressure and the current density are related by

$$J_\theta = r\frac{\mathrm{d}p}{\mathrm{d}\psi}$$

$$p(\psi) = p_0\psi \tag{5}$$

where

$$p_0 = \frac{\psi_0}{r_0^4}\frac{2(4+\alpha^2)}{\mu_0}$$

with $\mu_0 = 4\pi$ being the magnetic permeability. It should be noted that ψ_0 is related to r_0 by

$$r_0 = \left(-\frac{4\psi_0}{B_z}\right)^{1/2} \tag{6}$$

where B_z is the value of the magnetic field at the separatrix point defined by $r = R_0 = (\sqrt{2})\, r_0$ and $z = 0$.

The particle confinement time is introduced as

$$\tau_p = \frac{\int n \,\mathrm{d}V}{\Gamma_s} \tag{7}$$

where Γ_s is the particle flux defined by

$$\Gamma_s = \int_{\psi=0} n\mathbf{v}_\perp \cdot \mathrm{d}\mathbf{s}. \tag{8}$$

In the present zero dimensional model the surface integral shown in (8) is taken over the boundary (i.e. $\psi = 0$), whereas the density (and the perpendicular conductivity) is taken to have its volume-averaged value in order for the continuity equation to be satisfied in the steady state. Substituting from (3) into (8) we can write:

$$\Gamma_s = \frac{c^2\langle n\rangle p_0}{\langle\sigma_\perp\rangle}\left[1 - \frac{3}{2}\left\langle n\,\frac{\mathrm{d}T_e}{\mathrm{d}p}\right\rangle\right]\int_{\psi=0}\frac{2\pi r^2\,\mathrm{d}l}{B} \tag{9}$$

where we have let $\mathrm{d}s = 2\pi r\,\mathrm{d}l$, and denoted the volume average by

$$\langle F\rangle = \frac{\int F\,\mathrm{d}V}{V_s} \tag{10}$$

with V_s being the volume of the separatrix. Using $\nabla \cdot \mathbf{B} = 0$ along with equation (4) we find that

$$\int_{\psi=0}\frac{2\pi r^2\,\mathrm{d}l}{B} = \int\frac{2\pi r^2\,\mathrm{d}r}{B_r} = \frac{r_0^4}{2\alpha\,\psi_0}\int_{-\sqrt{2}\,r_0}^{\sqrt{2}\,r_0}\frac{r\,\mathrm{d}r}{(2r_0^2 - r^2)^{1/2}} = \frac{3\,V_s}{B_z} \tag{11}$$

where we have made use of (6). If we now utilize the last of equation (5) along with (11) in equation (9) we obtain for the particle flux

$$\Gamma_s = \frac{3(4+\alpha^2)c^2\langle n\rangle V_s}{8\pi r_0^2\langle\sigma_\perp\rangle}\left[1 - \frac{3}{2}\left\langle n\,\frac{\mathrm{d}T_e}{\mathrm{d}p}\right\rangle\right]. \tag{12}$$

In order to obtain the various averages shown, we assume that the density profie in the system is represented by

$$n = n_0\left(\frac{\psi}{\psi_0}\right)^{\mu}; \quad 0 \leqslant \mu \leqslant 1 \tag{13}$$

where n_0 is the value of the density at the vortex. Because of the linear dependence of the pressure on ψ as shown in (5), the temperature profile must assume the form

$$T = T_0 \left(\frac{\psi}{\psi_0}\right)^{1-\mu} \tag{14}$$

with T_0 being the value at the vortex. In view of these relations it is clear that

$$\langle n\rangle = \frac{n_0}{V_s}\int\int \left(\frac{\psi}{\psi_0}\right)^{\mu} \frac{\mathrm{d}s}{|\nabla\psi|}\,\mathrm{d}\psi \tag{15}$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}\psi} = \int \frac{\mathrm{d}s}{|\nabla\psi|} = \int \frac{2\pi\,\mathrm{d}l}{B} = \frac{3\sqrt{2}\,V_s K(k)}{8\psi_0(1+b)^{1/2}} \tag{16}$$

where $K(k)$ is the elliptic integral of the first kind, $b = (1-\psi/\psi_0)^{1/2}$, and $k^2 = 2b/(1+b)$. It might be noted at this point that to within 1% the quantity in (16) can be represented by

$$\frac{\mathrm{d}V}{\mathrm{d}\psi} \cong -0.83 \left(\frac{\psi}{\psi_0}\right)^{-0.17} \frac{V_s}{\psi_0} \tag{17}$$

so that equation (15) finally becomes

$$\begin{aligned} \langle n\rangle &= 0.83 n_0 \int_0^1 \left(\frac{\psi}{\psi_0}\right)^{\mu-0.17} \mathrm{d}\left(\frac{\psi}{\psi_0}\right) \\ &= \frac{0.83}{\mu+0.83}\, n_0\,. \end{aligned} \tag{18}$$

In a similar fashion we can show that

$$\begin{aligned} \langle \sigma_\perp\rangle &= 0.83\sigma_0 \int_0^1 \left(\frac{\psi}{\psi_0}\right)^{3/2(1-\mu)-0.17} \mathrm{d}\left(\frac{\psi}{\psi_0}\right) \\ &= \frac{0.83}{2.33-1.5\mu}\,\sigma_0 . \end{aligned} \tag{19}$$

In view of these results equation (12) becomes

$$\Gamma_s = \frac{3(4+\alpha^2)c^2 V_s n_0}{8\pi r_0^2 \sigma_0}\left(\frac{2.33-1.5\,\mu}{0.83+\mu}\right)\left[1 + \frac{3}{2}\frac{p_{e0}}{p_0}(\mu-1)\right]$$

where p_{e0} is the electron portion of the plasma pressure at the vortex. Combining this result with equations (18) and (7) we obtain for the particle confinement (with $\eta_0 = 1/\sigma_0$)

$$\tau_p = \frac{\langle n\rangle V_s}{\Gamma_s}$$

or

$$\tau_p = \frac{6.95 r_0^2}{(4+\alpha^2)c^2\eta_0}\left\{(2.33 - 1.5\,\mu)\left[1 + \frac{3}{2}\frac{p_{e0}}{p_0}(\mu - 1)\right]\right\}^{-1}. \quad (20)$$

Once again, the contribution to the particle confinement time of the temperature gradient is shown in the last term of (20). For isothermal plasma i.e. $\mu = 1$, there is no conductive heat transport and the energy transport will be only through convection. In that case the diffusing particles will carry (5/2) kT because of the work they do, and the convective energy confinement time takes on the value

$$\tau_{EcV} = \frac{3}{5}\tau_p. \quad (21)$$

We turn now to the calculation of the energy confinement that includes heat conduction. This process is carried out by the ions of the plasma through collisions with other ions. The conduction flux across is given by (BRAGINSKII, 1965)

$$\mathbf{q}_\perp = -K_\perp \nabla_\perp T_i \quad (22)$$

where the thermal conduction coefficient is expressed by

$$K_\perp = \frac{2p_i}{m_i\omega_{c_i}^2\tau_{ii}} \quad (23)$$

$$\tau_{ii} = \frac{3\sqrt{(m_i)}T_i^{3/2}}{4\sqrt{(\pi)}\lambda_i z^4 e^4 n_i}$$

and ω_{c_i} is the ion cyclotron frequency, τ_{ii} is the ion–ion collision frequency, λ_i is the Coulomb logarithm, and the remaining terms have their usual meaning. Substituting (23) into (22) and expressing the various terms through their ψ dependence we obtain

$$\mathbf{q}_\perp = \left[A\frac{p_i}{T_i^{3/2}}n_i\frac{\mathrm{d}T_i}{\mathrm{d}\psi}\frac{r}{B}\right]\hat{e}_\psi \quad (24)$$

where the constant A is given by

$$\mathrm{A} = \frac{2m_i c^2 T_i^{3/2}}{z^2 e^2 n_i \tau_{ii}}. \quad (25)$$

If we utilize the profiles given by (13) and (14), equation (24) becomes

$$\mathbf{q}_\perp = A\frac{p_i}{T_i^{3/2}}p_{i0}(1-\mu)\frac{r}{B}\,\hat{e}_\psi \quad (26)$$

where p_{i0} is the ion pressure at the vortex. The total heat flux, in analogy to equation (8), can be written as

$$\Gamma_q = \int_{\psi=0} \mathbf{q}_\perp \cdot \mathrm{d}s \tag{27}$$

where, as before, the surface integral is performed at the separatrix and the plasma parameters are replaced by their volume averages. In view of this, equation (27) reduces to

$$\Gamma_q = A \frac{\langle p_i \rangle}{\langle T_i^{3/2} \rangle} p_{i0}(1-\mu) \int_{\psi=0} \frac{r}{B} \mathrm{d}s \tag{28}$$

the averages in this case are given by

$$\langle F \rangle = \int_0^1 0.83F \left(\frac{\psi}{\psi_0}\right)^{-0.17} \mathrm{d}\left(\frac{\psi}{\psi_0}\right). \tag{29}$$

Carrying out the desired operations and making use of (5) and (11) we can put (28) in the form

$$\Gamma_q = 3AV_s \frac{\langle p_i \rangle}{\langle T_i^{3/2} \rangle} \left(\frac{p_{i0}}{p_0}\right)(1-\mu) \frac{(4+\alpha^2)}{8\pi r_0^2} \tag{30}$$

We recall from (21) and (20) that the convective heat flux can be expressed as

$$\Gamma_{\text{conv}} = \frac{5}{2} \frac{3(4+\alpha^2)c^2 V_s \langle p_i \rangle}{8\pi r_0^2 \langle \sigma_\perp \rangle} \left[1 + \frac{3}{2}\left(\frac{p_{e0}}{p_0}\right)(\mu - 1)\right] \tag{31}$$

so that upon combining (31) with (30) we can write for the general energy confinement time:

$$\tau_E = \frac{\frac{3}{2}\langle p_i \rangle V_s}{\Gamma_q + \Gamma_{\text{conv}}} \tag{32}$$

or more explicitly

$$\tau_E = \frac{3}{2} \frac{6.95 r_0^2}{(4+\alpha^2)c^2 \eta_0 h(\mu)} \tag{33}$$

where

$$h(\mu) = \left\{\frac{5}{2}\left[1 + \frac{3}{2}\left(\frac{p_{e0}}{p_0}\right)(\mu - 1)\right] + 60.5z\left(\frac{T_{e0}}{T_{i0}}\right)^{3/2} \times \left(\frac{p_{i0}}{p_0}\right)(1-\mu)\right\}(2.33 - 1.5\mu). \tag{34}$$

Clearly, the above result reduces to (21) when $\mu = 1$, i.e. for an isothermal plasma.

It is interesting to compare the results of this analysis with those of an existing field-reversed experiment. If, as indicated by numerical simulation (HAMASAKI and KRALL, 1980*b*), the temperature profile in the FRX-B experiment is constant (except at the edges which we ignore) then to obtain the desired confinement times we set $\mu = 1$ in the above results. The experiments also yield $\tau_p = 39 \pm 15\,\mu s$ and $\tau_E = 31 \pm 15\,\mu s$ for plasma parameters of $T_i = 200\,\text{eV}$, $T_e = 110\,\text{eV}$, $r_0 = 3.8\,\text{cm}$ and $n_e = 3.5 \times 10^{15}\,\text{cm}^{-3}$. Substituting these values in the appropriate expressions we find

$$\tau_p = \frac{132}{z}\,\mu s; \quad \tau_E = \frac{78}{z}\,\mu s.$$

If we now assume that the effective charge is $z = 2$, then we obtain $\tau_p \simeq 60\,\mu s$ and $\tau_E \simeq 40\,\mu s$ which are in good agreement with the experimental results.

It should be noted, however, that even though the above results are in good agreement with the experiments it does not necessarily follow that the Hill's Vortex model represents the plasma state in the FRX-experiments. This model assumes, among other things, that the plasma pressure varies linearly with the flux function thereby peaking at the vortex and vanishing at the separatrix. This further requires that the current density vary linearly with the radial distance and that beta assume a specific value that depends on geometry only. A numerical classical transport calculation utilizing a different equilibrium state (BYRNE and GROSSMAN, 1980) has also been carried out for the system under consideration and the results obtained were roughly five times larger than those observed in the experiment, pointing to the importance of the type of equilibrium assumed. Moreover, the analyses carried out by HAMASAKI and KRALL (1980*a*; 1980*b*) take into account transport along field lines due primarily to microinstabilities and the agreement with experimental results obtained seems to be mainly with the associated anomalous diffusion time and not with the classical transport as was done in this paper. In short, the results generated by the various models must be viewed in the context of the assumptions used until further information is obtained concerning the true applicability of these models.

REFERENCES

ARMSTRONG W. T., LINFORD R. K., LIPSON J., PLATTS D. A. and SHERWOOD E. G. (1980) Los Alamos Report LA-UR-80-1585. To be published in *Physics Fluids*.

BRAGINSKII S. I. (1965) *Reviews in Plasma Physics*, Vol. 1. Consultants Bureau, New York.

BYRNE R. N. and GROSSMANN W. (1980) Proc. 3rd Symp. on Physics and Technology of Compact Toroids, Los Alamos Report LA-8700-C, p. 138.

FANG Q. T. and MILEY G. H. (1980) Proc. 3rd Symp. on Physics and Technology of Compact Toroids, Los Alamos Report LA-8700-C, p. 144.

HAMASAKI S. and KRALL N. A. (1980*b*) JAYCOR Report, J510-80-034.

HAMASAKI S. and KRALL N. A. (1980*a*) Proc. 3rd Symp. on Physics and Technology of Compact Toroids, Los Alamos Report LA-8700-C, p. 152.

LINFORD R. K., ARMSTRONG W. T., PLATTS D. A. and SHERWOOD E. G. (1978) *Proc. 7th Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, Innsbruck, Vol. 2, p. 447. IAEA, Vienna.

THOMPSON W. B. (1962) *An Introduction to Plasma Physics*. Pergamon Press, Oxford.