NOTE:

Errata - follows page 336

TECHNICAL REPORT ECOM 01870-4-T July 1966

ON THE BEST CHEBYSHEV APPROXIMATION OF AN IMPULSE RESPONSE FUNCTION AT A FINITE SET OF EQUALLY-SPACED POINTS

Technical Report No. 172

CONTRACT NO. DA-28-043-AMC-01870(E)
DA Project No. 280-Mt-00287(E)

Prepared By

Robert Fischl

COOLEY ELECTRONICS LABORATORY

Department of Electrical Engineering
The University of Michigan
Ann Arbor, Michigan

for

U. S. ARMY ELECTRONICS COMMAND
FORT MONMOUTH, NEW JERSEY

ACKNOWLEDGEMENTS

The author is deeply indebted to Professor Keki B. Irani, the chairman of his doctoral committee, who not only aroused his interest in the problem of Chebyshev approximations in the time domain, but who also was a source of constant encouragement and of valuable suggestions during the course of the research. He also wishes to thank the other members of his doctoral committee for their helpful comments concerning the material. Gratitude is also expressed to Professor B. F. Barton and colleagues, both past and present, at the Cooley Electronics Laboratory who have contributed immeasurably to the author's education.

Finally, the author wishes to thank Miss Ann M. Rentschler, Mrs. Lillian M. Thurston, Mrs. Elizabeth W. Willsher, and Mr. Thomas E. Jennett for their extensive efforts in typing and preparing this manuscript for publication. In addition, he wishes to thank his wife, Ruth, for her patience and help in proofreading the final draft.

A part of the research reported in this dissertation was supported by the U. S. Army Signal Corps.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont.)

Page

TABLE OF CONTENTS (Cont.)

LIST OF TABLES

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

General Conventions

1. Lower case letters are used for constants, scalar-valued variables and functions, e.g., a, ρ, f(t).

2. Underlines lower case letters denote vectors, e.g., $\underline{f}$, $\underline{s}$, $\underline{z}$, $\underline{\alpha}$, $\underline{\beta}$.

3. Square-bracketed capital letters denote matrices, e.g., [A], $[E(\underline{s})]$, $[Z(\underline{z})]$, $[Z^{(k)}(\underline{z})]$.

4. Capital script letters denote sets, e.g., $\mathscr{A}_s$, $\mathscr{S}$, $\mathscr{B}_z$, $\mathscr{R}$.

5. g(·) or g(·,·) denotes a function, with the dot standing for an undesignated variable.

6. Braces denote a set or a family, e.g., $\{x_1, x_2, \ldots, x_n\}$ is the ordered set representing the components of $\underline{x}$. $\{\underline{x}: P\}$ is a set of $\underline{x}$'s having property P.

7. $(\underline{\alpha}, \underline{s})$ denotes the ordered pair of vectors $\underline{\alpha}$ and $\underline{s}$.

8. A bar (‾) over a symbol denotes the complex conjugate value.

9. A starred symbol denotes the optimum value, e.g., $f^*(t)$, $\underline{f}^{**}$, $\underline{r}^*$. Specifically, a single star denotes the optimum value with respect to one set of parameters and a double star denotes the optimum value with respect to two sets of parameters.

10. Superscript minus one ($^{-1}$) denotes the inverse of a matrix, e.g., $[E]^{-1}$.

11. Superscript dagger ($^{+}$) denotes the pseudo inverse of a matrix, e.g., $[E]^{+}$.

12. Superscript tilde ($\tilde{\ }$) denotes the estimate of the parameter vector, e.g., $\tilde{\underline{r}}$, $\tilde{\underline{z}}$.

13. Superscript hat ($\hat{}$) denotes the approximating function or vector, e.g., $\hat{\underline{f}}(t;\underline{\alpha}, \underline{s})$, $\hat{\underline{f}}(\underline{\alpha}, \underline{s})$.

14. Superscript T (^{T}) denotes the transpose of a matrix, e.g., $[A]^T$, $[R]^T$.

15. Vectors with letter superscripts denote the projections of the vectors onto a lower dimensional reference subspace having the same fixed basis, e.g., $\underline{f}^{(k)}$, $\underline{\epsilon}^{(k)}$.

16. A matrix with a letter superscript denotes (a) submatrix; or (b) the form of the matrix in a lower dimensional reference subspace, e.g., $[E^{(k)}]$, $[Z^{(k)}(\underline{z})]$, $[\Lambda^{(k)}(\underline{r})]$.

General Symbols and Abbreviations

$\triangleq$	equals by definition; denotes
ϵ	is an element of
$\notin$	does not belong to
$\cup$	union
$\oplus$	direct sum
x	Cartesian product
$(\cdot, \cdot)$	the inner product relation
$\lVert \cdot \rVert_p$	the p-th norm, where $p \geq 1$
$\{x : P\}$	the set of x's having property P
(a, b)	open interval $a < t < b$
$[a, b)$	semi-closed interval $a \leq t < b$

LIST OF SYMBOLS (Cont.)

$[a, b]$	closed interval $a \leq t \leq b$
$C_n(E)$	the n-dimensional column space of the matrix $[E]$
$Re\{s\}$	the real part of s
$\max_i$	maximum over i
$\max\{a, b\}$	$\max\{a, b\} = a$ if $a \geq b$, $\max\{a, b\} = b$ if $b > a$
sgn	signum function: $\text{sgn } x = \frac{x}{\lvert x \rvert}$ for all $x \neq 0$ $\text{sgn } 0 = 0$

Symbols with Special Meaning

Symbol	Description	Defined or first used on page
a	an arbitrary real constant	30
a	the real constant defined by Eq. 2.36	50
[A]	a q x n real matrix	40
$\mathcal{A}_s$	the set of all parameter vectors $\underline{\alpha}$ (Definition 1.6)	10
$\mathcal{A}_s \times \mathcal{S}$	the set of the parameter vector pairs $(\underline{\alpha}, \underline{s})$	11
b	an arbitrary real constant	30
$\mathcal{B}_z$	the set of all parameter vectors $\underline{\beta}$ for a fixed $\underline{z} \in \mathcal{Z}$. (Definition 1.8)	21
c	an arbitrary real constant	50
c	as a subscript or superscript denotes the index v which yields the best Chebyshev approximation in U^q	60

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
$C_n(E)$, $C_n(Z)$	the n-dimensional column space of the matrix [E] (or [Z])	53
d_1, d_2	arbitrary constants	114
$D_j(\underline{r})$	a polynomial in the components of $\underline{r} \in \mathscr{R}$	148
$e^{s_k t}$	the k-th exponential function	8
$\underline{e}_k$	the q-dimensional vector representing the ordered set $\{e^{s_k t_1}, e^{s_k t_2}, e^{s_k t_2}, \ldots, e^{s_k t_q}\}$ of the exponential function $e^{s_k t}$.	14
e_{ij}	the elements of the matrix [E] defined by Eq. 1.12	15
[E(s)], [E]	a q x n matrix whose column space defines $V^n(s)$ (see Eq. 1.12 or Eq. 1.12a)	15
$[E^{(k)}]$	an (n x 1) x n matrix submatrix of [E] given by Eq. 2.32	49
E^q	the real q-dimensional vector space	4
$\mathscr{E}_r$	the set of all vector $\underline{\epsilon}(\underline{r}) \in U^q$ which satisfy Eq. 3.16	99
f(t)	the prescribed function of t	30
$\underline{f}$	the prescribed real q-dimensional vector	27
$\hat{\underline{f}}(\underline{\beta}, \underline{z})$, $\hat{\underline{f}}(\underline{\alpha}, \underline{s})$	the real approximating vector in U^q	89

Symbol	Description	Defined or first used on page
$\underline{f}_p^{**}$, $\underline{f}^{**}$, $\hat{\underline{f}}(\underline{\beta}^{**}, \underline{z}^{*})$	the best real approximating vector in U^q	101
$\underline{f}^{(k)}$, $\hat{\underline{f}}^{(k)}$	the projections of the vectors in U^q and U_k^m	48
$\underline{f}^{(w)}$, $\hat{\underline{f}}^{(w)}$	the projection of the vectors in U^q onto the reference subspaces U_w^{2n+1}	133
$\hat{\underline{f}}^{(k)}(\underline{\beta}_k^{**}, \underline{z}_k^{*})$	the best Chebyshev real approximating vector in U_k^{2n+1}	134
[F]	a n x(n+1) real matrix defined by Eq. 2.71	75
[F]	the (q-n) x (n+1) real prescribed matrix defined by Eq. 4.34	120
$\underline{g}$	a q-dimensional real vector in U^q	28
$\hat{\underline{g}}$	the q-dimensional real vector in $C_{q-n}(R)$	92
h(t)	the prescribed impulse response function	2
$\hat{h}(t; \underline{\alpha}, \underline{s})$	the approximating impulse response function. (Eq. 1.3, Eq. 1.3a)	8
$\underline{h}$	the prescribed q-dimensional real vector--the ordered set of values $\{h(t_1), h(t_2), \ldots, h(t_q)\}$	14
$\underline{h}_n$, $\underline{h}_o$	the (q-n)-dimensional prescribed real vector defined by Eq. 2.80 (or Eq. 2.84)	79
$\hat{\underline{h}}(\underline{\alpha}, \underline{s})$	the q-dimensional real approximating vector	14

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
$\hat{h}(t;\underline{\alpha}_p^*, \underline{s})$, $h_p^*(t;\underline{s})$, $h^*(t;\underline{s})$, $h^*(t)$	the best L_p-approximating function of h(t) with respect to $\underline{\alpha} \in \mathscr{A}_s$ for a fixed $\underline{s} \in \mathscr{S}$. (Definition 1.1)	5
$h_p^{**}(t)$ $\hat{h}(t;\underline{\alpha}^{**}, \underline{s}^*)$, $\hat{h}(t;\underline{\alpha}_p^{**}, \underline{s}_p^*)$	the best L_p-approximating function with respect to $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \mathscr{S}$ (Definition 1.2)	5
$h^{**}(t)$	the best Chebyshev approximating function at a finite equally-space discrete value of t with respect to $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \mathscr{S}$	248
$\underline{h}^*(\underline{s})$, $\underline{\hat{h}}(\underline{\alpha}^*, \underline{s})$ $\underline{h}_p^*(s)$, $\underline{\hat{h}}(\underline{\alpha}_p^*, \underline{s})$	the best ℓ_p^q-approximating vector real vector with respect to $\underline{\alpha} \in \mathscr{A}_s$ for a fixed $s \in \mathscr{S}$. (see Definition 1.3)	6
$\underline{\hat{h}}(\underline{\alpha}_p^{**}, \underline{s}_p^*)$, $\underline{h}_p^{**}$, $\underline{h}^{**}$	the best ℓ_p^q-approximating real vector with respect to $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \mathscr{S}$ (Definition 1.4)	6
[H]	the (q-n) x (n+1) real matrix defined by Eq. 2.83	81
$[H_n]$, $[H_o]$	the (q-n) x n matrix defined by Eq. 2.80 (or Eq. 2.84)	79
$\hat{\mathscr{H}}$	the set of approximating functions $\{\hat{h}(t;\underline{\alpha}, \underline{s}) : t \in \mathscr{T}, \underline{\alpha} \in \mathscr{A}_s, \underline{s} \in \mathscr{S}\}$	2
i	the index $\{i = 1, \ldots, q\}$ or $\{i = 1, \ldots, n+1\}$	4
$[I_k]$	a q x m elementary matrix defined by Eq. 2.25	47

Symbol	Description	Defined or first used on page
j	the order of the repeated root	8
k	a set of n indices, $\{k = 1, 2, \ldots, n\}$	8
k	an arbitrary index in the set of $\binom{q}{m}$ indices	47
ℓ_p^q	Banach space of q-tuples	5
ℓ_p^q-approximation	the approximation which is measured by the ℓ_p^q-norm of the error vector $\underline{\epsilon}$	5
ℓ_p-projection	the best mapping of the prescribed vector in ℓ_p^q-space onto the approximating subspace V^n	27
L_p-space $L_p(\mathscr{T})$-space	Banach sapce of measurable function on interval $\mathscr{T}$	3
L_p-approximation	the approximation which is measured by the L_p-norm of the error function, $\epsilon(t)$	3
L_p-projection	the best mapping of a point in an L_p-space onto the approximation subspace	27
m	the dimension of the reference subspace	47
m	the number of components of the error vector whose absolute value are equal, $m \geq n+1$	146
m	arbitrary dimension of the vector space	50

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
n	the dimension of the parameter spaces $\mathscr{A}_s$, $\mathscr{S}$, $\mathscr{B}_z$, $\mathscr{Z}$	9
$N_j(\underline{r})$	a polynomial in the components of $\underline{r} \in \mathscr{R}$	148
p	a constant ≥ 1 used in related problems	3
$P_n(z)$	n-th order polynomial in z whose leading coefficient, $r_n = 1$. See Eq. 2.60	66
P(z)	the general n-th order polynomial equation defined by Eq. 2.65 or Eq. 3.4	72
P'(z)	the first derivative of the polynomial P(z)	95
P"(z)	the second derivative of the polynomial P(z)	95
q	the dimension of the prescribed vector space, i.e., the number of discrete values of t	4
$\underline{r}$	the (n+1)-dimensional parameter vector, defined by Eq. 2.66 (or Eq. 3.3), denoting the coefficients of the polynomial P(z)	72
$\tilde{\underline{r}}$	the vector $\underline{r} \in \mathscr{R}$ which yields the minimum value of $\Vert [F] \underline{r} \Vert_\infty$ in the set $\Vert \underline{r} \Vert_1 = 1$	122
$\underline{r}'$	an n-dimensional vector defined by Eq. 2.77	78

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
$\underline{r}'$	an arbitrary (n+1)-dimensional vector $\underline{r}$ in $\mathscr{R}$	123
$\tilde{\underline{r}}'$	the best estimate of the n-dimensional vector $\underline{r}'$ obtained by Ruston (or Yengst)	79
$\underline{r}^{(j)}$	the vector $\underline{r} \in \mathscr{R}$ whose j-th component is equal to one	128
$\underline{r}^*$	the best Chebyshev solution vector in $\mathscr{R}$ (see Definition 4. 1)	120
$\underline{r}_k{}^*$	the best Chebyshev k-th reference solution vector in $\mathscr{R}$ (see Definition 4. 2)	139
$\underline{r}_M$	the best k-th Chebyshev reference solution vector in $\mathscr{R}$ when all the components of $\underline{\epsilon}^{*(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value to $\Vert\underline{\epsilon}^{*(k)}(\underline{r})\Vert_\infty$ (see Definition 4. 3)	151
$\underline{r}_0$	the best Chebyshev k-th reference solution vector in $\mathscr{R}$ when 2n components of $\underline{\epsilon}^{*(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value. (see Definition 4. 4)	158
$[R(\underline{r})]$, $[R]$	is a $q\times(q-n)$ matrix defined by Eq. 3. 2, which is a function of $\underline{r}$. If $q = 2n$, then $[R]$ is defined by Eq. 2. 69	90
$\mathscr{R}$	the set of all parameter vectors $\underline{r}$ (see Definition 3. 1)	91
$\underline{s}$	the n-dimensional parameter vector denoting the n-exponents (Eq. 1. 5)	9
$\mathscr{S}$	the set of all parameter vectors $\underline{s}$ (Definition 1. 5)	10

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
t	time	2
Δt	time interval $(t_i - t_{i-1})$, $i = 1, 2, \ldots, q$	22
t_1	the initial sample time	4
t_q	the final sample time	4
T	as a superscript: indicates transposition	42
T	the finite point set $\{t_i : i = 1, 2, \ldots, q\}$	4
T_e	the finite equally-spaced point set $\{t_i = t_1 + (i-1)\Delta t : i = 1, 2, \ldots, q\}$	22
$[T]$	q x q diagonal matrix whose elements are $\{t_1, \ldots, t_q\}$	16
$\mathscr{T}$	the semi-infinite interval $[0, \infty)$	2
U^q	the complex q-dimensional vector space	4
U_k^m	the m-dimensional k-th reference subspace of U^q (Definition 2. 1)	47
U_v^{n+1}	the set of (n+1)-dimensional reference subspaces, where $v = 1, 2, \ldots, \binom{q}{n+1}$	48
U_w^{2n+1}	the set of (2n+1)-dimensional reference subspaces, where $w = 1, 2, \ldots, \binom{q}{2n+1}$	132
v	the set of indices, $v = 1, 2, \ldots, \binom{q}{n+1}$	48
$\underline{v}(\underline{z})$	a vector representing the orthogonal projection of $\underline{f}$ onto $C_n(Z)$	103
V^n, $V^n(\underline{s})$	the n-dimensional approximating subspace	15

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
w	the set of indices $\{1, 2, \ldots, \binom{q}{2n+1}\}$	107
$\underline{w}(\underline{r})$	a vector representing the orthogonal projection of $\underline{f}$ onto $C_{q-n}(R)$	103
$\underline{x}$	an arbitrary vector in E^m	50
$\underline{y}$	an arbitrary vector in E^m	50
$\underline{z}$	the n-dimensional parameter vector, defined by Eq. 1.19. The component $z_k = e^{s_k \Delta t}$	18
$\tilde{\underline{z}}$	the estimate of the vector $\underline{z} \in \mathcal{Z}$ obtained by Ruston (or Yengst)	80
$[Z(\underline{z})]$, $[Z]$	the q x n matrix which is a function of the vector $\underline{z} \in \mathcal{Z}$ and defined by Eq. 1.18 (or Eq. 1.18a, 1.18c)	18
$[Z^w(\underline{z})]$	is a (2n+1) x n submatrix of $[Z(\underline{z})]$, see Eq. 4.45	133
$\mathcal{Z}$	the set of all parameter vectors $\underline{z}$ Definition 1.7	21
$\underline{\alpha}$	the n-dimensional parameter vector denoting the n-coefficients of the n-exponential functions (Eq. 1.4)	9
$\underline{\alpha}_k^*$, $\underline{\alpha}_v^*$	the vector $\underline{\alpha} \in \mathcal{A}_s$ defining the best Chebyshev approximating vector in U_k^{n+1}	53
$(\underline{\alpha}, \underline{s})$	the ordered vector pair of the parameter vectors $\underline{\alpha}$ and $\underline{s}$	2

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
$(\underline{\alpha}_p^*, \underline{s})$, $(\underline{\alpha}^*, \underline{s})$	the best $L_p(\ell_p^q)$-approximating parameter vector $\underline{\alpha} \in \mathscr{A}_s$ for a fixed $\underline{s} \in \mathscr{S}$	5
$(\underline{\alpha}_p^{**}, \underline{s}_p^*)$, $(\underline{\alpha}^{**}, \underline{s}^*)$	the best $L_p(\ell_p^q)$-approximating parameter vector pair in $\mathscr{A}_s \times \mathscr{S}$	5
$\underline{\beta}$	the n-dimensional parameter vector defined by Eq. 1.22	21
$(\underline{\beta}, \underline{z})$	the ordered parameter vector pair $\underline{\beta}$ and $\underline{z}$	20
$(\underline{\beta}^{**}, \underline{z}^*)$	the best Chebyshev vector pair in $\mathscr{B}_z \times \mathscr{Z}$	106
$(\underline{\beta}_k^{**}, \underline{z}_k^*)$	the best Chebyshev k-th reference parameter vector pair in $\mathscr{B}_z \times \mathscr{Z}$	134
$\underline{\gamma}$	an arbitrary vector in E^{q-n}	92
$\underline{\delta}(\underline{r})$, $\underline{\delta}$	is a (q-n)-dimensional unknown real vector defined by Eq. 2.75 (or Eq. 2.76)	77
$\delta, \delta_1, \delta_2$	an arbitrary positive constant	111
$\epsilon(t)$, $\epsilon(t; \underline{\alpha}, \underline{s})$	the approximating error function	3
$\epsilon^*(t)$, $\epsilon_p^*(t; \underline{s})$, $\epsilon(t; \underline{\alpha}_p^*, \underline{s})$	the error function resulting from the best L_p-approximation with respect to $\underline{\alpha} \in \mathscr{A}_s$ for a fixed $\underline{s} \in \mathscr{S}$ (Definition 1.1)	5
$\epsilon_p^{**}(t)$	the error function resulting from the best L_p-approximation with respect to $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \mathscr{S}$ (Definition 1.2)	5

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
$\underline{\epsilon}$, $\underline{\epsilon}(\cdot)$, $\underline{\epsilon}(\cdot,\cdot)$	the q-dimensional real error vector, with the dots standing for the undesignated parameter vector	4
$\underline{\epsilon}^*(\underline{\cdot})$, $\underline{\epsilon}_p^*(\underline{\cdot})$	the real error vector resulting from the best ℓ_p^q-approximation with respect to one parameter	6
$\underline{\epsilon}_p^*(\underline{r})$, $\underline{\epsilon}^*(\underline{r})$	the best real error vector in U^q which satisfies Eq. 3.16 for a fixed $\underline{r} \in \mathscr{R}$	99
$\underline{\epsilon}^{**}$, $\underline{\epsilon}_p^{**}$	the real error vector resulting from the best ℓ_p^q-approximation with respect to $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \mathscr{S}$ (Definition 1.4)	6
$\underline{\epsilon}^*(\underline{r}^*)$	the error vector $\underline{\epsilon}^{**}$	101
$\underline{\epsilon}^{(k)}$, $\underline{\epsilon}^{(k)}(\cdot)$, $\underline{\epsilon}^{(k)}(\cdot,\cdot)$	the projection of the vectors in U^q onto the k-th reference subspace U_k^m, with the dots standing for the undesignated parameter vectors	53
$\underline{\epsilon}^{*(k)}(\underline{r})$	the best Chebyshev error vector in U_k^{2n+1} for a fixed $\underline{r} \in \mathscr{R}$	146
$\underline{\epsilon}^{*(k)}(\underline{r}_k^*)$, $\underline{\epsilon}^{(k)}(\underline{\beta}_k^{**}, \underline{r}_k^*)$, $\underline{\epsilon}^{**(k)}$	the best Chebyshev error vector in U_k^{2n+1} (see Definition 4.2)	139
ζ	a real constant	166
η	an arbitrary positive constant	111
$\tilde{\eta}$	the minimum value of $\lVert [F]\underline{r} \rVert_\infty$ defined by Eq. 4.38	121

Symbol	Description	Defined or first used on page
$\underline{\lambda}^{(k)}$	a real n+1 dimensional vector, defined by Eq. 2.42, which is orthogonal to the n-dimensional approximating subspace in U_k^{n+1}	54
$\lambda_{ij}^{(w)}(\underline{r})$	a nonzero element of the matrix $[\Lambda^{(w)}(\underline{r})]$	138
$\underline{\lambda}_j^{(k)}(\underline{r})$	the j-th column vector of the matrix $[\Lambda^{(k)}(\underline{r})]$	146
$[\Lambda^{(w)}(\underline{r})]$ $[\Lambda^{(k)}(\underline{r})]$	a (2n+1) x (n+1) matrix, defined by Eq. 4.59, which is a function of $\underline{r} \in \mathscr{R}$	138
$[\Lambda^{(\nu)}(\underline{r})]$	a 2nxn submatrix of $[\Lambda^{(k)}(\underline{r})]$	153
μ	an index	154
ν	the index denoting the set $\{1, 2, \ldots, 2n+1\}$	153
ξ_i	a fixed basis vector in U^q	47
ρ	a real parameter whose $\lvert\rho\rvert$ denotes the value of the components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ which are equal in absolute value	146
ρ_k	a real constant whose absolute denotes the absolute value of the component of the error vector $\underline{\epsilon}^{*(k)} \in U_k^{n+1}$ defined by Eq. 2.47	56
ρ_M	the value of ρ defined by Eq. 2.50	58
σ_k	the real part of s_k	10

LIST OF SYMBOLS (Cont.)

Symbol	Description	Defined or first used on page
$\underline{\sigma}^{(k)}$	an (n+1)-dimensional vector representing the sign configuration of the error vector $\underline{\epsilon}^{*(k)} \epsilon U_k^{n+1}$ defined by Eq. 2.46	55
$\underline{\sigma}^{(k)}$	a vector representing the prescribed sign configuration of the error vector $\underline{\epsilon}^{(k)} \epsilon U_k^{2n+1}$	146
$\underline{\sigma}^{(\mu)}$	the 2n dimensional vector representing a prescribed sign configuration of $\underline{\epsilon}^{(\mu)} \epsilon U_\mu^{2n}$	155
$\phi(\underline{x})$	a linear functional of $\underline{x}$	50
$\underline{\chi}$	an m-dimensional vector representing the sign configuration of $\underline{x} \epsilon E^m$ (Eq. 2.34)	50
ψ_k	the initial value of the exponential function $e^{s_k t}$, namely $e^{s_k t_1}$	18
$[\Psi]$	the nxn diagonal matrix whose elements are $\{\psi_1, \ldots, \psi_n\}$ defined by Eq. 1.20	18
ω_k	the imaginary part of s_k	10

LIST OF APPENDICES

ABSTRACT

This study considers the problem of approximating a prescribed impulse response function, h(t), at a finite number of equally-spaced discrete values of t, by a linear combination of exponential functions, such that the resulting error is minimum in the Chebyshev sense. Specifically, given a set of q values $\{h(t_i)\}$ of h(t), where $t_i = t_1 + (i-1)\Delta t$, for $i = 1,2,\ldots,q$; find the 2n complex constants $\{\alpha_k, s_k\}$, $k = 1,2,\ldots,n$, of the function $\hat{h}(t) = \sum_{k=1}^{n} \alpha_k e^{s_k t}$ so that the Chebyshev error

$$\|\epsilon(t_i)\|_\infty \triangleq \max_{1 < i < q} |h(t_i) - \hat{h}(t_i)|$$

is minimum when $q > 2n$.

The 2n complex constants $\{\alpha_k, s_k\}$ are limited to a set having the complex α_k's and s_k's occur in conjugate pairs. If the exponents $\{s_k\}$ are not distinct, then for each repeated exponent, s_k, the approximating function possesses terms of the form $\{e^{s_k t}, t\,e^{s_k t}, \ldots, t^{j-1} e^{s_k t}\}$, where j denotes the order of the repeated exponent,

$\{s_k\}$. The approximating impulse response function is such that its Laplace transform can be expressed in rational fraction form. If each exponent, s_k, of the approximating function, has a negative or zero real part, then this rational function can be realized as a linear, passive, lumped, bilateral R-L-C network.

The discrete time domain approximation problem is formulated in terms of an approximation in the finite dimensional vector space. Such a formulation depicts the relation between the exponents of the exponential functions and the orientation of the approximating subspace. Prony's method for determining the exponents, when $q = 2n$, is reviewed and is formally extended to the case when $q > 2n$. The formal extension of Prony's method is used to solve the Chebyshev approximation problem considered in this thesis.

A theorem guaranteeing the existence of the best Chebyshev approximation is proved. It is shown by means of examples, that the best Chebyshev approximation may not be unique. Several properties of the Chebyshev approximation are obtained, including the bounds within which the minimal value of $\|\epsilon(t_i)\|_\infty$ must lie. It is conjectured that, in general, the best Chebyshev approximation is characterized as that for which at least $(2n+1)$ values of $\{\epsilon(t_i)\}$ are equal in absolute value to $\|\epsilon(t_i)\|_\infty$. It is shown, however, that there are some special cases which do not possess this property.

Finally, a computational algorithm for solving the Chebyshev approximation problem is presented, along with some numerical examples. This algorithm determines the optimum α_k's and s_k's simultaneously.

CHAPTER I

INTRODUCTION

The time domain network synthesis problem consists of finding a practical network which yields the prescribed impulse response function. The solution to this problem is rarely an exact one because of the following limitations: (1) The resulting network must be physically realizable; and (2) it must employ only a finite number of elements. Thus, the synthesis problem is basically an approximation problem in which the physically realizable impulse response functions are the approximating functions. This study is concerned almost entirely with the approximation problem. Specifically, we shall be concerned with "discrete-approximations" where the prescribed impulse response function is approximated at a finite number of equally-spaced discrete values of the independent variable, time, by physically realizable impulse response functions.

In this chapter, after defining the notation used throughout this thesis, we shall present the two steps which are essential to the general approximation problem: (1) the selection of a class of approximating functions, and (2) the selection of a criterion which measures the degree of the approximation. Then we shall formulate the approximation problem and present the plan of this thesis.

1.1 Notation and Definitions

The prescribed impulse response function to be approximated is denoted by h(t), and the approximating function is denoted by $\hat{h}(t;\underline{\alpha},\underline{s})$. Both functions are defined for all t in the interval $\mathcal{T}$, where, if not otherwise mentioned, $\mathcal{T}$ denotes the semi-infinite interval $[0,\infty)$. The ordered pair of n-vectors $(\underline{\alpha},\underline{s})$ stand for the 2n parameters[1] of the approximating function $\hat{h}(t;\underline{\alpha},\underline{s})$. The approximating function $\hat{h}(t;\underline{\alpha},\underline{s})$ is selected from the set $\hat{\mathcal{H}} = \{\hat{h}(t;\underline{\alpha},\underline{s}) : t \in \mathcal{T}, \underline{\alpha} \in \mathcal{A}_s, \underline{s} \in \mathcal{S}\}$, where[2] $\mathcal{A}_s$ denotes the parameter space of the vector $\underline{\alpha}$ with respect to a fixed $\underline{s} \in \mathcal{S}$, and $\mathcal{S}$ denotes the parameter space of the vector $\underline{s}$. A precise definition of the spaces $\mathcal{A}_s$ and $\mathcal{S}$ is given in the next section.

At this point, it is noted that there are two different types of approximations available. One is called the "continuous-time" approximation in which we wish to approximate the prescribed function for all $t \in \mathcal{T}$; and the other is called the "discrete-time" approximation in which we wish to approximate the prescribed function only at a finite number of discrete values of $t \in \mathcal{T}$.

---

[1] For our purpose, it is necessary to denote the parameters of the approximating function by the two n-vectors $\underline{\alpha}$ and $\underline{s}$, because the 2n parameters are not independent. This point will be clarified in the next section, when the approximating function will be fully defined.

[2] The subscript s in $\mathcal{A}_s$ emphasizes the dependence of the parameter vector $\underline{\alpha}$ on the vector $\underline{s}$.

In the case of the "continuous-time" approximation, the approximation error function, at any time $t \in \mathscr{T}$, will be denoted by $\epsilon(t;\underline{\alpha},\underline{s})$, and defined by

$$\epsilon(t;\underline{\alpha},\underline{s}) \triangleq h(t) - \hat{h}(t;\underline{\alpha},\underline{s}) . \tag{1.1}$$

The criterion or measure of the degree of approximation will be given by the L_p-norm[3] of $\epsilon(t;\underline{\alpha},\underline{s})$, denoted by $\|\epsilon(t;\underline{\alpha},\underline{s})\|_p$, where $\|\epsilon(t;\underline{\alpha},\underline{s})\|_p$ is defined by[4]

$$\|\epsilon(t;\underline{\alpha},\underline{s})\|_p = \left[\int_{\mathscr{T}} |\epsilon(t;\underline{\alpha},\underline{s})|^p\right]^{1/p} \quad \text{for } 1 \leq p < \infty,$$

and

$$\|\epsilon(t;\underline{\alpha},\underline{s})\|_\infty \triangleq \operatorname*{essup}_{t \in \mathscr{T}} |\epsilon(t;\underline{\alpha},\underline{s})| .$$

It suffices to say, that in the case of the "continuous-time" approximation, we are assuming that the error function $\epsilon(t;\underline{\alpha},\underline{s})$ is in the L_p-space.[5] This approximation will be called the L_p-approximation.

In the case of the "discrete-time" approximation, we shall be concerned with the values of the functions $h(t)$ and $\hat{h}(t;\underline{\alpha},\underline{s})$ at a finite number of discrete values of $t \in \mathscr{T}$. We shall denote the discrete

---

[3] See Ref. 21, pp. 212-218.

[4] More accurately, the following integral should be a Lebesgue integral.

[5] See Footnote 3.

values of $t \in \mathcal{T}$ by the finite point set $T = \{t_i : i = 1,2,\ldots,q\}$, and represent the ordered set of values $\{h(t_1), h(t_2),\ldots,h(t_q)\}$ of h(t) and $\{\hat{h}(t_1;\underline{\alpha},\underline{s}), \hat{h}(t_2;\underline{\alpha},\underline{s}),\ldots,\hat{h}(t_q;\underline{\alpha},\underline{s})\}$ of $\hat{h}(t;\underline{\alpha},\underline{s})$, by the q-dimensional vectors $\underline{h}$ and $\underline{\hat{h}}(\underline{\alpha},\underline{s})$, respectively. Hence, in the "discrete-time" approximation, we shall be concerned with real vectors $\underline{h}$ and $\underline{\hat{h}}(\underline{\alpha},\underline{s})$ in the complex q-dimensional vector space[6] U^q. The approximation error vector will be denoted by $\underline{\epsilon}(\underline{\alpha},\underline{s})$ and defined by

$$\underline{\epsilon}(\underline{\alpha},\underline{s}) \triangleq \underline{h} - \underline{\hat{h}}(\underline{\alpha},\underline{s}) \tag{1.2}$$

Clearly, the vector $\underline{\epsilon}(\underline{\alpha},\underline{s})$ must be a real vector in U^q. The measure of the degree of approximation will be given by the ℓ_p^q-norm of the vector $\underline{\epsilon}(\underline{\alpha},\underline{s})$, denoted by $\|\underline{\epsilon}(\underline{\alpha},\underline{s})\|_p$, where $\|\underline{\epsilon}(\underline{\alpha},\underline{s})\|_p$ is given by

$$\|\underline{\epsilon}(\underline{\alpha},\underline{s})\|_p = \left[\sum_{i=1}^{q} |\epsilon_i(\underline{\alpha},\underline{s})|^p\right]^{1/p}, \quad \text{for } 1 \le p < \infty$$

and

$$\|\underline{\epsilon}(\underline{\alpha},\underline{s})\|_\infty = \max_{1 \le i \le q} |\epsilon_i(\underline{\alpha},\underline{s})|$$

where $\epsilon_i(\underline{\alpha},\underline{s})$ denotes $\epsilon(t_i;\underline{\alpha},\underline{s})$.

---

[6]The reason for being concerned with real vectors in U^q, rather than vectors in E^q, the real q-dimensional vector space, will become evident in Section 1.4, where we shall express the vector $\underline{\hat{h}}(\underline{\alpha},\underline{s})$ by a linear combination of complex vectors. Note that a real vector in U^q is a vector having only real components.

Note that in the case of the "discrete-time" approximation, we shall assume that the error vector, $\underline{\epsilon}(\underline{\alpha},\underline{s})$, is in the real ℓ_p^q-space. This approximation will be denoted by the ℓ_p^q-approximation.

Definition 1.1: The function $\hat{h}(t;\underline{\alpha}_p^*,\underline{s})$ is said to be the best L_p-approximating function of h(t), defined on $\mathcal{T}$, with respect to the parameter vector $\underline{\alpha} \in \mathcal{A}_s$ for a fixed $\underline{s} \in \mathcal{S}$, if it is selected from the class of functions $\{\hat{h}(t;\underline{\alpha},\underline{s}) : \underline{\alpha} \in \mathcal{A}_s\}$ and satisfies,

$$\|\epsilon_p^*(t;\underline{s})\|_p \triangleq \|h(t) - \hat{h}(t;\underline{\alpha}_p^*,\underline{s})\|_p \leq \|h(t) - \hat{h}(t;\underline{\alpha},\underline{s})\|_p$$

for all $(\underline{\alpha}, s) \in \mathcal{A}_s \times \underline{s}$. The function $\epsilon_p^*(t;\underline{s})$ is the resulting approximation error function, and $\underline{\alpha}_p^*$ is the corresponding "best" parameter vector in $\mathcal{A}_s$. For brevity, we shall denote the function $\hat{h}(t;\underline{\alpha}_p^*,\underline{s})$ by $h_p^*(t;\underline{s})$.

Definition 1.2: The function $\hat{h}(t;\underline{\alpha}_p^{**},\underline{s}_p^*)$ is said to be the best L_p-approximating function of h(t), defined on $\mathcal{T}$, with respect to the ordered parameter vector pair $(\underline{\alpha},\underline{s}) \in \mathcal{A}_s \times \mathcal{S}$, if it is selected from the class $\hat{\mathcal{H}} = \{\hat{h}(t;\underline{\alpha},\underline{s}) : \underline{\alpha} \in \mathcal{A}_s,\ \underline{s} \in \mathcal{S}\}$, and satisfies

$$\|\epsilon_p^{**}(t)\|_p \triangleq \|h(t) - \hat{h}(t;\underline{\alpha}_p^{**},\underline{s}_p^*)\|_p \leq \|h(t) - \hat{h}(t;\underline{\alpha},\underline{s})\|_p$$

for all $(\underline{\alpha},\underline{s}) \in \mathcal{A}_s \times \mathcal{S}$.

The function $\epsilon_p^{**}(t)$ is the resulting approximation error function, and $(\underline{\alpha}_p^{**},\underline{s}_p^*)$ is the corresponding "best" ordered vector pair in $\mathcal{A}_s \times \mathcal{S}$. For brevity, the function $\hat{h}(t;\underline{\alpha}_p^{**},\underline{s}_p^*)$ will be denoted by $h_p^{**}(t)$.

Definition 1.3: The real vector $\hat{\underline{h}}(\underline{\alpha}_p{}^*, \underline{s})$ is said to be the best ℓ_p^q-approximating vector of a real vector $\underline{h}$ in U^q with respect to the parameter vector $\underline{\alpha} \in \mathscr{A}_s$ for a fixed vector $\underline{s} \in \mathscr{S}$, if it is selected from the set of vectors $\{\hat{\underline{h}}(\underline{\alpha}, \underline{s}) : \underline{\alpha} \in \mathscr{A}_s\}$ and satisfies

$$\|\underline{\epsilon}_p{}^*(\underline{s})\|_p \triangleq \|\underline{h} - \hat{\underline{h}}(\underline{\alpha}_p{}^*, \underline{s})\|_p \leq \|\underline{h} - \hat{\underline{h}}(\underline{\alpha}, \underline{s})\|_p$$

for all $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \underline{s}$.

The vector $\underline{\epsilon}_p{}^*(\underline{s})$ is the resulting approximation error vector, and $\underline{\alpha}_p{}^*$ is the corresponding "best" parameter vector in $\mathscr{A}_s$. For brevity, the vector $\hat{\underline{h}}(\underline{\alpha}_p{}^*, \underline{s})$ will be denoted by $\underline{h}_p{}^*(\underline{s})$.

Definition 1.4: The real vector $\hat{\underline{h}}(\underline{\alpha}_p{}^{**}, \underline{s}_p{}^*)$ is said to be the best ℓ_p^q-approximating vector of the real vector $\underline{h} \in U^q$ with respect to the ordered parameter vector pair $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \mathscr{S}$, if it is selected from the set $\{\hat{\underline{h}}(\underline{\alpha}, \underline{s}) : \underline{\alpha} \in \mathscr{A}_s, \underline{s} \in \mathscr{S}\}$ and satisfies

$$\|\underline{\epsilon}_p{}^{**}\|_p = \|\underline{h} - \hat{\underline{h}}(\underline{\alpha}_p{}^{**}, \underline{s}_p{}^{**})\|_p \leq \|\underline{h} - \hat{\underline{h}}(\underline{\alpha}, \underline{s})\|_p$$

for all $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \mathscr{S}$.

The vector $\underline{\epsilon}_p{}^{**}$ is the resulting approximation error vector and $(\underline{\alpha}_p{}^{**}, \underline{s}_p{}^*)$ is the corresponding "best" order vector pair in $\mathscr{A}_s \times \mathscr{S}$.

For brevity, the vector $\hat{\underline{h}}(\underline{\alpha}_p{}^{**}, \underline{s}_p{}^*)$ will be denoted by $\underline{h}_p{}^{**}$.

Remark: From these definitions, it is clear that in general the values of the ordered pair of parameter vectors $(\underline{\alpha}_p{**}, \underline{s}_p{*})$ change as p is varied, a point which will be further clarified later. Furthermore, it should be mentioned that although we have used the same symbols $(\underline{\alpha}_p{**}, \underline{s}_p{*})$ to denote the best ordered vector pair corresponding to the best L_p-approximation of h(t) and the best ordered vector pair corresponding to the best ℓ_p^q-approximation of $\underline{h}$, the values of these two ordered vector pairs will usually differ!

In closing, it should be mentioned that to simplify our notation, we shall drop the subscript p in $\underline{\alpha}_p{**}$, $\underline{s}_p{*}$, $h_p{*}(t)$, $\epsilon_p{*}(t)$, etc., whenever there is no danger of ambiguity.

1.2 Selection of the Approximating Function, Exponential Representation

The first major problem encountered in obtaining an efficient network with an impulse response approximating the prescribed impulse response is one of selecting the class of approximating functions. In the network synthesis problem, the consideration of physical realizability enters at this point. The requirement that the transfer function of a physically realizable linear, passive, lumped, finite network be a rational function of polynomials in the complex frequency, s, is well-known (Ref. 10). The coefficients of these polynomials must be real; all roots of the denominator must have negative or zero real parts; and the roots having zero real parts must be simple.

Translating these requirements into the time domain shows that for physical realizability, it is sufficient that the approximating impulse response functions have the form

$$\sum_{k=1}^{n} \alpha_k \, e^{s_k t} ,$$

where α_k and s_k occur in conjugate pairs and each s_k must have a negative or zero real part.[7] Although this expression implies that the transfer function possesses only first order poles,[8] it should be noted that if the transfer function contains higher order poles,[9] then the approximating impulse response function must possess terms of the form $\{e^{st}, t\, e^{st}, \ldots, t^{(j-1)} e^{st}\}$, where j denotes the order of the repeated root.

It is convenient to define the approximating functions $\hat{h}(t;\underline{\alpha},\underline{s})$ by

$$\hat{h}(t;\underline{\alpha},\underline{s}) \triangleq \sum_{k=1}^{n} \alpha_k \, e^{s_k t} \tag{1.3}$$

---

[7] Clearly, the s_k's represent the pole locations and the α_k's represent the respective residues of the transfer function in the complex frequency domain.

[8] That is, the roots of the denominator, of the rational function of polynomials, are simple.

[9] Recall that the poles having zero real parts must be simple.

where the collection of the parameters $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{s_1, s_2, \ldots, s_n\}$ of the approximating function $\hat{h}(t; \underline{\alpha}, \underline{s})$ are denoted by the vectors $\underline{\alpha}$ and $\underline{s}$, respectively; that is

$$\underline{\alpha} \triangleq \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \tag{1.4}$$

and

$$\underline{s} \triangleq \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{bmatrix} \tag{1.5}$$

The number of parameters $\{\alpha_k\}$ or $\{s_k\}$ of approximating function is denoted by n.

It should be noted that in this thesis, the approximating function is not restricted to the form given by Eq. 1.3. In otherwords, if the procedure yields repeated values of s_k, then for each repeated s_k, the function $e^{s_k t}$ must be replaced by the set of functions $\left\{ e^{s_k t}, t\, e^{s_k t}, \ldots, t^{j-1}\, e^{s_k t} \right\}$, where j denotes the order of the repeated s_k. For example, if $s_1 = s_2 = \ldots = s_j$, then Eq. 1.3 is written as

$$\hat{h}(t;\underline{\alpha},\underline{s}) = (\alpha_1 + a_2 t + \ldots + \alpha_j t^{j-1}) e^{s_1 t} + \sum_{k=j+1}^{n} \alpha_k e^{s_k t} \tag{1.3a}$$

The set in which the parameter vector $\underline{s}$ lies is denoted by $\mathscr{S}$ and is defined as follows:

Definition 1.5: The set $\mathscr{S}$, of the parameter vector $\underline{s}$, is a set of all vectors $\underline{s} \in U^n$, the n-dimensional unitary space, with complex components occurring in conjugate pairs, i.e., for each $s_k = \sigma_k + j\omega_k$ there exists an $s_{k+1} = \sigma_k - j\omega_k$.

The set in which the parameter vector $\underline{\alpha}$ lies is denoted by $\mathscr{A}_s$ and is defined as follows:

Definition 1.6: The set $\mathscr{A}_s$ of the parameter vector $\underline{\alpha}$, is a subspace of the n-dimensional unitary space U^n which contains all vectors $\underline{\alpha} \in U^n$ such that when $\underline{s} \in \mathscr{S}$, then the function

$$\hat{h}(t;\underline{\alpha},\underline{s}) = \sum_{k=1}^{n} \alpha_k e^{s_k t}$$

is a real function of t.

From the above definition, it is seen that the set $\mathscr{A}_s$ depends on the vector $\underline{s} \in \mathscr{S}$. Hence, it is convenient to denote the 2n parameters of the approximating function $\hat{h}(t;\underline{\alpha},\underline{s})$ by the ordered vector pair $(\underline{\alpha},\underline{s})$ and the set in which the ordered pair $(\underline{\alpha},\underline{s})$ lies by $\mathscr{A}_s \times \mathscr{S}$.

At this point, it is noted that in defining the set $\mathscr{S}$ (Definition 1. 5), we have omitted the condition which guarantees that the approximating function $\hat{h}(t;\underline{\alpha},\underline{s})$ represents an impulse response function of a stable network, i. e. , the condition that $\mathrm{Re}\{s_k\} \leq 0$, for all $k = 1,2,\ldots,n$. Hence, the approximation problem in this thesis will not be concerned with the physical realizability of the resulting approximation.[10]

It should be mentioned that in the case of the ℓ_p^q-approximation, the real approximating vector $\hat{h}(\underline{\alpha},\underline{s}) \in U^q$ is taken to be the ordered set of values $\{\hat{h}(t_1;\underline{\alpha},\underline{s}), \hat{h}(t_2;\underline{\alpha},\underline{s}),\ldots,\hat{h}(t_q;\underline{\alpha},\underline{s})\}$ of $\hat{h}(t;\underline{\alpha},\underline{s})$ defined by Eq. 1. 3 (or Eq. 1. 3a). The set in which the ordered pair of vectors $(\underline{\alpha},\underline{s})$ lies is again the set $\mathscr{A}_s \times \mathscr{S}$, where the sets $\mathscr{S}$ and $\mathscr{A}_s$ are given by Definitions 1. 5 and 1. 6, respectively.

One further observation about the selection of the approximating function is that the complexity of the resulting network is directly proportional to n, the number of poles, s_k.

In summary, then, we shall be concerned with approximating functions selected from the class $\hat{\mathscr{H}} = \{\hat{h}(t;\underline{\alpha},\underline{s}) : \underline{\alpha} \in \mathscr{A}_s, \underline{s} \in \mathscr{S}\}$, where $\hat{h}(t;\underline{\alpha},\underline{s})$ is defined by Eq. 1. 3 (or Eq. 1. 3a).

---

[10]This point is discussed in Chapter VI.

1.3 Selection of the Criterion of Approximation

The second major step toward obtaining an efficient approximation is the definition of a measure of approximation. This measure is generally expressed by the norm of the error function. The two most widely used measures of approximation in the field of network synthesis are the least-squares, and the Chebyshev (or the uniform-norm) criteria.

The least-squares criterion is generally used when the specified data are known to contain random errors. In the case of the "continuous-time" approximation, the least-squares criterion is represented by the L_2-norm of the error function; that is,

$$\begin{aligned} \|\epsilon(t;\underline{\alpha},\underline{s})\|_2 &= \left[\int_{\mathscr{T}} |\epsilon(t;\underline{\alpha},\underline{s})|^2 \, dt\right]^{1/2} \\ &= \left[\int_{\mathscr{T}} |h(t) - \hat{h}(t;\underline{\alpha},\underline{s})|^2 \, dt\right]^{1/2} \end{aligned} \tag{1.6}$$

In the case of the "discrete-time" approximation, the least-squares criterion is represented by the ℓ_2^q-norm of the error vector, that is,[11]

$$\begin{aligned} \|\underline{\epsilon}(\underline{\alpha},\underline{s})\|_2 &= (\underline{\epsilon}(\underline{\alpha},\underline{s}),\ \underline{\epsilon}(\underline{\alpha},\underline{s}))^{1/2} \\ &= ([\underline{h} - \underline{\hat{h}}(\underline{\alpha},\underline{s})],\ [\underline{h} - \underline{\hat{h}}(\underline{\alpha},\underline{s})])^{1/2} \end{aligned} \tag{1.7}$$

---

[11]We use the inner product relation, $(\underline{\epsilon},\underline{\epsilon})$, to denote $\sum_{i=1}^{q} |\epsilon_i|^2$, (see Ref. 21, p. 245).

The Chebyshev criterion is generally used when a point by point replica of the specified data is desired. In the case of the "continuous-time" approximation, the Chebyshev criterion is represented by the L_∞-norm of the error function; that is,

$$\|\epsilon(t;\underline{\alpha},\underline{s})\|_\infty = \operatorname*{essup}_{t\in\mathcal{T}} |\epsilon(t;\underline{\alpha},\underline{s})|$$

$$= \operatorname*{essup}_{t\in\mathcal{T}} |h(t) - \hat{h}(t;\underline{\alpha},\underline{s})| \ . \tag{1.8}$$

In the case of the "discrete-time" approximation, the Chebyshev criterion is represented by the ℓ^q_∞-norm of the error vector; that is,

$$\|\underline{\epsilon}(\underline{\alpha},\underline{s})\|_\infty = \max_{1\le i\le q} |\epsilon_i(\underline{\alpha},\underline{s})|$$

$$= \max_{1\le i\le q} |h_i - \hat{h}_i(\underline{\alpha},\underline{s})| \ , \tag{1.9}$$

where $\epsilon_i(\underline{\alpha},\underline{s})$, h_i, and $\hat{h}_i(\underline{\alpha},\underline{s})$ denote the values of $\epsilon(t_i;\underline{\alpha},\underline{s})$, $h(t_i;\underline{\alpha},\underline{s})$, and $\hat{h}(t_i;\underline{\alpha}, s)$, respectively.

In this thesis, we shall be principally concerned with the best Chebyshev approximation of a real vector $\underline{h} \in U^q$, and so we shall use the measure of approximation defined by Eq. 1.19.

1.4 Formulation of the Problem

The problem of approximating a prescribed impulse response function h(t) by a linear combination of exponential functions, $\hat{h}(t;\underline{\alpha},\underline{s}) = \sum_{k=1}^{n} \alpha_k e^{s_k t}$, at a finite number of discrete values of t, may be formulated as follows: As mentioned before, the ordered set of real values $\{h(t_1), h(t_2), \ldots, h(t_q)\}$ of h(t), and $\{\hat{h}(t_1; \underline{\alpha}, \underline{s}), \hat{h}(t_2; \underline{\alpha}, \underline{s}), \ldots, \hat{h}(t_q; \underline{\alpha}, \underline{s})\}$ of $\hat{h}(t; \underline{\alpha}, \underline{s})$ are represented by q-dimensional real vectors in U^q; that is,

$$\underline{h} = \begin{bmatrix} h(t_1) \\ h(t_2) \\ \cdot \\ \cdot \\ \cdot \\ h(t_q) \end{bmatrix} ; \quad \text{and } \underline{\hat{h}}(\underline{\alpha}, \underline{s}) = \begin{bmatrix} \hat{h}(t_1; \underline{\alpha}, \underline{s}) \\ \hat{h}(t_2; \underline{\alpha}, \underline{s}) \\ \cdot \\ \cdot \\ \cdot \\ \hat{h}(t_q; \underline{\alpha}, \underline{s}) \end{bmatrix} \tag{1.10}$$

The ordered sets of values $\{e^{s_k t_1}, e^{s_k t_2}, \ldots, e^{s_k t_q}\}$ of the exponential function $e^{s_k t}$, are represented by vectors in U^q; that is,

$$\underline{e}_k = \begin{bmatrix} e^{s_k t_1} \\ e^{s_k t_2} \\ \cdot \\ \cdot \\ \cdot \\ e^{s_k t_q} \end{bmatrix} \tag{1.11}$$

where $k = 1, 2, \ldots, n$, and $q > 2n$. It is evident that if the exponents s_k's are complex, then the vectors, $\underline{e}_k$, are complex vectors in U^q. Furthermore, if the s_k's are distinct, then the vectors, $\underline{e}_k$, $k = 1, 2, \ldots, n$, are independent and form a basis of a complex n-dimensional subspace[12] $V^n(\underline{s})$ of U^q. A simple representation of the basis of $V^n(\underline{s})$ is given by the column space of a q x n matrix $[E(\underline{s})]$ defined by

$$[E(\underline{s})] \triangleq \left[\underline{e}_1\, \underline{e}_2 \cdots \underline{e}_n\right] \qquad (1.12)$$

where $[E(\underline{s})]$ is of rank n if the set $\{\underline{e}_k\}$ is an independent set of vectors. Therefore, any real vector $\underline{\hat{h}}(\underline{\alpha}, \underline{s})$ in $V^n(\underline{s})$ can be represented by

$$\underline{\hat{h}}(\underline{\alpha}, \underline{s}) = \sum_{k=1}^{n} \alpha_k\, \underline{e}_k \qquad (1.13)$$

or, alternately, by

$$\underline{\hat{h}}(\underline{\alpha}, \underline{s}) = [E(\underline{s})]\, \underline{\alpha} \qquad (1.14)$$

where $(\underline{\alpha}, \underline{s}) \in \mathcal{A}_s \times \mathcal{S}$.

It should be noted that if the approximating function $\hat{h}(t; \underline{\alpha}, \underline{s})$ is given by Eq. 1.3a (i.e., the case in which $s_1 = s_2 = \ldots = s_j$), then the vector $\underline{\hat{h}}(\underline{\alpha}, \underline{s})$ is given by

$$\underline{\hat{h}}(\underline{\alpha}, \underline{s}) = \left(\alpha_1 + \alpha_2[T] + \ldots + \alpha_j[T]^{j-1}\right) \underline{e}_1 + \sum_{k=j+1}^{n} \alpha_k\, \underline{e}_k\,, \qquad (1.13a)$$

---

[12] The notation $V^n(\underline{s})$ emphasizes the fact that this subspace is a function of the parameter vector $\underline{s} \in \mathcal{S}$, where $\underline{s}$ represents the ordered set of values $\{s_k\}$, $k = 1, \ldots, n$.

where $[T]$ is a $q \times q$-diagonal matrix defined by

$$[T] = \begin{bmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_q \end{bmatrix}$$

Clearly, the vector $\hat{\underline{h}}(\underline{\alpha}, \underline{s})$, given by Eq. 1.13a, can be represented by Eq. 1.14, where now the matrix $[E(\underline{s})]$ is given by

$$[E(\underline{s})] = \begin{bmatrix} e_{11} & t_1 e_{11} & \cdots & t_1^{j-1} e_{11} & e_{1,j+1} & \cdots & e_{1n} \\ e_{21} & t_2 e_{21} & \cdots & t_2^{j-1} e_{21} & e_{2,j+1} & \cdots & e_{2n} \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ e_{q1} & t_q e_{q1} & \cdots & t_q^{j-1} e_{q1} & e_{q,j+1} & \cdots & e_{qn} \end{bmatrix} \qquad (1.12a)$$

where e_{ik} represents the i-th component of the vector $\underline{e}_k$, defined by Eq. 1.11. Since we have assumed that the vectors $\{\underline{e}_{j+1}, \ldots, \underline{e}_n\}$ are independent, the matrix $[E(\underline{s})]$, defined by Eq. 1.12a, is of maximal rank n. Hence, in this thesis, the approximating subspace, spanned by the column space of $[E(\underline{s})]$, is n-dimensional for all $\underline{s} \in \mathcal{S}$.

When the real vector $\underline{h}$ in U^q is not in $V^n(\underline{s})$, then it is related to $\hat{\underline{h}}(\underline{\alpha},\underline{s})$ in $V^n(\underline{s})$ by

$$\underline{h} = \hat{\underline{h}}(\underline{\alpha},\underline{s}) + \underline{\epsilon}(\underline{\alpha},\underline{s}) \tag{1.15}$$

where $\underline{\epsilon}(\underline{\alpha},\underline{s})$ is some nonzero real vector in U^q. The real vector $\underline{\epsilon}(\underline{\alpha},\underline{s})$ is the error vector, represented by the ordered set of values $\{\epsilon(t_i;\underline{\alpha},\underline{s}) : i = 1,2,\ldots,q\}$, of the error function $\epsilon(t;\underline{\alpha},\underline{s})$.

Let us suppose that the discrete values of t are equally-spaced at intervals Δt, so that $t_i = t_1 + (i-1)\Delta t$. Then, each vector $\underline{e}_k$ can be written as:[13]

$$\underline{e}_k \triangleq \psi_k \underline{z}_k = \psi_k \begin{bmatrix} 1 \\ z_k \\ z_k^2 \\ \cdot \\ \cdot \\ \cdot \\ z_k^{q-1} \end{bmatrix} \tag{1.16}$$

[13] This restriction to equally-spaced sampling points will be discussed in Section 1. 5.

where $\psi_k = e^{s_k t_1}$ and $z_k = e^{s_k \Delta t}$. Furthermore, the matrix $[E(\underline{s})]$, defined by Eq. 1.12, may be written as

$$[E(\underline{s})] = [Z(\underline{z})] [\Psi] \tag{1.17}$$

where

$$[Z(\underline{z})] \triangleq \begin{bmatrix} 1 & \cdots & 1 \\ z_1 & & z_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ z_1^{q-1} & \cdots & z_n^{q-1} \end{bmatrix}, \tag{1.18}$$

$$\underline{z} \triangleq \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix} \tag{1.19}$$

and

$$[\Psi] \triangleq \begin{bmatrix} \psi_1 & & 0 \\ & \ddots & \\ 0 & & \psi_n \end{bmatrix} \tag{1.20}$$

In the case in which $[E(\underline{s})]$ is given by Eq. 1.12a, the matrix $[Z(\underline{z})]$ is defined by

$$[Z(\underline{z})] \triangleq \begin{bmatrix} 1 & \cdots & t_1^{j-1} & 1 & \cdots & 1 \\ z_1 & \cdots & [t_1+\Delta t]^{j-1} z_1 & z_{j+1} & \cdots & z_n \\ z_1^2 & \cdots & [t_1+2\Delta t]^{j-1} z_1^2 & z_{j+1}^2 & \cdots & z_n^2 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ z_1^{q-1} & \cdots & [t_1+(q-1)\Delta t]^{j-1} z_1^{q-1} & z_{j+1}^{q-1} & \cdots & z_n^{q-1} \end{bmatrix} \qquad (1.18a)$$

Note that if $t_1 = 0$, the matrix $[Z(\underline{z})]$ of Eq. 1.18a can be simplified as follows

$$[Z(\underline{z})] \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ z_1 & z_1 & \cdots & z_1 & z_{j+1} & \cdots & z_n \\ z_1^2 & 2z_1^2 & \cdots & 2^{j-1}z_1^2 & z_{j+1}^2 & \cdots & z_n^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ z_1^{q-1} & (q-1)z_1^{q-1} & \cdots & (q-1)^{j-1}z_1^{q-1} & z_{j+1}^{q-1} & \cdots & z_n^{q-1} \end{bmatrix} \quad (1.18b)$$

It should be noted that the matrix $[Z(\underline{z})]$ defined by Eq. 1.18a (or Eq. 1.18b) gives the form of the matrix $[Z(\underline{z})]$ when the first j-components of the vector $\underline{z}$ are identical, i.e., $z_1 = z_2 = \ldots = z_j$. Clearly, one can obtain, in a similar manner, a matrix $[Z(\underline{z})]$ for any other vector $\underline{z}$ with components that are not distinct. Rather than obtaining the general form of the matrix $[Z(\underline{z})]$, we shall say that when the components of the vector $\underline{z}$ are not distinct, then the matrix $[Z(\underline{z})]$ is defined by Eq. 1.18a.

At this point, it is convenient to replace the parameter vector pair $(\underline{\alpha}, \underline{s})$ of the approximating vector by another parameter vector pair, denoted by $(\underline{\beta}, \underline{z})$; thus, $\hat{\underline{h}}(\underline{\alpha}, \underline{s})$, defined by Eq. 1.14, will be denoted by $\hat{\underline{h}}(\underline{\beta}, \underline{z})$ and defined by

$$\underline{\hat{h}}(\underline{\beta},\underline{z}) \triangleq [Z(\underline{z})]\,\underline{\beta} \tag{1.21}$$

where

$$\underline{\beta} \triangleq [\Psi]\,\underline{\alpha}\,, \tag{1.22}$$

and where the q x n matrix $[Z(\underline{z})]$ is defined by Eq. 1. 18 (Eq. 1. 18a). The set in which the parameter vector $\underline{z}$ lies is denoted by $\mathcal{Z}$ and defined by:

Definition 1.7: The set $\mathcal{Z}$, of the parameter vector $\underline{z}$, is a set of all vectors $\underline{z} \in U^n$, the n-dimensional unitary space, with complex components occurring in conjugate pairs, i. e., for each complex z_k there exists a $z_j = \bar{z}_k$, $j \neq k$.

Remark: It should be noted that the (q x n) matrix $[Z(\underline{z})]$, defined by Eq. 1. 18 (Eq. 1. 18a), is of maximal rank n, for all $\underline{z} \in \mathcal{Z}$.

The set in which the parameter vector $\underline{\beta}$ lies is denoted by $\mathcal{B}_z$ and defined by:

Definition 1.8: The set $\mathcal{B}_z$ of the parameter vector $\underline{\beta}$ is a subspace of the n-dimensional unitary space U^n which contains all vectors $\underline{\beta} \in U^n$ so that if $\underline{z} \in \mathcal{Z}$, then the vector

$$\underline{\hat{h}}(\underline{\beta},\underline{z}) = [Z(\underline{z})]\,\underline{\beta}$$

is a real vector in U^q, where $[Z(\underline{z})]$ is the (q x n) matrix defined by Eq. 1.18 (Eq. 1.18a).

When the parameters of the approximating vector are given by the vector pair $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$, then the error vector is denoted by $\underline{\epsilon}(\underline{\beta},\underline{z})$. Furthermore, the vector relation of Eq. 1.15 becomes

$$\begin{aligned} \underline{h} &= \underline{\hat{h}}(\underline{\beta},\underline{z}) + \underline{\epsilon}(\underline{\beta},\underline{z}) \\ &= [Z(\underline{z})]\,\underline{\beta} + \underline{\epsilon}(\underline{\beta},\underline{z}) \, , \end{aligned} \tag{1.23}$$

where $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$.

1.5 Statement of the Problem

This thesis will be concerned with the problem of approximating a prescribed impulse response function, h(t), at a finite number of equally-spaced discrete values of t, by a linear combination of exponential functions, so that the resulting error is minimum in the Chebyshev sense. Specifically, we shall be concerned with the following approximation problem:

Given a real valued function h(t) defined on the interval $[t_1, t_q]$, select the function $\hat{h}(t;\underline{\alpha}^{**},\underline{s}^{*})$ from the class of functions $\hat{\mathscr{H}} = \{\hat{h}(t;\underline{\alpha},\underline{s}) : \underline{\alpha} \in \mathscr{A}_s, \ \underline{s} \in \mathscr{S}\}$ so that if

$t \in T_e \triangleq \{t_i = t_1 + (i-1)\Delta t : i = 1,2,\ldots,q, \ q > 2n, \ \Delta t = [(t_q - t_1)/(q-1)]\}$,

then

$$\max_{1 \leq i \leq q} |h(t_i) - \hat{h}(t_i;\underline{\alpha}^{**},\underline{s}^{*})| \leq \max_{1 \leq i \leq q} |h(t_i) - \hat{h}(t_i;\underline{\alpha},\underline{s})| \qquad (1.24)$$

for all $(\underline{\alpha},\underline{s}) \in \mathscr{A}_s \times \mathscr{S}$.

For the purpose of our investigation, this approximation problem is restated as follows:

Given a real vector $\underline{h} \in U^q$, find an ordered pair of vectors[14] $(\underline{\beta}^{**},\underline{z}^{*}) \in \mathscr{B}_z \times \mathscr{Z}$ so that if $\underline{h}^{**} \triangleq [Z(\underline{z}^{*})]\,\underline{\beta}^{**}$, then

$$\|\underline{h} - \underline{h}^{**}\|_\infty \leq \|\underline{h} - [Z(\underline{z})]\,\underline{\beta}\|_\infty \qquad (1.25)$$

for all $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$, and where $[Z(\underline{z})]$ in the $q \times n$ matrix is defined by Eq. 1.18 (Eq. 1.18a).

Some of the limitations imposed on our time domain approximation problem of network synthesis by this precise mathematical formulation are:

(1) The q values $\{h(t_i)\}$ of the prescribed impulse response function, $h(t)$, must be bounded; i.e.,

$$\max_{1 \leq i \leq q} |h(t_i)| < \infty .$$

(2) The form of the matrix $[Z]$ dictates that the q values $\{h(t_i)\}$ of $h(t)$ be equally spaced at intervals Δt, so that $t_i = t_1 + (i-1)\Delta t$.

---

[14] The ordered vector pair $(\underline{\beta}^{**},\underline{z}^{*}) \in \mathscr{B}_Z \times \mathscr{Z}$ corresponds to the optimum vector pair $(\underline{\alpha}^{**},\underline{s}^{*}) \in \mathscr{A}_s \times \mathscr{S}$.

(3) There is no guarantee that the approximating function $\hat{h}(t; \underline{\alpha}^{**}, \underline{s}^{*})$ corresponding to the optimum vector pair $(\underline{\beta}^{**}, \underline{z}^{*}) \in \mathscr{B}_z \times \mathscr{Z}$ will yield a network which is physically realizable, even though the prescribed impulse response function h(t) satisfies

$$\int_0^\infty |h(t)| \, dt < \infty .$$

(4) There is no control on the behavior of the error function, $\epsilon(t) \triangleq h(t) - h^{**}(t)$, for values of $t \notin T_e$, since the approximation is performed only at the values of t in the finite equally-spaced point set T_e.

1.6 Plan of the Thesis

To clarify the role which the theory of approximation is playing in many current investigations in the theory of network synthesis, Chapter II presents a brief summary of the theory of approximation in the language of linear spaces, giving special attention to the singular mappings involved. Furthermore, the previous attempts to apply the theory of approximation to the problem of network synthesis are reviewed. A thorough review of Prony's original work (Ref. 15), and of Ruston's method (Ref. 20) using the Chebyshev norm criterion, are also presented in this setting.

Chapter III presents the theory of extending Prony's original work for solving exponential approximation problems in the ℓ_p^q-space. The theory which leads to the solution of the Chebyshev approximation problem, defined by Eq. 1.24, is presented in Chapter IV. Special attention is given to the existence theorem and some special properties of the solution are considered. The computational methods leading to the solution of this approximation problem are given in Chapter V. This chapter also contains iterative procedures and illustrative examples which are worked out in detail.

In Chapter VI we apply the theory presented in the previous two chapters to the network synthesis problem. Procedure and illustrative examples are also given.

A general discussion of the results with recommendations for further study is presented in Chapter VII. In the Appendix we present, in the language of vector spaces, Stiefel's algorithm (see Ref. 22) for finding the best Chebyshev solution to an over-determined system of equations.

CHAPTER II

STATE-OF-THE-ART

2.1 Introduction

The theory of approximation in normed-linear spaces has been extensively studied by mathematicians for many years as is shown by the many excellent papers and books written on this subject (Refs. 2, 17, 18, 19, 22). In this chapter we shall first attempt to give the reader an intuitive feeling for this subject, so that he may be able to frame the whole problem more clearly. Then, we shall present the previous contributions to the time domain approximation problem of network synthesis. Specifically, we shall review the following contributions:

(1) Approximation techniques in the L_p-spaces; namely, the works of Aigrain and Williams (Ref. 1), Kautz (Ref. 10), and McDonough (Ref. 13), for $p = 2$; and the work of Tang (Ref. 23) for $p = \infty$.

(2) Approximation techniques in the ℓ_p^q-spaces; namely, the works of Yengst (Ref. 26) for $p = 2$; and the work of Ruston (Ref. 20) for $p = \infty$.

Let us begin by showing that the approximation problem in the L_p-space can be looked upon as an "L_p-projection," or rather as a singular transformation, which maps a point (or a function) in an L_p-space into some point in the approximating subspace of the L_p-space, so that the distance between these points is minimum in some sense. To illustrate this, we can use the finite dimensional ℓ_p^q-space, since many concepts of the approximation problem in the L_p-spaces can be visualized in finite dimensional ℓ_p^q-spaces. Recall that any vector in ℓ_p^q may be represented by a linear combination of a complete set of q basis vectors, where q denotes the dimension of the space (Ref. 21, Section 43). Furthermore, any n independent vectors in ℓ_p^q, where $n < q$, span an n-dimensional subspace V^n of ℓ_p^q.

We can now state the approximation problem in terms of the "ℓ_p-projection" problem as follows: Let us suppose that a vector $\underline{f}$ and a linear subspace V^n are given in ℓ_p^q. Then the vector $\underline{f}^*$ in V^n which best approximates $\underline{f}$ in ℓ_p^q with respect to the appropriate ℓ_p-norm, is the "ℓ_p-projection" of $\underline{f}$ onto V^n, when $p \geq 1$.

The "ℓ_p-projection" defined here is a generalization of the familiar orthogonal projection. The best way of illustrating this is to recall that the familiar orthogonal projection, which represents the best approximation of $\underline{f}$ in ℓ_2^q onto V^n with respect to the ℓ_2-norm, (i. e., the best least-squares approximation of $\underline{f}$ onto V^n in ℓ_2^q) can be obtained by placing a q-dimensional sphere at $\underline{f}$ and expanding it until it touches V^n. The point at which this hypersphere touches V^n is the orthogonal projection of $\underline{f}$ onto V^n. Furthermore, the radius of this sphere is the ℓ_2-norm of the error vector, i.e., $\|\underline{\epsilon}\|_2$. When $p \geq 1$, we can, in a similar way, define the ℓ_p-projection of $\underline{f}$ onto V^n as the point where the smallest q-dimensional convex body described by $\|\underline{\epsilon}\|_p$ and centered at $\underline{f}$ touches the subspace V^n. This ℓ_p-convex body of radius $\|\underline{\epsilon}\|_p$ and center $\underline{f}$, denoted by $K_p(\underline{f}, \|\underline{\epsilon}\|_p)$, is defined by

$$K_p(\underline{f}, \|\underline{\epsilon}\|_p) = \{\underline{g} : \|\underline{f}-\underline{g}\|_p \leq \|\underline{\epsilon}\|_p, \ \underline{g} \text{ in } \ell_p^q\} , \quad p \geq 1 . \qquad (2.1)$$

A simple illustration of the various shapes of K_p, for $p = 1, 2$, and ∞ is given in Fig. 1. In the Chebyshev approximation problem, the ℓ_∞-projection of $\underline{f}$ in ℓ_∞^q onto V^n is the point where the smallest q-dimensional cube, centered at $\underline{f}$ and having edges of length $2\|\underline{\epsilon}\|_\infty$, touches the subspace V^n.

A similar "projection" problem exists when considering the approximation problem in the L_p-normed linear space, where in the L_p-space a function represents a point in the L_p-space analogous to the point in ℓ_p^q-space described by a vector. Here, however, the "L_p-projection" is more difficult to visualize since both the n-dimensional approximating subspace V^n and the L_p-convex body are defined in terms of continuous functions in the L_p-space. Recall that any

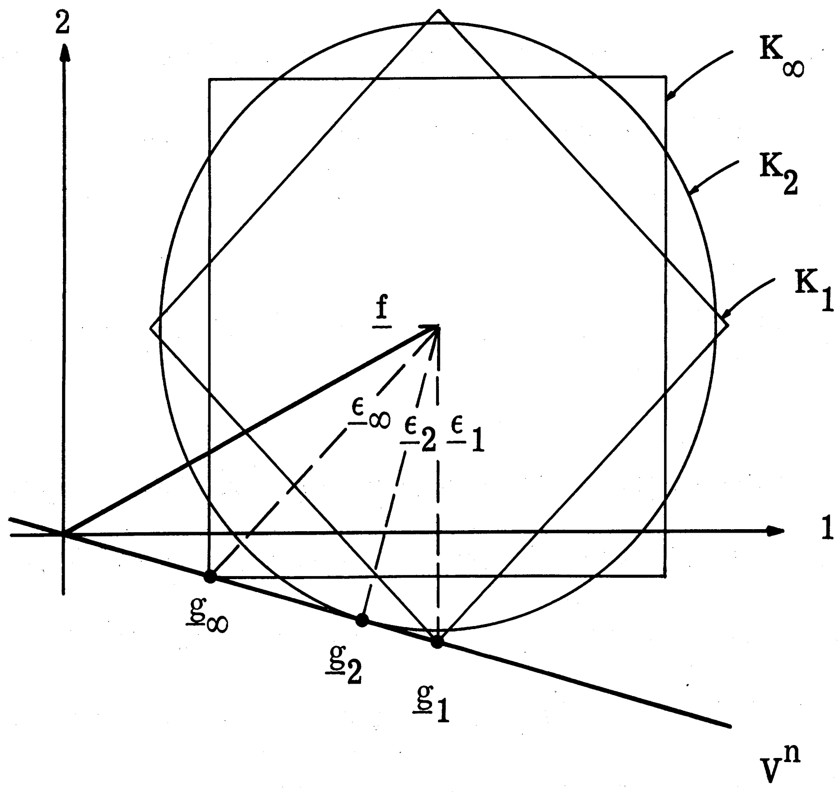


Fig. 1. The best approximation to $\underline{f}$ onto V^n in E^q for $p = 1, 2, \infty$, where $q = 2$, $n = 1$.

function f(t) in $L_p(a,b)$-space[1] may be represented by a linear combination of the complete set of basis functions which is infinite in number since the $L_p(a,b)$-space is of infinite dimensionality. Furthermore, the n-dimensional approximating subspace V^n in the $L_p(a,b)$-space is defined by the linear combination of n-independent functions. The L_p-convex body of radius $\|\epsilon(t)\|_p$ and center f(t), is defined by

$$K_p\left(f(t), \|\epsilon(t)\|_p\right) = \{g(t) : \|f(t) - g(t)\|_p \leq \|\epsilon(t)\|_p,\ g(t) \text{ in } L_p(a,b)\} \tag{2.2}$$

For further discussion on this subject, see Rice (Ref. 15, pp. 10-14).

These concepts are not new in engineering. A well-known example in engineering is the problem of approximating a function f(t) in $L_2(0, 2\pi)$-space by a finite set of sine and cosine functions. In the formal language of linear spaces, this problem may be stated as follows: Given a function $f(t) \in L_2(0, 2\pi)$ and a finite set of basis functions, $\{\cos kt, \sin kt : k = 1, 2, \ldots, n\}$ spanning a linear subspace V^{2n} in the $L_2(0, 2\pi)$-space; determine the 2n parameters $\{\alpha_k^*, \beta_k^*\}$, $k = 1, 2, \ldots, n$, so that the function f*(t), defined by

$$f^*(t) \triangleq \sum_{k=1}^{n} (\alpha_k^* \cos kt + \beta_k^* \sin kt) \tag{2.3}$$

---

[1] The notation $L_p(a,b)$-space, instead of $L_p(\mathscr{T})$-space, is used to emphasize that the function f(t) is defined for all t in the finite closed interval [a, b], rather than the semi-infinite interval $\mathscr{T} = [0, \infty)$.

best approximates f(t) in $L_2(0, 2\pi)$-space in the least-squares sense. Since this approximation is taken with respect to the L_2-norm of the error function, the solution is given by the orthogonal projection (i.e., "L_2-projection") of f(t) onto V^{2n}. Hence, the 2n parameters $\{\alpha_k^*, \beta_k^*\}$ can be obtained by taking the orthogonal projection of f(t) on the respective basis vectors, namely,

$$\alpha_k^* = (f(t), \cos kt), \quad \text{and} \tag{2.4}$$

$$\beta_k^* = (f(t), \sin kt), \quad k = 1, 2, \ldots, n \tag{2.5}$$

This is to say, that the set of best parameters $\{\alpha_k^*, \beta_k^*\}$ given by Eqs. 2.4 and 2.5, are equal to the coefficients of the appropriate cosine and sine functions resulting from the Fourier expansion of f(t). This result should be obvious because the infinite set of sine and cosine functions forms a complete orthonormal set of basis functions of the $L_2(0, 2\pi)$-space. Another evident result is that the sum of the squares of the coefficients of the sine and cosine functions which do not lie in V^{2n} gives the minimal value of the square of the L_2-norm of the approximating error function, i.e., $\|\epsilon(t)\|_2^2$. At this point, it should be mentioned that the best approximation of f(t) onto V^{2n} with respect to some other L_p-norm will <u>not</u> yield the above relationship between the Fourier series coefficient and the best parameters $\{\alpha_k^*, \beta_k^*\}$.

In recent years, under the impetus of the increasing need to handle complicated signals, it has become evident that the concepts involved may be best expressed in the language of linear space. For example, considerable attention has been devoted to the problem of representing signals in terms of various bases other than the familiar sine and cosine functions (Refs. 8, 12, 27). Thus, depending on the particular application, the approximating subspace V^n is defined in terms of different types of basis functions. In the field of network synthesis[2] the most efficient approximating subspace is the one spanned by a set of one-sided exponential functions[3]

$$\left\{ e^{s_k t} : t \geq 0, \ \mathrm{Re}\{s_k\} \leq 0 \right\}, \quad k = 1, 2, \ldots, n \tag{2.6}$$

Since the exponents $\{s_k\}$ are usually unknown, the approximating subspace is not fully prescribed, but given in terms of the n-unknowns $\{s_k\}$. Hence, the time domain approximation problem of network synthesis involves the determination of the best approximating function on an approximating subspace, V^n, which depends on the set $\{s_k : k = 1, 2, \ldots, n\}$. Recall that in Chapter I we have shown that such an approximation problem involves the determination

---

[2]See Section 1.2.

[3]That these functions are linearly independent when s_k's are distinct is obvious.

of the parameter vector pair $(\underline{\alpha}^{**}, \underline{s}^{*})$ of the best approximating function $\hat{h}(t; \underline{\alpha}^{**}, \underline{s}^{*})$ which is selected from the class of functions $\hat{\mathscr{H}} = \{\hat{h}(t; \underline{\alpha}, \underline{s}) : \underline{\alpha} \in \mathscr{A}_s, \ \underline{s} \in \mathscr{S}\}$, where $\hat{h}(t; \underline{\alpha}, \underline{s})$ is defined by Eq. 1.3 (Eq. 13.a). It, thus, becomes clear that the parameter vector $\underline{s}$ of the approximating function $\hat{h}(t; \underline{\alpha}, \underline{s})$ represents the orientation of the approximating subspace V^n, a fact which is emphasized by representing V^n by $V^n(\underline{s})$.

In the following paragraphs, we shall review the previous contributions to the problem of determining the best parameter vector pair $(\underline{\alpha}^{**}, \underline{s}^{*})$ which will be referred to as L_p-approximation or ℓ_p^q-approximation problem depending on the particular criterion used to measure the degree of approximation. In particular, we shall review the approximation techniques that were based on either the least-square, or the uniform norm (i. e., Chebyshev) criterion, since these are the most widely used criteria in the area of network synthesis.

It is appropriate to mention here that many of the previous works concentrated on the determination of the optimum parameter vector $\underline{\alpha} \in \mathscr{A}_s$, i. e., $\underline{\alpha}^{*}$, for some "good" estimate of the parameter vector $\underline{s} \in \mathscr{S}$ which was considered known for the approximation problem.

2.2 The L_p-Approximation Problem of Network Synthesis

2.2.1 L_2-Approximations. The first significant application of the approximation theory in the L_p-spaces to the time domain network synthesis problem can be considered to have been made by Aigrain and Williams (Ref. 1), and by Kautz (Ref. 10). They sought the best least-square approximation of h(t) where t is defined in the interval $\mathscr{T} = [0, \infty)$. If h(t) and the class of approximating functions $\hat{\mathscr{H}}$ are in the $L_2(\mathscr{T})$-space, then this is the familiar orthogonal projection problem. Here the measure of approximation is given by

$$\|\epsilon(t;\underline{\alpha},\underline{s})\|_2^2 = \int_0^\infty |\epsilon(t;\underline{\alpha},\underline{s})|^2 dt \tag{2.7}$$

$$= \int_0^\infty \left| h(t) - \sum_{k=1}^{n} \alpha_k e^{s_k t} \right|^2 dt \tag{2.8}$$

Aigrain and Williams recognized that the optimum ordered vector pair $(\underline{\alpha}^{**}, \underline{s}^*)$ denotes the stationary point[4] of the function $\|\epsilon(t;\underline{\alpha},\underline{s})\|_2^2$ defined in Eq. 2.8. Unfortunately, to obtain this stationary point, one has to solve a system of 2n simultaneous nonlinear equations a task which is, in itself, formidable. Furthermore, this is only a necessary condition, and a unique solution does not necees-sarily result.

---

[4]The stationary point $\underline{x}^*$ of a function $f(\underline{x})$ is the point where $\{[\partial f(\underline{x})]/[\partial x_i]\} = 0$ for all $i = 1, \ldots, n$.

About a decade later, McDonough (Ref. 13) simplified the solution of the system of simultaneous nonlinear equations by linearizing them. He used the orthogonality condition which exists in the L_2-space between the approximating function $\hat{h}(t;\underline{\alpha},\underline{s})$ and the error function $\epsilon(t;\underline{\alpha},\underline{s})$, and, thus, determined an expression for $\|\epsilon(t;\underline{\alpha},\underline{s})\|_2^2$ in terms of the parameter vector $\underline{s}$ alone. This new function is zero at the stationary point of the function $\|\epsilon(t;\underline{\alpha},\underline{s})\|_2^2$. His work on this subject contains an excellent review of other contributions to the time domain approximation problem of network synthesis.

A different approach to this problem, developed initially by Kautz (Ref. 10), uses a finite set of orthonormal functions constructed from the set of one-sided exponential functions given in Eq. 2.6. The choice of the parameter vector $\underline{s}$ is somewhat arbitrary so that the resulting approximating function is not necessarily the optimum approximating function, $h^{**}(t)$, of $h(t)$ in the $L_2(\mathcal{T})$-space. The choice of the parameter vector $\underline{s}$ is based on Tuttle's interpretation of Prony's work.[5] Tuttle (Ref. 25) realized that Prony's work, which constructs an n^{th}-order difference equation from a prescribed set of 2n equally-spaced values $\{h(t_i) : i = 1,2, \ldots, 2n\}$ of h(t), can be extended to construct an n^{th}-order differential equation (having constant coefficients) by confining oneself to

[5] A detailed analysis of Prony's original method is found in Section 2.3.2.

the single point $t = 0$. The important step of proceeding from the differential equation to the approximation problem in the $L_2(\mathscr{T})$-space was made by Kautz.[6] However, this method does not usually yield the optimum parameter vector $\underline{s}^*$. Thus, Kautz's method is basically a two-step approximation procedure. He first determines the pole locations (i. e., the parameter vector $\underline{s}$) which are not optimum and then determines the optimum residues (i. e., the parameter vector $\underline{\alpha}$) for this pole configuration.

The reason for the wide use of Kautz's method stems from the fact that the final approximation error $\|\epsilon^*(t)\|_2^2$ is relatively insensitive to a variation in the parameter vector $\underline{s}$. The significant contributions which stem from Kautz's method include:

(1) The generalization by Carr (Ref. 5) which extends Kautz's method to the approximation of any impulse response function $h(t) \epsilon L_2(\mathscr{T})$-space and not just those functions $h(t)$ which have derivatives through the n-th order in the $L_2(\mathscr{T})$-space.

(2) The methods which improve the approximation by changing the pole position (Ref. 4).

---

[6] Kautz's approximation method handles only functions $h(t)$ being everywhere smooth to a high-order in the interval $[0, \infty)$ of t, i. e., the set of functions $\left\{\frac{d^m h(t)}{dt^m} : m = 1, 2, \ldots, n\right\}$ must be in the $L_2(\mathscr{T})$-space.

<u>2.2.2 L_∞-Approximations.</u> The time domain approximation problem of network synthesis using the Chebyshev criterion has not been studied as extensively as that using the least-square criterion. One reason for this is that in $L_\infty(\mathscr{T})$-space, the "L_∞-projection" problem is not the familiar orthogonal projection problem defined in $L_2(\mathscr{T})$-space and, thus, is intuitively difficult to visualize. To the author's knowledge, the only contribution using the Chebyshev criterion has been made by Tang (Ref. 23). He shows how to obtain $h^*(t,\underline{s})$ with respect to the L_∞-norm if the vector $\underline{s} \in \mathscr{S}$ is prescribed to be a real n-dimensional vector. He, thus, determines the best RC network realization of h(t) in $L_\infty(\mathscr{T})$-space with respect to only the parameter vector $\underline{\alpha}$ in $\mathscr{A}_s$.

At this point, we conclude the review of the synthesis techniques in the L_p-spaces and turn to those in the ℓ_p^q-spaces.

2.3 The ℓ_p^q-Approximation Problem of Network Synthesis

The two significant contributions to the problem of network synthesis using approximation techniques at discrete points in the time domain were made by Yengst (Ref. 26) and Ruston (Ref. 20) who measured their approximations by the least-squares criterion and the Chebyshev criterion, respectively. However, in no case, for either criterion, was the optimum pole location (i.e., the vector $\underline{s}^*$) obtained. To clarify this point, we shall analyze the approximation problem in the ℓ_2^q-normed and ℓ_∞^q-normed vector spaces, in rather complete detail.

We shall begin by reviewing the approximation problem when the pole locations of the network are initially prescribed, i. e., the case when $\underline{s} \in \mathscr{S}$ is initially prescribed when using the formulation given in Section 1. 4. Clearly, when the components of the vector $\underline{s} \in \mathscr{S}$ are prescribed to be real, this problem is the usually considered ℓ_p^q-approximation problem in which a real vector in U^q is approximated on a prescribed subspace V^n of U^q. At this point, we note that when the components of $\underline{s}$ occur in conjugate pairs, and when the approximating vector must be real, then we need to make only a trivial extension to the case in which the components of $\underline{s}$ are real. Then we shall consider in detail the original work of Prony (Ref. 15) for the case when $q = 2n$, and interpret the significance of extending this formulation to the case in which $q > 2n$, in terms of operations in the vector space U^q. Finally, we shall analyze the works of Yengst and Ruston in the same context.

In summary, the specific topics which we shall consider are based on the three forms which the vector $[E(\underline{s})]\ \underline{\alpha}$ of the equation[7] can take.

$$\underline{h} = [E(\underline{s})]\ \underline{\alpha} + \underline{\epsilon}(\underline{\alpha}, \underline{s}) \tag{2.9}$$

---

[7]The vectors $\underline{h}$, $[E(\underline{s})]\ \underline{\alpha}$, and $\underline{\epsilon}(\underline{\alpha}, \underline{s})$ of this equation are defined in Section 1. 4.

These are:

(1) The case in which the vector $\underline{s} \in \mathscr{S}$ is initially prescribed, but the discrete values $\{ t_i : i = 1, 2, \ldots, q \}$ are not equally spaced. This is the approximation problem frequently considered in the theory of approximation in the ℓ_p^q-space.

(2) The case in which the vector $\underline{s} \in \mathscr{S}$ is not initially prescribed, but the $\{ t_i : i = 1, 2, \ldots, q \}$ are equally-spaced, $q = 2n$, and the error vector $\underline{\epsilon}(\underline{\alpha}, \underline{s}) = 0$, so that Eq. 2.9 can be replaced by[8]

$$\underline{h} = [Z(\underline{z})] \underline{\beta} \tag{2.10}$$

where $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$. This is the original problem considered by Prony (Ref. 15).

(3) The case in which the vector $\underline{s} \in \mathscr{S}$ is not initially prescribed, but the $\{ t_i : i = 1, 2, \ldots, q \}$ are equally-spaced and $q > 2n$, so Eq. 2.9 can be replaced by[9]

$$\underline{h} = [Z(\underline{z})] \underline{\beta} + \underline{\epsilon}(\underline{\beta}, \underline{z}) \tag{2.11}$$

where $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$. This is the problem considered by Yengst and Ruston using the least-square and Chebyshev criteria, respectively. Moreover, this case is also the subject of this dissertation.

---

[8]The vectors $\underline{h}$ and $[Z(\underline{z})] \underline{\beta}$ of this equation are defined in Section 1.4.

[9]See Footnote 8.

It is hoped that this approach helps to unify the material and to set the stage for the main contribution of this thesis which is presented in Chapters III and IV.

2.3.1 The Approximation Problem when the Matrix [E] is Initially Prescribed. When the q x n matrix[10] [E] of Eq. 2.9 is initially prescribed, the approximation problem may be stated as follows: Given a real vector $\underline{h}$ in U^q and a q x n matrix $[E(\underline{s})]$, defined by Eq. 1.12 (1.12a) of rank n $(n < q)$, and where $\underline{s} \in \mathscr{S}$, determine the vector $\underline{\alpha}_p{}^*$ in $\mathscr{A}_s$, so that if $\underline{h}_p{}^* \triangleq [E]\,\underline{\alpha}_p{}^*$, then,

$$\|\underline{\epsilon}_p{}^*\|_p \triangleq \|\underline{h} - \underline{h}_p{}^*\|_p \leq \|\underline{h} - [E]\,\underline{\alpha}\|_p \qquad (2.12)$$

for all $\underline{\alpha}$ in $\mathscr{A}_s$ and where $p = 2, \infty$.

This is the form which the approximation problem of Eq. 2.9 takes when the pole location (i.e., the vector $\underline{s}$) is initially prescribed and when $\{t_i : i = 1, 2, \ldots, q\}$ are not equally spaced. A problem of this nature, when instead of [E] we have any q x n real matrix [A], of rank n, $(n < q)$, has been discussed in the literature. It can be shown that if the prescribed vector $\underline{s}$ is any element of $\mathscr{S}$, then the theorems stated below are simple extensions of those given in the literature for the case

---

[10] Hereafter, we shall denote the matrix $[E(\underline{s})]$ by [E] if the parameter vector $\underline{s}$ is prescribed and if there is no danger of ambiguity.

in which [A] is a real matrix.[11]

2.3.1.1 The Least-Squares Approximation Problem. The most familiar approximation problem in the literature is the least-square approximation problem (i.e., the case when p = 2 in Eq. 2.12). Here the approximation criterion is the ℓ_2-norm of the error vector $\underline{\epsilon}$. The existence and uniqueness of the parameter vector $\underline{\alpha}^*$ in $\mathcal{A}_s$ is guaranteed by the Projection Theorem for a finite dimensional Unitary Space (Ref. 6). Furthermore, the vector $\underline{\alpha}^*$ may be determined directly with the aid of the well-known pseudo-inverse matrix (Refs. 7 and 28). These results can be summarized by the following theorem:

Theorem 2.1: For each real $\underline{f}$ in U^q and each q x n matrix $[E(\underline{s})]$, defined by Eq. 1.12 (Eq. 1.12a), of rank n, ($n < q$, $s \in \mathcal{S}$), there exists a unique n-dimensional vector $\underline{\alpha}^* \in \mathcal{A}_s$, such that if $\underline{f}^* \triangleq [E]\,\underline{\alpha}^*$, then

$$\|\underline{\epsilon}^*\|_2 \triangleq \|\underline{f} - \underline{f}^*\|_2 < \|\underline{f} - [E]\,\underline{\alpha}\|_2 \tag{2.13}$$

for all $\underline{\alpha} \neq \underline{\alpha}^*$ in $\mathcal{A}_s$. Furthermore, the resulting best least-square error vector, $\underline{\epsilon}^* \triangleq \underline{f} - \underline{f}^*$, is always orthogonal to the best approximating vector, $\underline{f}^*$, i.e.,

$$(\underline{f}^*, \underline{\epsilon}^*) = 0 \tag{2.14}$$

---

[11] To show this, it is sufficient to recall that the approximating vector, $\underline{h}(\underline{\alpha}^*, \underline{s}) = [E(\underline{s})]\underline{\alpha}^*$, must be selected from a set of real vectors, i.e., if the components of $\underline{\alpha}$ and $\underline{s}$ are complex, then they must occur in conjugate pairs.

This theorem is again a trivial extension to the approximation theorem which governs the least-squares approximation in the real vector space, ℓ_2^q. (Ref. 6)

The important result of this theorem is that the best least-square approximation of a real vector in U^q by a real vector in the subspace V^n, defined by the column vectors of the matrix [E], is the orthogonal projection of $\underline{f}$ onto V^n. Hence, the singular mapping involved is given by the projection operator $E^+ : U^q \rightarrow V^n$, so that the n-dimensional vector $\underline{\alpha}^*$ can be determined from

$$\underline{\alpha}^* = [E^+]\,\underline{f} \tag{2.15}$$

where $[E^+]$ is the pseudo-inverse matrix of [E], defined by

$$[E^+] = [E^T E]^{-1}[E^T] \tag{2.16}$$

where $[E^T]$ is the transpose of the matrix [E].

2.3.1.2 The Chebyshev Approximation Problem. The other interesting approximation is the one that seeks the best Chebyshev approximation of a real vector $\underline{f}$ in U^q in some subspace V^n of U^q defined by the column space of a prescribed q x n matrix $[E(\underline{s})]$, $\underline{s} \in \mathscr{S}$. Here, the approximation criterion is given by the ℓ_∞-norm, i. e., $\|\underline{\epsilon}\|_\infty = \max_{1 \leq i \leq q} |\epsilon_i|$. A similar approximation problem, when instead of [E] we have any q x n real matrix [A], has been discussed in the literature by Stiefel (Ref. 22) and Rivlin (Ref. 18). They have shown that the best Chebyshev approximate solution exists, and they also have developed an algorithm which yields this solution. Furthermore, they have obtained the conditions on the matrix [A] under which the best Chebyshev approximation is unique. Applying their results to the above approximation problem, we obtain the following theorem:[12]

Theorem 2. 2: For each real vector $\underline{f}$ in U^q and each (q x n) matrix $[E(\underline{s})]$, defined by Eq. 1. 12 (Eq. 1. 12a), of rank n ($n < q$, $\underline{s} \in \mathscr{S}$), there exists an n-dimensional vector $\underline{\alpha}^* \in \mathscr{A}_s$ such that if $\underline{f}^* \triangleq [E]\,\underline{\alpha}^*$, then

$$\|\underline{\epsilon}^*\|_\infty \triangleq \|\underline{f} - \underline{f}^*\|_\infty = \|\underline{f} - [E]\,\underline{\alpha}^*\|_\infty \leq \|\underline{f} - [E]\,\underline{\alpha}\|_\infty \qquad (2.17)$$

for n-dimensional vectors $\underline{\alpha} \in \mathscr{A}_s$. The resulting best Chebyshev error vector $\underline{\epsilon}^*$ has at least (n+1) components with absolute values equal to $\|\underline{\epsilon}^*\|_\infty$, namely,

$$\left|\epsilon^*_{i_v}\right| = \|\underline{\epsilon}^*\|_\infty, \quad \text{for } v = 1, 2, \ldots, n+1 \qquad (2.18)$$

Furthermore,

$$\left|\epsilon^*_{i_v}\right| \leq \|\underline{\epsilon}^*\|_\infty \quad \text{for } v = n+2, \ldots, q$$

Rather than proving this theorem, as is done in numerous places in the literature (Refs. 18, 19, and 22), we shall present the essential

---

[12]Note that again we make a trivial extension to the approximation problem which they considered.

steps of the proof in the form which is most suited to our future application. However, first let us offer some intuitive notions relating to the results given by Eq. 2.18. Consider the real error vector $\underline{\epsilon}$ in U^q, given by

$$\underline{\epsilon}(\underline{\alpha}) = \underline{f} - [E]\,\underline{\alpha} \tag{2.19}$$

Clearly this error vector is a function of the parameter vector $\underline{\alpha} \in \mathcal{A}_s$, since the vector $\underline{f}$ and the q x n matrix [E] have been initially prescribed. Let us assume that the function $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$ is continuous with respect to $\underline{\alpha}$. To satisfy Eq. 2.17, we must determine a vector $\underline{\alpha} = \underline{\alpha}^* \in \mathcal{A}_s$ so that the $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$ is minimum, that is,

$$\|\underline{\epsilon}^*\|_\infty \triangleq \|\underline{\epsilon}(\underline{\alpha}^*)\|_\infty = \min_{\underline{\alpha} \in \mathcal{A}_s} \|\underline{\epsilon}(\underline{\alpha})\|_\infty \tag{2.20}$$

That this vector, $\underline{\epsilon}^*$, will have at least (n+1) components with absolute values equal to $\|\underline{\epsilon}^*\|_\infty$ can be illustrated as follows:[13] Select some vector $\underline{\alpha} = \underline{\alpha}' \in \mathcal{A}_s$ and determine the vector $\underline{\epsilon}(\underline{\alpha}')$ from Eq. 2.19 and the value of $\|\underline{\epsilon}(\underline{\alpha}')\|_\infty$. Let us assume that the j^{th} component of $\underline{\epsilon}(\underline{\alpha}')$ has the largest absolute value; that is,

$$|\epsilon_j(\underline{\alpha}')| = \max_{1 \le i \le q} |\epsilon_i(\underline{\alpha}')| \triangleq \|\underline{\epsilon}(\underline{\alpha}')\|_\infty \; . \tag{2.21}$$

---

[13] The following method is called the "method of descent" (Ref. 19) rather than the "method of steepest descent" since we shall not use the maximum gradient of the function $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$.

Clearly, we can reduce the value of $\|\underline{\epsilon}(\underline{\alpha}')\|_\infty$ (or, equivalently, $|\epsilon_j(\underline{\alpha}')|$) by adjusting any one of the components of $\underline{\alpha}'$, say α_1', until the absolute values of two components of $\underline{\epsilon}(\underline{\alpha})$ are equal to $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$. Let us denote the corresponding vector $\underline{\alpha}$ by $\underline{\alpha}''$ and assume that the j^{th} and k^{th} component of $\underline{\epsilon}(\underline{\alpha}'')$ are equal to $\|\underline{\epsilon}(\underline{\alpha}'')\|_\infty$. Hence, we have the relation

$$|\epsilon_j(\underline{\alpha}'')| = |\epsilon_k(\underline{\alpha}'')| = \|\underline{\epsilon}(\underline{\alpha}'')\|_\infty < \|\underline{\epsilon}(\underline{\alpha}')\|_\infty \qquad (2.22)$$

We now reduce the value of $\|\underline{\epsilon}(\underline{\alpha}'')\|_\infty$ further, by adjusting two components of $\underline{\alpha}$, say α_1'' and α_2''. This process is continued until we have adjusted all the n-components of $\underline{\alpha}$. At this point it is found that the absolute values of at least (n+1) components of $\underline{\epsilon}(\underline{\alpha})$ are equal to $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$.[14] If any further adjustment in the components of the vector $\underline{\alpha}$ increases the value of $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$, then, we have achieved the minimum value of $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$; i.e., $\|\underline{\epsilon}^*\|_\infty$. The vector $\underline{\alpha}$ with which we have achieved the minimum value of $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$ is the vector $\underline{\alpha}^*$ of interest.

The characteristic property of the resulting Chebyshev error vector $\underline{\epsilon}^*$ (given by Eq. 2. 18) suggests an alternate method for the solution of the Chebyshev approximation problem. Such a method considers each one of the $\binom{q}{n+1}$ subsets containing only (n+1) equations out of the set of q equations defined by the vector equations in Eq. 2. 19. This method

---

[14] That this can be done is evident from the fact that only n components of $\underline{\epsilon}(\underline{\alpha})$ are changed independently by varying the n-components of $\underline{\alpha}$.

is based on the assumption that if, for each subset, the (n+1) components of $\underline{\epsilon}$ have their absolute values equal to some non-negative constant, say $|\rho|$, and have their signs chosen[15] to obtain the minimal value of $|\rho|$; then, the values of $|\rho|$ and the vector $\underline{\alpha}$ can be determined directly for each subset.[16] Then, out of the set of $\binom{q}{n+1}$ possible $\underline{\alpha}$'s thus determined, one selects the $\underline{\alpha} = \underline{\alpha}^*$ which corresponds to the greatest $|\rho|$. It can be shown[17] that this vector $\underline{\alpha}^*$ yields the minimum value of $\|\underline{\epsilon}(\underline{\alpha})\|_\infty$, i.e., the vector $\underline{\alpha}^*$ yields the vector $\underline{\epsilon}(\underline{\alpha}^*)$ in U^q with $\|\underline{\epsilon}(\underline{\alpha}^*)\|_\infty$ satisfying Eq. 2.20. This method of solution, sometimes called the "method of ascent", has been studied by Stiefel (Ref. 22) in a geometric setting. Since we shall use some of the results of this approach, let us now use it to sketch the proof of Theorem 2.2.

Let us begin by defining the selection of a subset of (n+1) equations out of the set of q equations, given by the vector relation

$$\underline{f} = [E]\,\underline{\alpha} + \underline{\epsilon} \quad \text{in } U^q \tag{2.23}$$

to represent a mapping of the vector space U^q onto an (n+1)-dimensional

---

[15] The appropriate choice of the signs of the (n+1) components of the vector $\underline{\epsilon}$ in each subset will be given in Theorem 2.3.

[16] That this can be done is evident from the fact that each subset contains (n+1) equations in (n+1) unknowns. These (n+1) unknowns consist of the n-components of $\underline{\alpha}$ and the one unknown representing the absolute value of all the n+1 components of $\underline{\epsilon}$ in this subset.

[17] See Corollary 2.1.

subspace of U^q, where the (n+1)-dimensional subspace of U^q is defined as follows:

Definition 2.1: (Reference subspace with respect to a fixed set of basis vectors.) Let $\{\underline{\xi}_i : i = 1, 2, \ldots, q\}$ be an orthonormal set of basis vectors of U^q, so that each vector $\underline{g}$ in U^q is defined by

$$\underline{g} = \sum_{i=1}^{q} g_i \, \underline{\xi}_i \tag{2.24}$$

and let $\left\{\underline{\xi}_{k_j} : j = 1, 2, \ldots, m < q\right\}$ be a subset of only m of these basis vectors, where k denotes the k^{th} subset out of the possible $\binom{q}{m}$ distinct subsets. These subsets are arbitrarily ordered, i.e., $k = 1, 2, \ldots, \binom{q}{m}$. Then the m-dimensional subspace, denoted by U_k^m, is said to be the k^{th} reference subspace if it is spanned by the basis vectors in the $\left\{\underline{\xi}_{k_j} : j = 1, 2, \ldots, m\right\}$. Furthermore, the projection operator $P_k : U^q \rightarrow U_k^m$ is denoted by a q x m elementary matrix[18] $[I_k]$ and defined by

$$[I_k] = \left[\underline{\xi}_{k_1} \; \underline{\xi}_{k_2} \; \cdots \; \underline{\xi}_{k_m}\right] \tag{2.25}$$

Hence, the projection $\underline{g}^{(k)}$ of $\underline{g}$ in U_k^m is related to $\underline{g}$ in U^q by

$$\underline{g}^{(k)} = [I_k]^T \underline{g} \tag{2.26}$$

---

[18] An elementary matrix represents a matrix having only one nonzero element in each row and column. This element is equal to one (Ref. 24, p. 96).

In a similar manner we can denote, in vector notation, the subsets of (n+1) equations out of the set of q equations defined by the vector equation of Eq. 2.23. Clearly, using Definition 2.1, one can form $\binom{q}{n+1}$ distinct (n+1)-dimensional reference subspaces, $\left\{U_v^{n+1} : v = 1, 2, \ldots, \binom{q}{n+1}\right\}$, from U^q in which the (n+1)-dimensional projections of the vectors $\underline{f}$, $[E]\,\underline{\alpha}$, and $\underline{\epsilon}$ in U^q of Eq. 2.23 are related by

$$\underline{f}^{(v)} = [E^{(v)}]\,\underline{\alpha} + \underline{\epsilon}^{(v)}, \qquad v = 1, 2, \ldots, \binom{q}{n+1} \tag{2.27}$$

where

$$\underline{f}^{(v)} = [I_v]^T\,\underline{f} \qquad \epsilon\ U_v^{n+1} \tag{2.28}$$

$$[E^{(v)}]\,\underline{\alpha} = [I_v]^T\,[E]\,\underline{\alpha} \qquad \epsilon\ U_v^{n+1} \quad \text{and} \tag{2.29}$$

$$\underline{\epsilon}^{(v)} = [I_v]^T\,\underline{\epsilon} \qquad \epsilon\ U_v^{n+1} \tag{2.30}$$

Let us now summarize the approach which we shall use to sketch the proof of Theorem 2.2. First, we shall state a theorem which will establish the existence of a unique Chebyshev approximation in an (n+1)-dimensional reference subspace U_k^{n+1}, where $k \in \left\{v : v = 1, 2, \ldots, \binom{q}{n+1}\right\}$. In other words, we shall show that for each real vector $\underline{f}^{(k)} \in U_k^{n+1}$ and each (n+1) x n matrix $[E^{(k)}]$ of rank n, there exists a unique vector $\underline{\alpha}_k^* \in \mathscr{A}_s$ so that

$$\| \underline{f}^{(k)} - [E^{(k)}]\,\underline{\alpha}_k \|_\infty < \| \underline{f}^{(k)} - [E^{(k)}]\,\underline{\alpha}_k^* \|_\infty \tag{2.31}$$

for all $\underline{\alpha} = \underline{\alpha}_k{}^* \in \mathscr{A}_s$. Then, using this result, we shall establish the existence of a unique Chebyshev approximation in U^q by showing that the parameter vector $\underline{\alpha}^* \in \mathscr{A}_s$, which defines the best Chebyshev approximating vector $\underline{f}^* \triangleq [E]\,\underline{\alpha}^*$ of the prescribed real vector $\underline{f} \in U^q$, lies in the set $\left\{\underline{\alpha}_v{}^* : v = 1, 2, \ldots, \binom{q}{n+1}\right\}$, where $\underline{\alpha}_v{}^*$ in $\mathscr{A}_s$ which satisfy Eq. 2.31.

To establish the existence of a unique Chebyshev approximation in U_k^{n+1} (or equivalently in U^q) we need the following assumption:

Assumption 2.1: Every $n \times n$ submatrix of the $q \times n$ matrix $[E]$, where $q > n$, is nonsingular.

It should be noted that this assumption will always hold when the prescribed matrix $[E]$ is of rank n and real.[19] Furthermore, if $[E]$ satisfies Assumption 2.1, then every $n \times n$ submatrix of the $(n \times 1) \times n$ matrix $[E^{(k)}]$, defined by

$$[E^{(k)}] \triangleq [I_k]^T [E] \tag{2.32}$$

must also be nonsingular.

We shall, henceforth, assume that the matrix $[E]$ satisfies Assumption 2.1. Let us now present the following lemma by de la Vallee Poussin (Refs. 14 and 18) which we shall use to establish the existence

[19] See Eq. 1.12 (Eq. 1.12a) when the s_k's are real.

of a unique Chebyshev approximation to $\underline{f}^{(k)}$ in U_k^{n+1}.

<u>Lemma 2.1:</u> For each real linear function $\phi(\underline{x}) \triangleq (\underline{x}, \underline{y})$ satisfying the equation $\phi(\underline{x}) = c$, where $\underline{x}$ and $\underline{y}$ are vectors in a real m-dimensional Euclidean space, E^m, and where c is a real nonzero constant and $\{y_i \neq 0 : i = 1, 2, \ldots, m\}$, there exists a unique vector $\underline{x}^*$ in E^m, with an ℓ_∞^m-norm that satisfies

$$\|\underline{x}^*\|_\infty < \|\underline{x}\|_\infty \tag{2.33}$$

for all $\underline{x} \neq \underline{x}^*$ in E^m satisfying $\phi(\underline{x}) = c$. Furthermore, the vector $\underline{x}^*$ is given by

$$\underline{x}^* = a\underline{\chi} \tag{2.34}$$

where[20]

$$\underline{\chi} = \begin{bmatrix} \operatorname{sgn} y_1 \\ \operatorname{sgn} y_2 \\ \vdots \\ \operatorname{sgn} y_m \end{bmatrix}, \text{ and} \tag{2.35}$$

$$a = \frac{c}{\|\underline{y}\|_1}, \quad \|\underline{y}\|_1 \triangleq \sum_{i=1}^{m} |y_i| \tag{2.36}$$

[20]By definition $\operatorname{sgn} y_i = \frac{y_i}{|y_i|}$ if $y_i \neq 0$, and $\operatorname{sgn} 0 = 0$.

The proof of this lemma is given in Refs. 14 and 18. However, an intuitive explanation can be given with the aid of Fig. 2, where m = 2. Here the linear function $\phi(\underline{x})$ satisfying the equation $\phi(\underline{x}) = c$ is a straight line in the 2-dimensional Euclidean space, E^2, which is orthogonal to the vector $\underline{y}$. Since the points along this line represent the various choices of the vectors $\underline{x}$, clearly then, the vector $\underline{x}$ with a minimal value of $\|\underline{x}\|_\infty$ will have components which satisfy:

a) $|x_1| = |x_2| = |a|$; and

b) $\operatorname{sgn} x_i = \operatorname{sgn} y_i$, $i = 1, 2$

This vector $\underline{x}$ has been denoted by $\underline{x}^*$.

Remark: Two points should be noted concerning Lemma 2.1. First, it should be noted that if $c = 0$ then the lemma is trivially true and $\|\underline{x}^*\|_\infty = 0$ since $a = 0$. The second point to be noted is that if $y_j = 0$, where $j \in \{i = 1, \ldots, m\}$, then although Eq. 2.35 gives the j^{th} component, x_j^*, of $\underline{x}^*$ to be equal to zero, the equation $\phi(\underline{x}) = c$ can be satisfied by using any value of x_j^* in the interval $\left[-\|\underline{x}^*\|_\infty, \|\underline{x}^*\|_\infty\right]$ without affecting the final value of $\|\underline{x}^*\|_\infty$. Hence, we shall say that if $y_i = 0$. for some i, then the uniqueness property of the vector $\underline{x}^*$ fails.

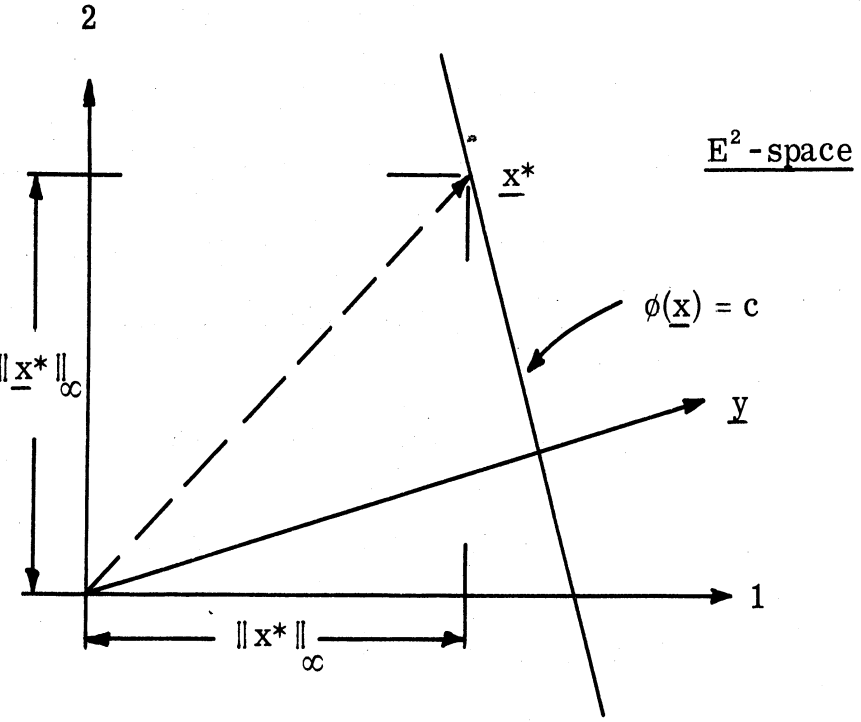


Fig. 2. Geometric interpretation of Lemma 2. 1.

Let us now establish the existence and uniqueness of the best Chebyshev approximation to $\underline{f}^{(k)}$ in U_k^{n+1}, where $k \in \left\{v = 1, 2, \ldots, \binom{q}{n+1}\right\}$, when the $(n+1) \times n$ matrix $[E^{(k)}]$, of rank n, is initially prescribed.

Theorem 2.3: For each real fector $\underline{f}^{(k)}$ in U_k^{n+1}, $k \in \left\{v = 1, 2, \ldots, \binom{q}{n+1}\right\}$, and an $(n+1) \times n$ matrix $[E^{(k)}]$, of rank n (defined by Eq. 2.32), there exists a unique n-dimensional vector $\underline{\alpha}_k{}^* \in \mathscr{A}_s$ so that if $\underline{f}^{*(k)} \triangleq [E^{(k)}]\, \underline{\alpha}_k{}^*$ is a real vector in U_k^{n+1}, then,

$$\|\underline{\epsilon}^{*(k)}\|_\infty \triangleq \|\underline{f}^{(k)} - \underline{f}^{*(k)}\|_\infty < \|\underline{f}^{(k)} - [E^{(k)}]\, \underline{\alpha}\|_\infty \tag{2.37}$$

for all n-dimensional vectors $\underline{\alpha} \neq \underline{\alpha}_k{}^* \in \mathscr{A}_s$. Furthermore, all the components of the (n+1)-dimensional error vector $\underline{\epsilon}^{*(k)}$ have absolute values equal to $\|\underline{\epsilon}^{*(k)}\|_\infty$; i.e.,

$$|\epsilon_i{}^{*(k)}| = \|\underline{\epsilon}^{*(k)}\|_\infty, \qquad i = 1, 2, \ldots, n+1 \tag{2.38}$$

Proof: The proof of this theorem is based on Lemma 2.1. Let us begin by considering the relation

$$\underline{f}^{(k)} = [E^{(k)}]\, \underline{\alpha} + \underline{\epsilon}^{(k)} \quad \text{in} \quad U^{n+1} \tag{2.39}$$

where the vector $\underline{f}^{(k)}$ and the $(n+1) \times n$ matrix $[E^{(k)}]$, of rank n, are prescribed; and the vectors $\underline{\alpha}$ and $\underline{\epsilon}^{(k)}$ are unknown. It is evident that the column space of $[E^{(k)}]$, $C_n(E^{(k)})$, describes an n-dimensional subspace in U_k^{n+1}. This implies that the orthogonal complement subspace of $C_n(E^{(k)})$ of U_k^{n+1} is simply a one-dimensional subspace. If $\underline{\lambda}^{(k)}$ is a real vector in this one-dimensional subspace, then,

$$([E^{(k)}]\,\underline{\alpha},\ \underline{\lambda}^{(k)}) = 0 \tag{2.40}$$

for all $\underline{\alpha} \in \mathscr{A}_s$; that is,

$$[E^{(k)}]^T\,\underline{\lambda}^{(k)} = 0 \tag{2.41}$$

where[21]

$$\underline{\lambda}^{(k)} = \begin{bmatrix} \lambda_1^{(k)} \\ \lambda_2^{(k)} \\ \vdots \\ \lambda_n^{(k)} \\ \lambda_{n+1}^{(k)} \end{bmatrix} \tag{2.42}$$

Equation 2.41 can be solved for $\underline{\lambda}^{(k)}$ with an $(n+1)^{st}$ component arbitrarily chosen to be equal to one;[22] i.e., a normalization of the magnitude of $\underline{\lambda}^{(k)}$ is made by taking $\lambda_{n+1}^{(k)} = 1$.

Let us now take the inner product of both sides of Eq. 2.39 with respect to $\underline{\lambda}^{(k)}$. This yields

---

[21] It should be noted that the vector $\underline{\lambda}^{(k)}$ determined from Eq. 2.41 will always be a real vector because we have assumed that if the column vectors of the matrix $[E^{(k)}]$ are complex, they must occur in conjugate pairs.

[22] Note that, by Assumption 2.1, $\lambda_i^{(k)} \neq 0$, for all $i = 1,2,\ldots,n+1$.

$$(\underline{f}^{(k)}, \underline{\lambda}^{(k)}) = (\underline{\epsilon}^{(k)}, \underline{\lambda}^{(k)}) \tag{2.43}$$

since $([E^{(k)}]\,\underline{\alpha}, \underline{\lambda}^{(k)}) = 0$ from Eq. 2.40. Furthermore, the vectors $\underline{f}^{(k)}$ and $\underline{\lambda}^{(k)}$ are both real and known, then Eq. 2.43 has the form

$$(\underline{\epsilon}^{(k)}, \underline{\lambda}^{(k)}) = c_k \tag{2.44}$$

where[23]

$$c_k = (\underline{f}^{(k)}, \underline{\lambda}^{(k)}) \tag{2.45}$$

The inner product $(\underline{\epsilon}^{(k)}, \underline{\lambda}^{(k)})$ is a linear functional of $\underline{\epsilon}^{(k)}$. Let us represent it by

$$\phi(\underline{\epsilon}^{(k)}) \triangleq (\underline{\epsilon}^{(k)}, \underline{\lambda}^{(k)})$$

The totality of all points $\underline{\epsilon}^{(k)}$ satisfying the equation $\phi(\underline{\epsilon}^{(k)}) = c_k$ is a hyperplane in the space E_k^{n+1}. The problem now is to determine the point $\underline{\epsilon}^{*(k)}$ in the hyperplane, $\phi(\underline{\epsilon}^{(k)}) = c_k$, so that $\|\underline{\epsilon}^{*(k)}\|_\infty$ is minimum.

Let us apply Lemma 2.1 to Eq. 2.44 taking $m = n+1$, $\underline{x} = \underline{\epsilon}^{(k)}$ and $\underline{y} = \underline{\lambda}^{(k)}$. From Lemma 2.1, the minimal value of $\|\underline{\epsilon}^{(k)}\|_\infty$ is attained when $\underline{\epsilon}^{(k)} = \underline{\epsilon}^{*(k)}$, where $\underline{\epsilon}^{*(k)}$ is defined by

$$\underline{\epsilon}^{*(k)} = \rho_k\, \underline{\sigma}^{(k)} \tag{2.46}$$

---

[23]If $c_k = 0$ (i.e., the vectors $\underline{f}^{(k)}$ and $\underline{\lambda}^{(k)}$ are orthogonal), then the vector $\underline{f}^{(k)}$ must lie in the approximating subspace, $C_n(E^{(k)})$.

and where

$$\rho_k = \frac{c_k}{\|\underline{\lambda}^{(k)}\|_1} \tag{2.47}$$

and

$$\underline{\sigma}^{(k)} = \begin{bmatrix} \operatorname{sgn} \lambda_1^{(k)} \\ \operatorname{sgn} \lambda_2^{(k)} \\ \vdots \\ \operatorname{sgn} \lambda_{n+1}^{(k)} \end{bmatrix} \tag{2.48}$$

Knowing $\underline{\epsilon}^{*(k)}$ and substituting it into Eq. 2.39, will give the vector $\underline{\alpha}_k^*$. Thus, the theorem is proved.

This theorem is illustrated geometrically in Fig. 3, where U_k^{n+1} is taken to be a real 2-dimensional space, E^2. The prescribed vector $\underline{f}^{(k)}$ and the prescribed column space of the (2x 1) matrix $[E^{(k)}]$ are depicted by the vector $\underline{f}$ and the line $C_1(E)$, respectively. The vector $\underline{\lambda}$, which lies along the perpendicular to the line $C_1(E)$, represents the vector $\underline{\lambda}^{(k)}$ which satisfies Eq. 2.41. In the figure, we show $\underline{\lambda}$ to be directed in the first quadrant.[24] The straight line, denoted by $\phi(\underline{\epsilon}) = c$, represents the relation of Eq. 2.43; namely, the line along which the orthogonal projections of $\underline{f}$ and $\underline{\epsilon}$ onto $\underline{\lambda}$ are equal. The dashed straight

[24] If $\underline{\lambda}$ is directed in the other direction, then $c_k < 0$ in Eqs. 2.45 and 2.47. However, this does not change the procedure.

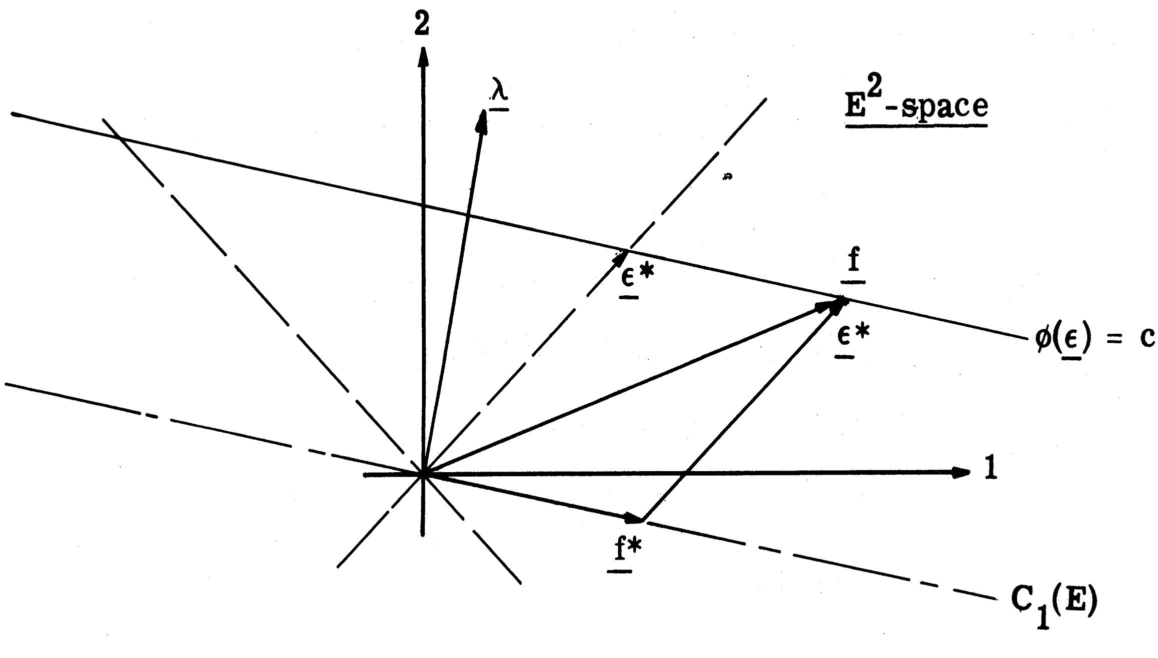


Fig. 3. Geometric interpretation of Theorem 2.3.

lines (at 45°) represent the direction of the vectors $\underline{\epsilon}$ having components equal in absolute value (see Lemma 2.1). Clearly, the vector $\underline{\epsilon}$ with $\|\underline{\epsilon}\|_{\infty}$ minimum and satisfying Eq. 2.43, must be the vector $\underline{\epsilon}^*$ shown in the figure. The vector $\underline{f}^*$, represents the best Chebyshev approximation and is given by the difference $\underline{f} - \underline{\epsilon}^*$.

In summary, Theorem 2.3 establishes the existence of a unique vector $\underline{\alpha}_v^* \in \mathscr{A}_s$ which defines the best Chebyshev real approximating vector $\underline{f}^{*(v)} \triangleq [E^{(v)}]\,\underline{\alpha}_v^*$ to the real vector $\underline{f}^{(v)}$ in U_v^{n+1}. The vector $\underline{f}^{(v)}$ and the $(n+1) \times n$ matrix $[E^{(v)}]$, $v = 1, 2, \ldots, \binom{q}{n+1}$, are obtained from the vector equation

$$\underline{f} = [E]\,\underline{\alpha} + \underline{\epsilon} \quad \text{in } U^q \tag{2.49}$$

The next step in the proof of Theorem 2.2 is to show that the above results imply the existence of a unique vector $\underline{\alpha}^* \in \mathscr{A}_s$ which defines the best Chebyshev approximating real vector, $\underline{f}^* \triangleq [E]\,\underline{\alpha}^*$ to the prescribed real vector $\underline{f} \in U^q$ and the prescribed $q \times n$ matrix $[E]$. Let $|\rho_M|$ be the largest value out of the set $\left\{|\rho_v| : v = 1, 2, \ldots, \binom{q}{n+1}\right\}$ where ρ_v is defined by Eq. 2.47; that is,

$$|\rho_M| \triangleq \max_{v} \left\{|\rho_v| : v = 1, 2, \ldots, \binom{q}{n+1}\right\} \tag{2.50}$$

Consider the relation

$$\|\underline{f} - [E]\,\underline{\alpha}\|_{\infty} \leq |\rho_M|$$

or alternately,

$$-|\rho_M| \leq f_i - \sum_{j=1}^{n} e_{ij}\,\alpha_j \leq |\rho_M|, \quad i = 1, 2, \ldots, q \qquad (2.51)$$

where $\{f_i : i = 1, 2, \ldots, q\}$ represents the q-components of $\underline{f} \in U^q$, and $\{e_{ij} : i = 1, 2, \ldots, q;\ j = 1, \ldots, n\}$ represents the elements of the q x n matrix [E]. The point sets in n-space, defined by Eq. 2.51, are convex sets. Recall, from Theorem 2.3, that any (n+1) of these convex sets have a point in common, namely $\underline{\alpha}_v^* \in \mathcal{A}_s$, where v = 1, 2, ..., $\binom{q}{n+1}$. Hence, by Helley's Theorem (Ref. 16), there exists a point which lies in all the convex sets of Eq. 2.61. This point, $\underline{\alpha}^*$, is equal to $\underline{\alpha}_M^*$ which corresponds to the value of $|\rho_M|$. Furthermore this $\underline{\alpha}^*$ is a unique vector in $\mathcal{A}_s$, since, by Assumption 2.1, any other $\underline{\alpha}_v^* \in \mathcal{A}_s$, v = 1, 2, ..., $\binom{q}{n+1}$, violates Eq. 2.51 for at least one $i \in \{i = 1, 2, \ldots, q\}$. This completes the sketch of Theorem 2.2.

We can now state the method which yields the best Chebyshev approximating real vector $\underline{f}^* \triangleq [E]\ \underline{\alpha}^*$ to a real vector $\underline{f}$ in U^q by using "the method of ascent" as follows:

<u>Corollary 2.1:</u> Let $\underline{f}^* \triangleq [E]\ \underline{\alpha}^*$ be the best Chebyshev approximation to $\underline{f} \in U^q$, where [E] is a given q x n matrix of rank n, defined by Eq. 1.12, and let $\underline{f}^{*(v)} \triangleq [E^{(v)}]\ \underline{\alpha}_v^*$ be the best Chebyshev approximation to $\underline{f}^{(v)} \in U_v^{n+1}$, where the matrix $[E^{(v)}]$ is a (n+1) x n matrix,

defined by Eq. 2.32, $\underline{f}^{(v)}$ is defined by Eq. 2.28 and $v = 1, 2, \ldots, \binom{q}{n+1}$. Then, there exists an (n+1)-dimensional reference subspace denoted by U_c^{n+1} so that the vector $\underline{\alpha}_c^*$ of the best Chebyshev approximation $\underline{f}^{*(c)} \triangleq [E^{(c)}]\,\underline{\alpha}_c^*$ to $\underline{f}^{(c)} \in U_c^{n+1}$ is equal to the vector $\underline{\alpha}^*$ of the best Chebyshev approximation $\underline{f}^* \triangleq [E]\,\underline{\alpha}^*$ to $\underline{f} \in U^q$; and

$$\|\underline{\epsilon}^{*(c)}\|_\infty \triangleq \|\underline{f}^{(c)} - [E^{(c)}]\,\underline{\alpha}_c^*\|_\infty = \|\underline{\epsilon}^*\|_\infty \triangleq \|\underline{f} - [E]\,\underline{\alpha}^*\|_\infty \tag{2.52}$$

where $\underline{f}^{(c)}$, $[E^{(c)}]\,\underline{\alpha}_c^*$, and $\underline{\epsilon}^{*(c)}$ are (n+1)-dimensional vectors in U_c^{n+1}. Furthermore, this reference space, U_c^{n+1}, is one which maximizes the deviation $\|\underline{\epsilon}^{*(v)}\|_\infty$ of the best Chebyshev approximation to $\underline{f}^{(v)}$ in U_v^{n+1} among all the (n+1)-dimensional subspaces $\left\{U_v^{n+1} : v = 1, 2, \ldots, \binom{q}{n+1}\right\}$; that is,

$$\|\underline{\epsilon}^{*(c)}\|_\infty \geq \|\underline{\epsilon}^{*(v)}\|_\infty, \quad v = 1, 2, \ldots, \binom{q}{n+1} \tag{2.53}$$

The proof is along the same lines as the sketch of the proof of Theorem 2.2 (see Ref. 19, p. 66).

This corollary states that the problem of minimizing the value of $\|\underline{\epsilon}(\underline{\alpha})\|_\infty \triangleq \|\underline{f} - [E]\,\underline{\alpha}\|_\infty$ with respect to $\underline{\alpha} \in \mathscr{A}_s$, that is,

$$\|\underline{\epsilon}^*\|_\infty = \min_{\underline{\alpha} \in \mathscr{A}_s} \|\epsilon(\underline{\alpha})\|_\infty \tag{2.54}$$

where $\underline{f}$, [E] $\underline{\alpha}$, and $\underline{\epsilon}$ are vectors in U^q, may be replaced by the problem of determining the largest value of $\|\underline{\epsilon}^{(v)}(\underline{\alpha}_v^*)\|_\infty$ from the set

$$\left\{ \|\underline{\epsilon}^{(v)}(\underline{\alpha}_v^*)\|_\infty \triangleq \|\underline{f}^{(v)} - [E^{(v)}]\,\underline{\alpha}_v^*\|_\infty : v = 1, 2, \ldots, \binom{q}{n+1} \right\}$$

that is,

$$\|\underline{\epsilon}^*\|_\infty = \max_v \left\{ \|\underline{\epsilon}^{(v)}(\underline{\alpha}_v^*)\|_\infty : v = 1, 2, \ldots, \binom{q}{n+1} \right\} \tag{2.55}$$

where $[E^{(v)}]\,\underline{\alpha}_v^*$ is the best Chebyshev approximation to $\underline{f}^{(v)}$ in U_v^{n+1}.

We are now in a position to illustrate a method of computing the best Chebyshev approximation to a real vector $\underline{f}$ in U^q, based on Corollary 2.1. One merely computes the set $\left\{\underline{\alpha}_v^*\right\}$, which defines the best approximation of $\underline{f}^{(v)}$ on $C_n(E^{(v)})$ in all the (n+1)-dimensional reference subspaces $\left\{U_v^{n+1}\right\}$, $v = 1, 2, \ldots, \binom{q}{n+1}$, and then chooses the one $\underline{\alpha}_v^*$ which yields the largest deviation $\|\underline{\epsilon}^{(v)}(\underline{\alpha}_v^*)\|_\infty$.

This is illustrated in Fig. 4, where U^q is assumed to be a real 3-dimensional space, E^3. The vector $\underline{f}$ and the line $C_1(E)$ represents the prescribed vector and approximating subspace, defined by a (3 x 1) matrix [E]. The three distinct 2-dimensional reference subspaces U_v^2, where $v = 1, 2, 3$, are the three coordinate planes 1-2, 2-3 and 3-1, respectively. The projections of $\underline{f}$ and $C_1(E)$ on the coordinate planes are depicted by $\underline{f}^{(v)}$ and $C_1(E^{(v)})$, respectively, where $v = 1, 2, 3$. The cube shown represents the smallest cube which can be fitted between $\underline{f}$ and $C_1(E)$, i.e., its edges are of length $\|\underline{\epsilon}^*\|_\infty$. By comparing the projection of this cube on the coordinate plane and the smallest square of length $\|\underline{\epsilon}^{(v)}(\underline{\alpha}_v^*)\|_\infty$ which can be fitted between $\underline{f}^{(v)}$ and $C_1(E^{(v)})$ for all $v = 1, 2, 3$, it is seen that the largest square is obtained when $v = 1$.

There are methods of selecting the appropriate (n+1)-dimensional subspace U_c^{n+1} out of the set of $\left\{U_v^{n+1}\right\}$ which are more systematic than the random search described above. One of these,[25] because of Stiefel (Ref. 22), uses a point-by-point exchange procedure which systematically converges to the solution. Another algorithm,[26] because of Remez (Ref. 17), which converges faster to the Chebyshev approximation, is based on exchanging all the points at each step.

At this point, we conclude our review of the approximation problem when the pole location is initially prescribed; i.e., the vector $\underline{s}$ in $\mathscr{S}$ of the matrix $[E(\underline{s})]$ is prescribed. However, before considering the contributions to the approximation problem when $\underline{s}$ is <u>not</u> prescribed initially, we shall review Prony's original work. We do this for two reasons. First, to introduce the formulation and illustrate how the constraint which requires that $\left\{t_i : i = 1, 2, \ldots, q\right\}$ be equally-spaced, simplifies the mathematics. Second, to demonstrate that when $q = 2n$,

---

[25]See Appendix A.

[26]The reader is referred to the treatment of this subject by Rice (Ref. 19, pp. 176-180).

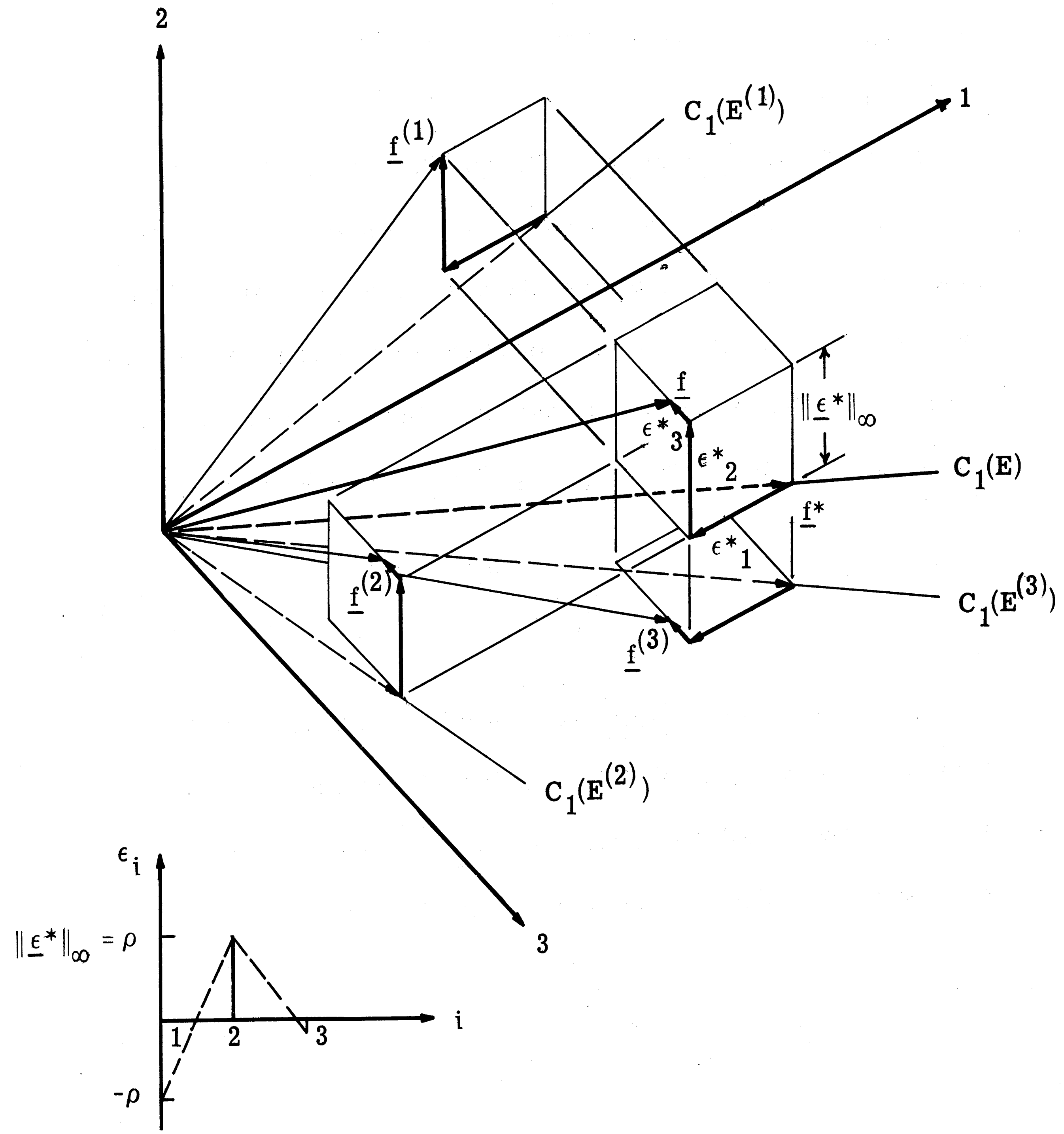


Fig. 4. Approximation of $\underline{f}$ in E^3 onto $C_1(E)$ in the Chebyshev sense.

we can obtain an exact representation of any real $\underline{f}$ in U^q, i. e., when $q = 2n$, then in general $\underline{\epsilon} = \underline{0}$.

2.3.2 The Original Method of Prony. Prony (Ref. 15) solved the following exponential interpolation problem: Given a set of measured data points of a process; determine the interpolation the values of the intermediate points when the behavior of the process is governed by a linear combination of exponential functions. In other words, he sought the values of the parameters $\{\alpha_k, s_k\}$, $k = 1, 2, \ldots, n$, of

$$f(t) = \sum_{k=1}^{n} \alpha_k e^{s_k t} \qquad (2.56)$$

when the set of 2n points $\{f(t_i)\}$ were specified. To carry out this interpolation, he suggested that the 2n points $\{t_i\}$ be equally spaced. Hence, he determined the 2n unknowns $\{\alpha_k, s_k\}$, $k = 1, 2, \ldots, n$ from the system of 2n simultaneous equations given by

$$f(t_i) = \sum_{k=1}^{n} \alpha_k e^{s_k t_i}, \quad i = 1, 2, \ldots, 2n \qquad (2.57)$$

where $t_i = t_1 + (i-1)\,\Delta t$, and Δt is the interval between the equally spaced $\{t_i\}$. At this point he observed that since there are 2n unknowns in Eq. 2.56 which must satisfy 2n independent conditions, an exact solution is possible.

Now before turning to a detailed consideration of Prony's method of solution, we observe that if the 2n unknowns $\{\alpha_k, s_k\}$ must satisfy q conditions, where $q > 2n$, then Eq. 2.57 represents a system of over-determined equations. Since, in general, an exact solution to such a problem cannot be obtained, then one must seek an approximate solution, by using approximation methods. The extension of Prony's original method to finding the approximate solution of such a system has been attempted in the literature and is sometimes misnamed, "the Prony method." To avoid this misconception, we shall refer to Prony's initial interpolation method as "Prony's Original Method," and the approximation method based on this method as "Prony's Extended Method."[27]

In this section, we shall review "Prony's Original Method" in terms of operations in the vector space U^q, when $q = 2n$. It is hoped that this approach will aid in clarifying the limitations of the previous works (discussed in Section 2.3.3) in which this method is used to solve approximation problems of network synthesis. Clearly, Eq. 2.57 may be stated in vector notation using the formulation of Section 1.4 as follows:

Given a real vector $\underline{f}$ in U^{2n}, determine the ordered pair of parameter vectors $(\underline{\beta}, \underline{z})$ in $\mathcal{B}_z \times \mathcal{Z}$ so that

---

[27] The "Prony's Extended Method" should not be confused with the previous methods (to be discussed in Section 2.3.3), which use the approach of "Prony's Original Method" to solve approximation problems. The formal definition of "Prony's Extended Method" will be given in Chapter III.

$$\underline{f} = [Z(\underline{z})]\,\underline{\beta} \tag{2.58}$$

where[28]

$$[Z(\underline{z})] = \begin{bmatrix} 1 & \dots & 1 \\ z_1 & & z_n \\ z_1^2 & & z_n^2 \\ \vdots & & \vdots \\ z_1^{2n+1} & \dots & z_n^{2n-1} \end{bmatrix}, \quad \underline{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \text{ and } \underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \tag{2.59}$$

The essential contribution of Prony is in the method he employed in finding the vector pair $(\underline{\beta}, \underline{z})$. He observed that if the (2nxn) matrix [Z] is of rank n, he could replace the 2n unknowns $\{\beta_k, z_k\}$, $k = 1, 2, \dots, n$ by a new set of n unknowns $\{r_i\}$, $i = 0, \dots, n-1$, which linearly relate any (n+1) successive rows of the matrix [Z]. Furthermore, this relation may be expressed by the n^{th}-order polynomial equation

$$P_n(z) \triangleq z^n + \sum_{i=0}^{n-1} r_i z^i = \prod_{k=1}^{n} (z - z_k) = 0 \tag{2.60}$$

---

[28] Note that the matrix $[Z(\underline{z})]$ defined here, represents the matrix defined by Eq. 1.18, where $q = 2n$, that is, it illustrates the form of $[Z(\underline{z})]$ when the components of $z \in \mathcal{Z}$ are distinct. If the components of $\underline{z}$ are not distinct then $[Z(\underline{z})]$ takes on the format of Eq. 1.18a, where $q = 2n$.

the roots of which are the n-unknowns $\{z_k\}$. Hence, this method consists of first determining the n unknowns $\{r_i\}$, $i = 0, 1, \ldots, n-1$, and then determining the 2n parameters, $\{\beta_k, z_k\}$, $k = 1, \ldots, n$. This is illustrated as follows:

If the 2nxn matrix [Z], defined by Eq. 2.59, is of rank n, then the format of [Z] reveals that any n consecutive rows of [Z] are independent. Hence, any $(n+1)^{st}$ row can be represented by the linear combination of the previous n rows, that is

$$z_k^{n+v-1} = \sum_{i=1}^{n} c_i z_k^{v+i-2} , \qquad \begin{array}{l} k = 1, 2, \ldots, n \\ v = 1, 2, \ldots, n \end{array} \tag{2.61}$$

Observe that the set $\{z_k : k = 1, 2, \ldots, n\}$ represents the n-roots of the n^{th} order polynomial equation given by

$$z^n - \sum_{i=1}^{n} c_i z^{i-1} = \prod_{k=1}^{n} (z - z_k) = 0 \tag{2.62}$$

Hence, if we let $c_i = -r_{i-1}$, then Eq. 2.62 yields the polynomial equation given by Eq. 2.60, and Eq. 2.61 may be written as

$$z^{v-1} \left\{P_n(z)\right\} = 0 , \qquad v = 1, 2, \ldots, n \tag{2.63}$$

where $P_n(z)$ is the n^{th}-order polynomial defined by Eq. 2.60.

At this point it should be noted that the polynomial $P_n(z)$ is invariant of the index v in Eq. 2.63. Furthermore, by specifying the n-coefficients $\{r_i : i = 0, 1, \ldots, n-1\}$ of $P_n(z)$ one also specifies the elements of the matrix $[Z(\underline{z})]$ defined by Eq. 2.59, since the set of values $\{z_k : k = 1, 2, \ldots, n\}$ can be determined directly from Eq. 2.60.

We now have the problem of determining the values of the n-coefficients $\{r_i : i = 0, 1, 2, \ldots, n-1\}$ of $P_n(z)$ from the 2n prescribed data points $\{f_i : i = 1, 2, \ldots, 2n\}$. Recall that since the set of n-unknowns $\{r_i\}$ linearly relate any (n+1) rows of the matrix $[Z(z)]$, then by the relation of Eq. 2.58 they must also linearly relate any (n+1) successive components of the vector $\underline{f}$, that is,

$$f_{v+n} + \sum_{i=0}^{n-1} r_i \, f_{i+v} = 0 , \qquad v = 1, 2, \ldots, n \tag{2.64}$$

Clearly, Eq. 2.64 represents a system of n simultaneous linear equations in n-unknowns. If these equations are independent, then they yield a unique set of $\{r_i\}$, which are real since the $\{f_i\}$ are real. Knowing $\{r_i\}$, the $\{z_k\}$ follow, being the roots of Eq. 2.60. (Note that since the $\{r_i\}$ are real, then the roots, $\{z_k\}$, of Eq. 2.60, are real or occur in conjugate pairs.) If all the roots of Eq. 2.60 are distinct, then the $\{\beta_k\}$ are determined from Eq. 2.58, where only the first n-relations need be considered. On the other hand, if some of the roots of Eq. 2.60 are repeated, then Prony shows that the $\{\beta_k\}$ can be determined by using a modified form of Eq. 2.58. This modified form of Eq. 2.58 is obtained from Eq. 2.56, by replacing the function $e^{s_k t}$, for each repeated root of Eq. 2.60, by the function $t^{j-1} e^{s_k t}$, where j denotes the order of the repeated root. For example, if $z_1 = z_2 = \ldots = z_j$ represents the j^{th} order root of Eq. 2.60, then Eq. 2.57 is written as[29]

$$f_i = (\alpha_1 + \alpha_2 t_i + \ldots + \alpha_j t_i^{j-1}) e^{s_1 t_i} + \sum_{k=j+1}^{n} \alpha_k e^{s_k t_i} ,$$

$$i = 1, 2, \ldots, 2n$$

Since $t_i = t_1 + (i-1)\Delta t$, then, if $t_1 = 0$, this expression yields the following modified form of Eq. 2.58,

$$f_i = [\beta_1 + (i-1)\Delta t\, \beta_2 + \ldots + (i-1)^{j-1} \Delta t^{j-1} \beta_j]\, z_1 + \sum_{k=j+1}^{n} \beta_k z_k^{\,i-1} ,$$

$$i = 1, 2, \ldots, 2n$$

This is illustrated in the following example:

Example 2.1: Let us fit n = 2 exponential functions to the function f(t) = 3 - t at the following 2n discrete points t = 0, 1, 2, 3, i.e., $\Delta t = 1$. In other words, given the vector

$$\underline{f} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

[29]See Eq. 1.13a.

find the vector pair $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$ when n = 2.

Equation 2.64 yields the following sets of equations

$$3\,r_0 + 2\,r_1 + 1 = 0$$

$$2\,r_0 + r_1 = 0$$

Solving this system of equations, yields $r_0 = 1$, $r_1 = -2$. Hence Eq. 2.60 becomes

$$z^2 - 2z + 1 = 0$$

the roots of which are $z_1 = 1$, $z_2 = 1$. Since we have a double root, i.e., $z_1 = z_2 = 1$ and since $t_1 = 0$ and $\Delta t = 1$, then from Eq. 1.18b the matrix $[Z(z)]$ becomes

$$[Z(z)] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solving for the vector $\underline{\beta}$ which satisfies Eq. 2.58, that is,

$$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

one obtains that $\underline{\beta} = [3, -1]$. Therefore, the vector pair $(\underline{\beta}, \underline{z})$ is given by

$$(\underline{\beta}, \underline{z}) = \left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

Mapping the vector pair $(\underline{\beta}, \underline{z})$ into $(\underline{\alpha}, \underline{s})$ yields[30]

$$(\underline{\alpha}, \underline{s}) = \left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

Hence, the interpolating function, given by

$$\hat{f}(t; \underline{\alpha}, \underline{s}) = [\alpha_1 + \alpha_2 t] \; e^{s_1 t}$$

becomes

$$\hat{f}(t) = (3 - t)$$

Clearly, the interpolating function $\hat{f}(t)$ does indeed pass through all four sample points. In fact, since the function $\hat{f}(t)$ is identical to f(t), the interpolating function $\hat{f}(t)$ will pass through any q-sampling points, where[31] $q > 2n = 4$.

At this point, we recast Prony's method in terms of vector operations in U^{2n}. First, let us redefine the polynomial of Eq. 2.60 as

---

[30] This is discussed in detail in Section 6.2 of Chapter VI.

[31] Note that this problem illustrates the case when the approximation problem yields a zero error.

$$P(z) \triangleq \sum_{i=0}^{n} r_i \; z^i \tag{2.65}$$

Clearly, when $r_n \neq 0$, then the zeros of this polynomial are identical to the zeros of the n^{th} order polynomial $P_n(z)$.. Hence, the choice of $r_n = 1$ in the definition of $P_n(z)$ is arbitrary. If we denote the ordered set of coefficients $\{r_0, r_1, \ldots, r_n\}$ by the vector $\underline{r} \in E^{n+1}$, i.e.,

$$\underline{r} = \begin{bmatrix} r_o \\ r_1 \\ \vdots \\ r_n \end{bmatrix} \tag{2.66}$$

then it suffices to say that Prony's method seeks <u>only</u> the direction of $\underline{r} \in E^{n+1}$, and <u>not</u> its magnitude. In other words, Prony's method seeks the vector $\underline{r} \in E^{n+1}$ which is restricted to the set defined by $\| \underline{r} \|_p = 1$. The implications of this formulation will become evident in the next sections, where the extensions to the "Prony's Original Method" are discussed. For the present discussion we shall follow "Prony's Original Method" and assume that $r_n = 1$.

Let us now represent the set of n polynomial equations of Eq. 2.63 in vector form, namely,

$$[Z(\underline{z})]^T \, \underline{r}^{(v)} = 0 \;, \qquad v = 1, 2, \ldots, n \tag{2.67}$$

where the 2n-dimensional vector $\underline{r}^{(v)}$ is given by

$$\underline{r}^{(v)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_0 \\ r_1 \\ \vdots \\ r_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow v^{th} \text{ components} \\ \\ \\ \\ \\ \\ \\ \end{matrix}$$

It is evident from Eq. 2.67 that the set of vectors $\left\{\underline{r}^{(v)}\right\}$, $v = 1, 2, \ldots, n$ are orthogonal to the column space[32] of $[Z(\underline{z})]$, $C_n(Z)$; that is,

$$([Z(\underline{z})]\,\underline{\beta}, \underline{r}^{(v)}) = 0 , \qquad v = 1, 2, \ldots, n \qquad (2.68)$$

for all $\underline{\beta} \in \mathscr{B}_z$. In fact, the set $\left\{\underline{r}^{(v)}\right\}$, $v = 1, 2, \ldots, n$, spans the orthogonal complement subspace of $C_n(Z)$ in U^{2n}. In matrix form, this orthogonal complement subspace may be represented by the column space of a $2n \times n$ matrix, $[R]$, which is defined by

[32]Recall from Section 1.4, that the matrix $[Z(\underline{z})]$, defined by Eq. 1.18 (or Eq. 1.18a), is of maximal rank n, for all $\underline{z} \in \mathscr{Z}$.

$$[R] \triangleq [\underline{r}^{(1)}\,\underline{r}^{(2)} \ldots \underline{r}^{(n)}] = \begin{bmatrix} r_0 & 0 & \ldots & 0 & 0 \\ r_1 & r_0 & & \vdots & 0 \\ r_2 & r_1 & & 0 & \vdots \\ \vdots & r_2 & & r_0 & 0 \\ r_n & \vdots & & r_1 & r_0 \\ 0 & r_n & & r_2 & r_1 \\ \vdots & 0 & & \vdots & r_2 \\ 0 & \vdots & & r_n & \vdots \\ 0 & 0 & \ldots & 0 & r_n \end{bmatrix} \qquad (2.69)$$

We have thus defined an n-dimensional subspace, $C_n(R)$, which is a function of the (n+1)-dimensional parameter vector $\underline{r}$ in E^{n+1}, defined by Eq. 2.66. Furthermore, this subspace is orthogonal complement subspace[33] of $C_n(Z)$ in U^{2n}.

In order to satisfy the equation $\underline{f} = [Z(\underline{z})]\,\underline{\beta}$, the vector $\underline{f}$ must lie in $C_n(Z)$, or alternatively $\underline{f}$ must be orthogonal to $C_n(R)$; that is,

$$[R]^T \underline{f} = 0 \qquad (2.70)$$

Clearly, when $r_n = 1$, this expression represents the system of simultaneous linear equations of Eq. 2.64. Observe that Eq. 2.70 may also be written as

---

[33] This will be discussed further in Chapter III.

$$[F]\,\underline{r} = 0 \tag{2.71}$$

where

$$[F] = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n+1} \\ f_2 & f_3 & & f_{n+2} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n+1} & \cdots & f_{2n} \end{bmatrix} = n\,x\,(n+1) \text{ matrix}$$

This equation can be solved for $\underline{r}$ when one of the components of $\underline{r}$ is arbitrarily chosen to be equal to one. For example, if we select $r_n = 1$, and if the first n columns of the matrix [F] yield an nxn nonsingular matrix, then the other n components of $\underline{r}$ can be obtained from

$$\begin{bmatrix} r_o \\ \vdots \\ r_{n-1} \end{bmatrix} = - \begin{bmatrix} f_1 & \cdots & f_n \\ \vdots & & \vdots \\ f_n & \cdots & f_{2n-1} \end{bmatrix}^{-1} \begin{bmatrix} f_{n+1} \\ \vdots \\ f_{2n} \end{bmatrix} \tag{2.72}$$

In conclusion, then, we have shown that Prony's original contribution consists of determining a new parameter vector $\underline{r}$ which is related to $\underline{z}$ by Eq. 2.65 and defines an n-dimensional orthogonal complement subspace to $C_n(Z)$ in U^{2n}. Using this setting, we are now in a position to summarize the contributions by Yengst and Ruston to approximation methods for network synthesis.

2.3.3 The Approximation Problem when the Matrix [E] is Not Initially Prescribed. We shall now consider the previous contributions to the problem of network synthesis using discrete approximation techniques. This is the case described in Eq. 2.9 when the vector $\underline{s} \in \mathscr{S}$ (i.e., the q x n matrix $[E(\underline{s})]$) is not initially prescribed and when $q > 2n$. The two significant contributions towards the solution of this problem were made by Yengst (Ref. 26) and Ruston (Ref. 20) who employed the least-squares and the Chebyshev criterion, respectively, to judge the degree of approximation. However, in no case, was the optimum pole location (i.e., the vector $\underline{s}^* \in \mathscr{S}$) determined.

Both Yengst and Ruston used Prony's approach to solve their respective approximation problems. Therefore, their methods require that the discrete values of the prescribed impulse response be taken at equally-spaced points of time. Although they did not formulate their problem in terms of the theory of approximations in U^q, it can be shown that they sought the best vector pair $(\underline{\beta}^{**}, \underline{z}^*) \in \mathscr{B}_z \times \mathscr{Z}$ which minimizes the ℓ_p^q-norm of the error vector (i.e., $\|\underline{\epsilon}(\underline{\beta}, \underline{z})\|_p$ with $p = 2$ for Yengst's problem and $p = \infty$ for Ruston's problem) of the equation

$$\underline{h} = [Z(\underline{z})]\,\underline{\beta} + \underline{\epsilon}(\underline{\beta}, \underline{z}) \quad \text{in} \quad U^q \tag{2.73}$$

where $q > 2n$. Their methods of solution involve a two-stage approximation process, by which they first determine an estimate of the pole location (or equivalently, the parameter vector $\underline{z}$) by extending Prony's original method; they then determine the optimum residues (or equivalently, the parameter vector $\underline{\beta}^*$) for this pole location.

Now before turning to detailed consideration of their methods of solution, let us show that their approximation procedure yields an error norm that is not necessarily the minimum possible. Recall, that when q = 2n, "Prony's Original Method" yields a system of n simultaneous equations, given by Eq. 2.70; i.e.,

$$[R(\underline{r})]^T \underline{h} = 0 \quad \text{in } E^n \tag{2.74}$$

This system can be directly solved for the vector $\underline{r}$. They argued that when $q > 2n$, one obtains a similar system of equations; however, now Eq. 2.74 characterizes an overdetermined system of equations since $[R]^T \underline{h}$ is a (q-n)-dimensional vector, where $(q-n) > n$. Being thus denied an exact solution, they sought the best approximate solution to this overdetermined system of equations, i.e., the vector $\tilde{\underline{r}}$ which satisfies the vector relation

$$[R(\underline{r})]^T \underline{h} = \underline{\delta}(\underline{r}) \quad \text{in } E^{q-n} \tag{2.75}$$

when $\|\underline{\delta}\|_p$ is minimum. By applying the operator $[R]^T$ to Eq. 2.73, one obtains for comparison, the result[34]

[34] To emphasize the dependence of the vector $\underline{\epsilon}$ on $\underline{r}$, the vector $\underline{\epsilon}(\underline{\beta}, \underline{z})$ is denoted by $\underline{\epsilon}(\underline{r})$.

$$\underline{\delta}(\underline{r}) = [R(\underline{r})]^T \underline{\epsilon}(\underline{r}) \quad \text{in } E^{q-n} \tag{2.76}$$

Clearly, minimizing the function $\|\underline{\delta}(\underline{r})\|_p$ with respect to $\underline{r}$ does not necessarily minimize the function $\|\underline{\epsilon}(\underline{r})\|_p$. Therefore, in general, the vector $\tilde{\underline{r}}$, which represents the solution vector $\underline{r}$ when $\|\underline{\delta}(\underline{r})\|$ is minimum, will not be equal to the vector $\underline{r}^*$, which represents the solution vector $\underline{r}$ when $\|\underline{\epsilon}(\underline{r})\|_p$ is minimum.

Furthermore, it should be mentioned that Ruston's and Yengst's approximation procedure considers only the vectors $\underline{r} \in E^{n+1}$ with the (n+1)st component equal to one, i.e., $r_n = 1$, rather than the vectors $\underline{r} \in E^{n+1}$ which[35] lie in the set defined by $\|\underline{r}\|_p = 1$. Hence, they minimize the function $\|\underline{\delta}(\underline{r}')\|_p$ with respect to $\underline{r}' \in E^n$, where the vector $\underline{r}'$ is defined by

$$\underline{r}' = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix} \tag{2.77}$$

The consequence of this assumption will be discussed further in the next sections, where their individual works are considered.

---

[35]See Eq. 2.66 of Section 2.3.2.

2.3.3.1 The Least-Square Approximation Method of Yengst.

Yengst (Ref. 26) considered the network synthesis problem using discrete least-square approximation techniques in the time domain. This problem is stated by Eq. 2.73. By using "Prony's Original Method" and the general least-square approximation theory, he was able to obtain an approximate solution to this problem.

In the setting introduced above, Yengst determines the n-dimensional vector $\underline{r}'$, defined by Eq. 2.77 which satisfies Eq. 2.75 in the best least-square sense; that is,

$$\|\underline{\delta}(\tilde{\underline{r}}')\|_2 \triangleq \|[R(\tilde{\underline{r}}')]^T \underline{h}\|_2 \leq \|[R(\underline{r}')]^T \underline{h}\|_2 \qquad (2.78)$$

for all $\underline{r}'$ in E^n. To elucidate the dependence of the vector $\underline{\delta}$ on the vector $\underline{r}'$, let us rewrite Eq. 2.75 as

$$\underline{\delta}(\underline{r}') = [H_n]\,\underline{r}' + \underline{h}_n \qquad (2.79)$$

where the $(q-n)\times n$ matrix $[H_n]$ and the $(q-n)$-dimensional vector $\underline{h}_n$ are defined by

$$H_n = \begin{bmatrix} h_1 & \cdots & h_n \\ h_2 & & h_{n+1} \\ \vdots & & \vdots \\ h_{q-n} & \cdots & h_{q-1} \end{bmatrix}, \quad \text{and} \quad \underline{h}_n = \begin{bmatrix} h_{n+1} \\ h_{n+2} \\ \vdots \\ h_q \end{bmatrix} \qquad (2.80)$$

Since the ℓ_2-norm of $\underline{\delta}$ is given by

$$\|\underline{\delta}(\underline{r}')\|_2^2 = ([H_n]\,\underline{r}' + \underline{h}_n\,,\ [H_n]\,\underline{r}' + \underline{h}_n)$$

then, it follows from $\frac{\partial}{\partial \underline{r}'}\ \|\underline{\delta}(\underline{r}')\|_2^2 = 0$ that

$$[H_n^T\,H_n]\,\tilde{\underline{r}}' = -H_n^T\,\underline{h}_n \tag{2.81}$$

This expression is the vector equivalent of Eq. 32 of Yengst's paper, (Ref. 26). The vector $\tilde{\underline{r}}'$ represents the stationary point of the function $\|\underline{\delta}(\underline{r}')\|_2^2$. If the nxn matrix $[H_n^T\,H_n]$ is nonsingular, $\tilde{\underline{r}}'$ can be determined from

$$\tilde{\underline{r}}' = -[H_n^T\,H_n]^{-1}\,H_n^T\,\underline{h}_n \tag{2.82}$$

Having determined $\tilde{\underline{r}}'$, Yengst proceeds to determine the vector $\tilde{\underline{z}}$ in terms of the roots of the polynomial equation

$$z^n + \sum_{i=0}^{n-1} \tilde{r}_i'\,z^i = \prod_{k=1}^{n} (z - \tilde{z}_k) = 0$$

From the above, it is evident that the resulting pole locations (or equivalently, the parameter vector $\tilde{\underline{z}}$) depend on the stationary point of the function $\|\underline{\delta}(\underline{r}')\|_2^2 = \|[R(\underline{r}')]^T\,\underline{\epsilon}(\underline{r}')\|_2^2$ rather than on the stationary point, $\underline{r}^*$, of the function $\|\underline{\epsilon}(\underline{r})\|_2^2$. Hence, Yengst's approximation technique does not necessarily yield an optimum least-square approximation to the prescribed real vector $\underline{h}$ in U^q.

Furthermore, since Yengst initially assumes that the (n+1)st component of $\underline{r} \in E^{n+1}$ is equal to one, then the vector given by $[\tilde{r}_1', \tilde{r}_2', \ldots, \tilde{r}_n', 1]^T$ may not represent the vector $\underline{r} \in E^{n+1}$ which minimizes the function $\|\underline{\delta}(\underline{r})\|_2$ along the set defined by $\|\underline{r}\|_p = 1$, when $\underline{\delta}(\underline{r})$ is defined by Eq. 2.75. This may be best illustrated by defining the vector $\underline{\delta}(\underline{r})$ of Eq. 2.75 in terms of $\underline{r} \in E^{n+1}$, namely

$$\delta(\underline{r}) = [H]\, \underline{r} \tag{2.83}$$

where

$$[H] = \begin{bmatrix} h_1 & \cdots & h_{n+1} \\ h_2 & \cdots & h_{n+2} \\ \vdots & & \vdots \\ h_{q-n} & \cdots & h_q \end{bmatrix}$$

If one initially assumes that some other component of $\underline{r}$ is equal to one, then Eq. 2.83 clearly yields a relation which differs from Eq. 2.72 only by the value of the elements of the matrix $[H_n]$ and the vector $\underline{h}_n$. For example, if the component $r_0 = 1$, then Eq. 2.83 may be written as

$$\underline{\delta}(\underline{r}'') = [H_0]\, \underline{r}'' + \underline{h}_0 \tag{2.84}$$

where

$$\underline{r}'' = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}, \qquad \underline{h}_0 = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{q-n} \end{bmatrix}$$

and

$$[H_0] = \begin{bmatrix} h_2 & \cdots & h_{n+1} \\ h_3 & \cdots & h_{n+2} \\ \vdots & & \vdots \\ h_{q-n+1} & \cdots & h_q \end{bmatrix} = (q-n) \times n \text{ matrix}$$

Note that while the vector $\underline{h}_0$ represents the vector to be approximated in Eq. 2.84, it represents a vector which spans the approximating subspace, defined by the column space of $[H_n]$, in Eq. 2.79. To illustrate that the relations given by Eq. 2.79 and 2.84 may yield two different best least-squares solutions (after making a normalization with respect to $\|\underline{r}\|_p$), let us suppose that in Eq. 2.84 the vector $\underline{h}_0$ lies in the column space of the matrix $[H_0]$. Clearly then, Eq. 2.84 may be solved exactly, so that the minimal value of $\|\underline{\delta}(\underline{r}'')\|_2 = 0$. On the other hand, Eq. 2.79 cannot be solved exactly (if the vector $\underline{h}_n$ does not lie in the column space of $[H_n]$), so that the minimal value of $\|\underline{\delta}(\underline{r}')\|_2 \neq 0$. Hence we note that, for this case, there exists an $\underline{r} \in E^{n+1}$ along the set $\|\underline{r}\|_p = 1$ which yields a $\|\underline{\delta}(\underline{r})\|_2 = 0$, where $\underline{\delta}(\underline{r})$ is defined by Eq. 4.84, a result which is not obtained by using Yengst's formulation of Eq. 2.79.

2.3.3.2 The Chebyshev Approximation Method of Ruston.

Ruston (Ref. 20) considered the Chebyshev approximation problem of network synthesis defined by Eq. 2.73. His method of solution is similar to that used by Yengst. By using "Prony's Original Method", and Stiefel's work (presented in Appendix A) he was able to obtain an approximate solution to this problem. His method involves a two-stage approximation where he first determines the parameter vector $\underline{z}$ and then determines the optimum parameter vector $\underline{\beta}^*$ for this $\underline{z}$. His procedure leads to an approximation which gives a Chebyshev type error, but as will be illustrated in the next paragraph, it is not the minimum possible Chebyshev error.

To determine the parameter vector $\underline{z}$, Ruston begins by finding the n-dimensional vector $\underline{\tilde{r}}'$ which satisfies Eq. 2.75, so that

$$\|\underline{\delta}(\underline{\tilde{r}}')\|_\infty \triangleq \|[R(\underline{\tilde{r}}')]^T \underline{h}\|_\infty \leq \|[R(\underline{r}')]^T \underline{h}\|_\infty \tag{2.85}$$

for all $\underline{r}'$ in E^n. Actually, to determine $\underline{\tilde{r}}'$ he uses the alternate representation of the vector $\underline{\delta}(\underline{r}')$ given by Eq. 2.79; that is,

$$\underline{\delta}(\underline{r}') = [H_n]\,\underline{r}' + \underline{h}_n \tag{2.86}$$

Since both the (q-n)xn matrix $[H_n]$ and the (q-n)-dimensional vector $\underline{h}_n$ are known (see Eq. 2.80), the vector $\underline{\delta}$ is strictly a function of the n-dimensional vector $\underline{r}'$, defined by Eq. 2.87. Hence, if $(q-n) > n$ and $[H_n]$ is of rank n, then the problem of finding the vector $\underline{\tilde{r}}' \epsilon E^n$ which minimizes

$\|\underline{\delta}(\underline{r}')\|_\infty$, where the vector $\underline{\delta}(\underline{r}')$ is defined by Eq. 2.86, is simply the Chebyshev approximation problem considered in Section 2.3.1.2.

The existence of the best Chebyshev approximate solution vector $\tilde{\underline{r}}' \epsilon E^n$ is given by Theorem 2.2, where now that the vector $\underline{h}_n$ and the n columns of the matrix $[H_n]$ describe vectors in a (q-n)-dimensional Euclidean vector space, E^{q-n}. At this point, Ruston applies Stiefel's algorithm to obtain the $\tilde{\underline{r}}' \epsilon E^n$ which satisfies Eq. 2.85. Once the vector $\tilde{\underline{r}}'$ is attained, then Ruston determines the parameter vector $\underline{z}$ from the roots of the polynomial equation

$$P_n(z) = z^n + \sum_{i=0}^{n-1} \tilde{r}_i \, z^i = 0 \qquad (2.87)$$

Let us denote the ordered set of roots of Eq. 2.87 by the vector $\tilde{\underline{z}}$. Consequently, from the vector $\tilde{\underline{z}}$, the q x n matrix $[Z(\tilde{\underline{z}})]$ can be fully determined. Denoting this matrix by $[\tilde{Z}]$, he arrives at the second approximation step which is defined by the equation

$$\underline{h} = [\tilde{Z}] \, \underline{\beta} + \epsilon \qquad (2.88)$$

where both the real vector $\underline{h}$ in U^q and the q x n matrix $[\tilde{Z}]$ are known. This step seeks the best Chebyshev approximate solution vector $\underline{\beta}^* \epsilon \mathscr{B}_z$ so that $\|\underline{\epsilon}\|_\infty$ is minimum. Clearly, the existence of a solution to this problem is given by Theorem 2.2, so that Ruston again applies Stiefel's algorithm to determine the vector $\underline{\beta}^* \epsilon \mathscr{B}_z$. It should be noted that the resulting error vector, given by

$$\underline{\epsilon}^*(\underline{\tilde{z}}) = \underline{h} - [\tilde{Z}]\,\underline{\beta}^* \qquad (2.89)$$

will satisfy the conditions given by Eq. 2.18 of Theorem 2.2; i.e., that the absolute values of at least (n+1) components[36] of $\underline{\epsilon}^*(\underline{\tilde{z}})$ are equal to $\|\underline{\epsilon}^*(\underline{\tilde{z}})\|_\infty$ and the absolute value of the others is less than $\|\underline{\epsilon}^*(\underline{\tilde{z}})\|_\infty$. To illustrate Ruston's method, consider the following example:[37]

<u>Example 2.2</u>: Given: q = 9, n = 2, and

$$\underline{h} = \begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.2500 \\ 0.1600 \\ 0.1110 \\ 0.0817 \\ 0.0625 \\ 0.0494 \\ 0.0400 \end{bmatrix}$$

Find the vector pair[38] $(\underline{\beta}^*, \underline{\tilde{z}}) \in \mathscr{B}_Z \times \mathscr{Z}$ such that the real vector $\underline{h}^*(\underline{\tilde{z}}) \triangleq [Z(\underline{\tilde{z}})]\,\underline{\beta}^*$ approximates $\underline{h}$ in the Chebyshev sense.

---

[36] In Chapter IV we shall show that in general the optimum solution to this Chebyshev approximation problem yields an error vector having at least (2n+1) components with absolute values equal to the ℓ_∞^q-norm of the error vector.

[37] The example we have chosen is presented on pp. 79-90 of Ruston's thesis (Ref. 20). However, we shall present this example using the notation developed above.

[38] It should be mentioned that Ruston intended to find the vector pair $(\underline{\beta}^{**}, \underline{z}^*) \in \mathscr{B}_Z \times \mathscr{Z}$, however, he only succeeded in finding the vector pair $(\underline{\beta}^*, \underline{\tilde{z}}) \in \mathscr{B}_Z \times \mathscr{Z}$. This is why we have stated the approximation problem as shown.

Let us begin by writing out the form of Eq. 2.86 as follows:

$$\begin{bmatrix} \delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta_7 \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.4450 \\ 0.4450 & 0.2500 \\ 0.2500 & 0.1600 \\ 0.1600 & 0.1110 \\ 0.1110 & 0.0817 \\ 0.0817 & 0.0625 \\ 0.0625 & 0.0494 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} + \begin{bmatrix} 0.2500 \\ 0.1600 \\ 0.1110 \\ 0.0817 \\ 0.0625 \\ 0.0494 \\ 0.0400 \end{bmatrix} \qquad (2.90)$$

This equation is Eq. 4.106 of Ruston's report (Ref. 20, p. 78). The vector $\underline{\tilde{r}}' \epsilon E^2$ which yields the minimum value of $\|\underline{\delta}\|_\infty$ is found using Steifel's algorithm (see Appendix A) to be

$$\underline{\tilde{r}}' = \begin{bmatrix} \tilde{r}_0 \\ \tilde{r}_1 \end{bmatrix} = \begin{bmatrix} 0.2033 \\ -1.0128 \end{bmatrix} \qquad (2.91)$$

and

$$\underline{\delta}(\underline{\tilde{r}}') = \begin{bmatrix} 0.002707 \\ -0.002707 \\ -0.000217 \\ 0.001801 \\ 0.002318 \\ 0.002707 \\ 0.002671 \end{bmatrix}$$

It is noted that (n+1) = 3 components of the vector $\underline{\delta}(\underline{\tilde{r}}')$ have absolute values which are equal to $\|\underline{\delta}(\underline{\tilde{r}}')\|_\infty = 0.0027$ (i.e., $|\delta_1| = |\delta_2| = |\delta_6| = 0.0027$) in accordance with Theorem 2.2. Substituting the values of $\{\tilde{r}_i'\}$, given by Eq. 2.91, into Eq. 2.87, and finding the roots of this polynomial equation yields

$$\underset{\sim}{\tilde{z}} = \begin{bmatrix} 0.7369 \\ 0.2759 \end{bmatrix}$$

Hence, Eq. 2.88 becomes

$$\begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.2500 \\ 0.1600 \\ 0.1110 \\ 0.0817 \\ 0.0625 \\ 0.0494 \\ 0.0400 \end{bmatrix} = \begin{bmatrix} 1.00000 & 1.00000 \\ 0.73690 & 0.27590 \\ 0.53402 & 0.07612 \\ 0.40015 & 0.02100 \\ 0.29487 & 0.00579 \\ 0.21729 & 0.00160 \\ 0.16012 & 0.00044 \\ 0.11799 & 0.00012 \\ 0.08695 & 0.00003 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_9 \end{bmatrix}$$

Solving this equation for $\underline{\beta}^*$, which minimizes $\|\underline{\epsilon}\|_\infty$, yields

$$\underline{\beta}^* = \begin{bmatrix} 0.38427 \\ 0.60917 \end{bmatrix}$$

$$\underline{\epsilon}^*(\underline{\tilde{z}}) = \begin{bmatrix} 0.00656 \\ -0.00424 \\ -0.00504 \\ -0.00656 \\ -0.00584 \\ -0.00277 \\ 0.00070 \\ 0.00399 \\ 0.00656 \end{bmatrix}$$

Again, it is noted that only $(n+1) = 3$ components of $\underline{\epsilon}^*(\underline{\tilde{z}})$ have absolute values equal to $\|\underline{\epsilon}^*(\underline{\tilde{z}})\|_\infty = 0.00656$, i.e., $|\epsilon_1| = |\epsilon_4| = |\epsilon_9| = 0.00656$.

This concludes the exposition of Ruston's method. However, the following question now arises: Is the point $\underline{\tilde{z}}$ the stationary point, $\underline{z}^*$, of the function $\|\underline{\epsilon}^*(\underline{z})\|_\infty$? It will be shown in Chapter IV that, in general, the answer is no, so that $\|\underline{\epsilon}^*(\underline{z}^*)\|_\infty \leq \|\underline{\epsilon}^*(\tilde{\underline{z}})\|_\infty$. Now, before turning to a detailed consideration of this problem, it should be noted that the comments, which were made at the end of the previous section, concerning the vector $[\tilde{r}_1', \tilde{r}_2', \ldots, \tilde{r}_n', 1]^T$ apply also to Ruston's method.

CHAPTER III

PRONY'S EXTENDED METHOD

The purpose of this chapter is to study "Prony's Extended Method"[1] for solving exponential approximation problems in the ℓ_p^q-space. Specifically, we shall consider "Prony's Extended Method" for solving the following approximation problem: Given a real vector $\underline{f} \in U^q$, and a set of approximating real vectors $\{\hat{\underline{f}}(\underline{\beta},\underline{z}) \triangleq [Z(\underline{z})]\underline{\beta} : (\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}\}$, where $[Z(\underline{z})]$ is a $q \times n$ matrix, defined by Eq. 1.18 (Eq. 1.18a), of rank $n(q > 2n)$. Find the best parameter vector pair $(\underline{\beta}_p^{**}, \underline{z}_p^*) \in \mathscr{B}_z \times \mathscr{Z}$ so that

$$\|\underline{f} - [Z(\underline{z}_p^*)]\underline{\beta}_p^{**}\|_p \leq \|\underline{f} - [Z(\underline{z})]\underline{\beta}\|_p \tag{3.1}$$

for all $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$, and where[2] $p \geq 1$.

The essential part of "Prony's Extended Method" for solving this approximation problem is that it replaces the parameter vector $\underline{z} \in \mathscr{Z}$ by a new real parameter vector $\underline{r}$ which is related to $\underline{z}$ by the n^{th} order polynomial equation $P(z) = 0$, defined by Eq. 2.65.

---

[1]See Section 2.3.2 for "Prony's Original Method." (Chapter II)

[2]Since "Prony's Extended Method" simply reformulates the approximation problem defined by Eq. 3.1, we may generalize the results obtained in this section by using ℓ_p^q-norm of $\underline{\epsilon}$ to judge the degree of approximation where $p \geq 1$.

Let us begin by defining a q x (q - n) matrix $[R(\underline{r})]$, of rank (q - n) which depends on an (n + 1)-dimensional vector $\underline{r}$, namely,[3]

$$[R(\underline{r})] \triangleq \begin{bmatrix} r_0 & 0 & \cdots & 0 \\ r_1 & r_0 & & \cdot \\ r_2 & r_1 & & \cdot \\ \vdots & r_2 & & 0 \\ r_n & \vdots & & r_0 \\ 0 & r_n & & r_1 \\ 0 & 0 & & \cdot \\ \vdots & \vdots & & \cdot \\ 0 & 0 & \cdots & r_n \end{bmatrix} \qquad (3.2)$$

where

$$\underline{r} = \begin{bmatrix} r_0 \\ r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{bmatrix} \qquad (3.3)$$

---

[3] Note that this matrix $[R(\underline{r})]$ becomes the matrix $[R(\underline{r})]$ defined by Eq. 2.69 when q = 2n.

The set of all the parameter vectors $\underline{r}$ is denoted by $\mathscr{R}$, and is defined as follows:

Definition 3.1: The set $\mathscr{R}$, of the parameter vector $\underline{r}$, is the set of all nonzero vectors $\underline{r} \in E^{n+1}$, the n+1-dimensional real Euclidean space, so that the roots $\{z_1, z_2, \ldots, z_n\}$ of the nth-order polynomial equation

$$P(z) = \sum_{i=0}^{n} r_i z^i = 0 \tag{3.4}$$

represent the vector $\underline{z} \in \mathscr{Z}$.

Clearly, Definition 3.1 establishes the relation[4] between the vector $\underline{z} \in \mathscr{Z}$ and the vector $\underline{r} \in \mathscr{R}$. Consequently, if either one of these vectors is known, then the other vector may be found from Eq. 3.4.

Let us now establish a lemma which relates the subspaces spanned by the columns of the matrices $[Z(\underline{z})]$ and $[R(\underline{r})]$.

Lemma 3.1: Let $[Z(\underline{z})]$ be a q x n matrix, defined by Eq. 1.18, (Eq. 1.18a), of rank $n(q > 2n)$, and let $[R(\underline{r})]$ be a q x (q - n) matrix (defined by Eq. 3.2) of rank (q - n). Then the n-dimensional subspace of U^q defined by the column space of $[Z(\underline{z})]$, $C_n(Z)$, and the (q - n)-dimensional subspace of U^q, defined by

---

[4]Note that this relation is not one-to-one, since the vector $\underline{z} \in \mathscr{Z}$, represents an arbitrary ordering of the roots of the polynomial equation of Eq. 3.4.

the column space of $[R(\underline{r})]$, $C_{q-n}(R)$, are orthogonal complement subspaces of U^q; that is,

$$C_n(Z) \oplus C_{q-n}(R) = U^q \qquad (3.5)$$

where

$$C_n(Z) \perp C_{q-n}(R)$$

if and only if, the components $\{z_1, z_2, \ldots, z_n\}$ of the vector $\underline{z} \in \mathscr{Z}$ are the roots of the nth-order real polynomial equation

$$P(z) = \sum_{i=0}^{n} r_i z^i = 0 \qquad (3.6)$$

where the ordered set $\{r_0, r_1, \ldots, r_n\}$ define the vector $\underline{r} \in \mathscr{R}$.

Proof: Let $\underline{\hat{f}} \triangleq [Z(\underline{z})]\underline{\beta}$ be a real q-dimensional vector in $C_n(Z)$, where $\underline{\beta} \in \mathscr{B}_z$, and let $\underline{\hat{g}} \triangleq [R(\underline{r})]\underline{\gamma}$ be a real q-dimensional vector in $C_{q-n}(R)$, where $\underline{\gamma} \in E^{q-n}$. To prove the "if" part of the lemma, we must show that when the vectors $\underline{z} \in \mathscr{Z}$ and $\underline{r} \in \mathscr{R}$ are related by Eq. 3.6, then the inner product $(\underline{\hat{f}}, \underline{\hat{g}}) = 0$, for all $\underline{\beta} \in \mathscr{B}_z$ and $\underline{\gamma} \in E^{q-n}$. Since the inner product between the vector $\underline{\hat{f}}$ and $\underline{\hat{g}}$ in U^q yields

$$(\underline{\hat{f}}, \underline{\hat{g}}) = \underline{\gamma}^T [R(\underline{r})]^T [Z(\underline{z})] \underline{\beta} \qquad (3.7)$$

then it suffices to show that the matrix product $[R(\underline{r})]^T[Z(\underline{z})] = 0$, for all $\underline{z} \in \mathcal{Z}$ and $\underline{r} \in \mathcal{R}$. To show this we shall consider the case in which the components of $\underline{z}$ are distinct. The case in which the components of $\underline{z}$ are not distinct will be considered separately.

(1) When the components of $\underline{z}$ are distinct, then the matrix $[Z(\underline{z})]$ is given by Eq. 1.18. In carrying out the matrix multiplication $[R]^T[Z]$, one obtains

$$[R]^T[Z] = \begin{bmatrix} P(z_1) & \cdots & P(z_n) \\ z_1 P(z_1) & \cdots & z_n P(z_n) \\ z_1^2 P(z_1) & \cdots & z_n^2 P(z_n) \\ \vdots & & \vdots \\ z_1^{q-n-1} P(z_1) & \cdots & z_n^{q-n-1} P(z_n) \end{bmatrix} \tag{3.8}$$

where

$$P(z_k) = \sum_{i=0}^{n} r_i \, z_k^{\,i}, \qquad k = 1,2,\ldots,n \tag{3.9}$$

Then, since by definition, $P(z_k) = 0$, for all $z_k \in \{z_k : k = 1,2,\ldots,n\}$, the matrix product of Eq. 3.8 becomes

$$[R]^T[Z] = [0] \tag{3.10}$$

(2) When the components of $\underline{z}$ are not distinct, it can be also shown that the matrix product $[R(\underline{r})]^T [Z(\underline{z})] = [0]$. Rather than considering the general case, we shall consider the specific case in which the vector $\underline{z}$ contains only three identical components.[5] Specifically, let $z_1 = z_2 = z_3$, and the components $\{z_4, z_5, \ldots, z_n\}$ are distinct. Furthermore, let us assume that $t_1 = 0$, so that the matrix $Z(\underline{z})$ is given by Eq. 1.18b, where $j = 3$. Denoting by a_{ij} the ij-th element of the matrix product $[R]^T[Z]$, then in carrying out the matrix multiplication $[R]^T[Z]$, one obtains

$$a_{i1} = z_1^{i-1} P(z_1) \, ,$$

$$a_{i2} = z_1^{i-1} \Delta t \, [(i-1) P(z_1) + z_1 P'(z_1)] \, ,$$

$$a_{i3} = z_1^{i-1} (\Delta t)^2 \, [(i-1)^2 P(z_1) + (2i-1) z_1 P'(z_1) + z_1^2 P''(z_1)] \, ,$$

$$a_{ij} = z_j^{i-1} P(z_j) \, , \qquad j = 4, 5, \ldots, n \tag{3.11}$$

$i = 1, 2, \ldots, q$; where $P(z_k)$ denotes the polynomial $P(z)$ evaluated at $z = z_k$ (see Eq. 3.9), and where

[5]The specific case considered here, illustrates the basic relations which are encountered when carrying out the matrix multiplication, $[R]^T[Z]$, in the case in which the components of $\underline{z}$ are not distinct. Although the generalization of this case is straightforward, the notation becomes unmanagable.

$$P'(z_1) \triangleq \left.\frac{dP(z)}{dz}\right|_{z=z_1} = \sum_{i=1}^{n} i\, r_i\, z_1^{i-1}$$

$$P''(z_1) \triangleq \left.\frac{dP(z)}{dz^2}\right|_{z=z_1} = \sum_{i=2}^{n} i(i-1)\, r_i\, z_1^{i-2}$$

Since, by definition, $P(z_k) = 0$ for all $z_k \in \{z_k : k = 1,2,\ldots,n\}$, and since[6] $P'(z_1) = 0$ and $P''(z_1) = 0$, then all the elements given by Eq. 3.11 must be equal to zero, i.e., $a_{ij} = 0$, for all $i = 1,2,\ldots,q$, $j = 1,2,\ldots,n$. Clearly, this implies that the matrix product $[R]^T [Z]$ satisfies Eq. 3.10.

Consequently, we have shown that for all $\underline{z} \in \mathcal{Z}$ and $\underline{r} \in \mathcal{R}$, related by Eq. 3.6, the inner product of Eq. 3.7 becomes

$$(\hat{\underline{f}}, \hat{\underline{g}}) = 0$$

for all $\underline{\beta} \in \mathcal{B}_Z$ and $\underline{\gamma} \in E^{q-n}$. Thus, the subspaces $C_n(Z)$ and $C_{q-n}(R)$ are orthogonal in U^q. Furthermore, since the dimensionality of $C_n(Z)$ is n and the dimensionality of $C_{q-n}(R)$ is (q-n), then $C_n(Z)$ and $C_{q-n}(R)$ are orthogonal complement subspaces in U^q. Hence,

$$C_n(Z) \oplus C_{q-n}(R) = U^q$$

where

$$C_n(Z) \perp C_{q-n}(R)$$

---

[6]Note that since z_1 represents the third order zero of the polynomial P(z), then it must represent a zero of the polynomials P'(z) and P''(z).

In order to establish the "only if" part of the lemma, we must show that if Eq. 3.5 is true, then

$$(\hat{\underline{f}},\hat{\underline{g}}) = 0$$

or alternately,

$$([Z]\underline{\beta},\ [R]\underline{\gamma}) = 0 \qquad (3.12)$$

For Eq. 3.12 to hold for all $\underline{\beta} \in \mathscr{B}_z$ and all $\underline{\gamma} \in E^{q-n}$, the matrix product $[R]^T[Z]$, given by Eq. 3.8, must satisfy

$$[R]^T[Z] = [0]$$

that is, the elements of the matrix $[R]^T[Z]$ must be zero. Clearly, this condition will be satisfied only if the set $\{z_1, z_2, \ldots, z_n\}$ represents the roots of the n-th order polynomial equation given by Eq. 3.6, that is,

$$P(z) = \sum_{i=0}^{n} r_i z^i = 0$$

Thus, the lemma is proved.

We have shown that the (q - n)-dimensional subspace, $C_{q-n}(R)$, which depends on the (n+1)-dimensional vector $\underline{r} \in \mathscr{R}$, is orthogonal to each approximating vector $\hat{\underline{f}} \triangleq [Z(\underline{z})]\underline{\beta}$ when $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$. Recall that this step is similar to that made in "Prony's Original Method," except that here the dimensionality of the orthogonal complement subspace $C_{q-n}(R)$ is higher, i.e., $(q-n) > n$. Let us show that with the aid of the subspace $C_{q-n}(R)$, we can establish an alternate method of solving the approximation problem defined by Eq. 3.1. This method is the so-called, "Prony's Extended Method."

Let us begin by considering the vector relation[7] in U^q, that is

$$\underline{f} = [Z(\underline{z})]\,\underline{\beta} + \underline{\epsilon} \tag{3.13}$$

where $\underline{f}$ is a prescribed real vector in U^q; $[Z(\underline{z})]$ is a $q \times n$ matrix, defined by Eq. 1.18 (Eq. 1.18a), of rank $n(q > 2n)$; and where both the vector pair $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$, and the real vector $\underline{\epsilon}$ in U^q are unknown.

Lemma 3.1 tells us that there exists a vector $[R(\underline{r})]\,\underline{\gamma}$ which is orthogonal to the vector $[Z(\underline{z})]\,\underline{\beta}$, when the vector $\underline{r}$ is in $\mathscr{R}$ (i.e., when $\underline{r}$ is related to the vector $\underline{z} \in \mathscr{Z}$ via the polynomial equation $P(z) = 0$, given by Eq. 3.4). The inner product of both sides of Eq. 3.13 with respect to the vector $[R(\underline{r})]\,\underline{\gamma}$ yields

$$([R(\underline{r})]\,\underline{\gamma}, \underline{f}) = ([R(\underline{r})]\,\underline{\gamma}, \underline{\epsilon}) \tag{3.14}$$

because $([R(\underline{r})]\,\underline{\gamma}, [Z(z)]\,\underline{\beta}) = 0$ from Eq. 3.12. Since Eq. 3.14 must be true for all $(q-n)$-dimensional vectors $\underline{\gamma}$, then

---

[7]Recall that this equation represents the relationship between the vectors in U^q for the approximation problem of Eq. 3.1.

$$[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon} \quad \epsilon \quad E^{q-n} \tag{3.15}$$

where $\underline{f}$ is a prescribed real vector in U^q, and where $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}$ in U^q are unknown real vectors. This equation (viz., Eq. 3.15) defines the basic relation which must be satisfied when using "Prony's Extended Method" to solve the approximation problem defined by Eq. 3.1. Note that since all the vectors in Eq. 3.15 are real, then it suffices to say that $\underline{f}$ and $\underline{\epsilon}$ are in E^q.

At this point we will denote the vector $\underline{\epsilon}$ given in Eq. 3.13 by $\underline{\epsilon}(\underline{\beta},\underline{z})$ and the vector $\underline{\epsilon}$ given by Eq. 3.15 by $\underline{\epsilon}(\underline{r})$. Clearly, these vectors are identical. However, we have chosen this notation to emphasize the parameters on which the vector $\underline{\epsilon}$ depends. Strictly speaking, the vector $\underline{\epsilon}$, given by Eq. 3.15, should be denoted by $\underline{\epsilon}(\underline{\beta},\underline{r})$ since only the vector $\underline{z}$ is mapped into the vector $\underline{r}$. This point can be best clarified by considering Eq. 3.15 for the case in which $\underline{z} \in \mathscr{Z}$ (i.e., $\underline{r} \in \mathscr{R}$) is initially prescribed in the approximation problem defined by Eq. 3.1. Recall that the best solution vector $\underline{\beta}_p{}^* \in \mathscr{B}_z$ of this approximation problem may be obtained directly from Eq. 3.13 when $\underline{\epsilon} = \underline{\epsilon}_p{}^*(\underline{z})$, where the vector $\underline{\epsilon}_p{}^*(\underline{z})$ is defined by

$$\|\underline{\epsilon}_p{}^*(\underline{z})\|_p = \|\underline{\epsilon}(\underline{\beta}_p{}^*,\underline{z})\|_p = \min_{\underline{\beta} \in \mathscr{B}_z} \|\underline{\epsilon}(\underline{\beta},\underline{z})\|_p$$

Let us now show that the vector $\underline{\epsilon}_p^*(\underline{z})$ may be also obtained from the vector relation given in Eq. 3.15. Recall that when the vector $\underline{z} \in \mathcal{Z}$ is prescribed, the vector $\underline{r} \in \mathcal{R}$ is also automatically prescribed. The set of all vectors $\underline{\epsilon}(\underline{r})$ in U^q which satisfy Eq. 3.15 for a prescribed vector $\underline{r} \in \mathcal{R}$, i.e., a prescribed matrix [R], is defined as follows:

Definition 3.2: For each vector $\underline{r} \in \mathcal{R}$ and a real vector $\underline{f} \in U^q$, the set $\mathcal{E}_r$ denotes the set of all real vectors $\underline{\epsilon}(\underline{r})$ in U^q which satisfy the relation

$$[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon}(\underline{r}) \tag{3.16}$$

Clearly, then, the vector $\underline{\epsilon}_p^*$ which equals the vector $\underline{\epsilon}_p^*(\underline{z})$ must satisfy

$$\|\underline{\epsilon}_p^*(\underline{r})\|_p = \min_{\underline{\epsilon}(\underline{r}) \in \mathcal{E}_r} \|\underline{\epsilon}(\underline{r})\|_p \tag{3.17}$$

This is summarized in the following lemma:

Lemma 3.2: For a specified real vector $\underline{f} \in U^q$ and a specified vector $\underline{z} \in \mathcal{Z}$, i.e., a specified vector $\underline{r} \in \mathcal{R}$, the real vectors $\underline{\epsilon}_p^*(\underline{z})$ and $\underline{\epsilon}_p^*(\underline{r})$ are equal in U^q, that is,

$$\underline{\epsilon}_p^*(\underline{z}) = \underline{\epsilon}_p^*(\underline{r}) \tag{3.18}$$

where $p \geq 1$; if, and only if,

(1) the vector $\underline{\epsilon}_p^*(\underline{z})$ is defined by

$$\|\underline{\epsilon}_p^*(\underline{z})\|_p \triangleq \min_{\underline{\beta} \epsilon \mathscr{B}_z} \|\underline{\epsilon}(\underline{\beta}, \underline{z})\|_p \tag{3.19}$$

where $\underline{\epsilon}(\underline{\beta}, \underline{z})$ is given in Eq. 3.13, when $\underline{z} \epsilon \mathscr{Z}$ is specified; and,

(2) the vector $\underline{\epsilon}_p^*(\underline{r})$ is defined by

$$\|\underline{\epsilon}_p^*(\underline{r})\|_p \triangleq \min_{\underline{\epsilon}(\underline{r}) \epsilon \mathscr{E}_r} \|\underline{\epsilon}(\underline{r})\|_p$$

where $\underline{\epsilon}(\underline{r})$ satisfies Eq. 3.16, when $\underline{r} \epsilon \mathscr{R}$ is specified by Definition 3.1.

We have thus shown that if for a specified real vector $\underline{f} \epsilon U^q$ and for a specified $\underline{r} \epsilon \mathscr{R}$ (i.e., a specified $\underline{z} \epsilon \mathscr{Z}$) one obtains the vector $\underline{\epsilon}_p^*(\underline{r})$ in $\mathscr{E}_r$, then, by Lemma 3.2, one can obtain the solution vector $\underline{\beta}_p^* \epsilon \mathscr{B}_z$ directly from Eq. 3.13, by taking $\underline{\epsilon} = \underline{\epsilon}_p^*(\underline{r})$. Now that we have discussed the preliminary ideas, we present the theorem which governs "Prony's Extended Method" for solving the approximation problem stated in Eq. 3.1.

Theorem 3.1: For each real vector $\underline{f} \epsilon U^q$, let $\underline{\epsilon}_p^{**}$ in U^q denote the vector $\underline{\epsilon}(\underline{r})$ which satisfies

$$[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon}$$

so that

$$\|\underline{\epsilon}_p^{**}\|_p \triangleq \min_{\underline{r} \in \mathscr{R}'} \|\underline{\epsilon}_p^{*}(\underline{r})\|_p$$

$$= \min_{\underline{r} \in \mathscr{R}'} \min_{\underline{\epsilon}(\underline{r}) \in \mathscr{E}_r} \|\underline{\epsilon}(\underline{r})\|_p$$

where $p \geq 1$. Then, the best ℓ_p^q-approximating real vector in U^q,

$$\underline{f}_p^{**} \triangleq [Z(\underline{z}_p^{*})]\, \underline{\beta}_p^{**}$$

where $(\underline{\beta}_p^{**}, \underline{z}_p^{*}) \in \mathscr{B}_z \times \mathscr{Z}$, is given by

$$\underline{f}_p^{**} = \underline{f} - \underline{\epsilon}_p^{**}$$

Proof: All of this theorem is contained in Lemma 3. 2, except for the assertion that

$$\min_{\underline{r} \in \mathscr{R}'} \|\underline{\epsilon}_p^{*}(\underline{r})\|_p = \min_{\underline{z} \in \mathscr{Z}} \|\underline{\epsilon}_p^{*}(\underline{z})\|_p$$

That this assertion is true should be evident from Definition 3. 1, where the relation between the vector $\underline{r} \in \mathscr{R}'$ and the vector $\underline{z} \in \mathscr{Z}$ was established.

Thus, we have shown that Prony's Extended Method gives an alternate method of solving the approximation problem defined by Eq. 3. 1. Such a method leads to the problem of determining the vectors $\underline{r}_p^* \in \mathscr{R}'$ and $\underline{\epsilon}_p^*(\underline{r}_p^*) \in U^q$ so that

$$\|\underline{\epsilon}_p^*(\underline{r}_p^*)\|_p \leq \|\underline{\epsilon}(\underline{r})\|_p , \qquad p \geq 1$$

for all $\underline{r} \in \mathscr{R}'$ and for all vectors $\underline{\epsilon}(\underline{r})$ which satisfy

$$[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon}(\underline{r})$$

It should be noted that the roots of the polynomial equation of Eq. 3.4 depend <u>only</u> on the direction and <u>not</u> on the magnitude of $\underline{r} \in E^{n+1}$. In other words, the vector $\underline{r} \in E^{n+1}$ may be normalized with respect to $\|\underline{r}\|_p$ without affecting the roots of Eq. 3.4 which represent the vector $\underline{z}$. Hence, the set of all vectors $\underline{r} \in \mathscr{R}'$ may be replaced by the set of all vectors $\underline{r} \in E^{n+1}$ with $\|\underline{r}\|_p = 1$. Since in Chapter IV we shall be interested in the ℓ_1^{n+1}-norm of the vector $\underline{r}$; namely $\|\underline{r}\|_1 \triangleq \sum_{i=0}^{n} |r_i|$, we make the following definition:

<u>Definition 3.3:</u> The set $\mathscr{R}$ of the parameter vector $\underline{r}$ is the set of all real vectors $\underline{r} \in E^{n+1}$ with $\|\underline{r}\|_1 = 1$.

<u>Remark:</u> Note that the set $\mathscr{R}$ contains all vectors $\underline{r}$ with the component $r_n = 0$. It should be noted, however, that for some prescribed vectors $\underline{f}$, if the vector $\underline{r}^*$ is in the set $\mathscr{R}$ with $r_n^* = 0$, then there is another vector $\underline{r}^* \in \mathscr{R}$ with $r_n^* \neq 0$ which yields an n-th order polynomial containing a root that does not contribute to the approximation. In other words, for some vector $\underline{r} \in E^{n+1}$, $\|\underline{r}\|_p = 1$, with component $r_n = 0$, there is another vector

$\underline{r} \in E^{n+1}$, $\|\underline{r}\|_p = 1$, with component $r_n \neq 0$,

that will yield the same real error vector

$\underline{\epsilon}_p*(\underline{r}) \in U^q$. This is illustrated in Example 5.6

of Chapter V.

Now before illustrating how "Prony's Extended Method" may be applied to solve exponential approximation problems, let us make the following observation about the singular mappings which are dictated by the matrices $[Z(\underline{z})]$ and $[R(\underline{r})]$. Recall that Lemma 3.1 tells us that the vector space U^q can be decomposed into two orthogonal complementary subspaces, $C_n(Z)$ and $C_{q-n}(R)$. Therefore, each real vector $\underline{f}$ in U^q can be uniquely represented by

$$\underline{f} = \underline{v}(\underline{z}) + \underline{w}(\underline{r}) \qquad (3.20)$$

where $\underline{v}(\underline{z})$ and $\underline{w}(\underline{r})$ represent the orthogonal projections of $\underline{f}$ onto the subspaces $C_n(Z)$ and $C_{q-n}(R)$, respectively.

Let us now show how "Prony's Extended Method" simplifies the least squares exponential approximation in the U^q space; i.e., the case in which $p = 2$ in Eq. 3.1. As is well-known,[8] for each fixed $\underline{z}$ in $\mathcal{Z}$ this approximation yields an error vector, $\underline{\epsilon}*(\underline{z})$, which is orthogonal to the vector $\underline{f}*(\underline{z})$. Hence, $\underline{\epsilon}*(\underline{z})$ represents the

[8] See Theorem 2.1. (Chapter II)

orthogonal projection of the prescribed $\underline{f}$ onto the orthogonal complement subspace of $C_n(Z)$ in U^q. Clearly then, in terms of Lemma 3.1, $\underline{\epsilon}^*(\underline{z})$ lies in $C_{q-n}(R)$. Hence, the vectors $\underline{f}^*(\underline{z})$ and $\underline{\epsilon}^*(\underline{z})$ have the relation of the vectors $\underline{v}(\underline{z})$ and $\underline{w}(\underline{r})$, respectively, of Eq. 3.20.

Therefore, we may represent the error vector, $\underline{\epsilon}^*$, as a function of either the parameter vector $\underline{z}$ or the parameter vector $\underline{r}$; that is,

$$\underline{\epsilon}^*(\underline{z}) = \underline{f} - [Z(\underline{z})]\, \underline{\beta}^* \tag{3.21}$$

where

$$\underline{\beta}^* = [Z^T Z]^{-1} Z^T \underline{f} \tag{3.22}$$

or, alternatively,

$$\underline{\epsilon}^*(\underline{r}) = [R(\underline{r})]\, \underline{\gamma}^* \tag{3.23}$$

where

$$\underline{\gamma}^* = [R^T R]^{-1} R^T \underline{f} \tag{3.24}$$

Thus, to find the vector $\underline{z}^*$ in $\mathcal{Z}$ which minimizes $\|\underline{\epsilon}\|_2$, we can either (1) determine the stationary point[9] $\underline{z}^*$ of the function

---

[9]That is, the point where $\dfrac{\partial \|\underline{\epsilon}(\underline{z})\|_2}{\partial z_k} = 0$ for all $k = 1, 2, \ldots, n$.

$\|\underline{\epsilon}^*(\underline{z})\|_2$, where $\underline{\epsilon}^*(\underline{z})$ is defined by Eq. 3.21; or, (2) use "Prony's Extended Method" to first determine the stationary point $\underline{r}^*$ in E^{n+1} of the function $\|\underline{\epsilon}^*(\underline{r})\|_2$, in the set $\|\underline{r}\|_p = 1$, where $\underline{\epsilon}^*(\underline{r})$ is defined by Eq. 3.23, and then find the n zeros of the polynomial equation

$$P(z) = \sum_{i=0}^{n} r_i^* z^i$$

In summary, we have shown that "Prony's Extended Method" gives an alternate approach to the solution of the exponential approximation problem. At this point, it should be mentioned again that to carry out this approximation method, it is necessary that the prescribed values of $\{f(t_i)\}$ be obtained at equal intervals of t_i.

In the next chapter, we shall use "Prony's Extended Method" to solve the Chebyshev exponential approximation problem in the U^q space, i.e., the case in which $p = \infty$ in Eq. 3.1.

CHAPTER IV

OPTIMAL CHEBYSHEV APPROXIMATIONS AT DISCRETE POINTS IN THE TIME DOMAIN

4.1 Introduction

The purpose of this chapter is to study the Chebyshev approximation problem stated in Section 1.5 of Chapter I, i.e., the Chebyshev approximation problem in a finite dimensional space occurring when the approximating subspace is not fully prescribed. Specifically, we shall consider the following approximation problem: Given a real vector $\underline{f}$ in U^q, and a $q \times n$ matrix $[Z(\underline{z})]$, defined by Eq. 1.18 (Eq. 1.18a), of rank $n(q > 2n)$; find the best Chebyshev parameter vector pair $(\underline{\beta}^{**}, \underline{z}^*) \in \mathscr{B}_z \times \mathscr{Z}$, such that if

$$\underline{f}^{**} \triangleq [Z(\underline{z}^*)]\, \underline{\beta}^{**} , \tag{4.1}$$

then,

$$\|\underline{\epsilon}^{**}\|_\infty \triangleq \|\underline{f} - \underline{f}^{**}\|_\infty \leq \|\underline{f} - [Z(\underline{z})]\, \underline{\beta}\|_\infty \tag{4.2}$$

for all $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$.

In this chapter we shall be primarily concerned with the existence and the properties of the best Chebyshev parameter vector pair $(\underline{\beta}^{**}, \underline{z}^*) \in \mathscr{B}_z \times \mathscr{Z}$. Most of the results will be presented in terms of "Prony's Extended Method" which was developed in Chapter III.

In Section 4.2, we shall offer some intuitive notions relating to the character of the resulting best Chebyshev error vector $\underline{\epsilon}^{**} \in U^q$. In Section 4.3, we shall give the existence theorem and the bounds within which the value of $\|\underline{\epsilon}^{**}\|_\infty$ must lie. The properties of the best Chebyshev vector pair $(\underline{\beta}^{**}, \underline{z}^*) \in \mathscr{B}_z \times \mathscr{Z}$ will be given in Section 4.4. Here, we shall show that the vector pair $(\underline{\beta}^{**}, \underline{z}^*) \in \mathscr{B}_z \times \mathscr{Z}$ represents the vector pair which defines the best Chebyshev approximating vector in some (2n+1)-dimensional reference subspace U_c^{2n+1} of U^q, where $c \in \{w : w = 1, 2, \ldots, \binom{q}{2n+1}\}$. In Section 4.5, we shall be concerned with finding the best Chebyshev approximation in each (2n+1)-dimensional reference subspace U_w^{2n+1} of U^q. Specifically, we shall give the theory which renders the computational method presented in Chapter V.

Although we are not primarily concerned with the uniqueness property of the best Chebyshev approximation in this chapter, a discussion of this subject is presented in Section 4.6.

4.2 General Approach to the Solution of this Chebyshev Approximation Problem

Before studying in detail the approximation problem considered here, we shall offer an intuitive notion relating to the character of the resulting Chebyshev error vector $\underline{\epsilon}^{**} \in U^q$.

Specifically, if we consider Theorem 2. 2 and note that our unknown parameters lie in a 2n-dimensional parameter space $\mathscr{B}_z \times \mathscr{Z}$, we should <u>expect</u> the absolute values of (2n + 1) components of $\underline{\epsilon}^{**}$ to be equal to $\|\underline{\epsilon}^{**}\|_\infty$. Admittedly, the expansion of the parameter space does not occur through the doubling of the dimensionality of the approximating subspace, but rather occurs through freeing the orientation of the approximating subspace, $C_n(Z)$. The achievement of at least (2n + 1) error components of equal magnitude is not a surprising result since when q = 2n, the error components are usually equal to zero. [1]

As a further digression, the approach to the solution of this Chebyshev approximation problem is now outlined. We begin by seeking the solution of the Chebyshev approximation problem for some fixed (i. e., prescribed) vector $\underline{z} \in \mathscr{Z}$. This problem is identical to the problem considered in Theorem 2. 2, since now the matrix [Z] is prescribed. From Theorem 2. 2 we find that the best Chebyshev approximation vector $\underline{f}^* \triangleq [Z]\, \underline{\beta}^*$, for a prescribed vector $\underline{z} \in \mathscr{Z}$, yields an error vector, $\underline{\epsilon}^* = \underline{f} - [Z]\, \underline{\beta}^*$, which will have at least (n + 1) components equal in absolute value to $\|\underline{\epsilon}^*\|_\infty$. It is evident that $\underline{\epsilon}^*$ depends on both the prescribed real vector $\underline{f}$ in U^q and on the prescribed q x n matrix [Z] (in other words, on

---

[1] See "Prony's Original Method" which is presented in Section 2. 3. 2, Chapter II.

the vector $\underline{z} \in \mathcal{Z}$). The fact that $\underline{\epsilon}^*$ is a function of the vector $\underline{z}$ will be emphasized by representing $\underline{\epsilon}^*$ by $\underline{\epsilon}^*(\underline{z})$. If now the vector $\underline{z} \in \mathcal{Z}$ is not prescribed, then we claim[2] that by varying $\underline{z}$, we can decrease the value of $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ so that the absolute values of at least (2n + 1) components of $\underline{\epsilon}^*(\underline{z})$ are equal to $\|\underline{\epsilon}^*(\underline{z})\|_\infty$. This claim is based on the fact that we have acquired extra n-degrees of freedom,[3] corresponding to the n-components of the parameter vector $\underline{z}$, with which to modify $\underline{\epsilon}^*(\underline{z})$. The resulting vector $\underline{\epsilon}^*(\underline{z})$ is denoted by $\underline{\epsilon}^{**} \triangleq \underline{\epsilon}^*(\underline{z}^*)$, where $\underline{z}^*$ denotes the $\underline{z}$ which gives the minimal value of $\|\underline{\epsilon}^*(\underline{z})\|_\infty$.

It should be mentioned that if some of the components of $\underline{z}^*$ are equal to zero, then the above mentioned character of the resulting error vector, $\underline{\epsilon}^{**}$, may not necessarily hold, i. e., the vector $\underline{\epsilon}^{**}$ may consist of less than (2n + 1) components equal in absolute value to $\|\underline{\epsilon}^{**}\|_\infty$. This will be discussed further in Section 4. 5.

4. 3 The Existence of the Best Chebyshev Approximation

We begin by proving that every prescribed real vector $\underline{f} \in U^q$ has a best Chebyshev approximating vector, $\underline{f}^{**}$, defined by Eq. 4. 1. Recall from Theorem 2. 2 that for each $\underline{z} \in \mathcal{Z}$ there exists a $\underline{\beta}^* \in \mathcal{B}_z$ such that

---

[2] This claim rests on continuity considerations, see Section 4.3.

[3] The other n-degrees of freedom correspond to the n-components of the parameter vector $\underline{\beta}$.

$$\|\underline{\epsilon}^*(\underline{z})\|_\infty \triangleq \|\underline{f} - [Z(\underline{z})]\,\underline{\beta}^*\|_\infty \leq \|\underline{f} - [Z(\underline{z})]\,\underline{\beta}\|_\infty \tag{4.3}$$

for all $\underline{\beta} \in \mathscr{B}_z$. Hence, to show that there exists a best approximating vector, $\underline{f}^{**} \triangleq [Z(\underline{z}^*)]\,\underline{\beta}^{**}$, to a prescribed real vector $\underline{f} \in U^q$, it suffices to show that there exists a $\underline{z}^* \in \mathscr{Z}$ such that

$$\|\underline{\epsilon}^*(\underline{z}^*)\|_\infty \leq \|\underline{\epsilon}^*(\underline{z})\|_\infty \tag{4.4}$$

for all $\underline{z} \in \mathscr{Z}$. Since our existence theorem is based on the fact that the function $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ is a continuous function of $\underline{z} \in \mathscr{Z}$, let us first establish the following lemma:

<u>Lemma 4.1</u>: Let the vector $\underline{\epsilon}^*(\underline{z}) \in U^q$ be a function of $\underline{z} \in \mathscr{Z}$, defined by

$$\underline{\epsilon}^*(\underline{z}) = \underline{f} - [Z(\underline{z})]\,\underline{\beta}^* \tag{4.5}$$

where $\underline{f}$ is a prescribed real vector in U^q; $[Z(\underline{z})]$ is a q x n matrix, defined by Eq. 1.18 (Eq. 1.18a), of rank n(q > 2n); and where $\underline{\beta}^*$ is a vector in $\mathscr{B}_z$ such that

$$\|\underline{\epsilon}^*(\underline{z})\|_\infty \triangleq \|\underline{f} - [Z(\underline{z})]\,\underline{\beta}^*\|_\infty \leq \|\underline{f} - [Z(\underline{z})]\,\underline{\beta}\|_\infty \tag{4.6}$$

for all $\underline{\beta} \in \mathscr{B}_z$. Then the function $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ is a continuous function of $\underline{z} \in \mathscr{Z}$.

Proof: To show that $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ is a continuous function of $\underline{z} \in \mathcal{Z}$, we must show that $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ is continuous at each point $\underline{z}_0 \in \mathcal{Z}$, that is, for every real number $\eta > 0$, there exists a real number $\delta > 0$, such that[4] $\underline{z} \in \mathcal{Z}$ and

$$\|\underline{z} - \underline{z}_0\|_1 < \delta \tag{4.7}$$

implies that

$$\Big| \|\underline{\epsilon}^*(\underline{z})\|_\infty - \|\underline{\epsilon}^*(\underline{z}_0)\|_\infty \Big| < \eta \tag{4.8}$$

Recall that

$$\underline{\epsilon}^*(\underline{z}) \stackrel{\Delta}{=} \underline{\epsilon}(\underline{\beta}^*, \underline{z})$$

and that the vector $\underline{\beta}^* \in \mathcal{B}_z$ is a function of $\underline{z} \in \mathcal{Z}$. Hence, let us denote the vector $\underline{\epsilon}^*(\underline{z})$ by $\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})$. Consider the relation between the function $\|\underline{\epsilon}(\underline{\beta}, \underline{z})\|_\infty$ at the four points in the $\mathcal{B}_z \times \mathcal{Z}$ space shown in Fig. 1, p. 112.

It can be shown (see Appendix B) that for $\underline{z}$ and $\underline{z}_0 \in \mathcal{Z}$:

(1) if $\underline{z}_0 \neq 0$ and if $\|\underline{z} - \underline{z}_0\|_1 \leq \frac{1}{2} \|\underline{z}_0\|_\infty$, then

$$\Big| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_0), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_0), \underline{z}_0)\|_\infty \Big| \leq c_1(\underline{z}_0) \|\underline{z} - \underline{z}_0\|_1 \tag{4.9}$$

---

[4] It should be noted that for each $\underline{z}$ and $\underline{z}_0 \in \mathcal{Z}$, Eq. 4.7 defines an open set in $\mathcal{Z}$, since Eq. 4.7 represents the intersection of the subspace $\mathcal{Z}$ of U^n with an open sphere in U^n (see Ref. 21, p. 64).

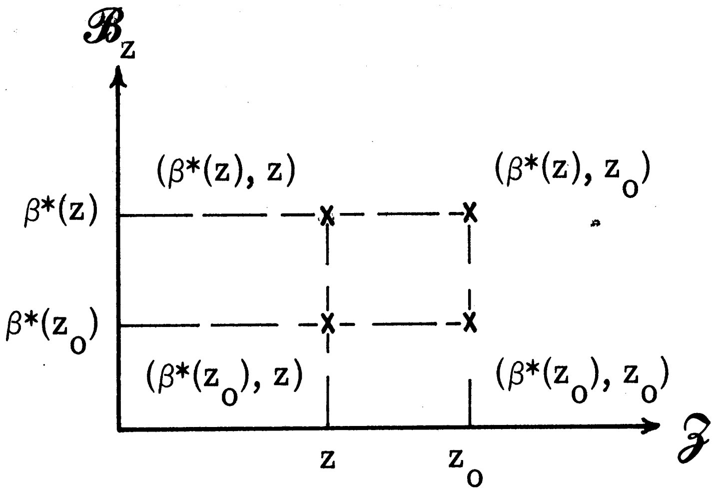


Fig. 5. The points in the $\mathscr{B}_z$ x $\mathscr{Z}$ space considered in Lemma 4. 1.

where

$$c_1(\underline{z}_o) = 2\,\|\underline{f}\|_\infty\,\|\underline{z}_o\|_\infty^{-1} \sum_{j=1}^{q-1} \binom{q-1}{j} \tag{4.10}$$

and

$$\Big|\,\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z}_o)\|_\infty\Big| \leq c_2(\underline{z}_o)\,\|\underline{z} - \underline{z}_o\|_1 \tag{4.11}$$

where

$$c_2(\underline{z}_o) = 2(n+1)\,\|\underline{f}\|_\infty\,\|\underline{z}_o\|_1^{-1} \sum_{j=1}^{q-1} \binom{q-1}{j} \tag{4.12}$$

(2) If $\underline{z}_o = 0$, and if $\|\underline{z}\|_1 \leq 1$, then

$$\Big|\,\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z}_o)\|_\infty\Big| \leq 2\,\|\underline{f}\|_\infty\,\|\underline{z}\|_1 \tag{4.13}$$

and

$$\Big|\,\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z}_o)\|_\infty\Big| \leq 2\,\|\underline{f}\|_\infty\,\|\underline{z}\|_1 \tag{4.14}$$

Hence, it can be seen from Eqs. 4.9 through 4.14, that for a given $\eta/2 > 0$, there exists a positive constant defined by

$$\delta_1 = \begin{cases} \min\left\{\dfrac{\eta}{2c_1(\underline{z}_o)},\ \dfrac{1}{2}\,\|\underline{z}_o\|_\infty\right\}, & \text{when } \underline{z}_o \neq 0 \\[2ex] \min\left\{\dfrac{\eta}{4\,\|\underline{f}\|_\infty},\ 1\right\}, & \text{when } \underline{z}_o = 0 \end{cases} \tag{4.15}$$

such that if $\|\underline{z} - \underline{z}_o\|_1 < \delta_1$, then

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z}_o)\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z})\|_\infty \right| < \frac{\eta}{2} \tag{4.16}$$

and there exists a positive constant δ_2, defined by

$$\delta_2 = \begin{cases} \min\left\{\dfrac{\eta}{2c_2(\underline{z}_o)}, \dfrac{1}{2}\|\underline{z}_o\|_\infty\right\}, & \text{when } \underline{z} \neq 0 \\ \min\left\{\dfrac{\eta}{4\|\underline{f}\|_\infty}, 1\right\}, & \text{when } \underline{z} = 0 \end{cases} \tag{4.17}$$

such that if $\|\underline{z} - \underline{z}_o\| < \delta_2$, then

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z}_o)\|_\infty \right| < \frac{\eta}{2} \tag{4.18}$$

Recall from Theorem 2.2 that for a specified $\underline{z} \in \mathcal{Z}$ we have the relation

$$\|\underline{\epsilon}(\underline{\beta},\underline{z})\|_\infty \geq \|\underline{\epsilon}(\underline{\beta}^*,\underline{z})\|_\infty$$

for all $\underline{\beta} \in \mathcal{B}_z$. Hence, we note that

$$\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z})\|_\infty = d_1 \geq 0 \tag{4.19}$$

and

$$\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z}_o)\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z}_o)\|_\infty = d_2 \geq 0 \tag{4.20}$$

Adding Eqs. 4.19 and 4.20, yields

$$d_1 + d_2 = \Big| \{ \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z}_o)\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})\|_\infty \}$$

$$+ \{ \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z}_o)\|_\infty \} \Big|$$

$$\leq \Big| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z}_o)\|_\infty \Big|$$

$$+ \Big| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z}_o)\|_\infty \Big| \quad (4.21)$$

If $\|\underline{z} - \underline{z}_o\|_1 < \delta$ where $\delta = \min\{\delta_1, \delta_2\}$, then the inequalities of Eqs. 4.16 and 4.18 are satisfied and the inequality of Eq. 4.21 yields

$$d_1 + d_2 < \frac{\eta}{2} + \frac{\eta}{2} = \eta \quad (4.22)$$

Let us now substitute the value of $\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z}_o)\|_\infty$, given by Eq. 4.20, into Eq. 4.18. This yields

$$\Big| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z}_o)\|_\infty - d_2 \Big| < \frac{\eta}{2}$$

or alternately

$$\Big| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z}_o)\|_\infty \Big| < \frac{\eta}{2} + d_2 \quad (4.23)$$

Similarly, substituting the value of $\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z})\|_\infty$, given by Eq. 4.19 into 4.16, yields

$$\Big| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z}_o)\|_\infty \Big| < \frac{\eta}{2} + d_1 \quad (4.24)$$

Adding Eqs. 4.23 and 4.24 and using the relation of Eq. 4.24 yields

$$\left| \|\underline{\epsilon}(\beta^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\beta^*(\underline{z}_o), \underline{z}_o)\|_\infty \right| < \frac{1}{2}\left(\frac{\eta}{2} + \frac{\eta}{2} + d_1 + d_2\right) < \eta \tag{4.25}$$

Hence, given an $\eta > 0$, we can find a $\delta > 0$, which is given by

$$\delta = \min\{\delta_1, \delta_2\} \tag{4.26}$$

such that if $\|\underline{z} - \underline{z}_o\|_1 < \delta$, then from Eq. 4.25 we have

$$\left| \|\underline{\epsilon}^*(\underline{z})\|_\infty - \|\underline{\epsilon}^*(\underline{z}_o)\|_\infty \right| < \eta \tag{4.27}$$

Thus, the lemma is proved.

Theorem 4.1 (Existence Theorem): For each real vector $\underline{f} \in U^q$, there exists a vector $\underline{r}^* \in \mathscr{R}$, where $\mathscr{R}$ is defined by Definition 3.3, such that

$$\|\underline{\epsilon}^{**}\|_\infty \triangleq \|\underline{\epsilon}^*(\underline{r}^*)\|_\infty \leq \|\underline{\epsilon}^*(\underline{r})\| \tag{4.28}$$

for all $\underline{r} \in \mathscr{R}$ and real vector $\underline{\epsilon}^*(\underline{r}) \in U^q$ which satisfy the relation

$$[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon}^*(\underline{r}) \tag{4.29}$$

where $[R(\underline{r})]$ is the q x (q - n) matrix defined by Eq. 3.2.

Proof: Since the set of vectors $\underline{r} \in \mathscr{R}$ (i.e., $\underline{r} \in E^{n+1}$ with $\|\underline{r}\|_1 = 1$) forms a compact set, then if the function $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ is a continuous function of $\underline{r} \in \mathscr{R}$, it attains its minimum on this set[5], i. e., there exists an $\underline{r}^*$ such that Eq. 4. 28 is satisfied. Hence it suffices to show that the function $\| \underline{\epsilon}^*(\underline{r}) \|_\infty$ is a continuous function of $\underline{r} \in \mathscr{R}$.

Since by Lemma 3. 2 the vector[6] $\underline{\epsilon}^*(\underline{r}) = \underline{\epsilon}^*(\underline{z})$, and since by Lemma 4. 1 the function $\| \underline{\epsilon}^*(\underline{z}) \|_\infty$ is a continuous function of $\underline{z} \in \mathscr{Z}$, then the function $\| \underline{\epsilon}^*(\underline{r}) \|_\infty$ is a continuous function of $\underline{r} \in \mathscr{R}'$. Recall from Definition 3. 1 that for each vector $\underline{r} \in \mathscr{R}'$, the vector $\underline{z} \in \mathscr{Z}$ can be obtained from the n-th order polynomial equation

$$P(z) = \sum_{i=0}^{n} r_i z^i = 0 \qquad (4.30)$$

Furthermore, if $r_n \neq 0$, the vector $\underline{r}$ may be normalized with respect to $\| \underline{r} \|_p$, as long as[7] $\| \underline{r} \|_p \neq 0$, without affecting the values of the roots of Eq. 4. 30 which represent the vector $\underline{z}$. In other words, for all $\underline{r} \in \mathscr{R}'$ for which $\| \underline{r} \|_p \neq 0$, the value of $\| \underline{\epsilon}^*(\underline{r}) \|_\infty$ is independent of $\| \underline{r} \|_p$. Furthermore, it can be shown that the continuity of the function $\| \underline{\epsilon}^*(\underline{r}) \|_\infty$ can be extended to all

---

[5] See Ref. 21, Chapter 4.

[6] Since in this chapter we shall consider only the case when $p = \infty$, we shall drop the subscript p in $\underline{\epsilon}^*_p$.

[7] Note that $\| \underline{r} \|_p = 0$ occurs only if $\underline{r} = 0$, a point in $\mathscr{R}'$ which is excluded by assumption, since $\underline{r} = 0$ implies that the polynomial P(z) has no roots, or $\underline{z} \notin \mathscr{Z}$.

$\underline{r} \in \mathcal{R}$. Thus the theorem is proved.

It should be noted that for some vectors $\underline{r}^* \in \mathcal{R}$ there exists no vector[8] $\underline{z} \in \mathcal{Z}$. For example, the case in which the minimal value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ occurs when the vector $\underline{r}^* \in \mathcal{R}$ has the component $r_n^* = 0$, and there exists no other vector $\underline{r}^* \in \mathcal{R}$ which yields the same value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ when[9] the component $r_n^* \neq 0$. This is illustrated in the following example:

Example: Given q = 5, n = 2, and

$$\underline{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 5 \end{bmatrix}$$

Since n = 2, then the vector $\underline{r} = [r_0 \; r_1 \; r_2]^T$. Let us assume that the initial approximation yields the following vectors $\underline{r}$, $\underline{z}$, and $\underline{\epsilon}^*(\underline{r})$

$$\underline{r} = \begin{bmatrix} 1 \\ 0 \\ -0.333 \end{bmatrix}, \quad \underline{z} = \begin{bmatrix} +1.732 \\ -1.772 \end{bmatrix}, \quad \underline{\epsilon}^*(\underline{r}) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ -0.5 \\ 0.5 \end{bmatrix}$$

---

[8] In other words, some roots of Eq. 4.30 are at infinity.

[9] See page 102.

If $\|\underline{\epsilon}^*(\underline{r})\|_\infty = 0.3$, then the vectors $\underline{r}$, $\underline{z}$, and $\underline{\epsilon}^*$ are

$$\underline{r} = \begin{bmatrix} 1 \\ -0.716 \\ 0.0785 \end{bmatrix}, \quad \underline{z} = \begin{bmatrix} 1.227 \\ -10.347 \end{bmatrix}, \quad \underline{\epsilon}^* = \begin{bmatrix} 0.3 \\ -0.033 \\ -0.3 \\ -0.3 \\ 0.3 \end{bmatrix}$$

If $\|\underline{\epsilon}^*(\underline{r})\|_\infty = 0.1$, then the vectors $\underline{r}$, $\underline{z}$, and $\underline{\epsilon}^*(\underline{r})$ are

$$\underline{r} = \begin{bmatrix} 1.000 \\ -0.875 \\ -0.028 \end{bmatrix}, \quad \underline{z} = \begin{bmatrix} 1.089 \\ -32.230 \end{bmatrix} \quad \underline{\epsilon}^* = \begin{bmatrix} 0.1 \\ 0.0068 \\ -0.1 \\ -0.1 \\ 0.1 \end{bmatrix}$$

The minimal value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty = 0$, which occurs when

$$\underline{r}^* = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{z}^* = \begin{bmatrix} 1 \\ -\infty \end{bmatrix}$$

Two points should be noted concerning the above example. First, although the example illustrates the nonexistence of a finite $\underline{z}$ for the best Chebyshev approximation, we can obtain a finite $\underline{z}$ for some prescribed neighborhood of $\|\underline{\epsilon}^{**}\|_\infty$. The second point to be noted is that the prescribed vector, $\underline{f}$, represents the sample values of a non-realizable impulse response function which is not generally encountered in network synthesis.

---

[10]This point will be discussed further in Chapter VI.

Let us now define the vector $\underline{r}^* \in \mathscr{R}$ which yields the minimum value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ as follows:

Definition 4.1: The vector $\underline{r}^*$ is said to be the best Chebyshev solution vector in $\mathscr{R}$, if

$$\|\underline{\epsilon}^*(\underline{r}^*)\|_\infty \leq \|\underline{\epsilon}^*(\underline{r})\|_\infty \qquad (4.31)$$

for all $\underline{r} \in \mathscr{R}$ which satisfy the relation

$$[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon}^*(\underline{r}) \qquad (4.32)$$

where $\underline{f}$ is a prescribed real vector in U^q, $[R(\underline{r})]$ is a $q \times (q - n)$ matrix defined by Eq. 3.2, and $\underline{\epsilon}^*(\underline{r})$ is defined by[11] Eq. 3.17.

Some further examination of Eq. 4.32 reveals that one can find the lower bound which the value of $\|\underline{\epsilon}^*(\underline{r}^*)\|_\infty$ can attain. First, we note that the left hand side of Eq. 4.32 may be written as

$$[R(\underline{r})]^T \underline{f} = [F]\,\underline{r} \qquad (4.33)$$

where

$$[F] = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n+1} \\ f_2 & f_3 & \cdots & f_{n+2} \\ \vdots & \vdots & & \vdots \\ f_{q-n} & f_{q-n+1} & \cdots & f_q \end{bmatrix} = (q - n) \times (n + 1) \text{ matrix}) \qquad (4.34)$$

[11]Recall from Lemma 3.2 that $\underline{\epsilon}^*(\underline{r}) = \underline{\epsilon}^*(\underline{z})$.

Let us substitute Eq. 4.33 into Eq. 4.32 and take the ℓ_∞-norm of both sides. This yields,

$$\|[F]\underline{r}\|_\infty = \|[R(\underline{r})]^T \underline{\epsilon}^*\|_\infty$$

$$= \max_{1\leq i\leq q-n} \left| \sum_{j=0}^{n} r_j \epsilon^*_{i+j} \right| \leq \max_{1\leq i\leq q-n} \sum_{j=0}^{n} |r_j \epsilon^*_{i+j}|$$

$$\leq \left(\sum_{j=0}^{n} |r_j|\right) \left(\max_{1\leq i+j\leq q} |\epsilon^*_{i+j}|\right) = \|\underline{r}\|_1 \|\underline{\epsilon}^*\|_\infty \tag{4.35}$$

Hence, from Eq. 4.35 one obtains the following relation:

$$\|\underline{\epsilon}^*(\underline{r})\|_\infty \geq \|[F] \frac{\underline{r}}{\|\underline{r}\|_1}\|_\infty \tag{4.36}$$

for all $\underline{r} \in \mathscr{R}$. The matrix $[F]$ is a prescribed real matrix, and the function $\|[F]\underline{r}\|_\infty$ is a continuous function of $\underline{r} \in E^{n+1}$. Hence, if we allow $\underline{r}$ to vary on the unit sphere defined by $\|\underline{r}\|_1 = 1$, then the function $\|[F]\underline{r}\|_\infty$ attains its minimum at some point on that sphere. Let us denote this minimum value by η. Thus, from Eqs. 4.31 and 4.36, we note that, for each[12] $\underline{r} \in \mathscr{R}$, the value of $\|\underline{\epsilon}^*(\underline{r}^*)\|_\infty$ lies in the interval defined by

$$\|\underline{\epsilon}^*(\underline{r})\|_\infty \geq \|\underline{\epsilon}^*(\underline{r}^*)\|_\infty \geq \eta \tag{4.37}$$

where

$$\eta = \min_{\|\underline{r}\|_1 = 1} \|[F]\underline{r}\|_\infty \tag{4.38}$$

---

[12] Note that again we assume that the minimum will not be attined at the point where the component $r_n = 0$ (see Chapter III).

<u>Remark</u>: Note that if $\eta = 0$, then the (n+1) colums of [F] are linearly dependent. This implies the existence of an exact solution, i.e., there exists an $\underline{r}^* \epsilon \mathscr{R}$, such that $\underline{\epsilon}^*(\underline{r}^*) = 0$. Clearly, this represents the case in which the interpolation of "Prony's Original Method" (Section 2.3.2, Chapter II) may be extended to q-points, where $q > 2n$. This case is illustrated in Example 5.6 of Chapter V.

The following lemma summarizes the above results:

<u>Lemma 4.2</u>: Let $\underline{r}^*$ be the best Chebyshev solution vector in $\mathscr{R}$ and let $\underline{\epsilon}^*(\underline{r}^*)$ be the corresponding real error vector in U^q. Furthermore, let $\underline{\tilde{r}}$ be the vector in $\mathscr{R}$ which yields the value of η defined by Eq. 4.38, that is

$$\eta \triangleq \| [F] \frac{\underline{\tilde{r}}}{\|\underline{\tilde{r}}\|_1} \|_\infty \leq \| [F] \frac{\underline{r}}{\|\underline{r}\|_1} \|_\infty \tag{4.39}$$

for all $\underline{r} \ \epsilon \ \mathscr{R}$, and where [F] is a prescribed $(q-n) \times (n+1)$ matrix defined by Eq. 4.34. Then the value of $\|\underline{\epsilon}^*(\underline{r}^*)\|_\infty$ lies in the interval

$$\|\underline{\epsilon}^*(\underline{\tilde{r}})\|_\infty \geq \|\underline{\epsilon}^*(\underline{r}^*)\|_\infty \geq \eta \tag{4.40}$$

and the equality holds, if and only if, any (n+1) consecutive components of the vector $\underline{\epsilon}^*(\underline{\tilde{r}})$ satisfy

(i)

$$|\epsilon^*_{M+j}(\underline{\tilde{r}})| = \|\underline{\epsilon}^*(\underline{\tilde{r}})\|_\infty , \qquad \text{for } j = 0, 1, \ldots, n \text{ and} \tag{4.41}$$

(ii) either

$$\text{sgn}\, \epsilon^*_{M+j}(\underline{\tilde{r}}) = \text{sgn}\, \tilde{r}_j , \qquad \text{for all } j = 0, 1, \ldots, n,$$

or

$$\text{sgn}\, \epsilon^*_{M+j}(\tilde{r}) = -\text{sgn}\, \tilde{r}_j , \qquad \text{for all } j = 0, 1, \ldots, n \tag{4.42}$$

where $M \epsilon \{i = 1, 2, \ldots, q-n\}$.

Proof: All of this lemma, except for the conditions of Eqs. 4.41 and 4.42, has been obtained in the discussion above. To show that the conditions given by Eqs. 4.41 and 4.42 are needed to make the inequality signs of Eq. 4.40 become equality signs, let us examine Eqs. 4.35 and 4.36. Clearly, if the condition given by Eq. 4.41 and 4.42 are satisfied, then the inequality sign of Eq. 4.36 becomes an equality sign.

To prove the "only if" part, let us suppose that there exists a vector $\underline{r}' \epsilon \mathcal{R}$ such that

$$\|\underline{\epsilon}^*(\underline{r}')\|_\infty = \eta$$

Then we can write

$$\epsilon_i^*(\underline{r}') = \zeta_i \eta , \qquad i = 1,2,\dots,q$$

where $-1 \leq \zeta_i \leq 1$. Substituting into Eq. 4.35, we obtain

$$\| [F] \underline{r}' \|_\infty = \| [R(\underline{r})]^T \underline{\epsilon}^*(\underline{r}') \|_\infty$$

$$= \eta \left(\max_{1 \leq i \leq q-n} \left| \sum_{j=0}^{n} r_j' \zeta_{i+j} \right| \right)$$

However, since

$$\eta = \frac{1}{\| \underline{r}' \|_1} \| [F] \underline{r}' \|_\infty$$

then we obtain

$$\| \underline{r}' \|_1 = \max_{1 \leq i \leq q-n} \left| \sum_{j=0}^{n} r_j' \zeta_{i+j} \right|$$

Denoting by M, the i for which the right hand side of the above expression attains its maximum, we have

$$\| \underline{r}' \|_1 = \left| \sum_{j=0}^{n} r_j' \zeta_{M+j} \right|$$

which can hold only if

$$\zeta_{M+j} = \operatorname{sgn} r_j' , \qquad j = 0,1,\dots,n$$

or

$$\zeta_{M+j} = -\operatorname{sgn} r_j' , \qquad j = 0,1,\dots,n$$

Thus, we have $|\epsilon^*_{M+j}(\underline{r}')| = \eta = \| \underline{\epsilon}^*(\underline{r}') \|_\infty$ and the condition of Eq. 4.42. Hence, the lemma is proved.

The following example[13] illustrates Lemma 4.2:

Example 4.1: Given $q = 5$, $n = 2$, and

$$\underline{f} = \begin{bmatrix} 3.9 \\ 3.1 \\ 1.9 \\ 1.1 \\ -0.1 \end{bmatrix}$$

Since $n = 2$, then the vector $\underline{r}$ is given by

$$\underline{r} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}$$

and from Eq. 4.34, the matrix [F] is given by

$$[F] = \begin{bmatrix} 3.9 & 3.1 & 1.9 \\ 3.1 & 1.9 & 1.1 \\ 1.9 & 1.1 & -0.1 \end{bmatrix}$$

The vector $\tilde{\underline{r}}$, which gives the minimum value of $\|[F]\,\underline{r}\|_\infty$, is found to be

[13]This example is considered in detail in Chapter V, Example 5.4.

$$\tilde{\underline{r}} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

and the corresponding vector $[F]\tilde{\underline{r}}$ is

$$[F]\tilde{\underline{r}} = \begin{bmatrix} -0.4 \\ 0.4 \\ -0.4 \end{bmatrix}$$

Therefore, the minimal value of $\|[F]\underline{r}\|_\infty$ in the set $\|\underline{r}\|_1 = 1$, i.e., the value of η given by Eq. 4.39, is

$$\eta \triangleq \frac{\|[F]\tilde{\underline{r}}\|_\infty}{\|\tilde{\underline{r}}\|_1} = \frac{0.4}{4} = 0.1$$

To find the best Chebyshev error vector $\underline{\epsilon}^*(\tilde{\underline{r}})$, we must first obtain the vector $\underline{z}$ from Eq. 3.4, namely from

$$P(z) = z^2 - 2z + 1 = 0$$

This yields

$$\tilde{\underline{z}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since the components of $\tilde{\underline{z}}$ are not distinct, then, if we assume that the initial sample of f(t) (i.e., the value represented by the component f_1)

is taken at $t = 0$, the matrix $[Z(\underline{z})]$ is given by[14] Eq. 1.18c, namely,

$$[Z(\tilde{\underline{z}})] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

Consider the vector relation

$$\begin{bmatrix} 3.9 \\ 3.1 \\ 1.9 \\ 1.1 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

The vector $\underline{\beta}^*$ which minimizes the value of $\|\underline{\epsilon}\|_\infty$ is found to be

$$\underline{\beta}^* = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

and the corresponding best Chebyshev error vector is

---

[14] The reader is referred to Example 5.4 where we have used the matrix $[Z(\underline{z})]$ to be given by Eq. 1.18. The results of the minimizing procedure are identical to those presented here.

$$\underline{\epsilon}^* \triangleq \underline{\epsilon}^*(\tilde{\underline{r}}) = \begin{bmatrix} -0.1 \\ 0.1 \\ -0.1 \\ 0.1 \\ -0.1 \end{bmatrix}.$$

Since the value of $\|\underline{\epsilon}^*(\tilde{\underline{r}})\|_\infty = 0.1 = \eta$, then the vector $\tilde{\underline{r}}$, found above, is the best Chebyshev solution vector $\underline{r}^*$. Furthermore, it is noted that the sign configuration of the vector $\underline{\epsilon}^*(\tilde{\underline{r}})$ satisfies Eq. 4.42 of Lemma 4.2.

It should be noted that the vector $\tilde{\underline{r}}$ defined in Lemma 4.2 may be obtained by extending Ruston's method (see Section 2.3.3.2, Chapter II) to the case when the vector $\underline{r} \epsilon E^{n+1}$ is restricted to the set defined by $\|\underline{r}\|_1 = 1$. Recall that Ruston's method yields the vector $\underline{r} \epsilon E^{n+1}$ for which the function[15] $\|[F]\underline{r}\|_\infty$ is minimum for all $\underline{r} \epsilon E^{n+1}$, when $r_n = 1$. It suffices to say[16] that if some other component of $\underline{r} \epsilon E^{n+1}$ is assumed to be equal to one, then the function $\|[F]\underline{r}\|_\infty$ may attain its minimum at some other $\underline{r} \epsilon E^{n+1}$. Let us denote by $\underline{r}^{(j)}$, the vector $\underline{r} \epsilon E^{n+1}$ with the j-th component equal to one, and denote by $\tilde{\underline{r}}^{(j)}$, the vector $\underline{r}$ for which the function $\|[F]\underline{r}^{(j)}\|_\infty$ is minimum. Since there are $(n+1)$ distinct vectors

---

[15]Note that the vector $[F]\underline{r}$ represents the vector $\underline{\delta}(\underline{r})$ defined by Eq. 2.93.

[16]See the discussion concerning Eq. 2.93 and Eq. 2.94.

$\underline{r}^{(j)} \in E^{n+1}$, then we have a set of (n+1) distinct vectors $\tilde{\underline{r}}^{(j)}$, namely, $\{\tilde{\underline{r}}^{(j)} : j = 1, \ldots, n+1\}$ and a corresponding set of values $\{\|[F]\tilde{\underline{r}}^{(j)}\|_\infty : j = 1, \ldots, n+1\}$. Hence η, defined by Eq. 4.38, is

$$\eta = \min_{1 \le j \le n+1} \left\{ \frac{\|[F]\tilde{\underline{r}}^{(j)}\|_\infty}{\|\tilde{\underline{r}}^{(j)}\|_1} \right\} \tag{4.43}$$

and the vector $\tilde{\underline{r}}$ defined by Lemma 4.2 is the vector $\underline{r}$ in the set $\{\tilde{\underline{r}}^{(j)} : j = 1, \ldots, n+1\}$ which yields η. This may be best illustrated by the following example:[17]

Example 4.2: Given q = 5, n = 2, and

$$\underline{f} = \begin{bmatrix} 0 \\ 2.8 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

Hence,

$$[F] = \begin{bmatrix} 0 & 2.8 & 2 \\ 2.8 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

[17] This example is considered in detail in Chapter V, Example 5.3.

and

$$\underline{r} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix} .$$

Let us denote the vector $[F]\,\underline{r}$ by the vector $\underline{\delta}(\underline{r})$, i.e., $\underline{\delta}(\underline{r}) = [F]\,\underline{r}$. The set of vectors $\{\underline{\tilde{r}}^{(j)} : j = 1,2,3\}$ which minimize the set of functions $\{\|\underline{\delta}(\underline{r}^{(j)})\|_\infty : j = 1,2,3\}$ is given in Table 4.1. From Eq. 4.43

j	1	2	3
$\tilde{r}_0^{(j)}$	1.0	-0.1346	-0.04098
$\tilde{r}_1^{(j)}$	-0.8000	1.0	-0.60109
$\tilde{r}_2^{(j)}$	0.3467	-1.1769	1.0
$\tilde{\delta}_1^{(j)}$	-1.5467	0.4462	0.31694
$\tilde{\delta}_2^{(j)}$	1.5467	0.4462	-0.31694
$\tilde{\delta}_3^{(j)}$	1.5467	-0.4462	0.31694
$\dfrac{\|\underline{\tilde{\delta}}^{(j)}\|_\infty}{\|\underline{\tilde{r}}^{(j)}\|_1}$	0.7205	0.1930149	0.1930116

Table 4.1. Results of Example 4.2.

and the set of values $\left\{ \frac{\|\underline{\tilde{\delta}}^{(j)}\|_\infty}{\|\underline{\tilde{r}}^{(j)}\|_1} : j = 1, 2, 3 \right\}$, where $\underline{\tilde{\delta}}^{(j)} \triangleq [F]\underline{\tilde{r}}^{(j)}$,

we obtain that $\eta = 0.1930116$. Furthermore, vector[18] $\underline{\tilde{r}}^{(3)}$ represents the vector $\underline{\tilde{r}}$ defined by Lemma 4.2.

The following two points should be noted concerning the above results: First, if the vector $\underline{r}$ obtained by Ruston's methods yields η, namely if $\underline{\tilde{r}}^{(n+1)} = \underline{\tilde{r}}$, then, by Eq. 4.40, Ruston's final ℓ_∞^q-norm of the error vector, $\|\underline{\epsilon}^*(\underline{\tilde{r}})\|_\infty$, represents the upper bound of the interval within which the optimum ℓ_∞^q-norm of the error vector, $\|\underline{\epsilon}^*(\underline{r}^*)\|_\infty$ lies. Furthermore, if the interval $[\eta, \|\underline{\epsilon}^*(\underline{\tilde{r}})\|_\infty]$ is small, then for a specific application, where only the final error of approximation is of interest, Ruston's solution vector $\underline{\tilde{r}} \in \mathscr{R}$ may be quite satisfactory (see Example 5.3). The second point to be noted is that if $\eta = 0$, then the vector $\underline{\tilde{r}} = \underline{r}^*$ and the best Chebyshev approximation yields $\|\underline{\epsilon}^*(\underline{r}^*)\|_\infty = 0$ (see Examples 5.5 and 5.6).

4.4 Properties of the Best Chebyshev Approximation

In this section we shall study the properties of the best Chebyshev approximation given by Eqs. 4.1 and 4.2. Recall, from Section 2.3.1.2,[19] that when the vector $\underline{z} \in \mathscr{Z}$ is initially prescribed (i.e., the q x n matrix $[Z(\underline{z})]$ is fully prescribed), then the vector $\underline{\beta}^* \in \mathscr{B}_z$,

---

[18] Note that the vectors $\underline{\tilde{r}}^{(2)}/\|\underline{\tilde{r}}^{(2)}\|_1$ and $\underline{\tilde{r}}^{(3)}/\|\underline{\tilde{r}}^{(3)}\|_1$ are almost identical.

[19] See Corollary 2.1.

which defines the best Chebyshev approximating vector, $\underline{f}^*$, in U^q, will also define the best Chebyshev approximating vector in some (n+1)-dimensional reference subspace of U^q taken from the set $\{U_v^{n+1} : v = 1, 2, \ldots, \binom{q}{n+1}\}$. In this section, we shall show that when the vector $\underline{z} \in \mathcal{Z}$ is not initially prescribed, then the vector pair $(\underline{\beta}^{**}, \underline{z}^*) \in \mathcal{B}_z \times \mathcal{Z}$, which defines the best Chebyshev approximating real vector in U^q, will also define the best Chebyshev approximating vector in some (2n+1)-dimensional reference subspace[20] of U^q, taken from the set $\{U_w^{2n+1} : w = 1, 2, \ldots, \binom{q}{2n+1}\}$. However, before turning to a detailed consideration of this property, we must first define the set of distinct (2n+1)-dimensional reference subspaces of U^q, given by $\{U_w^{2n+1} : w = 1, 2, \ldots, \binom{q}{2n+1}\}$, and establish the relationship between the prescribed and approximating vectors in each (2n+1)-dimensional reference subspace of U^q.

Let us define the (2n+1)-dimensional reference subspace, U_k^{2n+1}, of U^q according to Definition 2. 1. Clearly, when $m = 2n+1$ in Definition 2. 1, one can form a set of $\binom{q}{2n+1}$ distinct (2n+1)-dimensional reference subspaces from U^q, which will be denoted by the set $\{U_w^{2n+1} : w = 1, 2, \ldots, \binom{q}{2n+1}\}$. In each (2n+1)-dimensional reference subspace U_w^{2n+1} ; $w = 1, 2, \ldots, \binom{q}{2n+1}$, the (2n+1)-dimensional projections of the vectors $\underline{f}$, $[Z(\underline{z})]\underline{\beta}$, and $\underline{\epsilon}$ in U^q, are given by

---

[20] The increase in the dimensionality of the reference subspace is based on the fact that the parameter vector $\underline{z} \in \mathcal{Z}$ introduces at most n new degrees of freedom.

$$\underline{f}^{(w)} = [I_w]^T \underline{f} \quad \epsilon \; U_w^{2n+1} \tag{4.44}$$

$$[Z^{(w)}(\underline{z})] \underline{\beta} = [I_w]^T [Z(\underline{z})] \underline{\beta} \quad \epsilon \; U_w^{2n+1} \tag{4.45}$$

and

$$\underline{\epsilon}^{(w)} = [I_w]^T \underline{\epsilon} \quad \epsilon \; U_w^{2n+1} \tag{4.46}$$

where $[I_w]$ is a q x (2n+1) matrix defined by Eq. 2.26. Therefore, the basic relation between the prescribed and approximating vectors in U^q, namely,

$$\underline{f} = [Z(\underline{z})] \underline{\beta} + \underline{\epsilon}(\underline{\beta}, \underline{z}) \tag{4.47}$$

implies the following relation between the prescribed and approximating vectors in each (2n+1)-dimensional reference subspace, U_w^{2n+1}:

$$\underline{f}^{(w)} = [Z^{(w)}(\underline{z})] \underline{\beta} + \underline{\epsilon}^{(w)}(\underline{\beta}, \underline{z}) \tag{4.48}$$

where $w = 1, 2, \ldots, \binom{q}{2n+1}$.

The Chebyshev approximation problem in each (2n+1)-dimensional reference subspace U_k^{2n+1}, $k \; \epsilon \; \{w = 1, 2, \ldots, \binom{q}{2n+1}\}$ is stated as follows: Given a real vector $\underline{f}^{(k)} \; \epsilon \; U_k^{2n+1}$ and a real approximating vector $[Z^{(k)}(\underline{z})] \underline{\beta}$, defined by Eq. 4.45; find the best

Chebyshev k-th reference parameter vector pair[21]

$(\underline{\beta}_k^{**}, \underline{z}_k^*) \in \mathscr{B}_z \times \mathscr{Z}$, such that if

$$\hat{\underline{f}}^{(k)}(\underline{\beta}_k^{**}, \underline{z}_k^*) \triangleq [Z^{(k)}(\underline{z}_k^*)]\, \underline{\beta}_k^{**} \tag{4.49}$$

then

$$\|\underline{\epsilon}^{(k)}(\underline{\beta}_k^{**}, \underline{z}_k^*)\|_\infty \triangleq \|\underline{f}^{(k)} - \hat{\underline{f}}^{(k)}(\underline{\beta}_k^{**}, \underline{z}_k^*)\|_\infty \leq \|\underline{f}^{(k)} - [Z^{(k)}(\underline{z})]\, \underline{\beta}\|_\infty \tag{4.50}$$

for all $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$.

It should be noted that the approximating vector $\hat{\underline{f}}(\underline{\beta}_k^{**}, \underline{z}_k^*) \in U^q$ corresponding to the vector pair $(\underline{\beta}_k^{**}, \underline{z}_k^*) \in \mathscr{B}_z \times \mathscr{Z}$ might not necessarily define the best Chebyshev approximating vector in U^q, for the given vector $\underline{z}_k^* \in \mathscr{Z}$, since there might exist a $\underline{\beta}^* \neq \beta_k^{**} \in \mathscr{B}_{z_k^*}$ such that

$$\|\underline{\epsilon}(\underline{\beta}^*, \underline{z}_k^*)\|_\infty < \|\underline{\epsilon}(\underline{\beta}_k^{**}, \underline{z}_k^*)\|_\infty \tag{4.51}$$

where $\underline{\epsilon}(\underline{\beta}^*, \underline{z}_k^*)$ and $\underline{\epsilon}(\underline{\beta}_k^{**}, \underline{z}_k^*)$ in U^q are defined by

$$\underline{\epsilon}(\underline{\beta}^*, \underline{z}_k^*) = \underline{f} - \hat{\underline{f}}(\underline{\beta}^*, \underline{z}_k^*) = \underline{f} - [Z(\underline{z}_k^*)]\, \underline{\beta}^*$$

and

$$\underline{\epsilon}(\underline{\beta}_k^{**}, \underline{z}_k^*) = \underline{f} - \hat{\underline{f}}(\underline{\beta}_k^{**}, \underline{z}_k^*) = \underline{f} - [Z(\underline{z}_k^*)]\, \beta_k^{**}$$

---

[21] For clarity, we shall refer to the best Chebyshev parameter vector pair which defines the best Chebyshev approximation in U_k^{2n+1}, as the "best Chebyshev k-th reference parameter vector pair," and denote it by $(\underline{\beta}_k^{**}, \underline{z}_k^*)$.

The following theorem establishes the relation between the best Chebyshev parameter vector pair $(\underline{\beta}^{**},\underline{z}^*)$ and the best Chebyshev k-th reference parameter vector pair $(\underline{\beta}_k^{**},\underline{z}_k^*)$:

Theorem 4.2: If there exists a parameter vector pair $(\underline{\beta}',\underline{z}') \in \mathscr{B}_z \times \mathscr{Z}$ and a (2n+1)-dimensional reference subspace U_c^{2n+1}, where $c \in \{w = 1,2,\ldots, \binom{q}{2n+1}\}$, such that the ℓ_∞^{2n+1}-norm of the error vector $\underline{\epsilon}^{(c)}(\underline{\beta}',\underline{z}') \in U_c^{2n+1}$ satisfies

$$\|\underline{\epsilon}^{(c)}(\underline{\beta}',\underline{z}')\|_\infty \leq \|\underline{\epsilon}^{(c)}(\underline{\beta},\underline{z})\|_\infty \qquad (4.52)$$

for all $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$; and if the ℓ_∞^q-norm of the error vector $\underline{\epsilon}(\underline{\beta}',\underline{z}') \in U^q$ satisfies

$$\|\underline{\epsilon}(\underline{\beta}',\underline{z}')\|_\infty = \|\underline{\epsilon}^{(c)}(\underline{\beta}',\underline{z}')\|_\infty$$

then the vector pair $(\underline{\beta}',\underline{z}')$ is the best Chebyshev parameter vector pair in $\mathscr{B}_z \times \mathscr{Z}$, i.e.,

$$(\underline{\beta}',\underline{z}') = (\underline{\beta}^{**},\underline{z}^*)$$

Proof: First we note that for each parameter vector pair $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$

$$\|\underline{\epsilon}(\underline{\beta},\underline{z})\|_\infty = \max_{1 \leq w \leq \binom{q}{2n+1}} \{\|\underline{\epsilon}^{(w)}(\underline{\beta},\underline{z})\|_\infty\} \qquad (4.53)$$

or alternately,

$$\|\underline{\epsilon}(\underline{\beta},\underline{z})\|_\infty \geq \|\underline{\epsilon}^{(w)}(\underline{\beta},\underline{z})\|_\infty, \quad w = 1,2,\ldots,\binom{q}{2n+1} \tag{4.54}$$

where $\underline{\epsilon}^{(w)}(\underline{\beta},\underline{z})$ represents the projection of the error vector $\underline{\epsilon}(\underline{\beta},\underline{z}) \in U^q$ onto U_w^{2n+1} of U^q. Let us denote by U_c^{2n+1} the (2n+1)-dimensional reference subspace for which the ℓ_∞^{2n+1}-norm of the error vector $\underline{\epsilon}^{(c)}(\underline{\beta},\underline{z}) \in U_c^{2n+1}$, when $(\underline{\beta},\underline{z}) = (\underline{\beta}',\underline{z}')$, satisfies

$$\|\underline{\epsilon}^{(c)}(\underline{\beta}',\underline{z}')\|_\infty = \max_{1 \leq w \leq \binom{q}{2n+1}} \{\|\underline{\epsilon}^{(w)}(\underline{\beta}',\underline{z}')\|_\infty\}$$

Hence, from Eq. 4.53 we have

$$\|\underline{\epsilon}^{(c)}(\underline{\beta}',\underline{z}')\|_\infty = \|\underline{\epsilon}(\underline{\beta}',\underline{z}')\|_\infty \tag{4.55}$$

If $\|\underline{\epsilon}^{(c)}(\underline{\beta}',\underline{z}')\|_\infty$ satisfies Eq. 4.52, then from Eq. 4.55 we obtain

$$\|\underline{\epsilon}(\underline{\beta}',\underline{z}')\|_\infty = \|\underline{\epsilon}^{(c)}(\underline{\beta}',\underline{z}')\|_\infty \leq \|\underline{\epsilon}^{(c)}(\underline{\beta},\underline{z})\|_\infty \tag{4.56}$$

for all $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$. Moreover, by using the relation of Eq. 4.54, we have

$$\|\underline{\epsilon}^{(c)}(\underline{\beta},\underline{z})\|_\infty \leq \|\underline{\epsilon}(\underline{\beta},\underline{z})\|_\infty$$

Then the relation of Eq. 4.56 yields

$$\|\underline{\epsilon}(\underline{\beta}',\underline{z}')\|_\infty \leq \|\underline{\epsilon}(\underline{\beta},\underline{z})\|_\infty \tag{4.57}$$

for all $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$. However, since by definition the best Chebyshev parameter vector $(\underline{\beta}^{**},\underline{z}^*) \in \mathscr{B}_z \times \mathscr{Z}$ yields an error vector $\underline{\epsilon}(\underline{\beta}^{**},\underline{z}^{*}) \in U^q$ with a ℓ_∞^q-norm satisfying

$$\|\underline{\epsilon}(\underline{\beta}^{**},\underline{z}^{*})\|_\infty \leq \|\underline{\epsilon}(\underline{\beta},\underline{z})\|_\infty$$

for all $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$, then from Eq. 4. 57 we note that $(\underline{\beta}',\underline{z}') = (\underline{\beta}^{**},\underline{z}^{*})$. Thus, the theorem is proved.

Remark: Note that since the vector pair $(\underline{\beta}',\underline{z}')$, considered in Theorem 4. 2, must satisfy Eq. 4. 52, then $(\underline{\beta}',\underline{z}')$ represent the best Chebyshev k-th reference vector pair $(\underline{\beta}_k^{**},\underline{z}_k^{*})$ defined by Eq. 4. 50, when k = c.

It is convenient to formulate the above results in terms of "Prony's Extended Method." It can be shown (see Appendix C) that, in terms of "Prony's Extended Method," the relation of Eq. 4. 48, i. e. , the relation between the vectors in each (2n+1)-dimensional reference subspace U_w^{2n+1}, $w = 1,\ldots,\binom{q}{2n+1}$, is given by

$$[\Lambda^{(w)}(\underline{r})]^T \underline{f}^{(w)} = [\Lambda^{(w)}(\underline{r})]^T \underline{\epsilon}^{(w)}, \quad w = 1,\ldots,\binom{q}{2n+1} \qquad (4.58)$$

where $\underline{f}^{(w)}$ and $\underline{\epsilon}^{(w)}$ are given by Eqs. 4. 44 and 4. 46, respectively; and where $[\Lambda^{(w)}(\underline{r})]$ is a (2n+1) x (n+1) matrix defined by[22]

---

[22] See Appendix C, Definition C. 3, when m = 2n+1.

$$
[\Lambda^{(w)}(\underline{r})] = \begin{bmatrix}
\lambda^{(w)}_{1,1}(\underline{r}) & 0 & \cdots & 0 \\
\lambda^{(w)}_{2,1}(\underline{r}) & \lambda^{(w)}_{2,2}(\underline{r}) & & \vdots \\
\vdots & \lambda^{(w)}_{3,2}(\underline{r}) & & 0 \\
\lambda^{(w)}_{n+1,1}(\underline{r}) & \vdots & \cdots & \lambda^{(w)}_{n+1,n+1}(\underline{r}) \\
0 & \lambda^{(w)}_{n+2,2}(\underline{r}) & & \lambda^{(w)}_{n+2,n+1}(\underline{r}) \\
0 & 0 & & \vdots \\
\vdots & \vdots & & \\
0 & 0 & \cdots & \lambda^{(w)}_{2n+1,n+1}(\underline{r})
\end{bmatrix} \qquad (4.59)
$$

where the nonzero elements, $\lambda_{ij}^{(w)}(\underline{r})$, of the matrix $[\Lambda^{(w)}(\underline{r})]$, represent some polynomial in the n+1-variables $\{r_0, r_1, \ldots, r_n\}$ denoting the components of the vector $\underline{r} \in \mathscr{R}$.

The following three points should be noted concerning the matrix $[\Lambda^{(w)}(\underline{r})]$: First, the (n+1)-column vectors of $[\Lambda^{(w)}(\underline{r})]$ span an (n+1)-dimensional subspace, $C_{n+1}(\Lambda^{(w)})$, in U_w^{2n+1} which is orthogonal-complement to the n-dimensional subspace,[23] $C_n(Z^{(w)})$, spanned by the n column vectors of the (2n+1) x n matrix $[Z^{(w)}(\underline{z})]$. Second,

---

[23] Note, from Eq. 4.45, that since the matrix $[Z^{(w)}(\underline{z})] = [I_W]^T [Z(\underline{z})]$, and since the matrix $[Z(\underline{z})]$, defined by Eq. 1.18 (Eq. 1.18a), is of maximal rank, n, for all $\underline{z} \in \mathscr{Z}$, then the matrix $[Z^{(w)}(\underline{z})]$ is of maximal rank, n.

since the matrix $[\Lambda^{(w)}(\underline{r})]$ is a function $\underline{r} \in \mathscr{R}$, then the relation between the (q x n) matrix $[Z(\underline{z})]$ and the q x (q - n) matrix $[R(\underline{r})]$, given by Lemma 3. 1, is preserved. Third, for a given pair of vectors $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$, which satisfy Eq. 4. 58 for some $k \in \{w = 1, \ldots, \binom{q}{2n+1}\}$, one can obtain[24] a unique vector $\underline{\epsilon} \in U^q$ which satisfies the relation $[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon}$.

Let us now define the best Chebyshev k-th reference solution vector $\underline{r}_k^* \in \mathscr{R}$ as follows:

Definition 4. 2: The vector $\underline{r}_k^*$ is said to be the best Chebyshev k-th reference solution vector in $\mathscr{R}$, if

$$\|\underline{\epsilon}^{*(k)}(\underline{r}_k^*)\|_\infty \leq \|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty \tag{4.60}$$

for all $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}^{(k)}(\underline{r}) \in U_k^{2n+1}$ which satisfy the relation

$$[\Lambda^{(k)}(\underline{r})]^T \underline{f}^{(k)} = [\Lambda^{(k)}(\underline{r})]^T \underline{\epsilon}^{(k)}(\underline{r}) \tag{4.61}$$

where $\underline{f}^{(k)}$ and $\underline{\epsilon}^{(k)}(\underline{r})$ represent the projection of $\underline{f}$ and $\underline{\epsilon}(\underline{r}) \in U^q$ onto the (2n+1)-dimensional reference subspace U_k^{2n+1}, $k \in \{w = 1, 2, \ldots, \binom{q}{2n+1}\}$, according to Definition 2. 1; and where $[\Lambda^{(k)}(\underline{r})]$ is a (2n+1) x (n+1) matrix defined according to Definition C. 3, when m = 2n + 1.

---

[24] See Appendix C, Theorem C. 1, when m = 2n + 1.

<u>Remark</u>: Note that the vector $\underline{\epsilon}^{*(k)}(\underline{r}_k)$, defined by Eq. 4.60, and the vector $\underline{\epsilon}^{(k)}(\underline{\beta}_k^{**}, \underline{z}_k^{*})$, defined by Eq. 4.50, are identical, since the vectors $\underline{r}_k^*$ and $\underline{z}_k^*$ are related by Eq. 3.4. Hence, the vector $\underline{\epsilon}^{*(k)}(\underline{r}_k^*)$ may be denoted[25] by $\underline{\epsilon}^{(k)}(\underline{\beta}_k^{**}, \underline{r}_k^{*})$. This notation is particularly helpful when one wants to differentiate between the following two error vectors in U^q:

(1) the vector $\underline{\epsilon}(\underline{\beta}_k^{**}, \underline{r}_k^{*})$ which denotes the vector $\underline{\epsilon} \in U^q$ obtained from a given value of $\underline{r}_k^* \in \mathscr{R}$ and $\underline{\epsilon}^{*(k)}(\underline{r}_k^*) \in U_k^{2n+1}$, by using Theorem C. 1; and

(2) the vector $\underline{\epsilon}^*(\underline{r}_k^*)$ which denotes the vector $\underline{\epsilon} \in U^q$ obtained for a given value of $\underline{r}_k^* \in \mathscr{R}$ and with a ℓ_∞^q-norm satisfying

$$\|\underline{\epsilon}^*(\underline{r}_k^*)\|_\infty \triangleq \|\underline{\epsilon}(\underline{\beta}^*, \underline{r}_k^*)\|_\infty \leq \|\underline{\epsilon}(\underline{\beta}, \underline{r}_k^*)\|_\infty$$

for all $\underline{\beta} \in \mathscr{B}_z$. Clearly, the vector $\underline{\epsilon}^*(\underline{r}_k^*) \in U^q$ represents the vector $\underline{\epsilon}(\underline{\beta}^*, \underline{z}_k^*)$ discussed in Eq. 4.51.

---

[25]Note that our notation does not distinguish between the vector $\underline{\epsilon}^{*(k)}(\underline{r}_k^*)$ defined by Eq. 4.60 and the projection of the vector $\underline{\epsilon}^*(\underline{r}_k^*) \in U^q$ onto U_k^{2n+1}. Through this thesis we shall use $\underline{\epsilon}^{*(k)}(\underline{r}_k^*)$ to denote the vector defined by Eq. 4.60. If the projection of the vector $\underline{\epsilon}^*(\underline{r}_k^*) \in U_k^{2n+1}$ is considered and it does not satisfy Eq. 4.60, then for clarity we shall use the notation $\underline{\epsilon}^{(k)}(\underline{\beta}^*, \underline{r}_k^*)$

The following theorem establishes the relation between the best Chebyshev solution vector $\underline{r}^* \in \mathscr{R}$, defined by Definition 4.1, and the best Chebyshev k-th reference solution vector $\underline{r}_k^* \in \mathscr{R}$, defined by Definition 4.2:

Theorem 4.3: Let $\underline{r}_c^* \in \mathscr{R}$ be the best Chebyshev c-th reference solution vector in the (2n+1)-dimensional reference subspace U_c^{2n+1}, where $c \in \{w = 1, 2, \ldots, \binom{q}{2n+1}\}$, namely

$$\| \epsilon^{*(c)}(\underline{r}_c^*) \|_\infty \leq \| \underline{\epsilon}^{(c)}(\underline{r}) \|_\infty \tag{4.62}$$

for all $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}^{(c)} \in U_c^{2n+1}$ satisfying Eq. 4.61. If the ℓ_∞^q-norm of the error vector $\epsilon^*(\underline{r}_c^*) \in U^q$ satisfies

$$\| \underline{\epsilon}^*(\underline{r}_c^*) \|_\infty = \| \underline{\epsilon}^{*(c)}(\underline{r}_c^*) \|_\infty \tag{4.63}$$

then

$$\underline{r}_c^* = \underline{r}^*$$

where $\underline{r}^* \in \mathscr{R}$ is the best Chebyshev solution vector, defined by Definition 4.1.

The proof of this theorem is contained in Theorem 4.2 when the vector $\underline{z} \in \mathscr{Z}$ is replaced by the vector $\underline{r} \in \mathscr{R}$.

In summary, then, we have shown that the best Chebyshev approximation in U^q will also represent a best Chebyshev approximation in a (2n+1)-dimensional reference subspace U_c^{2n+1} of U^q. It should be mentioned that this result does not depend on the dimensionality of the reference subspace. It may be generalized to any m-dimensional reference subspace U_c^m of U^q where $c \in \{\mu = 1, 2, \ldots, \binom{q}{m}\}$ and[26] $m \geq n + 1$. We have selected m to be equal to (2n+1) because if, in each (2n+1)-dimensional reference subspace U_k^{2n+1} of U^q, the error vector, $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$, has all its (2n+1) components equal in absolute value to a parameter, $|\rho|$, then the k-th vector relation of Eq. 4.48 represents (2n+1) equations in (2n+1) unknowns, which are given by ρ and the 2n-parameters of $(\underline{\beta}, \underline{z})$. The problem of finding the best Chebyshev approximation in each (2n+1)-dimensional reference subspace of U^q will be considered in the next section.

4.5 The Best Chebyshev Approximation in a (2n+1)-Dimensional Reference Subspace

This section considers the problem of determining the best Chebyshev k-th reference solution vector, $\underline{r}_k^* \in \mathscr{R}$, which is defined by Definition 4.2, namely, to minimize[27] the function

[26] Recall, from Section 2.3.1.2 (Chapter II) that for each prescribed $\underline{z} \in \mathscr{Z}$ the reference subspace is (n+1)-dimensional.

[27] The term minimum is synonymous with the term "absolute minimum," unless otherwise indicated.

$\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$ with respect to $\underline{r} \in \mathscr{R}$, where $\underline{\epsilon}^{(k)}(\underline{r}) \in U_k^{2n+1}$, subject to

$$[\Lambda^{(k)}(\underline{r})]^T \underline{f}^{(k)} = [\Lambda^{(k)}(\underline{r})]^T \underline{\epsilon}^{(k)}(\underline{r}) \qquad (4.64)$$

where $\underline{f}^{(k)}$ is a fully prescribed real vector in U_k^{2n+1}, and $[\Lambda^{(k)}(\underline{r})]$ is a (2n+1) x (n+1) matrix defined by Eq. 4.59 (or equivalently Eq. C.23).

Our discussion will be primarily concerned with determining the vector $\underline{r} \in \mathscr{R}$ which yields the minimal value of $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$ subject to the constraint that all the (2n+1)-components of $\underline{\epsilon}^{(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value to $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$. We shall denote this vector $\underline{r}$ by[28] $\underline{r}_k^*$, if there exists no $\underline{r} \in \mathscr{R}$ which yields a smaller value of $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$ when only 2n-components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ are equal in absolute value. If, however, there exists an $\underline{r} \in \mathscr{R}$ which yields a smaller value of $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$ when 2n-components of $\underline{\epsilon}^{(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value, then we shall denote by $\underline{r}_k^*$ the vector $\underline{r} \in \mathscr{R}$ which yields the minimal value of $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$ for such a case.

The following points should be noted concerning the development presented in this section:

---

[28] Note that we use the same notation as that used in Definition 4.2 since we assume (although we were not able to prove it) that such a vector $\underline{r} \in \mathscr{R}$ is the best Chebyshev k-th reference solution vector in $\mathscr{R}$.

(1) The vector $\underline{r}$ represents a vector in E^{n+1}, the (n+1)-dimensional Euclidian space, subject to the constraint that one of its components is equal to one. Specifically, we shall assume that the (n+1)-st component of $\underline{r} \in E^{n+1}$ is equal to one,[29], i.e., $r_n = 1$, so that for each prescribed vector $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$, the (n+1)-equations given by Eq. 4.64 are functions of the n-variables $\{r_0, r_1, \ldots, r_{n-1}\}$. If under this assumption one cannot obtain a finite vector $\underline{r}$ which satisfies Eq. 4.64, then one must assume that some other component of $\underline{r}$ is equal to one.[30]

(2) The components of the vector $\underline{\gamma}(\underline{r}) \in E^{n+1}$, defined by

$$\underline{\gamma}(\underline{r}) = [\Lambda^{(k)}(\underline{r})]^T (\underline{f}^{(k)} - \underline{\epsilon}^{(k)})$$

are continuous functions of $\underline{r} \in \mathscr{R}$, for each real vector $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$. This condition follows from the fact that the nonzero elements of the matrix $[\Lambda^{(k)}(\underline{r})]$ are polynomials in terms of the components of $\underline{r}$.

At this point, let us note that Eq. 4.64 represents a set of (n+1)-simultaneous equations in (3n+1) unknowns, where the (3n+1) unknowns are given by the n-components of the vector $\underline{r}$ and the (2n+1) components of the vector $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$. Hence, it is reasonable that if the

---

[29] This constraint guarantees that the polynomial equation given by Eq. 3.4, yields n roots.

[30] Clearly, this assumption represents the constraint that the vector $\underline{r} \in E^{n+1}$ is restricted to the set $\|\underline{r}\|_1 = 1$.

values of 2n-unknowns are initially prescribed, then the value of the other (n+1)-unknowns can be determined from Eq. 4.64. For example, for each prescribed $\underline{r} \in \mathscr{R}$ and n-components of the vector $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$, the other (n+1)-components of $\underline{\epsilon}^{(k)}$ can always be determined from Eq. 4.64. It should be noted, however, that in the case when 2n-components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ are initially prescribed, a solution to Eq. 4.64 may not exist. This is because in such a case Eq. 4.64 may represent a system of (n+1)-simultaneous <u>nonlinear</u> equations in the n-components of $\underline{r}$ and the (2n+1)-st component of $\underline{\epsilon}^{(k)}$.

Now before presenting the method which we shall use to determine $\underline{r}_k^* \in \mathscr{R}$, let us consider the relations which we shall need. First, let us recall that for each $\underline{r} \in \mathscr{R}$ there exists a vector $\underline{\epsilon}^{*(k)} \in U_k^{2n+1}$ which satisfies Eq. 4.64 with a ℓ_∞^{2n+1}-norm that is minimum.[31] Hence, the problem of minimizing the function $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$ with respect to $\underline{r} \in \mathscr{R}$, subject to the constraint of Eq. 4.64, may be replaced by the problem of minimizing the function $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$ with respect to $\underline{r}$ subject to the same constraint. Therefore, we shall seek the vector $\underline{r}_k^* \in \mathscr{R}$, such that

[31] Recall that since Eq. 4.64 corresponds to the k-th relation of Eq. 4.47, then when $\underline{r}$ is initially prescribed in $\mathscr{R}$, i.e., $\underline{z}$ is initially prescribed in $\mathscr{Z}$, we have the Chebyshev approximation problem discussed in Section 2.3.1.2 (Chapter II) when q = 2n+1 and the matrix [E] is given by the matrix $[Z^{(k)}]$. Note that the vector $\underline{\epsilon}^{*(k)}$ denotes the vector $\underline{\epsilon}^{(k)}$ with $\|\underline{\epsilon}^{*(k)}\|_\infty \leq \|\underline{\epsilon}^{(k)}\|_\infty$ for all $\underline{\epsilon}^{(k)}$ satisfying Eq. 4.64, when $\underline{r}$ is initially prescribed.

the ℓ_∞^{2n+1}-norm of the corresponding error vector, $\underline{\epsilon}^{*(k)}(\underline{r}_k^*)$, is given by

$$\|\underline{\epsilon}^{*(k)}(\underline{r}_k^*)\|_\infty = \min_{\underline{r}\in\mathscr{R}} \|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty \tag{4.65}$$

Since the vector $\underline{\epsilon}^{*(k)}(\underline{r})$ is characterized by having at least (n+1)-component equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$, then the vector $\underline{\epsilon}^{*(k)}(\underline{r}_k^*)$ must be characterized by having m-components equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r}_k^*)\|_\infty$, where $n+1 \leq m \leq 2n+1$.

To show that one may obtain a vector $\underline{\epsilon}^{*(k)}(\underline{r})$, with (2n+1)-components equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$, let us consider Eq. 4.64 when the vector $\underline{\epsilon}^{(k)}(\underline{r})$ is given by

$$\underline{\epsilon}^{(k)} = \rho\, \underline{\sigma}^{(k)} \tag{4.66}$$

where now ρ is a real parameter and, the vector $\underline{\sigma}^{(k)}$ represents an initially prescribed sign configuration, i.e., $\sigma_i^{(k)} = \pm 1$, $i = 1,\ldots,2n+1$. Substituting Eq. 4.66 into Eq. 4.64, yields

$$[\Lambda^{(k)}(\underline{r})]^T (\underline{f}^{(k)} - \rho\, \underline{\sigma}^{(k)}) = 0 \tag{4.67}$$

or alternately

$$(\underline{\lambda}_j^{(k)}(\underline{r}), \underline{f}^{(k)}) = \rho(\underline{\lambda}_j^{(k)}(\underline{r}), \underline{\sigma}^{(k)}), \quad j = 1,\ldots,n+1 \tag{4.68}$$

where $\underline{\lambda}_j^{(k)}(\underline{r})$ is a vector in U_k^{2n+1} which represents the j-th column of the matrix $[\Lambda^{(k)}(\underline{r})]$. Observe, that Eq. 4.68 represents a system of (n+1)-equations in (n+1)-unknowns, where the (n+1)-unknowns are given by the parameter ρ and the n-unknowns of the vector $\underline{r}$. Since the parameter ρ is common to all the (n+1)-equations, then Eq. 4.68 may be written as

$$\frac{(\underline{\lambda}_j^{(k)}(\underline{r}),\ \underline{f}^{(k)})}{(\underline{\lambda}_j^{(k)}(\underline{r}),\ \underline{\sigma}^{(k)})} = \frac{(\underline{\lambda}_{n+1}^{(k)}(\underline{r}),\ \underline{f}^{(k)})}{(\underline{\lambda}_{n+1}^{(k)}(\underline{r}),\ \underline{\sigma}^{(k)})}, \quad j = 1,\ldots,n \tag{4.69}$$

and

$$\rho = \frac{(\underline{\lambda}_{n+1}^{(k)}(\underline{r}),\ \underline{f}^{(k)})}{(\underline{\lambda}_{n+1}^{(k)}(\underline{r}),\ \underline{\sigma}^{(k)})} \tag{4.70}$$

if $(\underline{\lambda}_{n+1}^{(k)}(\underline{r}),\ \underline{\sigma}^{(k)}) \neq 0$. Hence, the vector $\underline{r}$ is first determined[32] from Eq. 4.69 and then knowing $\underline{r}$ the parameter ρ is determined from Eq. 4.70. Since the nonzero components of the set of vectors $\{\underline{\lambda}_j^{(k)}(\underline{r}) \in U_k^{2n+1} : j = 1,\ldots,n+1\}$ are some polynomials in the variables $\{r_0, r_1,\ldots,r_n\}$ and the vectors $\underline{f}^{(k)}$ and $\underline{\sigma}^{(k)}$ are initially prescribed real vectors, then Eq. 4.69 can be represented as a system of n-polynomial equations

$$N_{n+1}(\underline{r})\, D_j(\underline{r}) - N_j(\underline{r})\, D_{n+1}(\underline{r}) = 0, \quad j = 1,\ldots,n \tag{4.71}$$

[32] For the present discussion we shall assume that there exists a solution vector $\underline{r}$ to Eq. 4.69 and that for this $\underline{r}$ the value of $(\underline{\lambda}_{n+1}^{(k)}(\underline{r}),\ \underline{\sigma}) \neq \bar{0}$.

where

$$N_j(\underline{r}) = (\underline{\lambda}_j^{(k)}(\underline{r}), \underline{f}^{(k)}) \qquad j = 1,\ldots,n+1$$

$$D_j(\underline{r}) = (\underline{\lambda}_j^{(k)}(\underline{r}), \underline{\sigma}^{(k)}) \qquad j = 1,\ldots,n+1$$

Let us now digress to present the following illustrative example:[33]

Example 4.3: Given the vectors

$$\underline{f}^{(k)} = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{2n} \\ f_{2n+1} \end{bmatrix} \quad \text{and} \quad \underline{\sigma}^{(k)} = \begin{bmatrix} +1 \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \\ +1 \end{bmatrix}$$

i.e., the components of $\underline{\sigma}^{(k)}$ are given by $\sigma_i^{(k)} = (-1)^{i+1}$, $i = 1,\ldots,2n+1$. Furthermore, the $(2n+1) \times (n+1)$ matrix $[\Lambda^{(k)}(\underline{r})]$ is given by

[33] This example considers the case when the subspace U_k^{2n+1} contains consecutive (2n+1) components of the vectors in U^q.

$$[\Lambda^{(k)}(\underline{r})] = \begin{bmatrix} r_0 & 0 & \cdots & 0 \\ r_1 & r_0 & & \cdot \\ \cdot & r_1 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \\ r_{n-1} & \cdot & & 0 \\ 1 & r_{n-1} & \cdots & r_0 \\ 0 & 1 & & r_1 \\ \cdot & 0 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ & & & r_{n-1} \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Substituting into Eq. 4.68, yields

$$\sum_{i=0}^{n-1} r_i f_{i+j} + f_{j+n} = \rho \left[\sum_{i=0}^{n-1} (-1)^{i+j+1} r_i + (-1)^{n+j+1} \right] , \quad j = 1,\ldots,n+1$$

Let

$$N_j(\underline{r}) = \sum_{i=0}^{n-1} r_i f_{i+j} + f_{j+n} , \qquad j = 1,\ldots,n+1$$

and let

$$D_j(\underline{r}) = (-1)^j \left[\sum_{i=0}^{n-1} (-1)^{i+1} r_i + (-1)^{n+1} \right]$$

then the above equation yields

$$N_j(\underline{r}) = \rho\, D_j(\underline{r}) , \qquad j = 1,\ldots,n+1$$

or

$$\rho = \frac{N_j(\underline{r})}{D_j(\underline{r})} , \qquad j = 1,\ldots,n+1$$

Therefore, the vector $\underline{r}$ must satisfy the following system of equations

$$\frac{N_j(\underline{r})}{D_j(\underline{r})} = \frac{N_{j+1}(\underline{r})}{D_{j+1}(\underline{r})} , \qquad j = 1,\ldots,n$$

Since $D_j(\underline{r}) = -D_{j+1}(\underline{r})$, then we have

$$N_j(\underline{r}) + N_{j+1}(\underline{r}) = 0 , \qquad j = 1,\ldots,n$$

which yield the following system of n-equation

$$\sum_{i=0}^{n-1} r_i (f_{i+j} + f_{i+j+1}) + (f_{n+j} + f_{n+j+1}) = 0 , \qquad j = 1,\ldots,n$$

Solving this system for the n-unknowns $\{r_0,\ldots,r_{n-1}\}$ yields an $\underline{r} \in \mathcal{R}$.

It suffices to say that if there exists <u>no</u> $\underline{r} \in \mathcal{R}$ which satisfies Eq. 4.71, then there exist no vector $\underline{\epsilon}^{*(k)}(\underline{r})$ with (2n+1)-components equal in absolute value for the prescribed sign configuration vector $\underline{\sigma}^{(k)}$. If, on the other hand, there is more than one possible solution vector $\underline{r}$ to Eq. 4.71, then only the one which gives the smallest value of $|\rho|$ is of interest. Furthermore, since the solution of Eq. 4.67 (or equivalently Eq. 4.71) depends on the choice of the sign configuration vector $\underline{\sigma}^{(k)}$, then to determine the vector $\underline{\epsilon}^{*(k)}(\underline{r})$ with (2n+1) components equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_{\infty}$ and with $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_{\infty}$ minimum, one must solve Eq. 4.67 for all possible sign configuration vectors,[34] and then select the one which yields the minimal value of $|\rho|$.

Definition 4.3: Let the set of vectors $\{\underline{r}_j \in \mathcal{R}\}$ and the set of vectors $\{\underline{\epsilon}^{*(k)}(\underline{r}_j) \in U_k^{2n+1}\}$ satisfy Eq. 4.64, when $\underline{\epsilon}^{*(k)}(\underline{r}_j) = \rho\underline{\sigma}^{(j)}$, where the set $\{\underline{\sigma}^{(j)} \in U_k^{2n+1}\}$ represents the set of 2^{n-1} distinct sign configuration vectors.[35] The vector $\underline{r}_M \in \mathcal{R}$ denotes the vector in the set $\{\underline{r}_j\}$ which yields the minimal value of $\|\underline{\epsilon}^{*(k)}(\underline{r}_j)\|_{\infty}$, and the vector $\underline{\epsilon}^{*(k)}(\underline{r}_M) \in U_k^{2n+1}$ denotes the corresponding vector in the set $\{\underline{\epsilon}^{*(k)}(\underline{r}_j)\}$.

Let us now give the conditions under which we say that the vector $\underline{r}_M \in \mathcal{R}$ is the vector $\underline{r}_k^* \in \mathcal{R}$ defined by Definition 4.2.

Conjecture 4.1 Given an error vector $\underline{\epsilon}^{*(k)}(\underline{r}_M) \in U_k^{2n+1}$, with (2n+1) components equal in absolute value, according to Definition 4.3, and the corresponding vector $\underline{r}_M \in \mathcal{R}$ which

---

[34] Note that there are 2^{n-1} distinct sign configuration vectors, $\underline{\sigma}^{(k)}$.

[35] Namely the i-th component of $\underline{\sigma}^{(j)}$ is given by +1 or -1.

satisfies Eq. 4.64. The vector $\underline{r}_M$ is the best Chebyshev k-th reference solution vector, $\underline{r}_k^*$, if for every set of 2n-components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$, denoted by $\{\epsilon_{i_v}^{(k)} : v = 1, \ldots, 2n\}$, and given by

$$\epsilon_{i_v}^{(k)} = \delta \, \epsilon_{i_v}^{*(k)}(\underline{r}_M), \quad v = 1, \ldots, 2n \tag{4.72}$$

where $0 \leq \delta < 1$; the solution of Eq. 4.64 yields either:

(1) a complex $\underline{r}$, i.e., an $\underline{r} \notin \mathscr{R}$; or

(2) a real vector $\underline{r}' \in \mathscr{R}$, for which $|\epsilon_{i_{2n+1}}^{(k)}|$, the absolute value of the (2n+1)-st component of $\underline{\epsilon}^{(k)}$, satisfies

$$\left|\epsilon_{i_{2n+1}}^{(k)}(\underline{r}')\right| > \|\underline{\epsilon}^{*(k)}(\underline{r}')\|_\infty \tag{4.73}$$

<u>Remark</u>: Conjecture 4.1 gives a test which guarantees that if there exists a vector $\underline{r}_M \in \mathscr{R}$ (and it has in every example considered), then there exists no other $\underline{r} \in \mathscr{R}$ which yields a smaller value of $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$ when only 2n-components of $\underline{\epsilon}^{*(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value. To prove whether such a test is sufficient for $\underline{r}_M \in \mathscr{R}$ to be the vector $\underline{r}_k^*$ defined by Definition 4.2, one must show that if Eq. 4.73 is satisfied, then there exists no $\underline{r} \in \mathscr{R}$

which yields a smaller value of $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_{\infty}$, when <u>less</u> than 2n-components of $\underline{\epsilon}^{*(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value. Although we were not able to prove this, such a case was not encountered in the examples considered.

Since the test, given by Conjecture 4.1, involves solving Eq. 4.64 when 2n-components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ are initially prescribed, we begin by separating the problem of finding the vector $\underline{r} \in \mathscr{R}$ from the problem of finding the unknown component of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ for such a case.

Let us denote by the vector $\underline{\epsilon}^{(\nu)}$ the 2n-components of the vector $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ which are initially prescribed. Clearly, $\underline{\epsilon}^{(\nu)}$ represents the projection of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ onto a 2n-dimensional subspace of U_k^{2n+1}, which will be denoted by U_ν^{2n}. There is no loss of generality in assuming that the vector $\underline{\epsilon}^{(\nu)} \in U_\nu^{2n}$ represents all but the ν-th component of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$. Since there are (2n+1) different ways of selecting 2n-components from the (2n+1)-components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$, then we must consider the (2n+1) vectors in the set $\{\underline{\epsilon}^{(\nu)} \in U_\nu^{2n} : \nu = 1, \ldots, 2n+1\}$. Denoting by $\underline{f}^{(\nu)}$ the projection of $\underline{f}^{(k)} \in U_k^{2n+1}$ onto U_ν^{2n}, then the relation between $\underline{f}^{(\nu)}$ and $\underline{\epsilon}^{(\nu)}$ in each U_ν^{2n}, $\nu = 1, \ldots, 2n+1$, is given by

$$[\Lambda^{(\nu)}(\underline{r})]^T \underline{f}^{(\nu)} = [\Lambda^{(\nu)}(\underline{r})]^T \underline{\epsilon}^{(\nu)} \tag{4.74}$$

where $[\Lambda^{(\nu)}(\underline{r})]$ is a 2n x n matrix representing a submatrix of the (2n+1) x (n+1) matrix which is equivalent[36] to $[\Lambda^{(k)}(\underline{r})]$. Furthermore, $\epsilon_\nu^{(k)}$, the ν-th component of the vector $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ (i. e., the unknown component of $\underline{\epsilon}^{(k)}$ when the other 2n-components are initially prescribed) is given by

$$\epsilon_\nu^{(k)} = f_\nu^{(k)} + \sum_{\substack{i=1 \\ i\neq\nu}}^{2n+1} \frac{\lambda_{i,j}^{(k)}(\underline{r})}{\lambda_{\nu,j}^{(k)}(\underline{r})} (f_i^{(k)} - \epsilon_i^{(k)}) \tag{4.75}$$

where $\{\lambda_{i,j}^{(k)}(\underline{r}) : i = 1,\ldots,2n+1\}$ represents the elements of the j-th column of the (2n+1) x (n+1) matrix $[\Lambda^{(k)}(\underline{r})]$ for which $\lambda_{\nu,j}^{(k)}(\underline{r}) \neq 0$ when $\underline{r}$ is known. Observe that when $\underline{\epsilon}^{(\nu)} = \underline{\epsilon}^{*(\nu)}(\underline{r}_M)$, then for all $\nu = 1,\ldots,2n+1$, the vector $\underline{r}_M$ must satisfy Eq. 4.74, and Eq. 4.75 must yield that $\epsilon_\nu^{(k)} = \epsilon_\nu^{*(k)}(\underline{r}_M)$. Furthermore, observe that when Eq. 4.74 has <u>no</u> solution, i. e., there exist no real $\underline{r} \in \mathscr{R}$, for some initially prescribed vector $\underline{\epsilon}^{(\nu)} \in U_\nu^{2n}$, then there exists no component ϵ_ν of the vector $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$.

At this point let us consider the following question: Suppose there exists an $\underline{r} \in \mathscr{R}$ such that the inequality of Eq. 4.73 is not satisfied for some subset of 2n components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$, i. e., suppose that for some $\mu \in \{\nu = 1,\ldots,2n+1\}$, the absolute value

---

[36] The relation between the matrices $[\Lambda^{(\nu)}(\underline{r})]$ and $[\Lambda^{(k)}(\underline{r})]$ is similar to the relation between the matrices $[\Lambda^{(k)}(\underline{r})]$ and $[R(\underline{r})]$ discussed in Appendix C.

of $\epsilon_{\mu}^{(k)}(\underline{r}_{\mu})$, the μ-th component of $\underline{\epsilon}^{(k)}$, which is determined from Eq. 4.75 when[37] $\underline{r} = \underline{r}_{\mu}$, satisfies

$$\left|\epsilon_{\mu}^{(k)}(\underline{r}_{\mu})\right| < \|\underline{\epsilon}^{(k)}(\underline{r}_{\mu})\|_{\infty} \tag{4.76}$$

Clearly, such a case must occur when the best Chebyshev k-th reference solution, $\underline{r}_k^* \in \mathscr{R}$, yields an error vector $\underline{\epsilon}^{*(k)}(\underline{r}_k^*) \in U_k^{2n+1}$ which is characterized by having at most 2n-components equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r}_k^*)\|_{\infty}$.

Let us now consider the case when 2n-components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ are equal in absolute value to $|\rho|$, where ρ is some real parameter in the interval $(-\infty, \infty)$. In other words, let the projection of the vector $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ onto the 2n-dimensional subspace U_{μ}^{2n} of U_k^{2n+1}, where $\mu \in \{\nu = 1,\ldots,2n+1\}$, is given by

$$\underline{\epsilon}^{(\mu)} = \rho\,\underline{\sigma}^{(\mu)} \in U_{\mu}^{2n} \tag{4.77}$$

where $\underline{\sigma}^{(\mu)}$ is a 2n-dimensional vector representing a prescribed sign configuration (i.e., $\sigma_i^{(\mu)} = \pm 1$, $i = 1,\ldots,2n$), and where ρ is a real parameter in $(-\infty, \infty)$. Substituting Eq. 4.77 into the μ-th relation of Eq. 4.74, yields

$$[\Lambda^{(\mu)}(\underline{r})]^T\,(\underline{f}^{(\mu)} - \rho\,\underline{\sigma}^{(\mu)}) = 0 \tag{4.78}$$

---

[37] Note that $\underline{r}_{\mu}$ represents a vector in the set $\{\underline{r}_{\nu}\}$, where each $\underline{r}_{\nu}$ represents the solution of the ν-th relation of Eq. 4.74, which yields the smallest value of $|\epsilon_{\nu}(\underline{r}_{\nu})|$, given by Eq. 4.75.

where $[\Lambda^{(\mu)}(\underline{r})]$ is a 2n x n matrix[38] and $\underline{f}^{(\mu)}$ is the projection of the prescribed real vector $\underline{f}^{(k)} \in U_k^{2n+1}$ onto U_μ^{2n}.

Observe that Eq. 4.78 represents a set of n-simultaneous equations in (n+1)-unknowns, which are given by ρ and the n-unknown components of $\underline{r}$, i.e., we have an extra degree of freedom when solving Eq. 4.78 for ρ and $\underline{r}$. However, since we seek the $\underline{r} \in \mathscr{R}$ which yields the minimal value of $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$ when $\underline{\epsilon}^{(k)}(\underline{r}) \in U_k^{2n+1}$ has 2n-components equal in absolute value to $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$, then we solve Eq. 4.78 for the vector $\underline{r} \in \mathscr{R}$ which yields the minimal value of $|\rho|$. It should be noted that although for each value of $\rho \in (-\infty, \infty)$, Eq. 4.78 represents a set of n-simultaneous equations in the n-unknowns of $\underline{r}$, there may be a case when for some $\rho \in (-\infty, \infty)$, there exists no real $\underline{r}$ (i.e., no $\underline{r} \in \mathscr{R}$) which satisfies Eq. 4.78, since in general Eq. 4.78 represents a set of n-nonlinear equations. This is illustrated in the following example.[39]

Example 4.4: Given n = 2,

$$\underline{f}^{(\mu)} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{\sigma}^{(\mu)} = \begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \end{bmatrix}$$

---

[38] See Eq. 4.74 for the definition of the matrix $[\Lambda^{(\mu)}]$.

[39] The example presented has been taken from Example 5.3 of Chapter V.

and

$$[\Lambda^{(\mu)}(\underline{r})] = \begin{bmatrix} r_0^2 & 0 \\ (r_0 - r_1^2) & r_0 \\ -r_1 & r_1 \\ 0 & 1 \end{bmatrix}.$$

Substituting the above value of $\underline{f}^{(\mu)}$, $\underline{\sigma}^{(\mu)}$, and $[\Lambda^{(\mu)}(\underline{r})]$ into Eq. 4.78, yields the following two equations:

$$\rho\, r_0^2 + (2 - \rho)(r_0 - r_1^2) + (1 + \rho)\, r_1 = 0$$

$$(2 - \rho)\, r_0 + (1 - \rho)\, r_1 + (1 - \rho) = 0$$

Solving these equations for r_0 in terms of ρ we have

$$r_0 = \frac{(5\rho^2 - 11\rho + 2) \pm (1 + \rho)\sqrt{(5\rho - 1)(2\rho^2 - 5\rho + 4}}{2\rho^3 - 4\rho^2 + 13\rho - 8}$$

Note that r_0 is complex if for some real $\rho \in (-\infty, \infty)$

$$(5\rho - 1)(2\rho^2 - 5\rho + 4) < 0$$

Since the term $(2\rho^2 - 5\rho + 4) > 0$ for all real $\rho \in (-\infty, \infty)$, and the term $(5\rho - 1) < 0$ for all real $\rho < 0.2$, then we note that there exists no $\underline{r} \in \mathscr{R}$ which satisfies the above equations for any $\rho < 0.2$.

Definition 4.4: Let ρ_0 denote the $\rho \in (-\infty, \infty)$ which yields the minimal value of $|\rho|$ for which there exists an $\underline{r} \in \mathscr{R}$ satisfying Eq. 4.78. Let the vector $\underline{r}_0$ denote the vector in $\mathscr{R}$ which satisfies Eq. 4.78 when $\rho = \rho_0$, and the vector $\underline{\epsilon}^{(k)}(\underline{r}_0) \in U_k^{2n+1}$ denotes the error vector which satisfies Eq. 4.64, when $\underline{r} = \underline{r}_0$ and when the 2n-components of $\underline{\epsilon}^{(k)}$ are equal in absolute value to $|\rho_0|$.

Note that if the relation of Eq. 4.76 is satisifed when $\underline{r} = \underline{r}_0$, then the ℓ_∞^{2n+1}-norm of $\underline{\epsilon}^{*(k)}(\underline{r}_0)$ is given by

$$\|\underline{\epsilon}^{*(k)}(\underline{r}_0)\|_\infty = |\rho_0|$$

Although one can obtain a test, similar to that given by Conjecture 4.1, which guarantees that there exists no $\underline{r} \in \mathscr{R}$ which yields a lower value of $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$ than $\|\underline{\epsilon}^{*(k)}(\underline{r}_0)\|_\infty$ when only (2n-1)-components of $\underline{\epsilon}^{*(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$. However, when (2n-1)-components of $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$ are initially prescribed to be equal in absolute value, then Eq. 4.64 represents a system of (n-1) equations[40] in the n-components of $\underline{r}$. Clearly, such a system contains more unknowns than equations, so that one has some freedom in selecting the solution vector $\underline{r} \in \mathscr{R}$. Although the solution vector $\underline{r}$ should be selected to give the minimal value of

---

[40]This system of (n-1)-equations is obtained from Eq. 4.64 in a manner similar to that which obtained the system of n-equations given by Eq. 4.74.

$\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$, it is not obvious how to select[41] such a vector $\underline{r}$. Hence, we shall say that if Eq. 4.76 is satisfied when $\underline{r} = \underline{r}_0$, then the vector $\underline{r}_0 \in \mathscr{R}$ represents the best Chebyshev k-th reference solution vector $\underline{r}_k^*$.

This is summarized in the following conjecture:

<u>Conjecture 4.2</u> Given an error vector $\underline{\epsilon}^{*(k)}(\underline{r}_0) \in U_k^{2n+1}$ according to Definition 4.4, i.e.,

$$\epsilon_i^{*(k)}(\underline{r}_0) = \rho_0 \sigma_i, \qquad i \neq \mu = 1, \ldots, 2n+1$$

$$\epsilon_\mu^{*(k)}(\underline{r}_0) < |\rho_0|$$

where $\sigma_i = \pm 1$, and the corresponding vector $\underline{r}_0 \in \mathscr{R}$ which satisfy Eq. 4.64. The vector $\underline{r}_0$ is the best Chebyshev k-th reference solution vector $\underline{r}_k^*$, if

$$\|\underline{\epsilon}^{*(k)}(\underline{r}_0)\|_\infty < \|\underline{\epsilon}^{*(k)}(\underline{r}_M)\|_\infty \tag{4.79}$$

where $\underline{\epsilon}^{*(k)}(\underline{r}_M)$ is defined by Definition 4.3.

---

[41] In Example 5.3 of Chapter V, we have selected the solution vector $\underline{r}$ which minimizes the absolute value of the unknown component of $\underline{\epsilon}^{(\mu)}(\underline{r}) \in U_\mu^{2n}$. However, it should be noted that we were unable to show that such a method of selection yields the minimal value of $\|\underline{\epsilon}^{(k)}(\underline{r})\|_\infty$

Now that the preliminary ideas have been discussed let us outline the procedure[42] we shall use to determine the vector $\underline{r}_k^* \in \mathscr{R}$: First, we determine the best Chebyshev error vector $\underline{\epsilon}^{*(k)} \in U_k^{2n+1}$ for some initial estimate of $\underline{r} \in \mathscr{R}$. Then we systematically decrease the value of $\|\underline{\epsilon}^{*(k)}\|_\infty$ and determine the corresponding value of $\underline{r} \in \mathscr{R}$. Eventually it may happen (and it has in every example) that all the (2n+1)-components of $\underline{\epsilon}^{*(k)} \in U_k^{2n+1}$ are equal in absolute value to $\|\underline{\epsilon}^{*(k)}\|_\infty$ for some $\underline{r} \in \mathscr{R}$. At this point we apply the test of Conjecture 4. 1. If the test fails[43] we then decrease the value of $\|\underline{\epsilon}^{*(k)}\|_\infty$ until either (1) the conditions of Conjecture 4. 1 are satisfied, or (2) the vector $\underline{r}_0 \in \mathscr{R}$, which yields the minimum value of $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$ when 2n-components of $\underline{\epsilon}^{*(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$, is obtained. When either one of these cases occurs we shall say that the resulting vector $\underline{r} \in \mathscr{R}$ is the vector $\underline{r}_k^* \in \mathscr{R}$.

Before concluding the discussion of the best Chebyshev approximation in U_k^{2n+1}, let us make some observations concerning the sign configuration of the components of $\underline{\epsilon}^{**(k)}$ which are equal in absolute

---

[42]The detailed consideration of this procedure is given in Section 5. 3 of Chapter V.

[43]Note that at this point the prescribed sign configuration of the vector $\underline{\epsilon}^{*(k)}$ may not represent the sign configuration which yields the minimal value of $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$ when (2n+1)-components of $\underline{\epsilon}^{*(k)}(\underline{r}) \in U_k^{2n+1}$ are equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$.

value to $\|\underline{\epsilon}^{**(k)}\|_\infty$. Recall from Corollary 2.1 that for each prescribed vector $\underline{z} \in \mathscr{Z}$ (or equivalently $\underline{r} \in \mathscr{R}$), the best Chebyshev approximation in U_k^{2n+1}, also represents the best Chebyshev approximation in some (n+1)-dimensional subspace of U_k^{2n+1}. In other words, if for a prescribed $\underline{r} \in \mathscr{R}$, $\underline{\epsilon}^{*(k)}$ represents the best Chebyshev error vector in U_k^{2n+1}, and if $\underline{\epsilon}^{*(\theta)}$ represents the best Chebyshev error vector in some (n+1)-dimensional reference subspace U_θ^{n+1} of U_k^{2n+1}, where[44] $\theta = 1, \ldots, \binom{2n+1}{n+1}$, then from Eq. 2.65, we have

$$\|\underline{\epsilon}^{*(k)}\|_\infty = \max_{1 \le \theta \le \binom{2n+1}{n+1}} \|\underline{\epsilon}^{*(\theta)}\|_\infty \tag{4.80}$$

Furthermore, recall from Theorem 2.3, that in each (n+1)-dimensional subspace U_θ^{n+1}, the vector $\underline{\epsilon}^{*(\theta)}$ is given by

$$\underline{\epsilon}^{*(\theta)} = \rho_\theta \, \underline{\sigma}^{(\theta)}$$

where

$$\rho_\theta = \frac{(\underline{f}^{(\theta)}, \underline{\lambda}^{(\theta)})}{\|\underline{\lambda}^{(\theta)}\|_1}, \tag{4.81}$$

and

$$\sigma_i^{(\theta)} = \operatorname{sgn} \lambda_i^{(\theta)} \quad , \quad i = 1, \ldots, n+1 \tag{4.82}$$

---

[44]Note that there are $\binom{2n+1}{n+1}$ distinct (n+1)-dimensional subspaces in the 2n+1-dimensional space U_k^{2n+1}.

and where $\underline{f}^{(\theta)}$ represents the projection of the vector $\underline{f}^{(k)} \in U_k^{2n+1}$ onto U_θ^{n+1}, and $\underline{\lambda}^{(\theta)}$ is a real vector in U_θ^{n+1} which is determined from Eq. 2.41. Since $\|\underline{\epsilon}^{*(\theta)}\|_\infty = |\rho_\theta|$ then Eq. 4.80 may be written as

$$\|\underline{\epsilon}^{*(k)}\|_\infty = \max_{1 \leq \theta \leq \binom{2n+1}{n+1}} \{ |\rho_\theta| \} \tag{4.83}$$

By examining the relation between the (2n+1) x n matrix $[Z^{(k)}(\underline{z})]$ and the (2n+1) x (n+1) matrix $[\Lambda^{(k)}(\underline{r})]$, it is noted that, for each $\underline{r} \in \mathscr{R}$, the nonzero elements of the (n+1)-column vectors of $[\Lambda^{(k)}]$ represent a particular[45] set of (n+1)-vectors in the set $\{ \underline{\lambda}^{(\theta)} : \theta = 1, \ldots, \binom{2n+1}{n+1}\}$. Furthermore, for each n+1-reference subspace in the set $\{ U_\theta^{n+1} : \theta = 1, \ldots, \binom{2n+1}{n+1}\}$, the corresponding vector $\underline{\lambda}^{(\theta)}$ can be obtained by taking the linear combination of the column vectors of the matrix $[\Lambda^{(k)}]$ so that the n-components of the resulting vector which do not lie in U_θ^{n+1} are equal to zero.[46] Observe that since the matrix $[\Lambda^{(k)}(\underline{r})]$ depends on the vector $\underline{r} \in \mathscr{R}$, then the set of vectors $\{ \underline{\lambda}^{(\theta)}\}$ will also depend on $\underline{r}$. To emphasize this dependence we shall denote $\underline{\lambda}^{(\theta)}$ by $\underline{\lambda}^{(\theta)}(\underline{r})$. Substituting $\underline{\lambda}^{(\theta)}(\underline{r})$ for $\underline{\lambda}^{(\theta)}$ in Eq. 4.81 yields

---

[45] Note that the nonzero elements of each column vector of $[\Lambda^{(k)}]$ represent the vector $\underline{\lambda}^{(\theta)}$ when the corresponding (n+1)-dimensional reference space contains the (n+1) consecutive components of the vectors in U_k^{2n+1}.

[46] This will be clarified in Chapter V, where numerical examples are presented.

$$\rho_\theta(\underline{r}) = \frac{(\underline{f}^{(\theta)}, \underline{\lambda}^{(\theta)}(\underline{r}))}{\|\underline{\lambda}^{(\theta)}(\underline{r})\|_1} \tag{4.84}$$

This expression relates the minimal value of $\|\underline{\epsilon}^{(\theta)}(\underline{r})\|_\infty \triangleq |\rho_\theta(\underline{r})|$ in each (n+1) dimensional subspace U_θ^{n+1}, $\theta = 1, \ldots, \binom{2n+1}{n+1}$, to the vector $\underline{r} \in \mathscr{R}$. Hence, when $\underline{r}$ is not initially prescribed, then the value of $\|\underline{\epsilon}^{*(k)}\|_\infty$, given by Eq. 4.83, becomes

$$\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty = \max_{1 \leq \theta \leq \binom{2n+1}{n+1}} \{|\rho_\theta(\underline{r})|\} \tag{4.85}$$

where $\rho_\theta(r)$ is given by Eq. 4.84. Hence, the problem of minimizing the function $\|\underline{\epsilon}^{*(k)}(\underline{r})\|_\infty$, stated in Eq. 4.65, may be written as

$$\|\underline{\epsilon}^{*(k)}(\underline{r}_k)\|_\infty = \min_{\underline{r} \in \mathscr{R}} \max_{1 \leq \theta \leq \binom{2n+1}{n+1}} \{|\rho_\theta(\underline{r})|\} \tag{4.86}$$

This representation of the minimization procedure yields the following geometrical interpretation: If we denote by $\tilde{\underline{\lambda}}^{(\theta)}(\underline{r})$ the vector $\underline{\lambda}^{(\theta)}(\underline{r})/\|\underline{\lambda}^{(\theta)}(\underline{r})\|_1$, then it is noted from Eq. 4.84 that each function $\rho_\theta(\underline{r})$ represents the orthogonal projection of the vector $\underline{f}^{(\theta)} \in U_\theta^{n+1}$ onto the vector $\tilde{\underline{\lambda}}^{(\theta)}(\underline{r})$. Since, by definition, the vector $\tilde{\underline{\lambda}}^{(\theta)}(\underline{r})$ is orthogonal to the approximating subspace, in U_θ^{n+1} (i.e., the column space of the (n+1) x n matrix $[Z^{(\theta)}(\underline{z})]$), then by varying the vector $\underline{r}$, one essentially rotates the vector $\tilde{\underline{\lambda}}^{(\theta)}(\underline{r})$ (or equivalently the approximating subspace). Clearly in each U_θ^{n+1} we may rotate $\underline{\tilde{\lambda}}^{(\theta)}(\underline{r})$ until $\rho_\theta(\underline{r}) = 0$; however this may result in $\rho_\theta(\underline{r}) \neq 0$ in some other (n+1)-dimensional subspace. Thus, the minimizing procedure seeks the $\underline{r} \in \mathscr{R}$ for which the vectors in the set $\{\underline{\lambda}^{(\theta)}(\underline{r}) : \theta = 1, \ldots, \binom{2n+1}{n+1}\}$ will yield a corresponding set of values $\{|\rho_\theta(\underline{r})|\}$ with the largest value being minimum. It has been found, in the examples considered, that when $\underline{r} = \underline{r}_k{}^*$, then this minimum will occur when at least (n+1) functions from the set $\{|\rho_\theta(\underline{r})|\}$ are equal.

Recall that since each $|\rho_\theta(\underline{r})|$ represents the minimal value of $\|\underline{\epsilon}^{*(\theta)}(\underline{r})\|_\infty$ in U_θ^{n+1}, then when (n+1) functions from the set $\{|\rho_\theta(\underline{r})|\}$ are equal for some given $\underline{r} \in \mathscr{R}$, they represent the minimal value of ℓ_∞-norm of the error vector in some higher dimensional subspace defined by

$$\bigcup_{j=1}^{n+1} U_{\theta_j}^{n+1} \tag{4.87}$$

where $\{U_{\theta_j}^{n+1}\}$ represents the set of (n+1)-dimensional subspaces in which the functions in the set $\{|\rho_{\theta_j}(\underline{r})|\}$ are equal. Furthremore, it has been found that the number of components of the error vector $\underline{\epsilon}^{*(k)}(\underline{r}_k{}^*) \in U_k^{2n+1}$ which are equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r}_k{}^*)\|_\infty$ is directly related to the dimension of the subspace defined by Eq. 4.87. For example, if only 2n-components of $\underline{\epsilon}^{*(k)}(\underline{r}_k{}^*)$ are equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r}_k{}^*)\|_\infty$, then Eq. 4.87 will define a 2n-dimensional subspace of U_k^{2n+1}.

In summary, then, it has been found that the projection of the components of $\underline{\epsilon}^{*(k)}(\underline{r}_k^*)$ which are equal in absolute value to $\|\underline{\epsilon}^{*(k)}(\underline{r}_k^*)\|_\infty$, for the given $\underline{r}_k^* \in \mathscr{R}$, represent the best Chebyshev error vectors in at least (n+1) subspaces $\{U_{\theta_j}^{n+1}\}$ of U_k^{2n+1}, where in each of these subspaces $U_{\theta_j}^{n+1}$, the sign configuration of the vector $\underline{\epsilon}^{*(\theta_j)}(\underline{r}_k^*)$ is given by Eq. 4.82, that is,[47]

$$\text{sgn}\ \epsilon_i^{*(\theta_j)}(\underline{r}_k^*) = [\text{sgn}\ \rho_{\theta_j}(\underline{r}_k^*)] \cdot [\text{sgn}\ \lambda_i^{(\theta_j)}(\underline{r}_k^*)], \quad i = 1,\ldots,n+1, \quad j = 1,\ldots,n+1 \qquad (4.88)$$

It should be mentioned that in the case when (2n+1) components of $\underline{\epsilon}^{**(k)} \in U_k^{2n+1}$ are equal in absolute value to $\|\underline{\epsilon}^{**(k)}\|_\infty$, it has been found[48] that the signs of these components alternate, namely,

$$\text{sgn}\ \epsilon_{i+1}^{**(k)} = -\ \text{sgn}\ \epsilon_i^{**(k)}, \qquad i = 1,\ldots,2n+1$$

Note that such a sign alternation property is not encountered in the case when the vector $\underline{r} \in \mathscr{R}$ is initially prescribed.[49]

---

[47] Note that this result is based on the examples considered in Section 5.4, (Chapter V) and it has not been proved whether the sign configuration given by Eq. 4.88 is sufficient for $\|\underline{\epsilon}^{*(k)}(\underline{r}_k^*)\|_\infty$ to be minimum.

[48] See Examples 5.1 and 5.2 of Section 5.4, Chapter V.

[49] See the sign configuration given by Eq. 4.82, or equivalently by Theorem 2.3 of Section 2.3.1.2, Chapter II.

4.6 On the Uniqueness of the Best Chebyshev Approximation

Although we are primarily interested in the existence and characterization of the best Chebyshev approximation, let us make some observations concerning the uniqueness problem. Since the uniqueness of the best Chebyshev approximating vector, $\underline{f}^{**} \epsilon U^q$, defined by Eq. 4.1, depends on the uniqueness of the vector pair $(\underline{\beta}^{**},\underline{z}^*) \epsilon \mathscr{B}_z \times \mathscr{Z}$, we state the following theorem:

Theorem 4.4: The best Chebyshev real approximating vector $\underline{f}^{**} \epsilon U^q$, defined by Eq. 4.1, is unique if the best Chebyshev parameter vector $\underline{z}^* \epsilon \mathscr{Z}$ is unique[50] and its n-components satisfy:

(1) $z_k^* \neq 0$ for all $k = 1,\ldots,n$; and

(2) $z_k^* \neq j\zeta$ for all $k = 1,\ldots,n$, where ζ is a real number, i.e., no component is purely imaginary.

Proof: Recall from Section 2.3.1.2 that if every (n x n) submatrix of the prescribed q x n matrix [E] is nonsingular, then the best Chebyshev approximating vector $\underline{f}^* \epsilon U^q$ is unique. Since the conditions (1) and (2) of the theorem guarantee Assumption 2.1 (i.e.,

---

[50]Since the vector $\underline{z} \epsilon \mathscr{Z}$ represents an arbitrary ordering of the set of n-parameters $\{z_k\}$, then the vector $\underline{z} \epsilon \mathscr{Z}$ is unique if the elements $\{z_k\}$ are unique.

that every (n x n) submatrix of the (q x n) matrix $[Z(\underline{z}^*)]$, defined by Eq. 1.18 (Eq. 1.18a), is nonsingular), then the theorem is proved.

In terms of "Prony's Extended Method" Theorem 4.4 becomes:

Theorem 4.5: The best Chebyshev real approximating vector $\underline{f}^{**} \epsilon U^q$, defined by Eq. 4.1, is unique if the best Chebyshev solution vector $\underline{r}^* \epsilon \mathscr{R}$, defined by Definition 4.1, is unique and it yields a unique error vector $\underline{\epsilon}^*(\underline{r}^*) \epsilon U^q$.

Proof: All of this theorem is contained in Theorems 4.4 and 3.1. Recall that Theorem 3.1 gives an alternate way of representing the best Chebyshev approximation in terms of the vector $\underline{r}^* \epsilon \mathscr{R}$ and $\underline{\epsilon}^*(\underline{r}^*) \epsilon U^q$.

By applying Theorem 4.3 to Theorem 4.5 we have the following theorem:

Theorem 4.6: The best Chebyshev real approximating vector $\underline{f}^{**} \epsilon U^q$, defined by Eq. 4.1, is unique if the best Chebyshev approximation in the (2n+1)-dimensional reference subspace U_c^{2n+1} is unique, i.e., the best Chebyshev reference c-th solution vector $\underline{r}_c^* \epsilon \mathscr{R}$ is unique and it yields a unique error vector $\underline{\epsilon}^{*(c)}(\underline{r}_c^*) \epsilon U_c^{2n+1}$.

It should be noted that Theorem 4.6 gives a stronger condition on the uniqueness of the best Chebyshev approximation. In other words, it is possible to obtain more than one best Chebyshev c-th reference solution vector $\underline{r}_c^* \in \mathscr{R}$ which yields a unique vector $\underline{\epsilon}^{*(c)}(\underline{r}_c^*)$ and still obtain a unique Chebyshev approximation in U^q if the solution vectors $\underline{r}_c^*$ are distinct.[51]

The following example[52] illustrates the case when the best Chebyshev approximation in U_k^{2n+1} yields two distinct vectors $\underline{r}_k^{*(1)}$ and $\underline{r}_k^{*(2)}$ in $\mathscr{R}$ which define the same error vector $\underline{\epsilon}^{*(k)}(\underline{r}_k^*) \in U_k^{2n+1}$:

Example 4.5: Consider the case when $n = 1$ (i.e., the vector $\underline{z} = z$), and the (2n+1)-dimensional reference subspace U_k^{2n+1} (i.e., U_k^3) in which the approximating vector is given by

$$\underline{\hat{f}}^{(k)}(\underline{\beta}, \underline{z}) = \begin{bmatrix} 1 \\ z^2 \\ z^4 \end{bmatrix} \beta$$

---

[51]This results from Theorem C.1, where we have shown that for each prescribed $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}^{(k)} \in U_k^{2n+1}$, there exists a unique vector $\underline{\epsilon} \in U^q$.

[52]The example considered here is based on the numerical example given by Example 5.1 of Chapter V.

Let the prescribed vector $\underline{f}^{(k)} \in U_k^3$ be given by

$$\underline{f}^{(k)} = \begin{bmatrix} 1.0000 \\ 0.2500 \\ 0.0625 \end{bmatrix}$$

then Eq. 4.48 yields the relation

$$\begin{bmatrix} 1.0000 \\ 0.2500 \\ 0.0625 \end{bmatrix} = \begin{bmatrix} 1 \\ z^2 \\ z^4 \end{bmatrix} \beta + \begin{bmatrix} \epsilon_1^{(k)} \\ \epsilon_2^{(k)} \\ \epsilon_3^{(k)} \end{bmatrix}$$

and Eq. 4.64 yields the relation

$$\begin{bmatrix} r_0^2 & -r_1^2 & 0 \\ 0 & r_0^4 & -r_1^4 \end{bmatrix} \begin{bmatrix} 1.0000 \\ 0.2500 \\ 0.0625 \end{bmatrix} = \begin{bmatrix} r_0^2 & -r_1^2 & 0 \\ 0 & r_0^4 & -r_1^4 \end{bmatrix} \begin{bmatrix} \epsilon_1^{(k)} \\ \epsilon_2^{(k)} \\ \epsilon_3^{(k)} \end{bmatrix} \qquad (4.89)$$

since the 3 x 2 matrix $[\Lambda^{(k)}(\underline{r})]$, obtained from Definition C.3, is given by

$$[\Lambda^{(k)}(\underline{r})] = \begin{bmatrix} r_0^2 & 0 \\ -r_1^2 & r_0^4 \\ 0 & -r_1^4 \end{bmatrix}$$

where

$$\underline{r} = \begin{bmatrix} r_0 \\ \\ r_1 \end{bmatrix}$$

Let us assume that the best Chebyshev error vector $\underline{\epsilon}^{*(k)}(\underline{r}_k^*) \in U_k^3$, is given by

$$\underline{\epsilon}^{*(k)}(\underline{r}_k^*) = \rho\, \underline{\sigma}^{(k)}$$

where

$$\underline{\sigma}^{(k)} = \begin{bmatrix} +1 \\ -1 \\ +1 \end{bmatrix}$$

Replacing the vector $\underline{\epsilon}^{(k)}$ in Eq. 4.89 by the vector $\underline{\epsilon}^{*(k)}(\underline{r}_k^*)$ given above, yields the following system of simultaneous equations:

$$\begin{aligned} \rho(r_0^2 + r_1^2) &= r_0^2 - 0.25\, r_1^2 \\ -\rho(r_0^4 + r_1^4) &= 0.25\, r_0^4 - 0.0625\, r_1^4 \end{aligned} \qquad (4.90)$$

Letting $r_1 = 1$ and solving Eq. 4.90 for r_0 and ρ yields $r_0 = \pm\ 0.546$ and $\rho = 0.037$. Hence, we have two best Chebyshev k-th reference solution vectors, given by

$$\underline{r}_k^{*(1)} = \begin{bmatrix} 0.546 \\ 1.000 \end{bmatrix}, \quad \text{and} \quad \underline{r}_k^{*(2)} = \begin{bmatrix} -0.546 \\ 1.000 \end{bmatrix}$$

which yields the following unique error vector in U_k^3:

$$\underline{\epsilon}^{*(k)}(\underline{r}_k^*) = \begin{bmatrix} +0.037 \\ -0.037 \\ +0.037 \end{bmatrix}$$

In conclusion, it should be mentioned that it has been found that when the best Chebyshev approximation did not yield an error vector $\underline{\epsilon}^{**} \in U^q$ which has (2n+1) components equal in absolute value to $\|\underline{\epsilon}^{**}\|_\infty$, then the best Chebyshev approximation was not unique. Specifically, Example 5.3, given in Section 5.4 (Chapter V) depicts the case when the best Chebyshev approximation yields an error vector $\underline{\epsilon}^{**} \in U^q$ which has only 2n-components equal in absolute value to $\|\underline{\epsilon}^{**}\|_\infty$. In this case, one of the components of the vector $\underline{z}^* \in \mathcal{Z}$ were found to be zero so that the corresponding component of the vector $\underline{\beta}^{**} \in \mathcal{B}_z$ may be selected arbitrarily. Hence, it may follow that the error vector $\underline{\epsilon}^{**}$, which has (2n+1)-components equal in absolute value to $\|\underline{\epsilon}^{**}\|_\infty$, characterizes a unique best Chebyshev approximation. Since we are primarily interested in the existence problem, we shall not examine this facet of the problem further.

4.7 Summary

The existence of the best Chebyshev approximation to the problem defined by Eqs. 4.1 and 4.2 has been shown in Section 4.3. Furthermore, we have obtained the bound within which the value of $\|\underline{\epsilon}^{**}\|_\infty$ must lie. Sections 4.4 and 4.5 discussed the theory behind the method of finding the best Chebyshev approximation, i.e., it has been shown that the best Chebyshev approximation can be obtained by systematically decreasing the value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$. This result will be used in Chapter V to obtain an algorithm for finding the best Chebyshev approximation.

CHAPTER V

COMPUTATIONAL METHOD AND EXAMPLES

5.1 Introduction

The purpose of this chapter is to give a computational method which will yield the best Chebyshev approximating real vector $\underline{f}^{**} = [Z(\underline{z}^*)]\,\underline{\beta}^{**}$ to the prescribed real vector $\underline{f} \in U^q$. Our approach to the solution of this problem is based on the theory developed in Chapter IV, namely a method of descent.

We shall begin this chapter by briefly discussing the methods of descent. Then, based on the theory of Chapter IV, an algorithm will be presented in Section 5.3. Numerical examples illustrating this algorithm will be given in Section 5.4.

5.2 The Method of Descent

As is well known, the method of descent is a systematic method used in solving minimization problems. It specifies a way of reaching the minimum of a multivariable function, by determining the downward direction of the function and the distance one must travel along this direction. One of its drawbacks is that it does not always distinguish between a local minimum and the overall minimum. This problem, however, is not encountered when minimizing a function which is convex, the case that is usually found in the linear $L_p(\ell_p)$-approximation theory, where $1 \leq p \leq \infty$. A discussion of the method of descent and its application to the theory of linear approximations is in Rice's book (Ref. 19, pp. 158-186). For example, he illustrates the use of the method of descent in solving Chebyshev approximation problems of the type discussed in Section 2. 3. 1. 2 (Chapter II), namely, the case when the vector $\underline{z}$ is initially prescribed.

Let us now consider the method of descent that we shall use in solving the Chebyshev approximation problem given by Eqs. 4. 1 and 4. 2. That is, given a real vector $\underline{f} \in U^q$, minimize the function $\|\underline{\epsilon}(\underline{\beta}, \underline{z})\|_\infty = \|\underline{f} - [Z(\underline{z})]\underline{\beta}\|_\infty$ with respect to the vector pair $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$, such that $(\underline{\beta}^{**}, \underline{z}^*)$ yields

$$\|\underline{\epsilon}^{**}\|_\infty \triangleq \|\underline{\epsilon}(\underline{\beta}^{**}, \underline{z}^*)\|_\infty = \min_{(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}} \|\underline{\epsilon}(\underline{\beta}, \underline{z})\|_\infty$$

Recall, from Section 4. 3, that since

$$\|\underline{\epsilon}^{**}\|_\infty = \min_{\underline{z} \in \mathscr{Z}} \|\underline{\epsilon}^*(\underline{z})\|_\infty$$

where $\underline{\epsilon}^*(\underline{z}) \triangleq \underline{\epsilon}(\underline{\beta}^*, \underline{z}) \in U^q$, then we want to find the minimum of the function $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ for all $\underline{z} \in \mathscr{Z}$. In seeking the minimum of the function $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ we shall use "Prony's Extended Method," namely we shall minimize the function $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ with respect to $\underline{r} \in \mathscr{R}$, where the vector $\underline{\epsilon}^*(\underline{r}) \in U^q$ must satisfy

$$[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon}^*(\underline{r}) \qquad (5.1)$$

and where the $q \times (q-n)$ matrix $[R(\underline{r})]$ is defined by Eq. 3.2. Hence, the method of descent will seek the vector $\underline{r}^* \in \mathscr{R}$, such that

$$\|\underline{\epsilon}^{**}\|_\infty \triangleq \|\underline{\epsilon}^*(\underline{r}^*)\|_\infty = \min_{\underline{r} \in \mathscr{R}} \|\underline{\epsilon}^*(\underline{r})\|_\infty \tag{5.2}$$

Although, in general, the function $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ is not convex with respect to $\underline{r} \in \mathscr{R}$, one can obtain a method of descent which yields the overall minimum of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ by using the fact that the vector $\underline{r}^* \in \mathscr{R}$ will yield the best Chebyshev approximation in some (2n+1)-dimensional reference subspace, U_c^{2n+1}, of U^q, where $U_c^{2n+1} \in \left\{ U_w^{2n+1} : w = 1, \ldots, \binom{q}{2n+1} \right\}$. In other words, the selection of the appropriate (2n+1)-dimensional reference subspace of U^q will govern the direction of descent and the distance one must travel along this direction.

Let us now outline the method of descent we shall use to determine the point $\underline{r}^*$ defined by Eq. 5.2. We begin by choosing an initial estimate of $\underline{r} \in \mathscr{R}$, for example, the vector $\underline{r}'$. Although the vector $\underline{r}'$ may be selected arbitrarily, recall from Lemma 4.2 that if $\underline{r}' = \underline{\tilde{r}}$, where $\underline{\tilde{r}}$ is defined by Eq. 4.39, then one can obtain the bounds[1] within which the value of $\|\underline{\epsilon}^{**}\|_\infty$ must lie. Knowing the vector $\underline{r}' \in \mathscr{R}$, the corresponding best Chebyshev error vector $\underline{\epsilon}^*(\underline{r}') \in U^q$ can be found.[2] Since the vector $\underline{\epsilon}^*(\underline{r}') \in U^q$ has at least (n+1)-components equal in absolute

---

[1] See Eq. 4.40, Chapter IV.

[2] Recall that for each prescribed $\underline{r}' \in \mathscr{R}$, there is a corresponding $\underline{z}' \in \mathscr{Z}$, and so the $q \times n$ matrix $[Z(\underline{z}')]$ is fully prescribed. Thus, we have the Chebyshev approximation problem discussed in Section 2.3.1.2, Chapter II.

value to $\|\underline{\epsilon}^*(\underline{r}')\|_\infty$, then let us denote these components by the set[3] $\left\{\epsilon^*_{i_v}(\underline{r}'): v = 1, \ldots, m;\ m \geq n+1\right\}$. If $m < 2n+1$, then we decrease, in small steps, the absolute values of the m-components in the set $\left\{\epsilon^*_{i_v}\right\}$ and calculate a new value of $\underline{r} \in \mathscr{R}$, and the other (q-m)-components of the vector $\underline{\epsilon} \in U^q$. When the absolute value of one of the (q-m) components of $\underline{\epsilon}$ is greater than or equal to that of the component in the set $\left\{\epsilon_{i_v} : v = 1, \ldots, m\right\}$, we add it to that set, i.e., we increase the value of m. Eventually, we obtain the case when $m = 2n+1$, i.e., we have an (2n+1)-dimensional reference subspace of U^q which satisfies Eq. 4.63 of Theorem 4.3. To see if it satisfies Eq. 4.62 of Theorem 4.3, we apply the test given by Conjecture 4.1. If the conditions of Conjecture 4.1 are satisfied, then we have obtained the minimal value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ and the corresponding $\underline{r}$ is the best Chebyshev solution vector $\underline{r}^* \in \mathscr{R}$. If, on the other hand, the conditions of Conjecture 4.1 are not satisfied, then we decrease the absolute values of the set of 2n-components for which Eq. 4.73 has failed. Eventually Conjecture 4.1 (or Conjecture 4.2) will be satisfied in some other (2n+1)-reference subspace.

5.3 The Computational Procedure

In this section we shall give an iterative procedure which yields the best Chebyshev approximation to the problem considered in Chapter IV.

---

[3] Note that this set can be used to define an m-dimensional reference subspace of U^q (see Appendix C).

Let us begin by listing the equations, in the form we shall need, as follows:

(1) $$\underline{f} = [Z(\underline{z})]\,\underline{\beta} + \underline{\epsilon}(\underline{\beta}, \underline{z}) \quad \text{in } U^q \tag{5.3}$$

where $\underline{f}$ is a prescribed real vector in U^q; $[Z(\underline{z})]$ is the $(q \times n)$ matrix, defined by Eq. 1.18 (or Eq. 1.18b); $(\underline{\beta}, \underline{z})$ is the parameter vector pair in $\mathscr{B}_z \times \mathscr{Z}$; and $\underline{\epsilon}(\underline{\beta}, \underline{z})$ is an unknown real error vector in U^q.

(2) $$\underline{f} = [Z]\,\underline{\beta} + \underline{\epsilon}(\underline{\beta}) \quad \text{in } U^q \tag{5.4}$$

where $[Z]$ is the $(q \times n)$ matrix $[Z(z)]$ when the vector $\underline{z} \in \mathscr{Z}$ is initially prescribed. Note that Eq. 5.4 represents the form of Eq. 5.3 when $\underline{z} \in \mathscr{Z}$ is initially prescribed.

(3) $$[R(\underline{r})]^T\,\underline{f} = [R(\underline{r})]^T\,\underline{\epsilon}(\underline{r}) \quad \text{in } E^{q-n} \tag{5.5}$$

where $\underline{\epsilon}(\underline{r}) \triangleq \underline{\epsilon}(\underline{\beta}, \underline{z})$ in E^q; $[R(\underline{r})]$ is the $q \times (q-n)$-matrix defined by Eq. 3.2; and where $\underline{r}$ is a vector in E^{n+1} given by

$$\underline{r} = \begin{bmatrix} r_o \\ r_1 \\ \vdots \\ r_n \end{bmatrix} \tag{5.6}$$

Since we are interested only in the value of $\underline{r}$ within a constant, we shall assume that the component[4] $r_n = 1$. An alternate representation of Eq. 5.5 is given by

$$\sum_{i=0}^{n} r_i(f_{i+j} - \epsilon_{i+j}) = 0 , \qquad j = 1, \ldots, q-n \qquad (5.5a)$$

(4)

$$\underline{\delta}(\underline{r}) = [F]\,\underline{r} \quad \text{in } E^{q-n} \qquad (5.7)$$

where [F] is the (q-n)x(n+1) matrix defined by Eq. 4.34; and $\underline{\delta}(\underline{r})$ is an unknown real vector in E^{q-n}. When seeking the minimum of $\|\underline{\delta}(\underline{r})\|_\infty$ with respect to $\underline{r} \in E^{n+1}$ when $\|\underline{r}\|_1 = 1$, we shall use the relation

$$\min_{\|\underline{r}\|_1 = 1} \|\underline{\delta}(\underline{r})\|_\infty = \min_{0 \leq j \leq n} \left\{ \frac{\displaystyle\min_{\underline{r}^{(j)} \in E^{n+1}} \|\underline{\delta}(\underline{r}^{(j)})\|_\infty}{\|\underline{r}^{(j)}\|_1} \right\} \qquad (5.8)$$

where $\underline{r}^{(j)}$ denotes the vector $\underline{r} \in E^{n+1}$ with the j-th component equal to one, i.e., $r_j^{(j)} = 1$.

(5) When m-components of the vector $\underline{\epsilon} \in U^q$ are equal in absolute value, where $n+1 \leq m \leq 2n+1$, then the m-dimensional k-th reference subspace which contains these m-components is denoted by U_k^m. Hence, according to Definition 2.1 (Chapter II) the projection of $\underline{\epsilon} \in U^q$ onto U_k^m is

---

[4] If the approximation procedure indicates that the other n-components of $\underline{r}$ tend to become large (i.e., tend to infinity), then we shall select some other component from the set $\{r_0, r_1, \ldots, r_{n-1}\}$ to be equal to one.

denoted by $\underline{\epsilon}^{(k)}$ and given by

$$\underline{\epsilon}^{(k)} = \rho\, \underline{\sigma}^{(k)} \tag{5.9}$$

where $\underline{\sigma}^{(k)}$ is the vector in U_k^m representing the prescribed sign configuration of $\underline{\epsilon}^{(k)}$, i.e., $\sigma_i^{(k)} = \pm 1$, $i = 1, \ldots, m$. Denoting by $\underline{f}^{(k)}$, the projection $\underline{f} \in U^q$ onto U_k^m, then the relation between $\underline{f}^{(k)}$ and $\underline{\epsilon}^{(k)}$ in U_k^m, is given by

$$[\Lambda^{(k)}(\underline{r})]^T \underline{f}^{(k)} = [\Lambda^{(k)}(\underline{r})]^T \underline{\epsilon}^{(k)} \tag{5.10}$$

where $[\Lambda^{(k)}(\underline{r})]$ is an $m \times (m-n)$ matrix defined by Definition C.3 (Appendix C). Its column space defines the orthogonal complement subspace of the approximating subspace in U_k^m, and it is obtained from the matrix $[R(\underline{r})]$ by appropriate column operations[5] and partitions.

Remark: When the vector $\underline{\epsilon}^{(k)} \in U_k^m$ is given by Eq. 5.9, then Eq. 5.10 will assume one of the following three forms:

(1) If $m = 2n+1$, then Eq. 5.10 represents a system of $(n+1)$-simultaneous equations. This system will be solved for ρ and the n-unknowns of the vector $\underline{r}$ (see Eqs. 4.69 and 70).

(2) If $m = 2n$, then Eq. 5.10 represents a system

---

[5] By appropriate column operations we mean the operations on the columns of the matrix $[R(\underline{r})]$ which yield a set of $(m-n)$ independent vectors with $(q-n)$-components, not lying in U_k^m, that are equal to zero.

of n-simultaneous equations. This system will be solved for the n-unknown components of the vector $\underline{r}$, by initially prescribing the value of ρ (see Eq. 4.78).

(3) If $m < 2n$, then Eq. 5.10 represents a system (m-n)-simultaneous equations. This system will be solved for (m-n) components or $\underline{r}$, by initially prescribing the value of ρ and the (2n+1-m) components of $\underline{r}$.

(6) $$P(z) = \prod_{k=1}^{n} (z - z_k) = \sum_{i=0}^{n} r_i z^i = 0 \qquad (5.11)$$

where the ordered sets $\{r_0, \ldots, r_n\}$ and $\{z_1, \ldots, z_n\}$ represent the vectors $\underline{r} \in \mathscr{R}$ and $\underline{z} \in \mathscr{Z}$, respectively.

(7) $$\Delta\epsilon_i = \epsilon_i(\underline{r}^{(2)}) - \epsilon_i(\underline{r}^{(1)}), \qquad i = 1, \ldots, q \qquad (5.12)$$

where $\epsilon_i(\underline{r}^{(2)})$ and $\epsilon_i(\underline{r}^{(1)})$ represent the i-th component of the vector $\underline{\epsilon}(\underline{r}) \in U^q$ evaluated at $\underline{r} = \underline{r}^{(2)}$ and $\underline{r} = \underline{r}^{(1)}$, respectively.

5.3.1 The Algorithm.

(1) Given the value of the real vector $\underline{f} \in U^q$, and the values of q and n.

(2) Using Eq. 5.7, determine the vector $\underline{r}$ in the set $\|\underline{r}\|_1 = 1$ for which $\|\underline{\delta}(\underline{r})\|_\infty$ is minimum. Denote this vector[6] by $\underline{r}'$ and the

[6] Note that since $\underline{r}'$ represents the initial estimate of $\underline{r}$, it may be picked arbitrarily.

corresponding vector $\underline{\delta}(\underline{r})$ by $\underline{\delta}(\underline{r}')$. If $\|\underline{\delta}(\underline{r}')\|_\infty = 0$, then $\underline{\epsilon}^*(\underline{r}') = 0$ and $\underline{r}' = \underline{r}^*$, go to Step(31). Otherwise, Step (3) is next.

(3) Determine the best Chebyshev real error vector $\underline{\epsilon}^*(\underline{r}') \in U^q$. This is done by first finding the vector $\underline{z}' \in \mathcal{Z}$ from Eq. 5.11, when $\underline{r} = \underline{r}'$ and forming the vector relation of Eq 5.4. By using the procedure given in Appendix A, obtain the best Chebyshev approximation to the problem given by Eq 5.4. The resulting error vector is the best Chebyshev error vector $\underline{\epsilon}^*(\underline{r}') \in U^q$.

(4) Is $\dfrac{\|\underline{\delta}(\underline{r}')\|_\infty}{\|\underline{r}'\|_1} = \|\underline{\epsilon}^*(\underline{r}')\|_\infty$? If it is, then $\underline{r}' = \underline{r}^*$ and proceed to Step (31). If it is not, go to Step (5).

(5) Form the set $\left\{\epsilon^*_{i_v}(\underline{r}') : v = 1, \ldots, m\right\}$ which contains the m-components of $\underline{\epsilon}^*(\underline{r}') \in U^q$ with absolute values that are equal to $\|\underline{\epsilon}^*(\underline{r}')\|_\infty$. Note that $m \geq n+1$.

(6) Is $m \geq 2n+1$? If it is, go to Step (19). If it is not, proceed to Step (7).

(7) Form an m-dimensional reference subspace, U_1^m, which contains all the components of $\underline{\epsilon}^*(\underline{r}')$ given by the set $\left\{\epsilon^*_{i_v}(\underline{r}') : v = 1, \ldots, m\right\}$ found in Step (5).

(8) Knowing U_1^m, determine the vectors $\underline{f}^{(1)}$ and $\underline{\epsilon}^{(1)}(\underline{r})$ in U_1^m, which represent the respective projections of $\underline{f}$ and $\underline{\epsilon}(\underline{r}) \in U^q$ onto U_1^m. Furthermore, determine the $m\times(m-n)$ matrix $[\Lambda^{(1)}(\underline{r})]$, which relates $\underline{f}^{(1)}$ and $\underline{\epsilon}^{(1)}(\underline{r})$ in U_1^m.

(9) Form a new vector $\underline{r}$, denoted by $\underline{r}''$ and defined by

$$\underline{r}'' = \begin{bmatrix} r_0 \\ \vdots \\ r_{m-n-1} \\ \hdashline r'_{m-n} \\ \vdots \\ r'_n \end{bmatrix}$$

where the last (2n+1-m)-components of $\underline{r}''$ take on the values of the last (2n+1-m)-components of $\underline{r}'$ found in the previous iteration step, and where the other (m-n)-components of $\underline{r}''$ are parameters.

(10) Let $\underline{\epsilon}^{(1)}(\underline{r}'') = 0.9\,[\underline{\epsilon}^{(1)}(\underline{r}')]$, where $\underline{\epsilon}^{(1)}(\underline{r}')$ denotes the vector $\underline{\epsilon}^{(1)}(\underline{r})$ found in the previous iteration step. Then, using $\underline{f}^{(1)}$ and $[\Lambda^{(1)}(\underline{r})]\big|_{\underline{r}=\underline{r}''}$ found in Step (8), determine the value of $\underline{r}''$ (i.e., evaluate the set of parameters $\{r_0, r_1, \ldots, r_{m-n-1}\}$) from the system of simultaneous equations given by Eq. 5.10. If there is a real solution, go to Step (12). If there is no real solution, go to Step (11).

(11) Is m = 2n? If it is, go to Step (31). If it is not, repeat the procedure from Step (10) where now the vector $\underline{r}''$ contains an extra parameters, i.e., it contains (m-n+1) parameters.[7]

---

[7]For example, the fixed component r'_{m-n} of the vector $\underline{r}''$, defined in Step 9, is replaced by a parameter denoted by r_{m-n}.

(12) Knowing the value of $\underline{r}''$ and the components of $\underline{\epsilon}^{(1)}(\underline{r}'')$ which are in U_1^m, determine the vector $\underline{\epsilon}(\underline{r}'')$ in U^q using Eq. 5.5 when $\underline{r} = \underline{r}''$ and $\underline{f}$ takes its prescribed value in U^q.

(13) Is $\|\underline{\epsilon}(\underline{r}'')\|_{\infty} = \|\underline{\epsilon}^{(1)}(\underline{r}'')\|_{\infty}$? If it is, go to Step (14). If it is not, go to Step (15).

(14) From the set $\{\epsilon_i(\underline{r}''): i = 1, 2, \ldots, q\}$ form the subset $\{\epsilon_{i_v}(\underline{r}''): v = 1, 2, \ldots, m\}$ which contains all the i's satisfying $|\epsilon_i(\underline{r}'')| = \|\underline{\epsilon}(\underline{r}'')\|_{\infty}$. If this set is identical to that used in the previous iterative step, then add to it the component $\epsilon_j(\underline{r}'')$ satisfying $|\Delta\epsilon_j| = \max_{1 \leq i \leq q} |\Delta\epsilon_i|$, where $\Delta\epsilon_i$ is given by Eq. 5.12. Step (16) is next.

(15) From the set $\{\epsilon_i(\underline{r}''): i = 1, 2, \ldots, q\}$ form the subset $\{\epsilon_{i_v}(\underline{r}''): v = 1, 2, \ldots, m\}$ which contains all the i's satisfying $|\epsilon_i(\underline{r}'')| \geq \|\underline{\epsilon}(\underline{r}'')\|_{\infty}$. Step (16) is next.

(16) Is $m = 2n+1$ (where the value of m was found in Step (14) or (15)? If it is, go to Step (19). If not, go to Step (17).

(17) Is $m > 2n+1$? If it is, repeat from Step (10), by using the vector $\underline{\epsilon}^{(1)}(\underline{r}'') = c\,\underline{\epsilon}^{(1)}(\underline{r}')$, where $0.9 < c < 1.0$. If it is not, go to Step (18).

(18) Form an m-dimensional reference subspace, U_2^m, which contains all the components of $\underline{\epsilon}(\underline{r}'') \epsilon U^q$ given by the set $\{\epsilon_{i_v}(\underline{r}''): v = 1, \ldots, m\}$ found in Step (14) or in Step (15). Step (8) is next, where the reference subspace U_1^m of Step (8) is replaced by the reference subspace U_2^m.

(19) Form a (2n+1)-dimensional reference subspace, U_2^{2n+1}, so that it contains all the components of $\underline{\epsilon}(\underline{r}'') \epsilon U^q$ given by the set $\{\epsilon_{i_v} : v = 1, 2, \ldots, 2n+1\}$ found in Step (14) or in Step (15).

(20) Knowing U_2^{2n+1}, determine the vectors $\underline{f}^{(2)}$, $\underline{\epsilon}^{(2)}(\underline{r}'')$ in U_2^{2n+1} and the (2n+1)x(n+1) matrix $[\Lambda^{(2)}(\underline{r})]$ which relate them in U_2^{2n+1}. Furthermore, determine the vector $\underline{\sigma}^{(2)} \epsilon U_2^{2n+1}$, from the relation $\sigma_i^{(2)} = \text{sgn}\, \epsilon_i^{(2)}(\underline{r}'')$, i = 1, ..., 2n+1.

(21) Let $\underline{\epsilon}^{(2)}(\underline{r}) = \rho\, \underline{\sigma}^{(2)}$. Then using $\underline{f}^{(2)}$ and $[\Lambda^{(2)}(\underline{r})]$ found in Step (20) solve Eq. 5.10 for $\underline{r}$ and ρ. If there are more than one solution vectors $\underline{r}$ for the same values of ρ determine them all.[8]

(22) Knowing $\underline{r}$ and $\underline{\epsilon}^{(2)}(\underline{r})$, determine the vectors $\underline{\epsilon}(\underline{r})$ in U^q using Eq. 5.5. Denote by $\hat{\underline{r}}$, the vector $\underline{r}$ which yields a vector $\underline{\epsilon}(\underline{r})$ in Step (22) with $\|\underline{\epsilon}(\underline{r})\|_\infty = |\rho|$, and by $\hat{\rho}$, the corresponding value of ρ found in Step (21). Proceed to Step (23) to test if $\hat{\underline{r}}$ is the optimum $\underline{r}^*$.

(23) Solve the equation used in Step (21) (i.e., Eq. 5.10) for the set $\{\hat{\underline{r}}^{(\nu)} : \nu = 1, 2, \ldots, 2n+1\}$ and $\{\rho^{(\nu)}\}$, when

$$\epsilon_i^{(2)} = 0.99\, \rho\, \sigma_i^{(2)} \quad \text{for } i \neq \nu$$
$$= \rho^{(\nu)} \quad \text{for } i = \nu,$$

where i = 1, 2, ..., 2n+1, and where the value of ρ and $\underline{\sigma}^{(2)}$ were found in Step (21).

---

[8] In the next chapter when considering the application of this computational method to the time domain network synthesis problem, we shall restrict $\underline{r}$ to give a physically realizable network.

(24) Is $|\rho^{(\nu)}| > |\hat{\rho}|$, for all $\nu = 1, 2, \ldots, 2n+1$? If it is, then $\hat{\underline{r}} = \underline{r}^*$ and go to Step (31). If it is not, then $\hat{\underline{r}} \neq \underline{r}^*$, and go to Step (25).

(25) Let μ denote the set of indices ν for which $|\rho^{(\nu)}| < |\hat{\rho}|$. Determine the vectors $\left\{\underline{\epsilon}(\hat{\underline{r}}^{(\mu)})\right\}$ in U^q from Eq. 5.5, by using $\underline{r} = \hat{\underline{r}}^{(\mu)}$ and the values of $\underline{\epsilon}^{(2)}(\hat{\underline{r}}^{(\mu)})$ determined in Step (23).

(26) Out of the set of vectors $\left\{\underline{\epsilon}(\hat{\underline{r}}^{(\mu)})\right\}$ is there a vector $\underline{\epsilon}(\hat{\underline{r}}^{(j)})$ such that $\|\underline{\epsilon}(\hat{\underline{r}}^{(j)})\|_\infty > |\hat{\rho}|$? If there is, repeat from Step (15) using the vector $\underline{\epsilon}(\hat{\underline{r}}^{(j)})$. If not, go to Step (27).

(27) Out of the set $\left\{\underline{\epsilon}(\hat{\underline{r}}^{(\mu)})\right\}$, pick any vector and denote it by $\underline{\epsilon}(\underline{r}')$ and repeat from Step (5).

(28) Determine the vector $\underline{\sigma}^{(1)}$ from the relation $\sigma_i^{(1)} = \operatorname{sgn} \epsilon_i^{(1)}(\underline{r}')$, $i = 1, \ldots, 2n$, where $\underline{\epsilon}^{(1)}(\underline{r}')$ denotes the vector $\underline{\epsilon}^{(1)}(\underline{r})$ found in the previous step.

(29) Let $\underline{\epsilon}^{(1)}(\underline{r}) = \rho\, \underline{\sigma}^{(1)}$ where $\underline{\sigma}^{(1)}$ is given in Step (28). Then using $\underline{f}^{(1)}$ and $[\Lambda^{(1)}(\underline{r})]$ found in the previous step, solve Eq. 5.10 for $\underline{r}$ and ρ so that the value of $|\rho|$ is minimum.

(30) Knowing $\underline{r}$ and $\underline{\epsilon}^{(1)}(\underline{r})$, determine the vector $\underline{\epsilon}(\underline{r})$ in U^q by using Eq. 5.5. If $\|\underline{\epsilon}(\underline{r})\|_\infty > \|\underline{\epsilon}^{(1)}(\underline{r})\|_\infty = |\rho|$, then repeat from Step (15). If $\|\underline{\epsilon}(\underline{r})\|_\infty = \|\underline{\epsilon}^{(1)}(\underline{r})\|_\infty = |\rho|$ then the vector $\underline{r}$ found in Step (29) is the optimum $\underline{r}^*$. Step (31) is next.

(31) Knowing $\underline{r}^*$ in E^{n+1}, determine $\underline{z}^*$ from Eq. 5.11.

(32) Knowing $\underline{z}^*$, determine[9] $\underline{\beta}^{**}$ from any set of n equations defined by Eq. 5.3, where $\underline{\epsilon}(\underline{\beta}, \underline{z}) = \underline{\epsilon}(\underline{r}^*)$ found in Step (22), or Step (30).

(33) From $(\underline{\beta}^{**}, \underline{z}^*)$, determine the vector $\underline{f}^{**} \triangleq [Z(\underline{z}^*)]\, \underline{\beta}^{**}$.

5.3.2 Comments on the Algorithm. The algorithm presented in Section 5.3.1 is based on the theoretical results obtained in Chapter IV. To program this algorithm, for digital computer use, is no simple task. The difficulty lies in attempting to program the steps containing the formation of the $m \times (m-n)$ matrix $[\Lambda^{(k)}(\underline{r})]$, i.e., the steps which construct the system of simultaneous equations. To perform these steps the computer program must be able to generate its own program, i.e., the system of simultaneous equations in variable form. On the other hand, to specify initially all the possible forms of the elements of the matrix $[\Lambda^{(k)}(\underline{r})]$ is impractical when $n > 2$ and the value of $(q-n)$ is large.

A more suitable procedure for use with a digital computer would be the one which minimizes the function $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ with respect to $\underline{r} \in E^{n+1}$, along the set $\|\underline{r}\|_1 = 1$; or equivalently, the function $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ with respect to $\underline{z} \in \mathcal{Z}$. However, to develop such a procedure one must first obtain a method of varying $\underline{r}$ (or $\underline{z}$) so that the value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ or $\|\underline{\epsilon}^*(\underline{z})\|_\infty$ is systematically decreased. An alternate procedure requires

---

[9] Note that the vector $\underline{\beta}^{**}$ is not unique if some of the components of $\underline{z}^*$ are identical, and/or equal to zero. Hence, if some of the components of $\underline{\beta}^{**}$ can be made to equal zero without affecting the value of $\|\underline{\epsilon}^{**}\|_\infty$, then we may neglect the corresponding components of $\underline{z}^{**}$, i.e., lower the value of n.

the development of an exchange method [of the type developed by Stiefel (Ref. 22)] which will systematically exchange the (2n+1)-dimensional reference subspaces until the reference subspace U_c^{2n+1} of Theorem 4.3 (or Theorem 4.2) is attained.

5.4 Numerical Examples

The following examples illustrate the algorithm presented in Section 5.3.1. The first two examples chosen (i.e., Examples 5.1 and 5.2), are identical to those considered by Ruston (Ref. 20). It will be seen that a considerable improvement over his result is achieved using the above procedure. Furthermore, these two examples will illustrate the effect of increasing the dimensions of the parameter space on the resulting Chebyshev approximating error. The first example considers the simple case when n = 1, and it will be worked out in extensive detail. The second example, the case when n = 2, will be presented using somewhat less detail. Example 5.3 illustrates the case when the best Chebyshev approximation is characterized by an error vector $\underline{\epsilon}^{**} \in U^q$ which has only 2n-components equal in absolute value to $\|\underline{\epsilon}^{**}\|_\infty$. Examples 5.4 and 5.5 illustrate the case when the components are not distinct. Example 5.6 illustrates the case when the minimum value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ is attained at the vector $\underline{r}$, with the component $r_n = 0$.

Now before turning to the examples, it should be mentioned that a digital computer was used to solve these examples, however, it was used simply as a desk calculator because of the difficulty encountered in programming the algorithm of Section 5.3.1. In the following examples, we shall denote by "( )" the computational steps, and by "step-" when referring to the steps of the algorithm of Section 5.3.1.

Example 5.1

Consider the problem of approximating the real vector

$$\underline{f} = \begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.2500 \\ 0.1600 \\ 0.1110 \\ 0.0817 \\ 0.0625 \\ 0.0494 \\ 0.0400 \end{bmatrix} \text{ by the real vector } \hat{\underline{f}}(\beta, z) = \beta \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \\ z^5 \\ z^6 \\ z^7 \\ z^8 \end{bmatrix}$$

so that $\|\underline{f} - \hat{\underline{f}}(\beta^{**}, z^*)\|_\infty$ is minimum for all (β, z) in $\mathscr{B}_z \times \mathscr{Z}$. The computation procedure is as follows:

(1) Step 1: Substitute the prescribed vectors into Eq. 5.3. This yields

$$\begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.2500 \\ 0.1600 \\ 0.1110 \\ 0.0817 \\ 0.0625 \\ 0.0494 \\ 0.0400 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \\ z^5 \\ z^6 \\ z^7 \\ z^8 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \end{bmatrix} \text{ in } U^9 \tag{5.13}$$

Note that $q = 9$ and $n = 1$.

(2) Step 2: Eq. 5.7 becomes

$$\underline{\delta}(\underline{r}) = \begin{bmatrix} 1.0000 & 0.4450 \\ 0.4450 & 0.2500 \\ 0.2500 & 0.1600 \\ 0.1600 & 0.1110 \\ 0.1110 & 0.0817 \\ 0.0817 & 0.0625 \\ 0.0625 & 0.0494 \\ 0.0494 & 0.0400 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$$

The vector $\underline{r}^{(1)}$, and the corresponding vector $\underline{\delta}(\underline{r}^{(1)})$ for which $\|\underline{\delta}(\underline{r})\|_\infty$ is minimum for all $\underline{r}$ in the set $\|\underline{r}\|_1 = 1$ are

$$\underline{r}^{(1)} = \begin{bmatrix} -0.484 \\ 1.000 \end{bmatrix}, \quad \underline{\delta}(\underline{r}^{(1)}) = \begin{bmatrix} -0.039 \\ 0.035 \\ 0.039 \\ 0.0336 \\ 0.0241 \\ 0.0230 \\ 0.0192 \\ 0.0162 \end{bmatrix}$$

Note that when $n = 1$, Ruston's method will always yield the vector $\underline{r}$ for which $\|\underline{\delta}(\underline{r})\|_\infty$ = minimum.

(3) Step 3: From Eq. 5.11 we get $z^{(1)} = 0.484$. Substituting this value of z into Eq. 5.13 and determining the best Chebyshev approximation, by using Stiefel's method (see Appendix A), yields

$$\underline{\epsilon}^*(\underline{r}^{(1)}) = \underline{\epsilon}^*(z^{(1)}) = \begin{bmatrix} -0.0331 \\ -0.05454 \\ 0.0085 \\ 0.0432 \\ 0.05454 \\ 0.0544 \\ 0.0493 \\ 0.043 \\ 0.037 \end{bmatrix}$$

(4) Step 4: $\dfrac{\|\underline{\delta}(\underline{r}^{(1)})\|_\infty}{\|\underline{r}^{(1)}\|_1} = 0.0263$; $\|\underline{\epsilon}^*(\underline{r}^{(1)})\|_\infty = 0.05454$.

Since $\dfrac{\|\underline{\delta}(\underline{r}^{(1)})\|_\infty}{\|\underline{r}^{(1)}\|_1} < \|\underline{\epsilon}^*(\underline{r}^{(1)})\|_\infty$, go to Step (5).

(5) Step 5: From (4) it is seen that components $|\epsilon_2| = |\epsilon_5| =$ 0.05454, so that the set $\left\{ \epsilon^*_{i_v} (\underline{r}^{(1)}) \right\} = \left\{ \epsilon_2, \epsilon_5 \right\}$.

(6) Step 6: $m = 2$, and $2n+1 = 3$. Since $m < 2n+1$, go to Step 7.

(7) Step 7: The reference subspace U_1^m (i.e., U_1^2, since $m = 2$) containing ϵ_2 and ϵ_5 is the 2-dimensional subspace defined by the mapping $I_1 : U^q \rightarrow U_1^m$, where

$$[I_1] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(8) Step 8 yields

$$\underline{\epsilon}^{(1)}(\underline{r}^{(1)}) = [I_1]^T \underline{\epsilon}(\underline{r}^{(1)}) = \begin{bmatrix} -0.05454 \\ 0.05454 \end{bmatrix}$$

$$\underline{f}^{(1)} = [I_1]^T \underline{f} = \begin{bmatrix} 0.4450 \\ 0.1110 \end{bmatrix}$$

To determine $[\Lambda^{(1)}(\underline{r})]$ begin with the matrix

$$[R(\underline{r})] = \begin{bmatrix} r_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r_1 & r_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_1 & r_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 & r_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1 & r_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_1 & r_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & r_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_1 & r_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1 \end{bmatrix}$$

Find the matrix $[R_1(\underline{r})]$ which is equivalent to $[R(\underline{r})]$ according to Eq. C.20 of Appendix C. This yields

$$[R_1(\underline{r})] = \left[\begin{array}{c|ccccccc} r_0^3 & r_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ r_1^3 & 0 & 0 & r_1 & r_0 & 0 & 0 & 0 \\ \hline 0 & r_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 & r_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_1 & r_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_1 & r_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_1 & r_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1 \end{array}\right] = \left[\begin{array}{c|c} R_{11}^{(1)} & R_{12}^{(1)} \\ \hline R_{21}^{(1)} & R_{22}^{(1)} \end{array}\right]$$

Hence,

$$[\Lambda^{(1)}(\underline{r})] = [R_{11}^{(1)}] = \begin{bmatrix} r_0^{\,3} \\ r_1^{\,3} \end{bmatrix}$$

(9) Step 9:

$$\text{Let } \underline{r}^{(2)} = \begin{bmatrix} r_0 \\ 1 \end{bmatrix}$$

(10) Step 10: Let

$$\underline{\epsilon}^{(1)}(\underline{r}^{(2)}) = \begin{bmatrix} -0.053 \\ 0.053 \end{bmatrix}$$

and substituting it into Eq. 5. 10 when k = 1, m = 1, and where the values of $\underline{f}^{(1)}$ and $[\Lambda^{(1)}(\underline{r})]\Big|_{\underline{r}=\underline{r}^{(2)}}$ are those found in (8). This yields

$$-0.053\, r_o^3 + 0.053 = 0.445\, r_o^3 + 0.111$$

Solving this equation for r_o one obtains

$$\underline{r}^{(2)} = \begin{bmatrix} -0.48835 \\ 1.00000 \end{bmatrix} .$$

(11) Step 12: Using the values obtained in (10) and the prescribed values of $\underline{f} \in U^9$, the unknown components $\underline{\epsilon} \in U^9$ can be determined from Eq. 5.5 or equivalently from Eq. 5.5a. For example, knowing the set of values $\{f_i : i = 1, \ldots, 9\}$, $\{\epsilon_1, \epsilon_2\}$, and $\{r_o, r_1\}$, then the unknown components $\{\epsilon_1, \epsilon_3, \epsilon_4, \epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9\}$ can be obtained from Eq. 5.5a as follows:

$$\epsilon_1 = f_1 + (f_2 - \epsilon_2)\,\frac{r_1}{r_o}$$

$$\epsilon_i = f_i + (f_{i-1} - \epsilon_{i-1})\,\frac{r_o}{r_1}\ , \qquad i = 3, 4, 6, 7, 8, 9$$

Solving for the unknown $\{\epsilon_i\}$ yields the vector

$$\underline{\epsilon}(\underline{r}^{(2)}) = \begin{bmatrix} -0.01976 \\ -0.053 \\ 0.0068 \\ 0.04123 \\ 0.0530 \\ 0.05337 \\ 0.04866 \\ 0.04264 \\ 0.03669 \end{bmatrix}$$

(12) Step 13: Since

$$\|\underline{\epsilon}(\underline{r}^{(2)})\|_{\infty} = 0.05337 > \|\underline{\epsilon}^{(1)}(\underline{r}^{(2)})\|_{\infty} = 0.053$$

go to Step 15.

(13) Step 15: Form the set $\left\{\epsilon_2, \epsilon_5, \epsilon_6\right\}$.

(14) Step 16: Since m = 3 = 2n + 1 go to Step 19.

(15) Step 19: The reference subspace U_2^{2n+1}, containing the components $\left\{\epsilon_2, \epsilon_5, \epsilon_6\right\}$, is defined by the mapping $I_2 : U^9 \rightarrow U_2^3$, where

$$I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(16) Step 20 yields

$$\underline{f}^{(2)} = [I_2]^T \underline{f} = \begin{bmatrix} 0.4450 \\ 0.1110 \\ 0.0817 \end{bmatrix}$$

$$\underline{\sigma}^{(2)} = \begin{bmatrix} -1 \\ +1 \\ +1 \end{bmatrix}, \quad \text{and}$$

$$[\Lambda^{(2)}(\underline{r})] = \begin{bmatrix} r_0^3 & 0 \\ r_1^3 & r_0 \\ 0 & r_1 \end{bmatrix}$$

(17) Step 21: Let $\hat{\underline{r}} = \begin{bmatrix} r_o \\ 1 \end{bmatrix}$, and let $\underline{\epsilon}^{(2)}(\underline{r}) = \rho\, \underline{\sigma}^{(2)}$. Substituting it into Eq. 5. 10, when k = 2, and where the value of the vector $\underline{f}^{(2)}$, $\underline{\sigma}^{(2)}$, and $[\Lambda^{(2)}(\underline{r})]$, are given in (16), yields

$$\rho(-r_o^3 + 1) = 0.445\, r_o^3 + 0.1110$$

$$\rho(r_o + 1) = 0.111\, r_o + 0.0817$$

Solving for r_o and ρ yields,

$$\hat{\underline{r}} = \begin{bmatrix} -0.4847 \\ 1.0000 \end{bmatrix}$$

$$\rho = 0.0541$$

(18) Step 22: Solving Eq. 5. 5, when $\underline{r} = \hat{\underline{r}}$, $\epsilon_2 = -0.0541$, $\epsilon_5 = 0.0541$, and $\epsilon_6 = 0.0541$, yields

$$\underline{\epsilon}(\hat{\underline{r}}) = \begin{bmatrix} -0.0296 \\ -0.0541 \\ 0.0081 \\ 0.0426 \\ 0.0541 \\ 0.0541 \\ 0.0491 \\ 0.0429 \\ 0.0373 \end{bmatrix}$$

Remark: One could have skipped (17) and (18) and have processed directly to (19) (i.e., Step 23 of the algorithm), since it is self-evident that the value of $\|\underline{\epsilon}(\underline{r})\|_{\infty} \leq 0.05454$. Clearly, if the test of Step 23 indicates that $\hat{\underline{r}} = \underline{r}^*$, i.e., that the value of $\|\underline{\epsilon}(\hat{\underline{r}})\|_{\infty}$ is the minimum possible value of $\|\underline{\epsilon}(\underline{r})\|_{\infty}$, then one must return to Steps 21 and 22 of the algorithm and calculate $\underline{r}^*$ and $\underline{\epsilon}(\underline{r}^*)$, respectively.

(19) Step 23: Consider the equation

$$\begin{bmatrix} r_0^3 & r_1^3 & 0 \\ 0 & r_0 & r_1 \end{bmatrix} \begin{bmatrix} -\eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} r_0^3 & r_1^3 & 0 \\ 0 & r_0 & r_1 \end{bmatrix} \begin{bmatrix} 0.4450 \\ 0.1110 \\ 0.0817 \end{bmatrix}$$

Solve for $\{r_0^{(\nu)} : \nu = 1, 2, 3\}$, when $r_1^{(\nu)} = 1$,

$\eta_i = 0.052$ for $i \neq \nu$, and

$\eta_i = \rho^{(\nu)}$ for $i = \nu$, where $i = 1, 2, 3$.

The results are given in Table 5.1.

ν	1	2	3
$\hat{r}_0^{(\nu)}$	-0.5035	-0.4944	-0.4917
$\hat{r}_1^{(\nu)}$	1.0000	1.0000	1.0000
ϵ_1	0.08234	-0.0052	
ϵ_2	-0.01694	-0.052	-0.052
ϵ_3	0.01746	0.00426	
ϵ_4	0.0428	0.0385	
ϵ_5	0.052	0.05094	0.052
ϵ_6	0.052	0.052	0.05253
ϵ_7	0.04755	0.0478	
ϵ_8	0.04084	0.04213	
ϵ_9	0.03569	0.03639	

Table 5.1. Results of Step 19 of Example 5.1.

(20) Step 24: From Table 5.1 it is seen that $\hat{\underline{r}}$ found in (17) is <u>not</u> optimum, because $|\rho^{(\nu)}| < |\rho|$ when $\nu = 1, 2$.

(21) Step 25: The two vectors $\underline{\epsilon}(\hat{\underline{r}}^{(1)})$ and $\underline{\epsilon}(\hat{\underline{r}}^{(2)})$ are given in Columns 1 and 2 of Table 5.1, respectively.

(22) Step 26: From Column 1 of Table 5.1, it is noted that when $\underline{r} = \hat{\underline{r}}^{(1)}$, then $|\epsilon_1| > 0.052$. Hence, return to Step 15 of the algorithm using the vector $\underline{\epsilon}(\hat{\underline{r}}^{(1)})$.

(23) Step 15 yields the set $\left\{\epsilon_1, \epsilon_5, \epsilon_6\right\}$. Since m = 3, proceed to Step 19.

(24) Step 19: The reference subspace U_3^{2n+1} contains the components $\left\{\epsilon_1, \epsilon_5, \epsilon_6\right\}$.

(25) Step 20 yields

$$\underline{f}^{(3)} = \begin{bmatrix} 1.0000 \\ 0.1110 \\ 0.0817 \end{bmatrix}, \qquad \underline{\sigma}^{(3)} = \begin{bmatrix} +1 \\ +1 \\ +1 \end{bmatrix}, \text{ and}$$

$$[\Lambda^{(3)}(\underline{r})] = \begin{bmatrix} r_0^{4} & 0 \\ -r_1^{4} & r_0 \\ 0 & r_1 \end{bmatrix}$$

(26) Step 21: Let $\underline{r} = [r_0 \;\; 1]^T$ and let $\underline{\epsilon}^{(3)} = \rho\, \underline{\sigma}^{(3)}$, then $[\Lambda^{(3)}(\underline{r})]^T\, \underline{\epsilon}^{(3)} = [\Lambda^{(3)}(\underline{r})]^T\, \underline{f}^{(3)}$ yields

$$\rho(r_0^{4} - 1) = 1.00\; r_0^{4} - 0.111$$

$$\rho(r_0 + 1) = 0.111\, r_0 + 0.0817$$

Solving for r_0 and ρ, yields

$$\hat{\underline{r}} = \begin{bmatrix} -0.49826 \\ 1.00000 \end{bmatrix}$$

$$\rho = 0.0526$$

(27) Step 22: Solving Eq. 5.5, when $\underline{r} = \underline{r}$, and when $\underline{\epsilon}^{(3)}(\hat{\underline{r}}) = \begin{bmatrix} 0.0526 \\ 0.0526 \\ 0.0526 \end{bmatrix}$

$$\underline{\epsilon}(\hat{\underline{r}}) = \begin{bmatrix} 0.0526 \\ -0.02705 \\ 0.0098 \\ 0.0428 \\ 0.0526 \\ 0.0526 \\ 0.048 \\ 0.0422 \end{bmatrix}$$

(28) Step 23: Solve the equation

$$\begin{bmatrix} r_0^{4} & -r_1^{4} & 0 \\ 0 & r_0 & r_1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} r_0^{4} & -r_1^{4} & 0 \\ 0 & r_0 & r_1 \end{bmatrix} \begin{bmatrix} 1.0000 \\ 0.1110 \\ 0.0817 \end{bmatrix}$$

for $\left\{\hat{r}_0^{(\nu)} : \nu = 1, 2, 3\right\}$, when $\hat{r}_1^{(\nu)} = 1$, $\eta_i = 0.052$ for $i \neq \nu$,

and $\eta_i = \rho^{(\nu)}$ for $i = \nu$,

where $i = 1, 2, 3$.

The results are tabulated in Table 5.2.[10]

(29) Step 24: Clearly, from Table 5.2, $\underline{r}$ found in (26) is not optimum.

(30) Step 25 through 27: Using $\underline{\epsilon}(\hat{\underline{r}}^{(2)})$ repeat Step 5.

(31) Step 6: Since $m = 2 < 2n+1 = 3$ proceed to Step 7.

(33) Step 7 yields the reference subspace U_4^{2} containing the set $\left\{\epsilon_1, \epsilon_6\right\}$.

---

[10] Note that in Table 5.2 the vector $\hat{\underline{r}}^{(1)}$ is identical to $\hat{\underline{r}}^{(1)}$ of Table 5.1.

ν	1	2	3
$\hat{r}_0^{(\nu)}$	-0.5035	-0.50025	-0.49948
$\hat{r}_1^{(\nu)}$	1.0000	1.00000	1.00000
ϵ_1	0.08234	0.052	0.052
ϵ_2		-0.02924	
ϵ_3		0.01276	
ϵ_4		0.04132	
ϵ_5	0.052	0.05163	0.052
ϵ_6	0.052	0.052	0.05223
ϵ_7		0.04764	
ϵ_8		0.04197	
ϵ_9		0.03628	

Table 5.2. Results of Step 28 of Example 5.1.

(34) Step 8 yields

$$\underline{f}^{(4)} = \begin{bmatrix} 1.0000 \\ 0.0817 \end{bmatrix}, \qquad \sigma^{(4)} = \begin{bmatrix} +1 \\ +1 \end{bmatrix}, \quad \text{and}$$

$$[\Lambda^{(4)}(\underline{r})] = \begin{bmatrix} r_0^{5} \\ r_1^{5} \end{bmatrix}$$

(35) Step 9: Take $\underline{r}^{(3)} = \begin{bmatrix} r_0 \\ 1 \end{bmatrix}$

(36) Step 10: Let $\underline{\epsilon}^{(4)}(\underline{r}^{(3)}) = 0.05\,[\underline{\sigma}^{(4)}] = \begin{bmatrix} 0.05 \\ 0.05 \end{bmatrix}$ and substituting it into Eq. 5.10, when $\underline{f}^{(4)}$ and $[\Lambda^{(4)}(\underline{r})]\Big|_{\underline{r}=\underline{r}^{(3)}}$, have the values found in (34), yields

$$0.05\,(r_0^{\,5} + 1) \;=\; 1.000\, r_0^{\,5} + 0.0817$$

Solving this equation for r_0, yields $\underline{r}^{(3)} = \begin{bmatrix} -0.50661 \\ 1.0000 \end{bmatrix}$

(37) Step 12 yields

$$\underline{\epsilon}(\underline{r}^{(3)}) = \begin{bmatrix} 0.05 \\ -0.03628 \\ 0.00618 \\ 0.03648 \\ 0.04843 \\ 0.05 \\ 0.0464 \\ 0.04126 \\ 0.03588 \end{bmatrix}$$

(38) Since Step 13 yields a "yes", Step 14 is next.

(39) Step 14: Form the set $\left\{\epsilon_1, \epsilon_2, \epsilon_6\right\}$ from the components of $\underline{\epsilon}(\underline{r}^{(3)})$. The component ϵ_2 was added to the previous set since $|\Delta\epsilon_2| = \max_{1 \leq i \leq 9} |\Delta\epsilon_i|$, where $|\Delta\epsilon_i| \triangleq |\epsilon_i(\underline{r}^{(3)}) - \epsilon_i(\underline{r}^{(2)})|$ and the value of $\underline{\epsilon}(\underline{r}^{(3)})$ and $\underline{\epsilon}(\underline{r}^{(2)})$ are given in (37) and in Table 5.2, respectively.

(40) Since Step 16 yields a "yes", Step 19 is next.

(41) Step 19: The reference subspace U_5^{2n+1} contains the components

$$\left\{\epsilon_1, \epsilon_2, \epsilon_6\right\} \quad \text{of} \quad \underline{\epsilon} \quad \text{in} \quad U^9$$

(42) Step 20 yields

$$\underline{f}^{(5)} = \begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.0817 \end{bmatrix}, \qquad \underline{\sigma}^{(5)} = \begin{bmatrix} +1 \\ -1 \\ +1 \end{bmatrix}, \quad \text{and}$$

$$[\Lambda^{(5)}(\underline{r})] = \begin{bmatrix} r_0 & 0 \\ r_1 & r_0^4 \\ 0 & -r_1^4 \end{bmatrix}$$

(43) Step 21: Let $\underline{r} = [r_0 \ 1]^T$, and let $\underline{\hat{\epsilon}}^{(5)} = \rho \, \underline{\sigma}^{(5)}$, then $[\Lambda^{(5)}]^T \underline{\epsilon}^{(5)} = [\Lambda^{(5)}]^T \underline{f}^{(5)}$ yields

$$\rho(r_0 - 1) = 1.00 \ r_0 + 0.445$$

$$\rho(-r_0^4 - 1) = 0.445 \, r_0^4 - 0.0817$$

Solving for r_0 and ρ, yields $\underline{\hat{r}} = \begin{bmatrix} -0.516 \\ 1.000 \end{bmatrix}$

$$\rho = 0.046831$$

(44) Step 22 yields

$$\underline{\epsilon}(\underline{\hat{r}}) = \begin{bmatrix} 0.04683 \\ -0.04683 \\ -0.00381 \\ 0.02904 \\ 0.04342 \\ 0.04683 \\ 0.04451 \\ 0.04012 \\ 0.0353 \end{bmatrix}$$

(45) Step 23: Solve the equation

$$\begin{bmatrix} r_0 & r_1 & 0 \\ 0 & r_0^4 & -r_1^4 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ -\epsilon_2 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} r_0 & r_1 & 0 \\ 0 & r_0^4 & -r_1^4 \end{bmatrix} \begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.0817 \end{bmatrix}$$

for $\left\{\hat{r}_0^{(\nu)} : \nu = 1, 2, 6\right\}$ when $\hat{r}_1^{(\nu)} = 1$ and $\epsilon_i = 0.045$ for $i \neq \nu$,

$$\epsilon_i = \rho^{(\nu)} \quad \text{for } i = \nu \text{ , where } i = 1, 2, 6.$$

The results are given in Table 5.3.

ν	1	2	6
$\hat{r}_0^{(\nu)}$	-0.523	-0.5208	-0.5131
$\hat{r}_1^{(\nu)}$	1.000	1.0000	1.0000
ϵ_1	0.06312	0.045	0.045
ϵ_2	-0.045	-0.0524	-0.045
ϵ_6	0.045	0.045	0.04773

Table 5.3. Results of Step 45 of Example 5.1.

(46) Step 24: Clearly from Table 5.3, $\hat{\underline{r}} = \underline{r}^* = \begin{bmatrix} -0.516 \\ 1.000 \end{bmatrix}$. Therefore, $\underline{\epsilon}^{**} = \underline{\epsilon}(\underline{r})$ found in (44).

(47) Step 31: Eq. 5.11 becomes

$$-0.516 + z = 0$$

Therefore, $z^* = 0.516$.

(48) Steps 32 and 33: Using the first component of Eq. 5.13 when $z = z^*$ found in (47) and when $\underline{\epsilon}(\underline{\beta}, \underline{z}) = \underline{\epsilon}^{**}$ found in (44), one obtains

$$1.000 = \beta^* + 0.04683$$

Hence, $\beta^{**} = 0.95317$. Therefore, the optimum parameter pair for the Chebyshev approximation problem is given by

$$\boxed{(\beta^{**}, z^*) = (0.95317, \ 0.516)}$$

Furthermore, the best Chebyshev approximating vector, $\underline{f}^{**}$, to

$$\underline{f} = \begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.2500 \\ 0.1600 \\ 0.1110 \\ 0.0817 \\ 0.0625 \\ 0.0494 \\ 0.0400 \end{bmatrix}, \text{ is given by } \underline{f}^{**} = \begin{bmatrix} 0.95317 \\ 0.49183 \\ 0.25381 \\ 0.13096 \\ 0.06758 \\ 0.03487 \\ 0.01799 \\ 0.00928 \\ 0.00470 \end{bmatrix}$$

where

$$\underline{\epsilon}^{**} = \begin{bmatrix} 0.04683 \\ -0.04683 \\ -0.00381 \\ 0.02904 \\ 0.04342 \\ 0.04683 \\ 0.04451 \\ 0.04012 \\ 0.03530 \end{bmatrix}$$

and where $\|\underline{\epsilon}^{**}\|_\infty = 0.04683$.

Figure 6 depicts graphically the components of the final error vector $\underline{\epsilon}^{**}$. For comparison, the initial error vector, $\underline{\epsilon}^*(\underline{r}^{(1)})$, (i. e., Ruston's final error vector) is also shown. In Fig. 7, we have attempted to illustrate graphically the results of using the method of descent in minimizing the function $\|\underline{\epsilon}(\underline{r})\|_\infty$. The numbers in parentheses refer to the steps of the iteration used in the above numerical example.

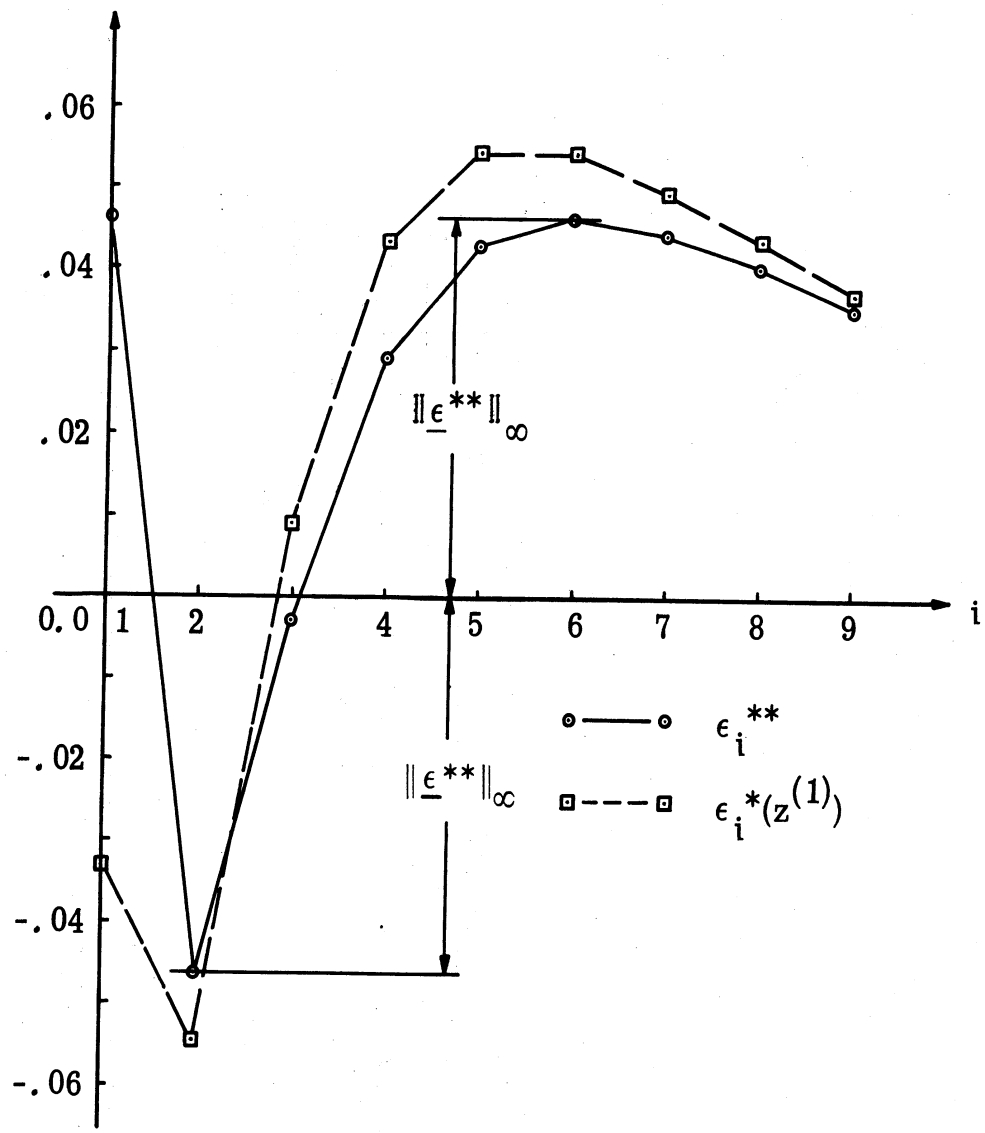


Fig. 6. The Chebyshev approximation error, optimum approximation yields $\|\underline{\epsilon}^{**}\|_\infty = 0.0468$

Ruston's approximation yields $\|\underline{\epsilon}^*(z^{(1)})\|_\infty = 0.0545$

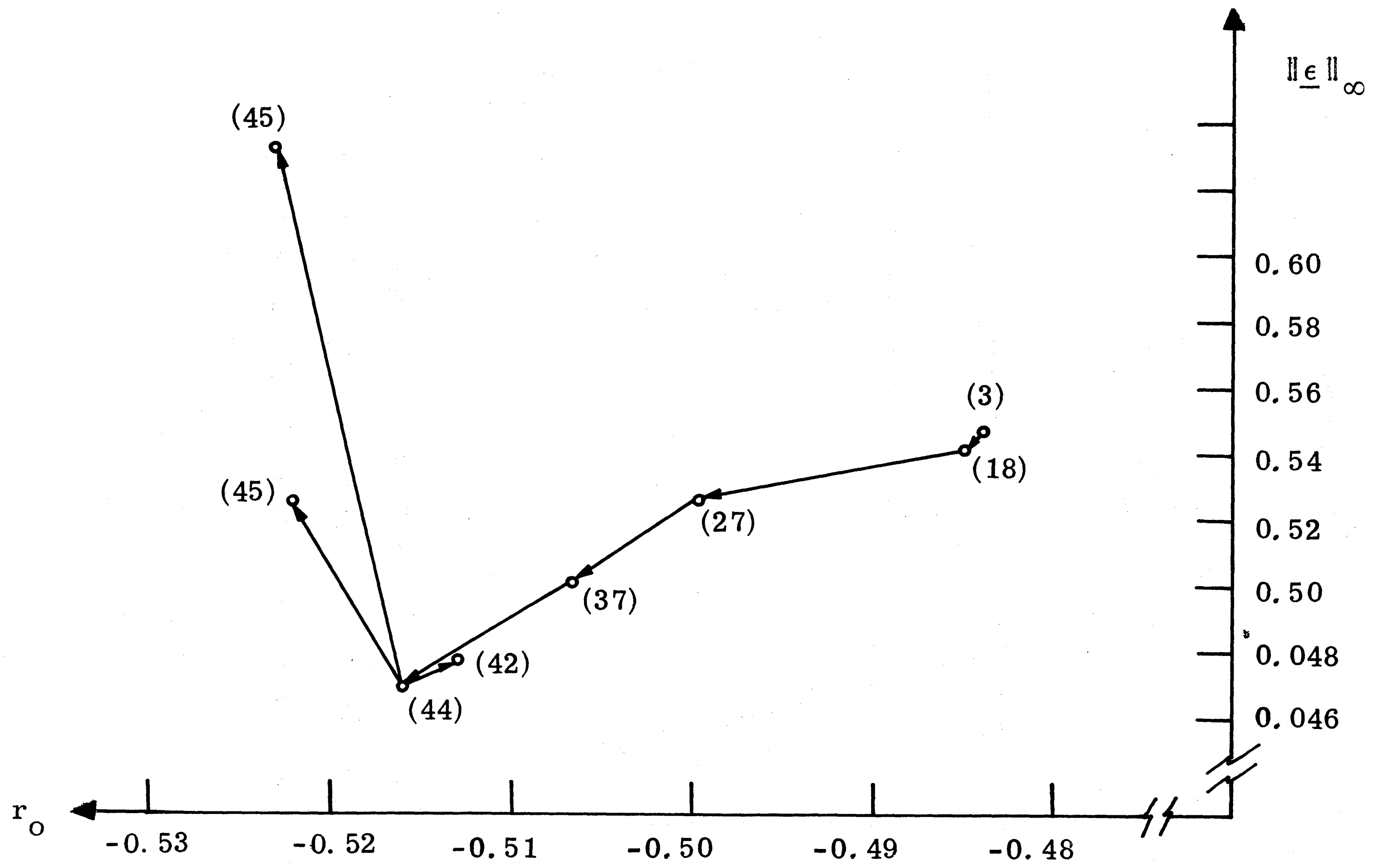


Fig. 7. $\|\underline{\epsilon}(\underline{r})\|_\infty$ vs. r_o for Example 5.1.

Example 5. 2:

Consider again the problem of approximating the vector $\underline{f}$ given in the previous example, but this time let n = 2.

(1) Step 1: Eq. 5. 3 becomes

$$\begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.2500 \\ 0.1600 \\ 0.1110 \\ 0.0817 \\ 0.0625 \\ 0.0494 \\ 0.0400 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \\ z_1^2 & z_2^2 \\ z_1^3 & z_2^3 \\ z_1^4 & z_2^4 \\ z_1^5 & z_2^5 \\ z_1^6 & z_2^6 \\ z_1^7 & z_2^7 \\ z_1^8 & z_2^8 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \end{bmatrix} \qquad (5.14)$$

Note that q = 9 and n = 2.

(2) Step 2: Eq. 5. 7 becomes

$$\underline{\delta}(\underline{r}) = \begin{bmatrix} 1.0000 & 0.4450 & 0.2500 \\ 0.4450 & 0.2500 & 0.1600 \\ 0.2500 & 0.1600 & 0.1110 \\ 0.1600 & 0.1110 & 0.0817 \\ 0.1110 & 0.0817 & 0.0625 \\ 0.0817 & 0.0625 & 0.0494 \\ 0.0625 & 0.0494 & 0.0400 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}$$

Let $\underline{r}^{(j)}$ denote the vector $\underline{r}$ having the j-th component equal to one. Minimizing $\|\underline{\delta}(\underline{r}^{(j)})\|_\infty$ with respect to $\underline{r}^{(j)}$, where j = 1, 2, 3, yields the results given in Table 5.4.

j	1	2	3
$\tilde{r}_0^{(j)}$	1	-0.20092	0.20331
$\tilde{r}_1^{(j)}$	-4.976793	1	-1.0128
$\tilde{r}_2^{(j)}$	4.911880	-0.98695	1
$\tilde{\delta}_1^{(j)}$	0.013297	-0.0026718	0.0027072
$\tilde{\delta}_2^{(j)}$	-0.013297	0.0026718	-0.0027072
$\tilde{\delta}_3^{(j)}$	-0.001068	0.0002148	-0.000217
$\tilde{\delta}_4^{(j)}$	0.008877	-0.0017835	0.0018072
$\tilde{\delta}_5^{(j)}$	0.001139	-0.0022883	0.0023186
$\tilde{\delta}_6^{(j)}$	0.013297	-0.0026718	0.0027072
$\tilde{\delta}_7^{(j)}$	0.013122	-0.0026365	0.0026714
$\dfrac{\Vert \underline{\tilde{\delta}}^{(j)} \Vert_\infty}{\Vert \underline{\tilde{r}}^{(j)} \Vert_1}$	0.00122121	0.00122118	0.00122121

Table 5.4. Results of Step 2 of Example 5.2.

From Table 5.4 it is seen that all the vectors $\left\{\tilde{\underline{r}}^{(j)}\right\}$ give the same minimal value of $\|\underline{\delta}(\underline{r})\|_\infty$ in the set $\|\underline{r}\|_1 = 1$. We select for the initial estimate the vector, $\tilde{\underline{r}}^{(3)}$, namely,

$$\underline{r}^{(1)} = \begin{bmatrix} 0.20331 \\ -1.0128 \\ 1.0000 \end{bmatrix}$$

The corresponding value of $\|\underline{\delta}(\underline{r}^{(1)})\|_\infty = 0.0027072$.

(3) Step 3: From Eq. 5.11, one gets $\underline{z}^{(1)} = \begin{bmatrix} 0.7369 \\ 0.2759 \end{bmatrix}$. Substituting this value of $\underline{z}$ into Eq. 5.14, yields the relation of Eq. 5.4. The best Chebyshev approximation yields the error vector $\underline{\epsilon}^*(\underline{r}^{(1)})$, the value of which is given in Table 5.5, Column I.

(4) Step 4: $\dfrac{\|\underline{\delta}(\underline{r}^{(1)})\|_\infty}{\|\underline{r}^{(1)}\|_1} = 0.00122, \quad \|\underline{\epsilon}^*(\underline{r}^{(1)})\|_\infty = 0.00656.$

Since $\dfrac{\|\underline{\delta}\|_\infty}{\|\underline{r}^{(1)}\|_1} < \|\underline{\epsilon}^*\|_\infty$, go to Step 5.

(5) Step 5: From Column I of Table 5.5 we obtain $\left\{\epsilon_1, \epsilon_4, \epsilon_9\right\}$.

(6) Step 6: $m = 3$, and $2n+1 = 5$. Since $m < 2n+1$, go to Step 7.

(7) Steps 7 and 8: Form the reference subspace U_1^3 containing $\left\{\epsilon_1, \epsilon_4, \epsilon_9\right\}$. Hence, $\underline{f}^{(1)}$, $\underline{\epsilon}^{(1)}(\underline{r}^{(1)})$, and $[\Lambda^{(1)}(\underline{r})]$ in U_1^3 are:

	I	II
$r_0^{(j)}$	.20331	0.20316
$r_1^{(j)}$	-1.0128	-1.0128
$r_2^{(j)}$	1.0000	1.0000
ϵ_1	0.00656	0.0065
ϵ_2	-0.00424	-0.00591
ϵ_3	-0.00504	-0.004846
ϵ_4	-0.00656	-0.0065
ϵ_5	-0.00584	-0.00586
ϵ_6	-0.00277	-0.002826
ϵ_7	0.00070	0.000632
ϵ_8	0.00399	0.003913
ϵ_9	0.00656	0.0065

Table 5.5. Values of $\epsilon_i(\underline{r})$ for Steps 3 and 10 of Example 5.2.

$$\underline{f}^{(1)} = \begin{bmatrix} 1.00 \\ 0.16 \\ 0.04 \end{bmatrix}, \quad \underline{\epsilon}^{(1)}(\underline{r}^{(1)}) = \begin{bmatrix} 0.00656 \\ -0.00656 \\ 0.00656 \end{bmatrix}, \text{ and}$$

$$[\Lambda^{(1)}(\underline{r})] = \begin{bmatrix} (r_0^3 r_1^4 + r_0^5 r_2^2 - 3 r_0^4 r_1^2 r_2) \\ (r_1^7 - 6 r_0 r_1^5 r_2 + 10 r_0^2 r_1^3 r_2^2 - 4 r_0^3 r_1 r_2^3) \\ (r_1^2 r_2^5 - r_0 r_2^6) \end{bmatrix}$$

(8) Step 9: Let $\underline{r}^{(2)} = \begin{bmatrix} r_0 \\ -1.0128 \\ 1.0000 \end{bmatrix}$

(9) Step 10: Let $\underline{\epsilon}^{(1)}(\underline{r}^{(2)}) = \begin{bmatrix} 0.0065 \\ -0.0065 \\ 0.0065 \end{bmatrix}$

Solve Eq. 5.10 for r_0, when k = 1, m = 1 and using the values of $\underline{f}^{(1)}$ and $[\Lambda^{(1)}(\underline{r})]\Big|_{\underline{r}=\underline{r}^{(2)}}$ found in (7). This yields

$$\underline{r}^{(2)} = \begin{bmatrix} 0.20316 \\ -1.0128 \\ 1.0000 \end{bmatrix}$$

(10) Steps 12 and 13: Solve Eq. 5.5 for $\underline{\epsilon}(\underline{r}^{(2)}) \in U^9$, using $\underline{r} = \underline{r}^{(2)}$ and the vector $\underline{\epsilon}^{(1)}(\underline{r}^{(2)})$. The values of the components of the resulting vector $\underline{\epsilon}(\underline{r}^{(2)})$ are given in Table 5.5, Column II. Note that $\|\underline{\epsilon}(\underline{r}^{(2)})\|_\infty = \|\underline{\epsilon}^{(1)}(\underline{r}^{(2)})\|_\infty$.

(11) Step 14: Form the set $\{\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5, \epsilon_9\}$. The component ϵ_2 is chosen because $\Delta|\epsilon_2| = \max_{1 \leq i \leq 9} |\Delta\epsilon_i|$, where $\Delta\epsilon_i$ is the difference between the values of i-th components given in Columns I and II of

Table 5.5. The component ϵ_5 is chosen[11] because $|\epsilon_5(\underline{r}^{(2)})| > |\epsilon_5(\underline{r}^{(1)})|$.

(12) Step 16: Since m = 5 = 2n+1 = 5, go to Step 19.

(13) Steps 19 and 20: Form the reference subspace U_2^5 containing the set of components $\{\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5, \epsilon_9\}$. Hence, $\underline{f}^{(2)}$, $\underline{\sigma}^{(2)}$ and $[\Lambda^{(2)}(\underline{r})]$ in U_2^5 are:

$$\underline{f}^{(2)} = \begin{bmatrix} 1.000 \\ 0.445 \\ 0.160 \\ 0.111 \\ 0.040 \end{bmatrix} \qquad \underline{\sigma}^{(2)} = \begin{bmatrix} +1 \\ -1 \\ -1 \\ -1 \\ +1 \end{bmatrix}$$

$$[\Lambda^{(2)}(\underline{r})] = \begin{bmatrix} r_0 r_1 & 0 & 0 \\ (r_1^2 - r_0 r_2) & r_0^2 & 0 \\ -r_2^2 & (r_0 r_2 - r_1^2) & (2r_0^2 r_1 r_2 - r_0 r_1^3) \\ 0 & -r_1 r_2 & (3r_0 r_1^2 r_2 - r_1^4 - r_0^2 r_2^2) \\ 0 & 0 & r_2^4 \end{bmatrix}$$

(14) Let $\underline{r}^{(3)} = \begin{bmatrix} r_0 \\ r_1 \\ 1 \end{bmatrix}$ and $\underline{\epsilon}^{(2)}(\underline{r}^{(3)}) = \rho\, \underline{\sigma}^{(2)}$. Then, according to Step 21, solve for $\{r_0, r_1, \rho\}$ from the relation,

---

[11] Note that we have skipped an iteration step by selecting ϵ_5 here. If this results in an $\underline{\epsilon}(\underline{r}) \in U^q$ with $\|\underline{\epsilon}(\underline{r})\|_\infty > \|\underline{\epsilon}^{(2)}(\underline{r})\|_\infty$, where $\underline{\epsilon}^{(2)} \in U_2^5$, then we can always return and add the missing step.

$$\begin{bmatrix} r_0 r_1 & (r_1^2 - r_0) & -1 & 0 & 0 \\ 0 & r_0^2 & (r_0 - r_1^2) & -r_1 & 0 \\ 0 & 0 & (2r_0^2 r_1 - r_0 r_1^3) & (3r_0 r_1^2 - r_1^4 - r_0^2) & 1 \end{bmatrix} \begin{bmatrix} 1.000 - \rho \\ 0.445 + \rho \\ 0.160 + \rho \\ 0.111 + \rho \\ 0.040 - \rho \end{bmatrix} = 0$$

This yields that $r_0 = 0.20921$, $r_1 = -1.02457$, and $\rho = 0.005628$.

(15) Step 22: Using $\underline{r}^{(3)}$ and ρ found in (14), Eq. 5.5 yields the vector $\underline{\epsilon}(\underline{r}^{(3)})$ tabulated in Column I of Table 5.6.

(16) Test $\underline{r}^{(3)}$ to see if it is the optimum vector $\underline{r}^*$, according to Steps 23 through 27. The results are given in Columns II, III, IV, V, and VI of Table 5.6. From these results it is seen that $\underline{r}^{(3)}$ is not the optimum (see Column IV of Table 5.6).

(17) Steps 5 and 8: From Column IV of Table 5.6, form a new sub-space $U_3^{\ 4}$ which contains the set of components $\{\epsilon_1, \epsilon_2, \epsilon_5, \epsilon_9\}$ of $\underline{\epsilon}$. Hence, $\underline{f}^{(3)}$, $\underline{\sigma}^{(3)}$ and $[\Lambda^{(3)}(\underline{r})]$ in $U_3^{\ 4}$ are:

$$\underline{f}^{(3)} = \begin{bmatrix} 1.000 \\ 0.445 \\ 0.111 \\ 0.040 \end{bmatrix} ; \qquad \underline{\sigma}^{(3)} = \begin{bmatrix} +1 \\ -1 \\ -1 \\ +1 \end{bmatrix}$$

$$[\Lambda^{(3)}(\underline{r})] = \begin{bmatrix} (r_0 r_1^2 - r_0^2 r_2) & 0 \\ (r_1^3 - 2r_0 r_1 r_2) & (r_0^3 r_1^3 - 2r_0^4 r_1 r_2) \\ r_2^3 & (r_1^6 - 5r_0 r_1^4 r_2 + 6r_0^2 r_1^2 r_2^2 - r_0^3 r_2^3) \\ 0 & (r_0 r_2^5 - r_1^2 r_2^4) \end{bmatrix}$$

	I	II	III	IV	V	VI	VII
r_0	.20921	0.212997	0.212004	0.21036	0.209907	0.209907	0.218972
r_1	-1.02457	-1.03046	-1.02907	-1.02676	-1.02602	-1.02396	-1.043247
r_2	1.0	1.0	1.0	1.0	1.0	1.0	1.0
ϵ_1	0.005628	0.011717	0.0055	0.0055	0.0055	0.0055	0.0045
ϵ_2	-0.005628	-0.0055	-0.006547	-0.0055	-0.0055	-0.0055	-0.0045
ϵ_3	-0.003670	--	--	-0.003353	--	--	0.000957
ϵ_4	-0.005628	-0.0055	-0.0055	-0.005366	-0.0055	-0.0055	-0.003385
ϵ_5	-0.005628	-0.0055	-0.0055	-0.0055	-0.005601	-0.0055	-0.0045
ϵ_6	-0.003143	--	--	-0.00313	--	--	-0.003019
ϵ_7	-0.000028	--	--	0.000094	--	--	-0.000593
ϵ_8	0.003085	--	--	0.002975	--	--	0.002129
ϵ_9	0.005628	0.0055	0.0055	0.0055	0.0055	0.005723	0.0045

Table 5. 6. **Results of Steps (14) through (16) of Example 5. 2.**

(18) Steps 9, 10, and 12: Let $\underline{r}^{(4)} = \begin{bmatrix} r_0 \\ r_1 \\ 1 \end{bmatrix}$, and let $\underline{\epsilon}^{(3)}(\underline{r}^{(4)}) =$ $0.0045\ \underline{\sigma}^{(3)}$. Then, solving Eq. 5.10, when k = 3, yields the results given in Column VII of Table 5.6. The values of the components of $\underline{\epsilon}(\underline{r}^{(4)}) \in U^9$ have been obtained from Eq. 5.5.

(19) Since $\|\underline{\epsilon}(\underline{r}^{(4)})\|_{\infty} = \|\underline{\epsilon}^{(3)}(\underline{r}^{(4)})\|_{\infty}$, then according to Step 13, calculate $|\Delta\epsilon_i|$ using the values of the ϵ_i's given in Columns VII and IV of Table 5.6. From the set of $|\Delta\epsilon_i|$, it is seen that $|\Delta\epsilon_3| = \max\limits_{1 \le i \le 9} |\Delta\epsilon_i|$. Hence, we form a new reference subspace U_4^5 which will contain $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_5, \epsilon_9\}$.

(20) Step 20: $\underline{f}^{(4)}$, $\underline{\sigma}^{(4)}$ and $[\Lambda^{(4)}(\underline{r})]$ in U_4^5 are as follows:

$$\underline{f}^{(4)} = \begin{bmatrix} 1.000 \\ 0.445 \\ 0.250 \\ 0.111 \\ 0.040 \end{bmatrix}; \quad \underline{\sigma}^{(4)} = \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ +1 \end{bmatrix}; \text{ and}$$

$$[\Lambda^{(4)}(\underline{r})] = \begin{bmatrix} r_0 & 0 & 0 \\ r_1 & r_0 r_1 & 0 \\ r_2 & (r_1^2 - r_0 r_2) & (r_0^2 r_1^2 - 2r_0^3 r_2) \\ 0 & -r_2^2 & (-r_1^4 + 4r_0 r_1^2 r_2 - 3r_0^2 r_2^2) \\ 0 & 0 & r_2^4 \end{bmatrix}$$

(21) Steps 21 and 22: Let $\underline{r}^{(5)} = \begin{bmatrix} r_0 \\ r_1 \\ 1 \end{bmatrix}$ and $\underline{\epsilon}^{(4)}(\underline{r}^{(5)}) = \rho\, \underline{\sigma}^{(4)}$.

Then, solving Eq. 5.10, when k = 4 and $\underline{r} = \underline{r}^{(5)}$, yields the results tabulated in Column I of Table 5.7.

(22) Test this vector $\underline{r}^{(5)}$ to see if it is the optimum vector $\underline{r}^*$, according to Step 23. The results are given in Columns II, III, IV, V, and VI of Table 5.7. From these results it is seen that $\underline{r}^{(5)} = \underline{r}^*$. Therefore,

$$\underline{r}^* = \begin{bmatrix} 0.23212 \\ -1.06873 \\ 1.0 \end{bmatrix}$$

(23) Step 31: Using the value of $\underline{r}^*$ in (22), determine the vector $\underline{z}^*$ from Eq. 5.11. This yields

$$\underline{z}^* = \begin{bmatrix} 0.76552 \\ 0.30317 \end{bmatrix}$$

(24) Substitute this value of $\underline{z}^*$ and the value of $\underline{\epsilon}^{**}$ given in Column I of Table 5.7 into Eq. 5.14. Then, determine $\underline{\beta}^{**}$ from the first two components of this vector relation, namely

$$\begin{bmatrix} 1.000 \\ 0.445 \end{bmatrix} = \begin{bmatrix} 1.00000 & 1.00000 \\ 0.76552 & 0.30317 \end{bmatrix} \begin{bmatrix} \beta_1^{**} \\ \beta_2^{**} \end{bmatrix} + \begin{bmatrix} 0.00284 \\ -0.00284 \end{bmatrix}$$

	I	II	III	IV	V	VI
r_0	0.232118	0.233236	0.232803	0.232408	0.2319098	0.231352
r_1	-1.068733	-1.070449	-1.069855	-1.069302	-1.068469	-1.067228
r_2	1.00	1.00	1.00	1.00	1.00	1.00
ϵ_1	0.0028387	0.0046685	0.0028	0.0028	0.0028	0.0028
ϵ_2	-0.0028387	-0.0028	-0.0030519	-0.0028	-0.0028	-0.0028
ϵ_3	0.0028387	0.0028	0.0028	0.0029247	0.0028	0.0028
ϵ_4	-0.0001983	--	--	--	--	--
ϵ_5	-0.0028387	-0.0028	-0.0028	-0.0028	-0.0029222	-0.0028
ϵ_6	-0.0027783	--	--	--	--	--
ϵ_7	-0.0013608	--	--	--	--	--
ϵ_8	0.0007587	--	--	--	--	--
ϵ_9	0.0028387	0.0028	0.0028	0.0028	0.0028	0.0029934

Table 5.7. Results of Steps 21 and 22 of Example 5.2.

Hence,

$$\underline{\beta}^{**} = \begin{bmatrix} 0.31476 \\ 0.68240 \end{bmatrix}$$

Therefore, the optimum vector pair is,

$$(\underline{\beta}^{**}, \underline{z}^{*}) = \left(\begin{bmatrix} 0.31476 \\ 0.68240 \end{bmatrix}, \begin{bmatrix} 0.76552 \\ 0.30317 \end{bmatrix} \right)$$

and the best Chebyshev approximating vector to the prescribed vectors $\underline{f}$ is given by

$$\underline{f}^{**} \triangleq [Z(\underline{z}^{*})]\, \underline{\beta}^{**} = \begin{bmatrix} 0.9972 \\ 0.4478 \\ 0.2472 \\ 0.1602 \\ 0.1138 \\ 0.0845 \\ 0.0639 \\ 0.0486 \\ 0.0372 \end{bmatrix}$$

The resulting Chebyshev error vector is given by

$$\underline{\epsilon}^{**} = \begin{bmatrix} 0.0028387 \\ -0.0028387 \\ 0.0028387 \\ -0.0001983 \\ -0.0028387 \\ -0.0027783 \\ -0.0013608 \\ 0.0007587 \\ 0.0028387 \end{bmatrix}$$

and $\|\underline{\epsilon}^{**}\|_{\infty} = 0.0028387$.

Figure 8 compares the components of the final error vector, $\underline{\epsilon}^{**}$, and those of the initial error vector, $\underline{\epsilon}^{*}(\underline{r}^{(1)})$. The improvement in the approximation is evident and it will be discussed in Section 5.5.

Example 5.3

Consider the problem of approximating the real vector

$$\underline{f} = \begin{bmatrix} 0 \\ 2.8 \\ 2.0 \\ 1.0 \\ 1.0 \end{bmatrix} \text{ by the real vector } \hat{\underline{f}}(\underline{\beta}, \underline{z}) = \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \\ z_1^2 & z_2^2 \\ z_1^3 & z_2^3 \\ z_1^4 & z_2^4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

so that $\|\underline{f} - \hat{\underline{f}}(\underline{\beta}^{**}, \underline{z}^{*})\|_{\infty}$ is minimum for all $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$.

(1) Step 1: The following vector relation in U^5 must be considered

$$\begin{bmatrix} 0 \\ 2.8 \\ 2 \\ 1.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \\ z_1^2 & z_2^2 \\ z_1^3 & z_2^3 \\ z_1^4 & z_2^4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix} \qquad (5.15)$$

Note that $q = 5$ and $n = 2$.

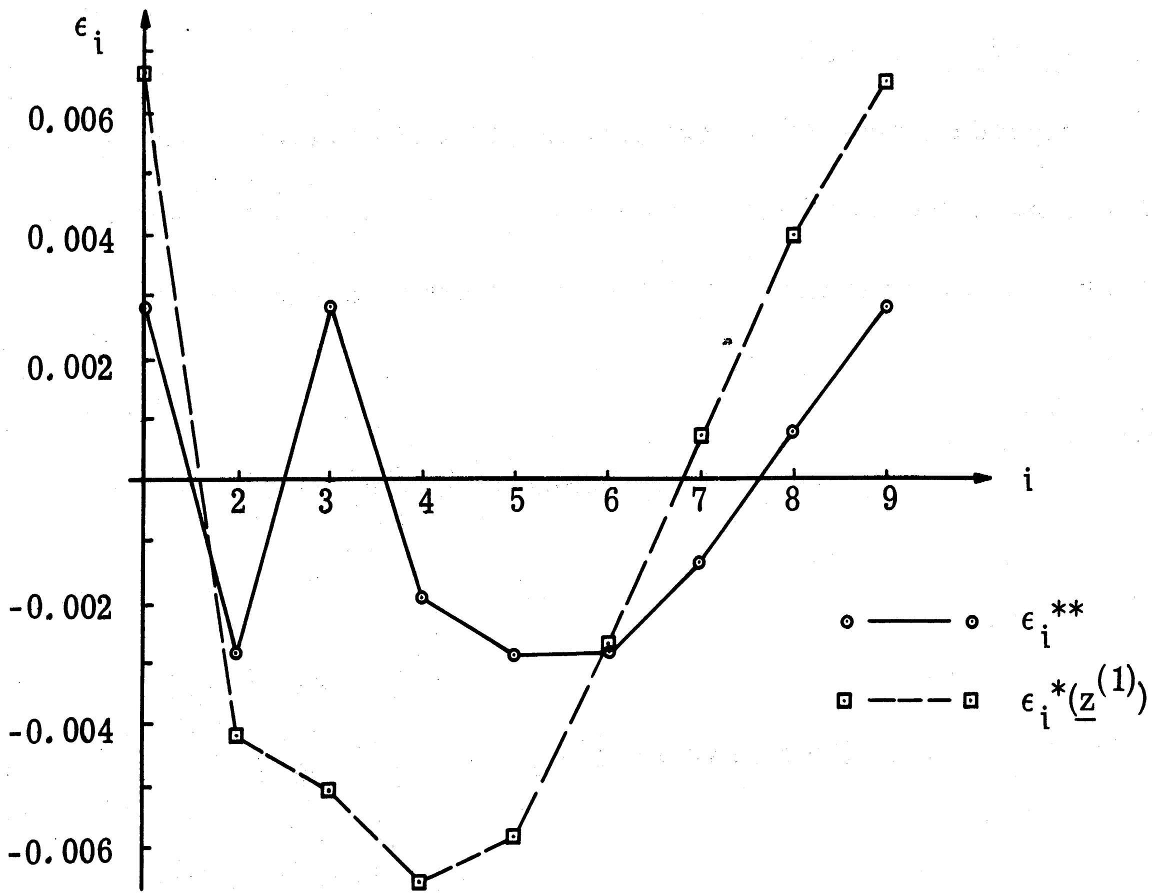


Fig. 8. Chebyshev approximation error,
optimum approximation yields $\|\underline{\epsilon}^{**}\|_\infty = 0.00284$
Ruston's approximation yields $\|\underline{\epsilon}^*(\underline{z}^{(1)})\|_\infty = 0.00656$

(2) The minimal value of $\|\underline{\delta}(\underline{r})\|_\infty$, in the set $\|\underline{r}\|_1 = 1$, was obtained in Example 4.1 of Chapter IV. The vector $\underline{r}^{(1)}$ is taken to be the vector $\tilde{\underline{r}}^{(3)}$ of Table 4.1, namely

$$\underline{r}^{(1)} = \begin{bmatrix} 0.04098 \\ -0.60109 \\ 1.0 \end{bmatrix}$$

and the corresponding $\|\underline{\delta}\|_\infty = 0.3169$.

(3) Step 3:

$$\underline{z}^{(1)} = \begin{bmatrix} 0.662916 \\ -0.061823 \end{bmatrix}$$

The best Chebyshev error vector $\underline{\epsilon}^*(\underline{r}^{(1)}) \in U^5$ is given in Column I of Table 5.8.

(4) Step 4: Since

$$\frac{\|\underline{\delta}\|_\infty}{\|\underline{r}^{(1)}\|_1} = 0.193 < \|\underline{\epsilon}^*(\underline{r}^{(1)})\|_\infty = 0.20315, \text{ go to Step 5.}$$

(5) Steps 5 through 8: Form the reference subspace U_1^3 containing the set of components $\{\epsilon_1, \epsilon_4, \epsilon_5\}$. Hence $\underline{f}^{(1)}$, $\underline{\epsilon}^{(1)}(\underline{r}^{(1)})$ and $[\Lambda^{(1)}(\underline{r})]$ in U_1^3 are:

	I	II
r_0	-0.04098	-0.04425
r_1	-0.60109	-0.60109
r_2	1.0	1.0
ϵ_1	-0.203152	-0.2
ϵ_2	-0.17827	-0.163
ϵ_3	0.201457	0.2217
ϵ_4	-0.203152	-0.2
ϵ_5	0.203152	0.2

Table 5.8. Results of Steps 3 and 6 of Example 5.3.

$$\underline{f}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \underline{\epsilon}^{(1)}(\underline{r}^{(1)}) = \begin{bmatrix} -0.203152 \\ -0.203152 \\ 0.203152 \end{bmatrix}$$

$$\Lambda^{(1)}(\underline{r}) = \begin{bmatrix} r_0^3 \\ (r_1^3 - 2r_0 r_1 r_2) \\ (r_1^2 r_2 - r_0 r_2^2) \end{bmatrix}$$

(6) Steps 9 and 10: Let $\underline{r}^{(2)} = \begin{bmatrix} r_0 \\ -0.60109 \\ 1 \end{bmatrix}$ and let $\underline{\epsilon}^{(1)}(\underline{r}^{(2)}) = \begin{bmatrix} -0.2 \\ -0.2 \\ 0.2 \end{bmatrix}$.

Substituting into Eq. 5.10, when k = 1 and $\underline{r} = \underline{r}^{(3)}$ yields

$$r_0^3 + 3.213116\, r_0 + 0.142156 = 0$$

Therefore,

$$r_0 = -0.04425$$

The value of $\underline{r}^{(2)}$ and the corresponding value of $\underline{\epsilon}(\underline{r}^{(2)}) \epsilon\ U^5$ is given in Column II of Table 5.8.

(7) Since $\|\underline{\epsilon}(\underline{r}^{(2)})\|_\infty > \|\underline{\epsilon}^{(1)}(\underline{r}^{(2)})\|_\infty$ go to Step 15 and from the set $\{\epsilon_1, \epsilon_3, \epsilon_4, \epsilon_5\}$.

(8) Since $m = 4 < 2n+1 = 5$ go to Step 18 and form the reference subspace U_2^4 containing the components $\{\epsilon_1, \epsilon_3, \epsilon_4, \epsilon_5\}$.

(9) $\underline{f}^{(2)}$, $\underline{\epsilon}^{(2)}(\underline{r}^{(2)})$ and $[\Lambda^{(2)}(\underline{r})]$ in U_2^4 are:

$$\underline{f}^{(2)} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \underline{\epsilon}^{(2)}(\underline{r}^{(2)}) = \begin{bmatrix} -0.2 \\ 0.2217 \\ -0.2 \\ 0.2 \end{bmatrix}$$

$$[\Lambda^{(2)}(\underline{r})] = \begin{bmatrix} r_0^2 & 0 \\ (r_0 r_2 - r_1^2) & r_0 \\ -r_1 r_2 & r_1 \\ 0 & r_2 \end{bmatrix}$$

(10) Let $\underline{r}^{(3)} = \begin{bmatrix} r_0 \\ r_1 \\ 1 \end{bmatrix}$ and let $\underline{\epsilon}^{(2)}(\underline{r}^{(3)}) = \begin{bmatrix} -0.19 \\ 0.19 \\ -0.19 \\ 0.19 \end{bmatrix}$

Substituting into Eq. 5.10, when k = 2 and $\underline{r} = \underline{r}^{(3)}$, yields

$$0.19\, r_0^2 + 1.81\, r_0 - 1.81\, r_1^2 - 1.19\, r_1 = 0$$

$$1.81\, r_0 + 1.19\, r_1 + 0.81 = 0$$

Solving for r_0 and r_1 yields that both r_0 and r_1 are complex. Therefore the above equations have no real solution. Go to Step 28.

(11) Steps 28 and 29: The vector $\underline{f}^{(2)} \epsilon U_2^{\,4}$ and the matrix $[\Lambda^{(2)}(\underline{r})]$ are given in (9). The vector $\underline{\sigma}^{(2)} \epsilon U_2^{\,4}$ is

$$\underline{\sigma}^{(2)} = \begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \end{bmatrix}$$

Hence Eq. 5.10, when k = 2, $\underline{r} = \underline{r}^{(3)}$ and $\underline{\epsilon}^{(2)}(\underline{r}) = \rho\, \underline{\sigma}^{(2)}$ becomes

$$2(r_0 - r_1^2) - r_1 = \rho(-r_0^2 + r_0 - r_1^2 + r_1)$$

$$2r_0 + r_1 + 1 = \rho(r_0 - r_1 + 1)$$

Solving for r_0 and r_1, which yields the minimal value of $|\rho|$, one obtains[12] $r_0 = 0$, $r_1 = -2/3$, and $\rho = 0.2$. The vector $\underline{r}^{(3)}$ and

---

[12] See Example 4.3 of Chapter IV.

the corresponding value of $\underline{\epsilon}(\underline{r}^{(3)}) \epsilon U^5$ is given in Column I of Table 5.9.

(12) Step 30: Since $\|\underline{\epsilon}(\underline{r}^{(3)})\|_\infty = |\rho|$, then $\underline{r}^{(3)} = \underline{r}^*$, namely

$$\underline{r}^* = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

(13) Let us now deviate from the procedure and determine the solution vector $\underline{r}$ when the (2n+1) components of $\underline{\epsilon} \epsilon U^5$ are equal in absolute value. Hence, let us solve

$$\begin{bmatrix} r_0 & r_1 & 1 & 0 & 0 \\ 0 & r_0 & r_1 & 1 & 0 \\ 0 & 0 & r_0 & r_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2.8 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r_0 & r_1 & 1 & 0 & 0 \\ 0 & r_0 & r_1 & 1 & 0 \\ 0 & 0 & r_0 & r_1 & 1 \end{bmatrix} \rho \underline{\sigma}$$

for r_0, r_1, and ρ, when

$$\underline{\sigma} = \begin{bmatrix} -1 \\ +1 \\ +1 \\ -1 \\ -1 \end{bmatrix}, \quad \text{or} \quad \underline{\sigma} = \begin{bmatrix} -1 \\ -1 \\ +1 \\ -1 \\ +1 \end{bmatrix}$$

The solutions which yield the minimal value of $\|\underline{\epsilon}\|_\infty$ are given in Columns II and III of Table 5.9. It is noted that the resulting value

	I	II	III	IV	V	VI	VII
r_0	0	0.018717	-0.0447	-0.4475	1	0	0.888
r_1	-2	-0.6937	-0.5951	0	0	-0.669	-2.433
r_2	3	1	1	1	-0.7846	1	1
ϵ_1	-0.2	-0.20043	-0.20346	-4.045	-0.19	-0.19	-0.19
ϵ_2	0.1	0.20043	-0.20346	0.141	1.364	1.589	-0.819
ϵ_3	0.2	0.20043	0.20346	0.19	1.866	0.19	0.19
ϵ_4	-0.2	-0.20043	-0.20346	-0.19	-0.19	-0.211	-0.19
ϵ_5	0.2	0.20043	0.20346	0.19	0.19	0.19	-0.288

Table 5.9. Results of Steps 11 and 13 of Example 5.3.

of $\|\underline{\epsilon}\|_\infty$ is greater than the value of $\|\underline{\epsilon}\|_\infty$ given in Column I of Table 5.9, the case when 2n components of $\underline{\epsilon}$ are equal in absolute value.

As a further digression, let us test if there exists an $\underline{r}$ which yields a lower value of $\|\underline{\epsilon}(\underline{r})\|_\infty$ than $\|\underline{\epsilon}(\underline{r}^{(3)*})\|_\infty$, when (2n-1) components of $\underline{\epsilon}(\underline{r}) \in U^q$ are equal in absolute value. Following the procedure given in Step 23 of the algorithm, we decrease the values of each subset of 3-components out of the set $\{\epsilon_1, \epsilon_3, \epsilon_4, \epsilon_5\}$ and solve the resulting relations for $\underline{r}$. The results are given in Columns IV, V, VI, and VII of Table 5.9.

It should be noted that when performing this test, one must solve an under determined equation. For example, when $\epsilon_3 = -\epsilon_4 = \epsilon_5 = 0.19$, i.e., the case which yields the values given in Column IV of Table 5.9, one obtains the relation

$$1.81\, r_0 + 1.19\, r_1 + 0.81 = 0 \tag{5.16}$$

where r_0 and r_1 are unknown. Although this equation can be solved, by arbitrarily selecting one of the unknowns, one is interested in the solution which minimizes the value of $\|\underline{\epsilon}(\underline{r})\|_\infty$. We have assumed (without proof) that such a solution occurs when the value of $|\epsilon_1|$ is minimum. Hence, we have solved Eq. 5.16 subject to the constraint $\epsilon_1^2 =$ minimum. Note that the component ϵ_1 is related to the prescribed components $\{\epsilon_3, \epsilon_4, \epsilon_5\}$ by

$$\epsilon_1 = \frac{(f_3 - \epsilon_3)(r_0 - r_1^2) + (f_4 - \epsilon_4) r_1}{r_0^2}$$

$$\frac{1.81 (r_0 - r_1^2) + 1.19 r_1}{r_0^2}$$

By using the relation between r_0 and r_1, given by Eq. 5.16, we can obtain the component ϵ_1 as a function of r_0 (or r_1), and the r_0 for which ϵ_1^2 = minimum, is obtained from

$$\frac{d\,\epsilon_1^2(r_0)}{d\,r_0} = 0$$

The resulting values of r_0, r_1, and ϵ_1 are given in Column IV of Table 5.9. From Table 5.9 we note that the vector $\underline{r}^*$ obtained in (12) yields the minimal value of $\|\underline{\epsilon}(\underline{r})\|_\infty$.

(14) Step 31 yields $\underline{z}^* = \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}$

(15) Step 32: Substituting the value of $\underline{z}^*$ in Eq. 5.15 yields

$$\begin{bmatrix} 0 \\ 2.8 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0.6667 \\ 0 & 0.4444 \\ 0 & 0.2963 \\ 0 & 0.1975 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} -0.2 \\ 0.1 \\ 0.2 \\ -0.2 \\ 0.2 \end{bmatrix}$$

Solving for β_1, and β_2 yields

$$\underline{\beta}^{**} = \begin{bmatrix} -3.85 \\ 4.05 \end{bmatrix} \tag{5.17}$$

It should be noted that the component β_1^{**} may take on any value in the interval [-4.25, -3.85] without affecting the value of $\|\underline{\epsilon}^{**}\|_\infty = 0.2$. Clearly then, the best Chebyshev parameter vector pair $(\underline{\beta}^{**}, \underline{z}^*) \in \mathcal{B}_z \times \mathcal{Z}$ is not unique.

(16) The optimum vector pair is

$$(\underline{\beta}^{**}, \underline{z}^*) = \left(\begin{bmatrix} -3.85 \\ 4.05 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.6667 \end{bmatrix} \right)$$

The optimum Chebyshev approximating vector and error vector are:

$$\underline{f}^{**} = \begin{bmatrix} 0.2 \\ 2.7 \\ 1.8 \\ 1.2 \\ 0.8 \end{bmatrix}, \quad \text{and} \quad \underline{\epsilon}^{**} = \begin{bmatrix} -0.2 \\ 0.1 \\ 0.2 \\ -0.2 \\ 0.2 \end{bmatrix}$$

<u>Example 5.4</u>

Consider the problem of approximating the function $f(t) = (4 - t - 0.1 \cos \pi t)$ by the set of $n = 2$ exponential functions in the finite point set

$$T_e = \{0, 1, 2, 3, 4\}$$

(1) Step 1: Given q = 5, n = 2, and

$$\underline{f} = \begin{bmatrix} 3.9 \\ 3.1 \\ 1.9 \\ 1.1 \\ -0.1 \end{bmatrix}$$

(2) Step 2: The minimal value of $\|\underline{\delta}(\underline{r})\|_{\infty}$, in the set $\|\underline{r}\|_1 = 1$, was obtained in Example 4.1 of Chapter IV. The minimal value of $\|\underline{\delta}(\underline{r})\|_{\infty} =$ 0.4, which is obtained when $\underline{r}$ is given by

$$\underline{r}^{(1)} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(3) Step 3: The corresponding vector $\underline{z}^{(1)}$, obtained from Eq. 5.11, is

$$\underline{z}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since the two components of $\underline{z}^{(1)}$ are equal, the matrix $[Z(\underline{z})]$ is given by Eq. 1.18b (Chapter I). This problem was considered in Example 4.1 of Chapter IV. In this example let us assume that the matrix $[Z(\underline{z})]$ is given by Eq. 1.18 so that we shall consider the relation

$$\begin{bmatrix} 3.9 \\ 3.1 \\ 1.9 \\ 1.1 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

The best Chebyshev parameter vector $\underline{\beta}^*$ is found to be

$$\underline{\beta}^* = \begin{bmatrix} 1.9 \\ 0 \end{bmatrix}$$

and the corresponding error vector $\underline{\epsilon}^*(\underline{r}^{(1)})$ is given in Column I of Table 5.10.

	I	II	III	IV	V	VI
r_0	1	0.933	1	1.02	1.016	0.8081
r_1	-2	-2	-2	-2	-1.986	-1.6903
r_2	1	1	1	1	1	1
ϵ_1	2.0	0.9	0.4	0.2	0.15	0.1
ϵ_2	1.2	0.9	0.4	0.2	0.15	0.1
ϵ_3	0	0.3	0	-0.126	-0.15	-0.1
ϵ_4	-0.8	-0.047	0	0.006	0.0251	0.144
ϵ_5	-2.0	-0.9	-0.4	-0.2	-0.15	-0.1

Table 5.10. Results of Steps 4 and 5 of Example 5.4.

(4) Step 4: Since $\frac{\|\underline{\delta}\|_\infty}{\|\underline{r}^{(1)}\|_\infty} = 0.1 < \|\underline{\epsilon}^*(\underline{r}^{(1)})\|_\infty = 2.0$, go to Step 5.

(5) The results of the different steps of the procedure are given in Columns II through VI of Table 5.10.

(6) Steps 21 and 22: Taking $\underline{r} = \begin{bmatrix} r_0 \\ r_1 \\ 1 \end{bmatrix}$ and $\underline{\epsilon}(\underline{r}) = \rho\,\underline{\sigma}$,

where

$$\underline{\sigma} = \begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \\ -1 \end{bmatrix}$$

then the solution of Eq. 5.10 (namely, Eq. 5.5 since $q = 2n+1$) yields the results given in Column I of Table 5.11.

(7) Testing if this vector $\underline{r}$ is the optimum $\underline{r}^*$ according to Step 23 yields the results given in Columns II through VI of Table 5.11. From these results it is seen that

$$\underline{r}^* = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \underline{\epsilon}^*(\underline{r}^*) = \begin{bmatrix} -0.1 \\ +0.1 \\ -0.1 \\ +0.1 \\ -0.1 \end{bmatrix}$$

	I	II	III	IV	V	VI
r_0	1	1	1	1	0.9269	1
r_1	-2	-1.9615	-1.9612	-2.016	-1.8898	-2.130
r_2	1	0.8845	0.9227	1.015	1	1.217
ϵ_1	-0.1	-0.2438	-0.09	-0.09	-0.09	-0.09
ϵ_2	0.1	0.09	0.129	0.09	0.09	0.09
ϵ_3	-0.1	-0.09	-0.09	-0.15	-0.09	-0.09
ϵ_4	0.1	0.09	0.09	0.09	0.129	0.09
ϵ_5	-0.1	-0.09	-0.09	-0.09	-0.09	-0.233

Table 5.11. Results of Steps 6 and 7 of Example 5.4.

It is noted that this result is identical to that given in Example 4.1 of Chapter IV. Therefore, the matrix $[Z(\underline{z}^*)]$ is given by

$$[Z(\underline{z}^*)] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

and the vector $\underline{\beta}^{**}$, found from Eq. 5.3, is

$$\underline{\beta}^{**} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

It can be shown[13] that the best Chebyshev vector pair

$$(\underline{\beta}^{**}, \underline{z}^{*}) = \left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

yields the vector pair

$$(\underline{\alpha}^{**}, \underline{s}^{*}) = \left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

Thus, the approximating function

$$f^{**}(t) = (\alpha_1^{**} + \alpha_2^{**}\, t)\, e^{s_1^{*} t}$$

becomes

$$f^{**}(t) = 4 - t$$

It should be noted that since the error function

$$\epsilon(t) \triangleq f(t) - f^{**}(t) = -0.1 \cos \pi t$$

then the approximating function $f^{**}(t)$ represents the best Chebyshev approximating function for a larger point set, i.e., for the point set

---

[13] See Section 6.2 of Chapter VI.

$$T_e = \{ t = (i-1) : i = 1, 2, \ldots, q;\ q > 2n \}.$$

Example 5.5

Consider the problem of approximating the function $f(t) = t\, e^{-t}$, in the finite point set $T_e = \{ 0, 1, 2, 3, 4 \}$, by the set of n-exponential function where (a) n = 1, and (b) n = 2. Hence, we have q = 5, and

$$\underline{f} = \begin{bmatrix} 0.0 \\ 0.36788 \\ 0.27067 \\ 0.14936 \\ 0.07326 \end{bmatrix}$$

(a) When n = 1, the vector $\underline{r}$ is defined by

$$\underline{r} = \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$$

Equation 5.7 becomes

$$\underline{\delta}(\underline{r}) = \begin{bmatrix} 0.0 & 0.36788 \\ 0.36788 & 0.27067 \\ 0.27067 & 0.14936 \\ 0.14936 & 0.07326 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$$

The vector $\underline{r}^{(1)}$ which yields the minimum value of $\| \underline{\delta}(\underline{r}) \|_{\infty}$, in the set $\| \underline{r} \|_1 = 1$, is found to be

$$\underline{r}^{(1)} = \begin{bmatrix} -1.7358 \\ 1 \end{bmatrix}$$

and the corresponding value of $\|\underline{\delta}(\underline{r}^{(1)})\|_\infty = 0.36788$. Equation 5.11 becomes

$$P(z) = z - 1.7358 = 0$$

so that $z^{(1)} = 1.7358$. Equation 5.4 becomes

$$\begin{bmatrix} 0.0 \\ 0.36788 \\ 0.27067 \\ 0.14936 \\ 0.07326 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.7358 \\ 3.0129 \\ 5.2296 \\ 9.0773 \end{bmatrix} \beta + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

The β which yields the minimal value of $\|\underline{\epsilon}\|_\infty$ is found to be $\beta^* = 0.040797$. The corresponding value of $\underline{\epsilon}^*$ is given in Column I of Table 5.12. Since

$$\|\underline{\epsilon}^*\|_\infty = 0.2971 > \frac{\|\underline{\delta}(\underline{r}^{(1)})\|_\infty}{\|\underline{r}^{(1)}\|_1} = 0.13447$$

we minimize $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ according to the algorithm. The results of the different steps of the procedure are given in Columns II and III of Table 5.12. The optimum vector $\underline{r}^*$ is found to be

$$\underline{r}^* = \begin{bmatrix} -1.0624 \\ 1 \end{bmatrix}$$

	I	II	III	IV	V	VI	VII
r_0	-1.7358	-1.3997	-1.1763	-1.0624	-1.0392	-1.0988	-1.2993
r_1	1	1	1	1	1	1	1
ϵ_1	-0.0408	-0.0842	-0.1427	-0.1673	-0.200	-0.16	-0.16
ϵ_2	0.2971	0.25	0.2	0.1673	0.16	0.192	0.16
ϵ_3	0.1478	0.1057	0.0732	0.0576	--	--	--
ϵ_4	-0.0640	-0.0816	-0.0829	-0.0771	--	--	--
ϵ_5	-0.2971	-0.25	-0.2	-0.1673	-0.16	-0.16	-0.38

Table 5.12. The results of the minimization procedure

and the values of the components of the error vector $\underline{\epsilon}^*(\underline{r}^*)$ are given in Column IV of Table 5. 12. The results of the test are shown in Columns V, VI and VII of Table 5. 12.

Therefore, the best Chebyshev pair is

$$(\beta^{**}, z^*) = (0.1673, \quad 1.0624)$$

and the corresponding pair (α^{**}, s^*) is

$$(\alpha^{**}, s^*) = (0.1673, \quad 0.0607)$$

Thus, the approximating function $f^{**}(t) = \alpha^{**} e^{s^* t}$ becomes

$$f^{**}(t) = 0.1673 \; e^{0.0607 t}$$

(b) When n = 2, the vector $\underline{r}$ is defined by

$$\underline{r} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}$$

Equation 5. 7 becomes

$$\underline{\delta}(\underline{r}) = \begin{bmatrix} 0.0 & 0.36788 & 0.27067 \\ 0.36788 & 0.27067 & 0.14936 \\ 0.27067 & 0.14936 & 0.07326 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}$$

The vector $\underline{r}^{(1)}$ which yields the minimal value of $\|\underline{\delta}(\underline{r})\|_{\infty}$ in the set $\|\underline{r}\|_1 = 1$ is found to be

$$\underline{r}^{(1)} = \begin{bmatrix} 0.13534 \\ -0.73576 \\ 1 \end{bmatrix}$$

and the corresponding value of $\|\underline{\delta}(\underline{r}^{(1)})\|_{\infty} = 0$. Since $\|\underline{\delta}(\underline{r}^{(1)})\|_{\infty} = 0$, the vector $\underline{r}^{(1)} = \underline{r}^*$. Equation 5.11 becomes

$$P(z) = z^2 - 0.73576\,z + 0.13534 = 0$$

the roots of which are given by

$$\underline{z}^* = \begin{bmatrix} 0.36788 \\ 0.36788 \end{bmatrix}$$

Since the components of $\underline{z}^*$ are identical the matrix $[Z(\underline{z}^*)]$ is given by Eq. 1.18(b), i.e.,

$$[Z(\underline{z}^*)] = \begin{bmatrix} 1.0 & 0.0 \\ 0.36788 & 0.36788 \\ 0.13534 & 0.27067 \\ 0.04979 & 0.14936 \\ 0.01832 & 0.07326 \end{bmatrix}$$

Calculating the value of $\underline{\beta}^{**}$ which satisfies $\underline{f} = [Z(\underline{z}^*)]\,\underline{\beta}^{**}$ yields

$$\underline{\beta}^{**} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, the best Chebyshev vector pair $(\underline{\beta}^{**}, \underline{z}^{*})$ is given by

$$(\underline{\beta}^{**}, \underline{z}^{*}) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.36788 \\ 0.36788 \end{bmatrix} \right)$$

It can be shown[14] that this yields the vector pair

$$(\underline{\alpha}^{**}, \underline{s}^{*}) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right)$$

Therefore, the approximating function

$$f^{**}(t) = (\alpha_1^{**} + \alpha_2^{**}\, t)\, e^{s_1^{*} t}$$

becomes

$$f^{**}(t) = t\, e^{-t}$$

which is identical to the prescribed function f(t).

Example 5. 6

Consider the problem of approximating the function $f(t) = (2e^{-t} - e^{-t/2})$, in the finite point set $T_e = \left\{0, 1, 2, \ldots, 6\right\}$, by the set of n = 3 exponential functions. Hence, we are given q = 7, n = 3, and

[14] See Section 6. 2 of Chapter VI.

$$\underline{f} = \begin{bmatrix} 1 \\ 0.12923 \\ -0.09721 \\ -0.12356 \\ -0.09870 \\ -0.06861 \\ -0.04483 \end{bmatrix}$$

Since n = 3, the vector $\underline{r}$ is defined by

$$\underline{r} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

and Eq. 5.7 becomes

$$\underline{\delta}(\underline{r}) = [F]\,\underline{r} = \begin{bmatrix} 1 & 0.12923 & -0.09721 & -0.12356 \\ 0.12923 & -0.09721 & -0.12356 & -0.09870 \\ -0.09721 & -0.12356 & -0.09870 & -0.06861 \\ -0.12356 & -0.09870 & -0.06861 & -0.04483 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

Note that the 4x4-matrix [F] is of rank 2. Therefore, the minimal value of $\|\underline{\delta}(\underline{r})\|_\infty = 0$, in the set $\|\underline{r}\|_1 = 1$. Furthermore, the vector $\underline{\tilde{r}}$, which yields $\|\underline{\delta}(\underline{\tilde{r}})\|_\infty = 0$, may be obtained by making one of the components of $\underline{r}$ equal to zero. The value of the various vectors $\underline{\tilde{r}}$, thus obtained, are given in Table 5.13, where the values of the corresponding vector pairs $(\underline{\beta}, \underline{z})$ are also given. Note that each set of vectors $\left\{\underline{r}, \underline{z}, \underline{\beta}\right\}$, given in Table 5.13, represents the set of best Chebyshev

	I	II	III	IV
r_0	0.22313	0.2174	0.05109	0.0
r_1	-0.97441	-0.7260	0	0.22313
r_2	1	0	-0.7454	-0.97441
r_3	0	1	1	1
z_1	0.36788	0.36788	0.36788	0.36788
z_2	0.60653	0.60653	0.60653	0.60653
z_3	--	-0.97441	-0.22899	0
β_1	2	2	2	2
β_2	-1	-1	-1	-1
β_3	--	0	0	0

Table 5.13. The best Chebyshev parameter vectors of Example 5.6.

parameter vectors $\left\{\underline{r}^*, \underline{z}^*, \underline{\beta}^{**}\right\}$, since $\|\underline{\delta}\|_\infty = 0$ implies $\|\underline{\epsilon}^{**}\|_\infty = 0$.

It should be noted that this example illustrates the case when the minimum value of $\|\underline{\epsilon}^*(\underline{r})\|_\infty$ is attained at the point $\underline{r} \epsilon E^{n+1}$ with the component $r_n = 0$ (see Column I of Table 5.13). As was mentioned in Chapter IV, in such a case, there exists another vector $\underline{r} \epsilon E^{n+1}$, with $r_n \neq 0$ and which yields the same approximation. This is shown in Columns II, III, and IV of Table 5.13, where we note that although the n-th order polynomial equation contains an extra root, this root does not contribute to the approximating vector since $\beta_3 = 0$. Therefore, we could have selected n=2 rather than n=3 at the start of the approximation.

Now before concluding this example, let us make some observations concerning the values of the extra root z_3, given in Table 5.13. First, note that $\{z_1, z_2\}$ represent the two roots of the polynomial equation

$$P(z) = r_3 z^3 + r_2 z^2 + r_1 z + r_0 = 0$$

when $r_3 = 0$. Recall that in the case when $r_3 \neq 0$, the coefficients $\{r_0, r_1, r_2\}$ are related to the roots of $P(z) = 0$ by

$$\frac{r_0}{r_3} = -z_1 z_2 z_3$$

$$\frac{r_1}{r_3} = (z_1 z_2 + z_2 z_3 + z_1 z_3), \text{ and}$$

$$\frac{r_2}{r_3} = -(z_1 + z_2 + z_3)$$

where z_3 denotes the extra root which does not contribute to the approximation. From these relations, one can obtain the value of z_3 in terms of z_1 and z_2 as follows:

When $r_0 = 0$, then $z_3 = 0$

when $r_1 = 0$, then $z_3 = -\dfrac{z_1 z_2}{z_1 + z_2}$ and

when $r_2 = 0$, then $z_3 = -(z_1 + z_2)$

It is observed that the values of z_3, given in Columns II, III, and IV of Table 5. 13, satisfy these relations.

5. 5 Analysis of Examples

The method of approximation used in this thesis represents an improvement over that method used by Ruston (Ref. 20). This improvement is summarized in Table 5. 14.

Example Number	Ruston's $\|\underline{\epsilon}^*(\tilde{\underline{r}})\|_\infty$	$\|\underline{\epsilon}^{**}\|_\infty$	Percent Decrease
5.1	0. 05454	0. 04683	14 percent
5. 2	0. 00656	0. 00284	57 percent
5. 3	0. 20315	0. 2	1. 5 percent
5. 4	2. 0	0. 1	95 percent
5. 5a	0. 297	0. 167	44 percent
5. 5b	0. 0	0. 0	--
5. 6	0. 0	0. 0	--

Table 5. 14. Comparison between the initial and final values of $\|\underline{\epsilon}\|_\infty$.

The percent of decrease, given in Table 5. 14, is defined by

$$\text{Percent Decrease} = \frac{\|\underline{\epsilon}^*(\tilde{\underline{r}})\|_\infty - \|\underline{\epsilon}^{**}\|_\infty}{\|\underline{\epsilon}^*(\tilde{\underline{r}})\|_\infty} \times 100$$

The improvement achieved by increasing the dimension, n, of the parameter spaces is also illustrated in Table 5.14 by a comparison of the results of Example 5.1 (where n = 1) and Example 5.2 (where n = 2); or of the results of Example 5.5a (where n = 1) and Example 5.5b (where n = 2).

Examples 5.1, 5.2, 5.4, and 5.5a illustrate the case when the best Chebyshev approximation is characterized by an error vector $\underline{\epsilon}^{**}$ which has (2n+1)-components equal in absolute value to $\|\underline{\epsilon}^{**}\|_{\infty}$. It is seen that the vector pair $(\underline{\beta}^{**}, \underline{z}^{*}) \in \mathscr{B}_z \times \mathscr{Z}$ is unique. Furthermore, the signs of the (2n+1) components of $\underline{\epsilon}^{**}$ with absolute values equal to $\|\underline{\epsilon}^{**}\|_{\infty}$ alternate (2n+1) times, i.e.,

$$\operatorname{sgn} \epsilon^{**}_{i_{v+1}} = -\operatorname{sgn} \epsilon^{**}_{i_v}, \qquad v = 1, 2, \ldots, 2n+1$$

where the components $\left\{\epsilon^{**}_{i_v}\right\}$ satisfy

$$\left|\epsilon^{**}_{i_v}\right| = \|\underline{\epsilon}^{**}\|_{\infty}, \qquad v = 1, 2, \ldots, 2n+1$$

Example 5.3 illustrates the case when the best Chebyshev approximation is characterized by an error vector $\underline{\epsilon}^{**}$ which has 2n-components equal in absolute value to $\|\underline{\epsilon}^{**}\|_{\infty}$. First, let us note that since the interval

$$\left[\frac{\|\underline{\delta}(\underline{r}^{(1)})\|_{\infty}}{\|\underline{r}^{(1)}\|_{1}}, \quad \|\underline{\epsilon}^{*}(\underline{r}^{(1)})\|_{\infty}\right] \quad \text{is small}$$

then the initial estimate of $\underline{r}$, given by $\underline{r}^{(1)}$, may represent the vector $\underline{r}^*$. The second point to be noted is that since one of the components of the vector $\underline{z}^*$ is equal to zero (or equivalently, the component $r_0 = 0$, of the vector $\underline{r}$), then the best Chebyshev approximation is not unique. Furthermore, if $-4.25 < \beta_1^{**} < -3.85$, where β_1^{**} is the first component of the vector $\underline{\beta}^{**}$ given by Eq. 5.17, then the resulting error vector $\underline{\epsilon}^{**}$ will have only (2n-1)-components equal in absolute value to $\|\underline{\epsilon}^{**}\|_\infty$, without affecting the value of $\|\underline{\epsilon}^{**}\|_\infty = 0.2$. For example, if $\beta_1^* = -4.05$, then the vector pair

$$(\underline{\beta}^{**}, \underline{z}^*) = \left(\begin{bmatrix} -4.05 \\ -4.05 \end{bmatrix}, \begin{bmatrix} 0.0 \\ 0.667 \end{bmatrix} \right)$$

yields the error vector

$$\underline{\epsilon}^{**} = \begin{bmatrix} 0.0 \\ 0.1 \\ 0.2 \\ -0.2 \\ 0.2 \end{bmatrix}$$

It should be mentioned that, in the examples considered above, the projections of the components of $\underline{\epsilon}^{**} \in U^q$, which are equal in absolute value to $\|\underline{\epsilon}^{**}\|_\infty$, represent the best Chebyshev error vectors in at least n+1 reference subspaces [15] $\left\{ U_{\theta_j}^{n+1} : j = 1, 2, \ldots, n+1 \right\}$.

---

[15] See the discussion relating to Eq. 4.87 of Section 4.5, Chapter IV.

Specifically, in Example 5.1, the projections of the components $\{\epsilon_1^{**}, \epsilon_2^{**}, \epsilon_6^{**}\}$ represent the best Chebyshev error vectors in the 2-dimensional subspaces $\{U_{\theta_1}^2, U_{\theta_2}^2\}$, where $U_{\theta_1}^2$ contains the components $\{\epsilon_1^{**}, \epsilon_2^{**}\}$, and $U_{\theta_2}^2$ contains the components $\{\epsilon_2^{**}, \epsilon_6^{**}\}$. In Example 5.2, the projections of the components $\{\epsilon_1^{**}, \epsilon_2^{**}, \epsilon_3^{**}, \epsilon_5^{**}, \epsilon_9^{**}\}$ represent the best Chebyshev error vectors in the 3-dimensional subspaces $\{U_{\theta_1}^3, U_{\theta_2}^3, U_{\theta_3}^3\}$, where $U_{\theta_1}^3$ contains the components $\{\epsilon_1^{**}, \epsilon_2^{**}, \epsilon_3^{**}\}$, $U_{\theta_2}^3$ contains the components $\{\epsilon_2^{**}, \epsilon_3^{**}, \epsilon_5^{**}\}$, and where $U_{\theta_3}^3$ contains the components $\{\epsilon_3^{**}, \epsilon_5^{**}, \epsilon_9^{**}\}$. In Example 5.3, the projections of the components $\{\epsilon_1^{**}, \epsilon_3^{**}, \epsilon_4^{**}, \epsilon_5^{**}\}$ represent the best Chebyshev error vectors in $\{U_{\theta_1}^3, U_{\theta_2}^3, U_{\theta_3}^3\}$, where $U_{\theta_1}^3$ contains the components $\{\epsilon_1^{**}, \epsilon_3^{**}, \epsilon_4^{**}\}$, $U_{\theta_2}^3$ contains the components $\{\epsilon_1^{**}, \epsilon_4^{**}, \epsilon_5^{**}\}$, and where $U_{\theta_3}^3$ contains the components $\{\epsilon_3^{**}, \epsilon_4^{**}, \epsilon_5^{**}\}$.

CHAPTER VI

THE DISCRETE TIME DOMAIN CHEBYSHEV APPROXIMATION PROBLEM OF NETWORK SYNTHESIS

6.1 Introduction

In this chapter we shall consider the application of the results of the previous chapters to the discrete time domain approximation problem of network synthesis. Specifically, we shall consider the following problem: Given a prescribed impulse response, h(t), find the best vector pair $(\underline{\alpha}^{**}, \underline{s}^*)$ in $\mathcal{A}_s \times \mathcal{S}$ such that if the approximating impulse response function is given by[1]

$$h^{**}(t) = \sum_{k=1}^{n} \alpha_k^{**} e^{s_k^* t}$$

then

$$\| \epsilon^{**}(t_i) \|_\infty \triangleq \| h(t_i) - h^{**}(t_i) \|_\infty , \quad i = 1, 2, \ldots, q \tag{6.1}$$

is minimum, at the q equally-spaced discrete values of t, $\{ t_i = t_1 + (i-1)\Delta t : i = 1, \ldots, q, \; q > 2n \}$.

Recall that in Section 1.4 (Chapter I), we reformulated this problem in terms of an approximation problem in ℓ_∞^q-space which we then

---

[1]Recall that in the case when the components of $\underline{s}^*$ are not distinct, the function h**(t) has the form given by Eq. 1.3a (Chapter I).

solved in Chapter IV. We shall now take the results of that chapter, i. e., the vector pair $(\underline{\beta}^{**}, \underline{z}^{*})$ in $\mathscr{B}_z \times \mathscr{Z}$ and determine the equivalent vector pair $(\underline{\alpha}^{**}, \underline{s}^{*})$ in $\mathscr{A}_s \times \mathscr{S}$. The condition, which must be satisfied by the vector pair $(\underline{\alpha}^{**}, \underline{s}^{*})$, or $(\underline{\beta}^{**}, \underline{z}^{*})$, so that the approximating function $h^{**}(t)$ represents an impulse response of a physically realizable network is given in Section 6. 3. In Sections 6. 4 and 6. 5 we shall consider the problem of selecting the finite approximation point set and the dimensions of the parameter vectors. Then, in Section 6. 6, we shall give a procedure which outlines the application of these results to the R-L-C network synthesis problem. Finally, we shall present a simple illustrative example of the synthesis procedure.

6. 2 Determination of the Vector Pair $(\underline{\alpha}, \underline{s})$ from the Vector Pair $(\underline{\beta}, \underline{z})$

In Section 1. 4, we have given the mapping which takes the vector pair $(\underline{\alpha}, \underline{s})$ in $\mathscr{A}_s \times \mathscr{S}$ into the vector pair $(\underline{\beta}, \underline{z})$ in $\mathscr{B}_z \times \mathscr{Z}$. This mapping is defined by the following equations

$$\beta_k = \alpha_k e^{s_k t_1}, \qquad k = 1, 2, \ldots, n \tag{6. 2}$$

and

$$z_k = e^{s_k \Delta t}, \qquad k = 1, 2, \ldots, n \tag{6. 3}$$

where t_1 is the initial value of the finite set of discrete equally-spaced values of t, taken at an interval Δt.

In this section, we shall consider the inverse mapping, i. e., the mapping which takes the vector pair $(\underline{\beta}, \underline{z})$ in $\mathscr{B}_z \times \mathscr{Z}$ into the vector pair $(\underline{\alpha}, \underline{s})$ in $\mathscr{A}_s \times \mathscr{S}$. This inverse mapping is not one-to-one as shown by the equations

$$s_k = \frac{1}{\Delta t} [\ell n |z_k| + j(\arg z_k + 2\pi m)], \quad m = 0, +1, +2, \ldots \tag{6.4}$$

$$\alpha_k = \beta_k e^{-s_k t_1} \tag{6.5}$$

where $k = 1, 2, \ldots, n$. However, it should be noted that if only the principal value (i. e., if $m = 0$), is considered, then this mapping is one-to-one.[2]

There is one special case, specifically, the case when z_k is real and negative. Let z_0 denote the real negative component of the vector $\underline{z}$. Then, when $z_k = z_0$ and $m = 0$, Eq. 6.4 yields the exponent

$$s_0 = \frac{1}{\Delta t} [\ell n |z_0| + j\pi] \tag{6.6}$$

where

$$z_0 = -|z_0| = |z_0| e^{j\pi} \tag{6.7}$$

This implies that a single negative real component of $\underline{z}$ yields a single complex exponent, s_0. Clearly, this s_0 is not in the set $\mathscr{S}$,

---

[2]Note that one may choose some other value of $m \neq 0$. However, if $m \neq 0$, then, one must consider both the positive and negative values of m, to obtain a complex conjugate pair of the exponents s_k.

since it yields a single <u>complex</u> exponential approximating function, namely,

$$\alpha_0 \, e^{s_0 t} = \beta_0 \, e^{s_0(t-t_1)} = \beta_0 \, e^{(\sigma_0 + j\omega_0)(t-t_1)} \tag{6.8}$$

where

$$\sigma_0 = \frac{1}{\Delta t} \, [\ell n \, |z_0|], \qquad \text{and} \tag{6.9}$$

$$\omega_0 = \frac{\pi}{\Delta t} \tag{6.10}$$

We shall now show how to obtain a complex conjugate pair of exponents from a negative real component z_0. Let us represent z_0 by the sum of two components z_{0_1} and z_{0_2}, i.e.,

$$z_0 = \frac{1}{2} \, (z_{0_1} + z_{0_2})$$

where

$$z_{0_1} = -|z_0| \triangleq |z_0| \, e^{j\pi} \qquad \text{and} \tag{6.11}$$

$$z_{0_2} = -|z_0| \triangleq |z_0| \, e^{-j\pi} \tag{6.12}$$

Hence, the pair (β_0, z_0) may be represented by the vector pair

$$\left(\begin{bmatrix} \beta_0/2 \\ \beta_0/2 \end{bmatrix} , \begin{bmatrix} z_{0_1} \\ z_{0_2} \end{bmatrix} \right) \tag{6.13}$$

Applying the mapping defined by Eqs. 6.4 and 6.5 yields the vector pair

$$\left(\begin{bmatrix} \alpha_{0_1} \\ \alpha_{0_2} \end{bmatrix} , \begin{bmatrix} s_{0_1} \\ s_{0_2} \end{bmatrix} \right) \tag{6.14}$$

where

$$s_{0_1} = \sigma_0 + j\omega_0$$

$$s_{0_2} = \sigma_0 - j\omega_0$$

and

$$\alpha_{0_1} = \frac{\beta_0}{2} e^{-s_{0_1} t_1}$$

$$\alpha_{0_2} = \frac{\beta_0}{2} e^{-s_{0_2} t_1}$$

and where σ_0 and ω_0 are given by Eqs. 6.9 and 6.10, respectively. Clearly, the vector pair given by Eq. 6.14 is in $\mathscr{A}_s \times \mathscr{S}$, since the resulting exponential function is real, that is,

$$\begin{aligned} \hat{h}_0(t) &= \alpha_{0_1} e^{s_{0_1} t} + \alpha_{0_2} e^{s_{0_2} t} \\ &= \frac{\beta_0}{2} e^{\sigma_0(t-t_1)} \left[e^{j\omega_0(t-t_1)} + e^{-j\omega_0(t-t_1)} \right] \\ &= \beta_0 \, e^{\sigma_0(t-t_1)} \cos \omega_0(t-t_1) \end{aligned} \tag{6.15}$$

Some reflection reveals that Eq. 6.15 is simply the real part of Eq. 6.8, namely, $\mathrm{Re}\{\alpha_0\, e^{s_0 t}\}$.

In summary, the vector pair $(\underline{\alpha}, \underline{s}) \in \mathscr{A}_s \times \mathscr{S}$ of the approximating function[3]

$$\hat{h}(t;\underline{\alpha},\underline{s}) = \sum_{k=1}^{n} \alpha_k\, e^{s_k t} \tag{6.16}$$

or alternately,

$$\hat{h}(t;\underline{\alpha},\underline{s}) = (\alpha_1 + \alpha_2 t + \ldots + \alpha_j t^{j-1})\, e^{s_1 t} + \sum_{k=j+1}^{n} \alpha_k\, e^{s_k t} \tag{6.16a}$$

is obtained from a given vector pair $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$ as follows:

(1) For each real non-negative[4] or complex z_k, the components $\{\alpha_k, s_k\}$ are obtained from Eqs. 6.4 and 6.5, when $m = 0$.

(2) For each real negative z_k, one has to use the equivalent exponential function given by Eq. 6.15.

6.3 Physical Realizability

In the previous section we have determined the vector pair $(\underline{\alpha}^{**}, \underline{s}^{*})$ in $\mathscr{A}_s \times \mathscr{S}$ from the vector pair $(\underline{\beta}^{**}, \underline{z}^{*})$ in $\mathscr{B}_z \times \mathscr{Z}$.

---

[3] Note that Eq. 6.16 gives the format of $\hat{h}(t;\underline{\alpha},\underline{s})$ when the components of $\underline{s}$ are distinct, and Eq. 6.16a gives the format of $\hat{h}(t;\underline{\alpha},\underline{s})$ when the components of $\underline{s}$ are not distinct, namely when $s_1 = s_2 = \ldots = s_j$.

[4] Note that when $z_k = 0$, for some k, then the function $e^{s_k t}$ represents a unit impulse function.

We shall now consider the necessary condition which must be satisfied by the vector pair $(\underline{\alpha}^{**},\underline{s}^{*})$, or $(\underline{\beta}^{**},\underline{z}^{*})$, so that the approximating function[5]

$$h^{**}(t) = \hat{h}(t;\underline{\alpha}^{**},\underline{s}^{*}) = \sum_{k=1}^{n} \alpha_k^{**} e^{s_k^{*}t} \tag{6.17}$$

may be realized as a network function.

Recall from Section 1.2 that the approximating function, selected from the set $\hat{\mathscr{H}} = \{\hat{h}(t;\underline{\alpha},\underline{s}) : \underline{\alpha} \in \mathscr{A}_s, \underline{s} \in \mathscr{S}\}$, represents an impulse response function of a linear, passive, lumped, finite network, if all the s_k's have negative real parts and if each s_k having zero real part is simple. Hence, we shall say that the approximating function $h^{**}(t)$, defined by Eq. 6.17, is physically realizable if the vector pair $(\underline{\alpha}^{**},\underline{s}^{*}) \in \mathscr{A}_s \times \mathscr{S}$ satisfies the following condition:

<u>Condition PR_s</u>: The vector pair $(\underline{\alpha},\underline{s}) \in \mathscr{A}_s \times \mathscr{S}$ is said to satisfy Condition PR_s if the components of the vector $\underline{s} \in \mathscr{S}$ satisfy

(1) $\mathrm{Re}\{s_k\} \leq 0$, $k = 1,2,\ldots,n$, and

(2) when $\mathrm{Re}\{s_k\} = 0$, then $s_k \neq s_j$ for all $k \neq j$

Since the approximation problem, considered in this thesis, seeks the parameter vector pair $(\underline{\beta}^{**},\underline{z}^{*}) \in \mathscr{B}_z \times \mathscr{Z}$, it is convenient to

---

[5]Note again that in the case when the components of $\underline{s}^{*}$ are not distinct, then $h^{**}(t)$ has the form given by Eq. 6.16a. The results presented in this section apply for all $\underline{s} \in \mathscr{S}$.

translate Condition PR_s in terms of the vector pair $(\underline{\beta},\underline{z})$. This yields the following condition:

Condition PR_z: The vector pair $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$ is said to satisfy Condition PR_z if the components of the vector $\underline{z} \in$ satisfy

(1) $|z_k| \leq 1$, $k = 1,2,\ldots,n$, and

(2) when $|z_k| = 1$, then $z_k \neq z_j$ for all $k \neq j$.

Hence, if the vector pair $(\underline{\beta}^{**},\underline{z}^*) \in \mathscr{B}_z \times \mathscr{Z}$ satisfies Condition PR_z, then the corresponding approximating function h**(t) is physically realizable. It stands to reason that, by using Definition 3. 1, one can obtain a condition on the vector $\underline{r}^* \in \mathscr{R}$, which guarantees that h**(t) is physically realizable. However, since such a condition is quite involved (Refs. 4, 9), we shall not examine it here.

The assumption of Condition PR_z immediately bridges the gap between the approximation problem considered in Chapter IV and the time domain approximation problem of network sy nthesis. It is desirable, however, to know at the start of the approximation problem that Condition PR_z will be satisfied. Clearly, such a problem inovlves the solution of the Chebyshev approximation problem, of Chapter IV, with constraints. In other words, since the Condition PR_z restricts the vector $\underline{z}$ to some subset, $\mathscr{Z}_{PR}$, of the set $\mathscr{Z}$, then one must seek the vector pair $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$, but among those $(\underline{\beta},\underline{z}) \in \mathscr{B}_z \times \mathscr{Z}_{PR}$.

However, since Condition PR_z depends also on the selection of the finite point set $T_e = \{t_i : i = 1,\ldots,q\}$ with respect to the behavior of the prescribed h(t), we shall not examine the approximation problem with constraints here. Instead, we shall attempt to choose the point set T_e to guarantee the assumption of Condition PR_z. This will be discussed in the next section.

6.4 Selection of the Finite Approximation Point Set, T_e

Let us now state some of the considerations which dictate the choice of the finite point set $T_e = \{t_i : i = 1,2,\ldots,q\}$ over which the approximation is to be performed. Recall that since the elements $\{t_i\}$ of T_e are related by

$$t_i = t_1 + (i-1)\Delta t\,, \qquad i = 1,2,\ldots,q \tag{6.18}$$

where Δt is the interval between the equally-spaced values of t, then the point set T_e is fully specified, once (1) the approximation interval $[t_1, t_q]$, and (2) the number q have been selected. Let us consider these two problems separately.

(1) Approximation interval $[t_1, t_q]$ should be chosen so that the solution of the approximation problem will give a vector pair $(\underline{\beta}^{**}, \underline{z}^*)$ which will satisfy Condition PR_z. For example, assume that the

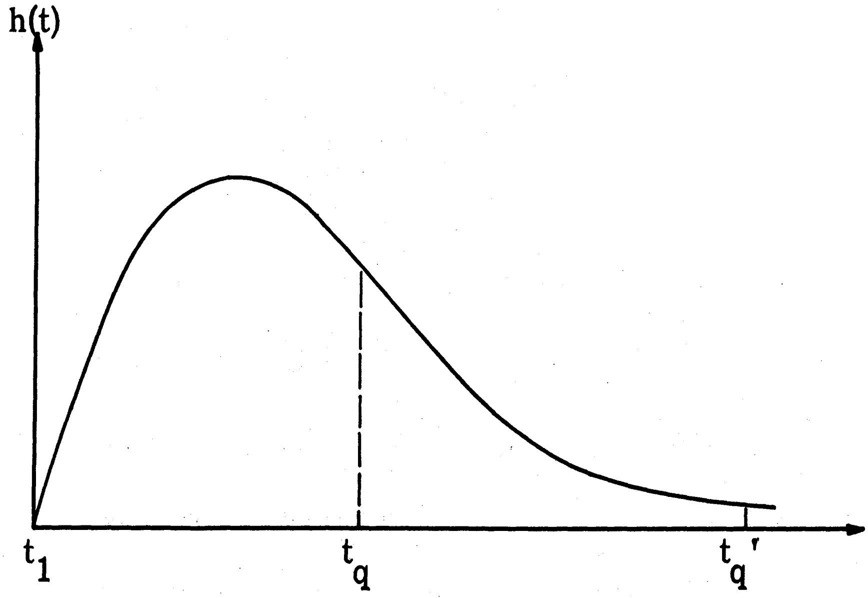


Fig. 9. Example of a prescribed impulse response function h(t).

impulse response,[6] h(t), shown in Fig. 9 is to be approximated over a specified interval $[t_1, t_q]$. After performing the approximation if the vector pair $(\underline{\beta}^{**}, \underline{z}^*)$ does not satisfy Condition PR_z, then by enlarging the approximation interval to $[t_1, t_q']$ one might assure that the new $(\underline{\beta}^{**}, \underline{z}^*)$ will satisfy Condition PR_z.

(2) The number q should be chosen so that the value of $\|\epsilon(t)\|_\infty \triangleq \|h(t) - h^{**}(t)\|_\infty$ is small[7] in between the discrete values of $\{t_i\}$. This implies that the interval Δt must be smaller than the smallest interval in which h(t) varies between relative maximums and minimums.[8] However, since the number q is directly related to the amount of numerical work involved in performing the approximation, then it should be as small as possible.

In summary, then, the choice of the finite point set T_e dictates the value of $\|\epsilon(t)\|_\infty$ for all $t \in [t_1, t_q]$, and the physical realizability of the approximating function $h^{**}(t)$.

---

[6] Note that since Fig. 9 depicts a physically realizable impulse response, h(t), i. e., $\int_0^\infty |h(t)|\, dt < \infty$, one expects to obtain a $(\underline{\beta}^{**}, \underline{z}^*)$ which satisifes Condition PR_z.

[7] Recall that ideally one would like to approximate h(t) over the continuous interval $[t_1, t_q]$.

[8] Note that the value of Δt can be obtained from the Sampling Theorem, i. e., $\Delta t = \frac{1}{2W}$, where W is the highest frequency of the waveform h(t).

6.5 Choice of the Dimensions of the Parameter Spaces

In choosing the dimension n, of parameter space $\mathscr{B}_z \times \mathscr{Z}$ one must consider the effect of n on the complexity of the resulting network, the tolerable error, $\|\underline{\epsilon}^{**}\|_\infty$, and the amount of computational work. With regard to the complexity of the network, it is seen that since the approximating function is a linear combination of n exponential functions, then the resulting network will have between 2n and 3n elements. The relation between n and the value of $\|\underline{\epsilon}^{**}\|_\infty$ is more complicated. All that one can say is that the number n is inversely proportional to the value of $\|\underline{\epsilon}^{**}\|_\infty$. Hence, when the value of n is increased, the value of $\|\underline{\epsilon}^{**}\|_\infty$ is decreased. However, this holds only up to the point when $n = \frac{q}{2}$ since then, $\|\underline{\epsilon}^{**}\|_\infty = 0$.

Another difficult relationship to define is the one between the amount of computational work and the number n. However, in general the larger the number n, the more computational work will be involved in solving the approximation problem. Therefore, the selection of the number n depends on a balance between these three factors.

6.6 Synthesis Procedure

Now that the preliminary ideas have been discussed, let us consider the computational procedure to be followed in approximating a prescribed impulse response, h(t), which is given in analytic or graphical form:

(1) Select the finite point set $T_e = \{t_i = t_1 + (i-1)\Delta t : i = 1, 2, \ldots, q\}$ according to Section 6. 4.

(2) Knowing the finite points set T_e, form the real vector $\underline{h}$ in U^q from the ordered set of values $\{h(t_i)\}$ of $h(t)$.

(3) Choose the number n according to Section 6. 5. (At this point, it is helpful to write out the relation $\underline{h} = [Z(\underline{z})]\,\underline{\beta} + \underline{\epsilon}$, using the value of $\underline{h}$ determined in Step 2.)

(4) Calculate the parameter vectors $(\underline{\beta}^{**}, \underline{z}^{*})$ in $\mathcal{B}_z \times \mathcal{Z}$ using the procedure of Section 5. 3. 1.

(5) Does $(\underline{\beta}^{**}, \underline{z}^{*}) \in \mathcal{B}_z \times \mathcal{Z}$ satisfy Condition PR_z? If yes, go to Step 6. If no, repeat from Step 1 by using a new finite point set T_e if possible.

(6) Determine the vector pair $(\underline{\alpha}^{**}, \underline{s}^{*})$ using Eqs. 6. 4 and 6. 5 (and/or Eq. 6. 15) of Section 6. 2.

(7) Form the approximating function $h^{**}(t)$ using Eq. 6. 16 (or Eq. 6. 16a).

(8) In the prescribed interval $[t_1, t_q]$, determine the over-all Chebyshev error $\|\epsilon(t)\|_\infty \triangleq \sup_{t \in [t_1, t_q]} |\epsilon(t)|$, where $\epsilon(t) = h(t) - h^{**}(t)$. If this $\|\epsilon(t)\|_\infty$ is not tolerable, repeat from Step 1 using a greater number q and/or n.

(9) Determine the equivalent network function by taking the Laplace transform of $h^{**}(t)$, namely $H^{**}(s) = L[h^{**}(t)]$.

(10) Realize a suitable network having the network function $H^{**}(s)$ as its driving point or transfer function depending on the desired performance of the network.

6.6.1 A Simple Example of the Synthesis Procedure. The following example illustrates the application of the above procedure.

Example 6.1: Let $h(t)$ be given by

$$h(t) = \frac{1}{(1+t)^2} \quad \text{for } t \geq 0 ,$$

$$= 0 \quad \text{for } t < 0 \qquad (6.19)$$

Synthesize a network whose output-to-input voltage impulse response function, given by Eq. 6.16 (or Eq. 6.16a), approximates $h(t)$ at a finite discrete value of t, in the Chebyshev sense.

Solution:

(1) The function $h(t)$ is plotted in Fig. 10. The finite point set T_e is selected to be

$$T_e = \{0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0\}$$

where $\Delta t = 0.5$, and $q = 9$.

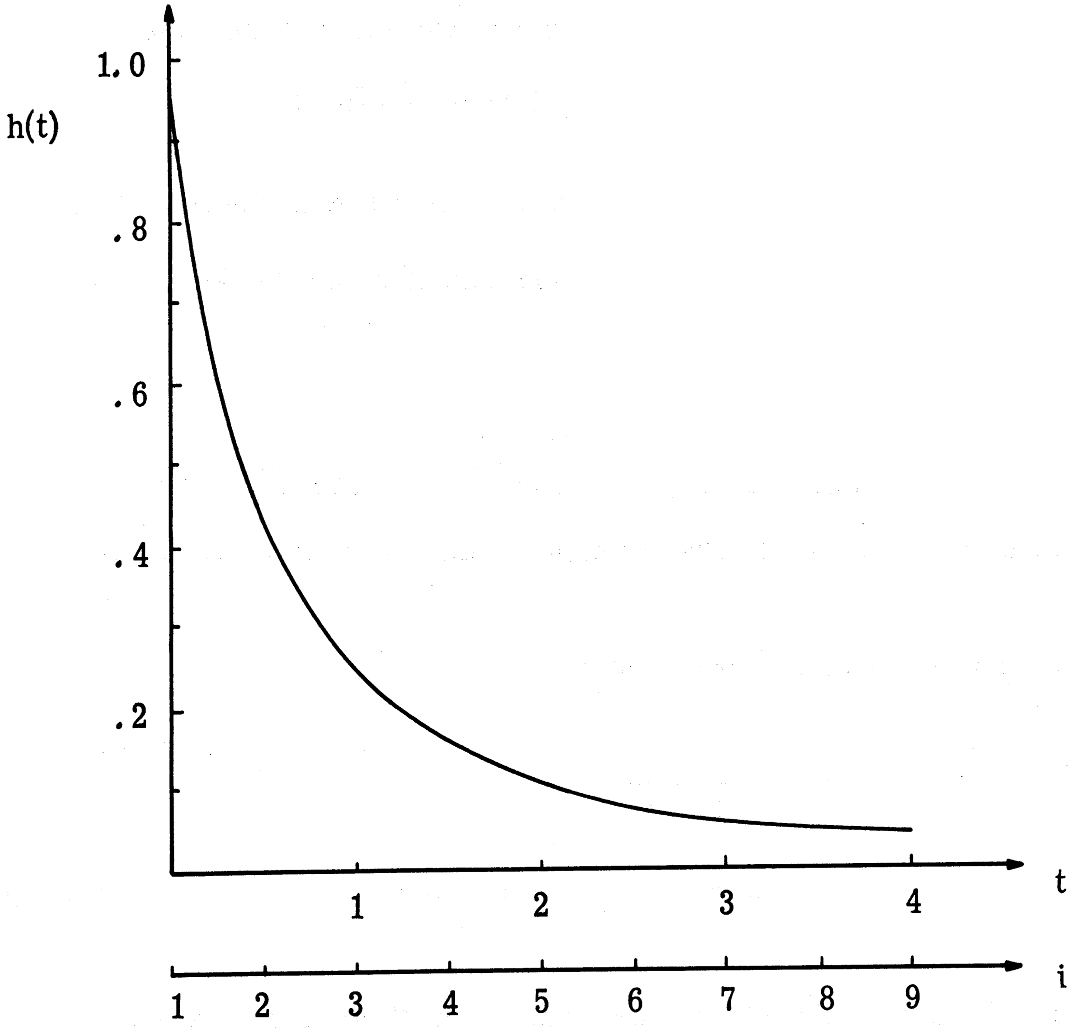


Fig. 10. Impulse response function, $h(t) = \dfrac{1}{(1+t)^2}$.

(2)

$$\underline{h} = \begin{bmatrix} 1.0000 \\ 0.4450 \\ 0.2500 \\ 0.1600 \\ 0.1110 \\ 0.0817 \\ 0.0625 \\ 0.0494 \\ 0.0400 \end{bmatrix}$$

(3) Select n = 2. Then the relation $\underline{h} = [Z(\underline{z})]\,\underline{\beta} + \underline{\epsilon}$ yields Eq. 5.14 of Section 5.4.

(4) The solution of this problem is presented in Example 5.2 of Section 5.4. It yields

$$\underline{\beta}^{**} = \begin{bmatrix} 0.31476 \\ 0.68240 \end{bmatrix}$$

$$\underline{z}^{*} = \begin{bmatrix} 0.76552 \\ 0.30317 \end{bmatrix}$$

(5) The vector pair $(\underline{\beta}^{**}, \underline{z}^{*})$ satisfies Condition PR_z.

(6) When $\Delta t = 0.5$, and $m = 0$, Eqs. 6.4 and 6.5 yield

$$s_1^* = 2\,\ell n\,(0.76552) = -0.53444,$$
$$s_2^* = 2\,\ell n\,(0.30317) = -2.38692, \text{ and}$$
$$\alpha_1^{**} = 0.31476$$
$$\alpha_2^{**} = 0.68240$$

Hence,

$$(\underline{\alpha}^{**}, \underline{s}^{*}) = \left(\begin{bmatrix} 0.31476 \\ 0.68240 \end{bmatrix}, \begin{bmatrix} -0.53444 \\ -2.38692 \end{bmatrix} \right)$$

(7) The approximating function, given by Eq. 6.16, becomes

$$h^{**}(t) = 0.31476\, e^{-0.534t} + 0.6824\, e^{-2.387t}$$

(8) The plot of $\epsilon(t) = h(t) - h^{**}(t)$ is shown in Fig. 11. It is observed that

$$\|\epsilon(t)\|_{\infty} = \sup_{t \in [0,4]} |\epsilon(t)| = 0.0118 > \|\underline{\epsilon}^{**}\|_{\infty} = 0.00284$$

Furthermore, it is seen that if one would perform another approximation over the interval $[0, 4]$ when $\Delta t = 0.25$, then the resulting $\|\epsilon(t)\|_{\infty}$ would lie in the interval $0.00284 \leq \|\epsilon(t)\|_{\infty} \leq 0.0118$ and the value of $\|\epsilon(t)\|_{\infty}$ will be closer to the value of $\|\underline{\epsilon}^{**}\|_{\infty}$. However, we shall consider this error function, $\epsilon(t)$, to be satisfactory and proceed to the next step.

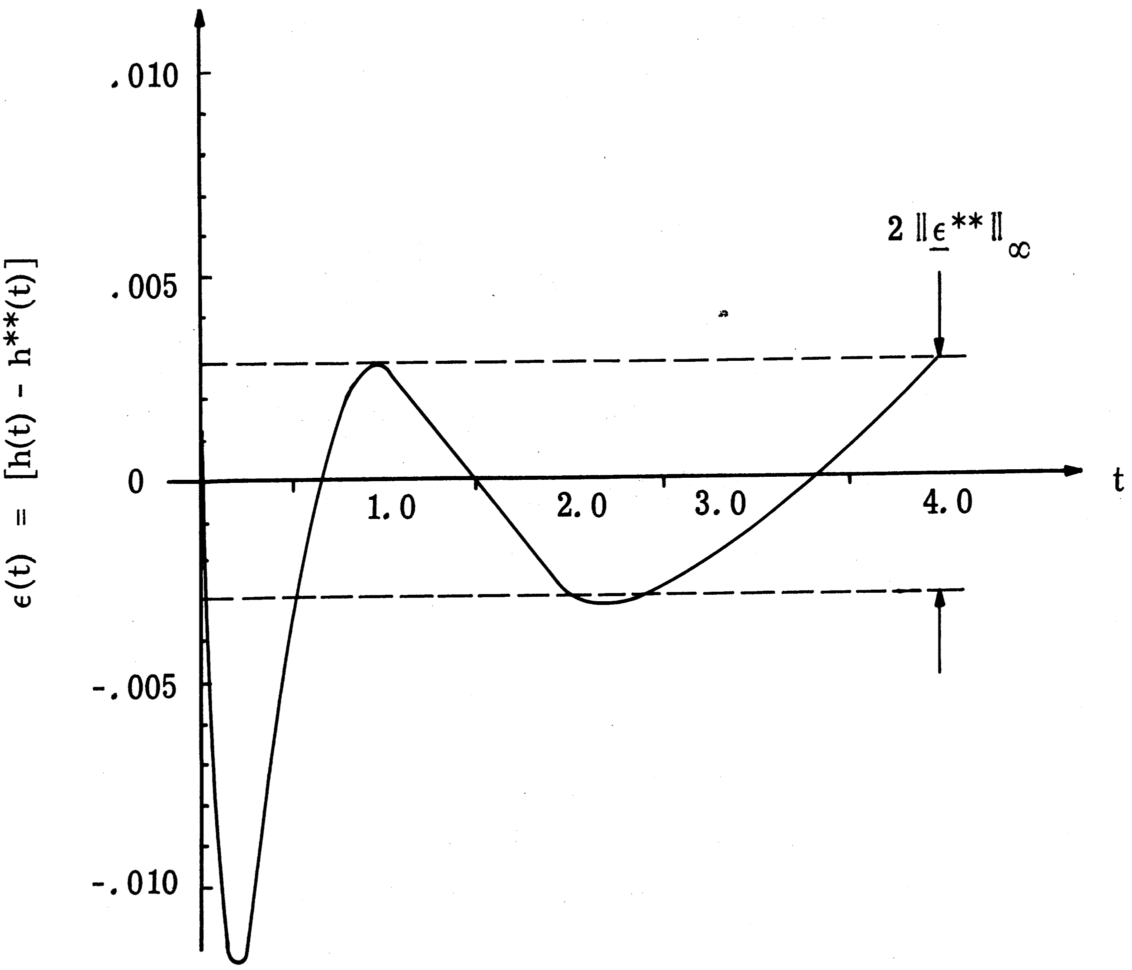


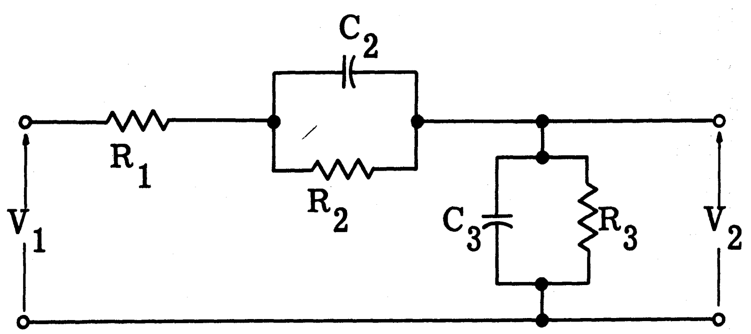
Fig. 11. Approximation error function,
$\epsilon(t) = h(t) - h^{**}(t)$.

(9) The Laplàce transform of h**(t) yields

$$H^{**}(s) = \frac{0.31476}{s + 0.534} + \frac{0.6824}{s + 2.387}$$

$$= \frac{0.9972\,(s + 1.1186)}{(s + 0.534)(s + 2.387)}$$

(10) A network realizing H**(s) as an input-to-output voltage ratio, i.e., $\frac{V_2}{V_1}(s) = H^{**}(s)$, is shown in Fig. 12.



$R_1 = 1\,\Omega$ $\quad C_2 = 1.393$ fd

$R_2 = 0.642\,\Omega$ $\quad C_3 = 1.0028$ fd

$R_3 = 11.538\,\Omega$

Fig. 12. Network realizing h**(t) as an input-to-output voltage ratio.

CHAPTER VII

CONCLUSIONS AND AREAS OF FUTURE WORK

The discrete time domain approximation problem of network synthesis was originally posed in Chapter I. In particular, the problem of determining the coefficients and exponents of the exponential functions when a prescribed impulse response function, h(t), is approximated by a linear combination of exponential functions so that the resulting error is minimum in the Chebyshev sense, at a finite number of equally-spaced discrete values of t. This led to the Chebyshev approximation problem considered in Chapter IV, where both the best orientation of the approximating subspace and the approximating vector in it were determined for any prescribed vector in a finite dimensional vector space. In this chapter we shall summarize and evaluate these results and point out the various extensions which can be carried out in the future.

7.1 Discussion of Results

The primary contributions to the solution of the problem considered in this thesis are:

(1) The formulation of the time domain approximation problem of network synthesis in the language of the theory approximation in

linear spaces. In particular, we have formulated the discrete approximation problem in terms of an approximation problem in the finite dimensional vector space. Such a formulation depicts the direct relation between the pole position of the network function (i. e., the exponents of the exponential function) and the orientation of the approximating subspace.

(2) The formal extension of "Prony's Original Method" to the solution of discrete exponential approximation problems, which is presented in Chapter III. Such an extension gives an alternate approach to the solution of exponential approximation problems. It illustrates the limitations of the previous works by Ruston (Ref. 20) and by Yengst (Ref. 26), which use similar approaches.

(3) The solution of the Chebyshev approximation problem in the finite dimensional vector space when the approximating subspace is a function of a collection of n-parameters. The main results of the theory, presented in Chapter IV, are: (1) the existence theorem; (2) the bounds within which the minimal value of $\|\underline{\epsilon}\|_\infty$ must lie; (3) the property which states that the best Chebyshev parameters define the best Chebyshev approximation in some (2n+1)-dimensional reference subspace; and (4) the conjecture which states that the minimal value of $\|\underline{\epsilon}\|_\infty$ occurs when at least (2n+1)-components of $\underline{\epsilon}$ are equal in absolute value to $\|\underline{\epsilon}\|_\infty$ if there exists no other

approximation which yields a smaller value of $\|\underline{\epsilon}\|_\infty$ when 2n-components of $\underline{\epsilon}$ are equal in absolute value. If, however, there exists an approximation which yields a smaller value of $\|\underline{\epsilon}\|_\infty$ when 2n-components of $\underline{\epsilon}$ are equal in absolute value, then it is conjectured that such an approximation represents the best Chebyshev approximation.

(4) The method of descent used to solve the Chebyshev approximation problem. It is based on "Prony's Extended Method," and is presented in Chapter V, where a computational algorithm is given. This method systematically decreases the value of $\|\underline{\epsilon}\|_\infty$ until (2n+1)-components of the error vector, $\underline{\epsilon}$, are equal in absolute value to $\|\underline{\epsilon}\|_\infty$. At this point, the error vector is tested to determine whether or not there exists another error vector, $\underline{\epsilon}$, which gives a lower value of $\|\underline{\epsilon}\|_\infty$, when only 2n-components of $\underline{\epsilon}$ are equal in absolute value. The computations required for this test lead to the next step of descent, if it is required.

(5) The approximating function is such that its Laplace transform can be expressed in rational function form. This rational function is <u>not</u> restricted to a class of functions containing only first-order poles, since for each repeated exponent, s_k, the approximating function possesses terms of the form $\{e^{s_k t}, t\, e^{s_k t}, \ldots, t^{j-1} e^{s_k t}\}$, where j denotes the order of the repeated exponent, s_k. This result

follows directly from "Prony's Extended Method," presented in Chapter III. If each exponent, of the approximating function, has a negative or zero real part, then the rational function is suitable for realization as a driving point or transfer function.

7.2 Limitation of the Results

The significant limitations of the results presented in this thesis are concentrated in their application to the time domain approximation problem of network synthesis. The approximation theory presented in Chapter IV is limited only in the fact that the resulting approximation might not be unique. However, the significance of this limitation is diminished by the fact that in network synthesis the problem of realization is usually not unique either.

The limitations encountered in applying the approximation theory developed here to the time domain approximation problem of network synthesis are:

(1) The form of the matrix $[Z(\underline{z})]$ dictates that the prescribed impulse response function must be sampled at equally-spaced intervals of t.

(2) There is no control of the behavior of the error function, $\epsilon(t) \triangleq h(t) - h^{**}(t)$, in between the equally-spaced discrete values of t.

(3) The solution of the approximation problem does not necessarily yield a physically realizable impulse response function. Recall that the requirement that the approximating impulse response function be physically realizable was considered only after the approximating function had been determined.

7.3 Future Areas of Study

The following areas of further study were revealed during the course of the investigation reported in this thesis:

(1) The problem of proving Conjectures 4.1 and 4.2. This is the problem of showing that Conjectures 4.1 and 4.2 give the necessary and sufficient conditions for the best Chebyshev approximation in each (2n+1)-dimensional reference subspace. It involves the study of the behavior of the nonlinear equations which relate the (2n+1)-components of the error vector in each (2n+1)-dimensional reference subspace.

(2) The problem of determining the necessary and sufficient conditions for the uniqueness of the solution of the Chebyshev approximation problem, considered in Chapter IV. This investigation should yield a certain class of prescribed vectors which will always result in a unique Chebyshev approximation.

(3) The problem of determining the necessary and sufficient conditions for the approximating function to be physically realizable.

This is the approximation problem of Chapter IV with constraints, where the constraints are imposed on the parameter vector $\underline{z}$, i.e., $\underline{z}$ must lie in the parameter space $\mathcal{Z}_{PR}$, defined in Section 6.3.

(4) The problem of selecting the finite approximation point set, T_e, and the dimension, n, of the parameter space, so that the value of $\|\epsilon(t)\|_\infty \triangleq \|h(t) - h^{**}(t)\|_\infty$ is small in between the equally-spaced discrete values of t.

(5) The problem of approximating a prescribed impulse response, h(t), over a continuous bounded interval of t, [a, b], so that the resulting error $\|\epsilon(t)\|_\infty$ is minimum. Some thought should be given as to whether the approach of using the orthogonal complement subspace to the approximating subspace may be used to solve this problem. A similar approach was used by McDonough (Ref. 11) in determining the best least-square approximation over the continuous interval $[0, \infty)$. It should be mentioned that in the case of the Chebyshev approximation the resulting error function should, in general, alternate (2n+1) times in the approximation interval [a, b].

(6) The problem of finding an efficient computational method should be examined so that this method can be directly adapted for digital computer use.

APPENDIX A

STIEFEL'S ALGORITHM

A. 1 Theoretical Analysis

In Section 2. 3. 1. 2 of Chapter II, we mentioned the "method of ascent" used by Stiefel (Ref. 22) to solve the Chebyshev approximation problem. We shall now present this method in considerable detail, giving the iterative computational method and an illustrative example.

Stiefel sought the best Chebyshev approximate solution, $\{x_j^* : j = 1, 2, \ldots, n\}$ of the overdetermined system of linear equations given by

$$f_i = \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, 2, \ldots, q \tag{A. 1}$$

where the sets $\{f_i\}$ and $\{a_{ij}\}$ are initially prescribed, and where $q > n$. Instead of presenting Stiefel's method of solving Eq. A. 1 verbatim, we shall present his method in the setting of this thesis, i. e., in terms of operations in a finite dimensional real vector space E^q. Recall, from Section 2. 3. 1. 2, that the problem of solving the overdetermined system of equations of Eq. A. 1 may be stated as follows:

Given a vector $\underline{f}$ in E^q and a $q \times n$ matrix $[A]$, of rank n $(q > n)$; find the parameter vector $\underline{x}^*$ in E^n such that if $\underline{f}^* \triangleq [A]\, \underline{x}^*$, then

$$\|\underline{\epsilon}^*\| \equiv \|\underline{f} - \underline{f}^*\|_\infty < \|\underline{f} - [A]\,\underline{x}\|_\infty \tag{A.2}$$

for all $\underline{x} \neq \underline{x}^*$ in E^n. Hence, we shall be concerned with the vector relation

$$\underline{f} = [A]\,\underline{x} + \underline{\epsilon}, \quad \text{in } E^q \tag{A.3}$$

where the initially prescribed vector $\underline{f}$ and the $q \times n$ matrix $[A]$, of rank n $(n < q)$, are given by

$$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_q \end{bmatrix}, \quad \text{in } E^q, \quad \text{and} \tag{A.4}$$

$$[A] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{q1} & \cdots & a_{qn} \end{bmatrix} \tag{A.5}$$

and where the unknown vectors $\underline{\epsilon}$ and $\underline{x}$ are defined by

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_q \end{bmatrix}, \quad \text{in } E^q, \quad \text{and} \tag{A.6}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{in } E^n \tag{A.7}$$

Furthermore, since the method of solving Eq. A. 2, proposed by Stiefel, is based on Corollary 2. 1 above, we shall be also concerned with the vector relation

$$\underline{f}^{(v)} = [A^{(v)}]\,\underline{x} + \underline{\epsilon}^{(v)}, \quad \text{in } E_v^{n+1} \tag{A.8}$$

where the (n+1)-dimensional vectors $\underline{f}^{(v)}$, $[A^{(v)}]\,\underline{x}$, and $\underline{\epsilon}^{(v)}$ in E_v^{n+1} are the respective projections of the q-dimensional vectors $\underline{f}$, $[A]\,\underline{x}$, and $\underline{\epsilon}$ in E^q onto the v-th reference subspace, E_v^{n+1} of E^q, according to Definition 2. 1.

Recall that Corollary 2. 1 states that the problem of minimizing the value of $\|\underline{\epsilon}\|_\infty$, i. e.,

$$\|\underline{\epsilon}^*\|_\infty = \min_{\underline{x} \text{ in } E^n} \|\underline{\epsilon}(\underline{x})\|_\infty \tag{A.9}$$

where $\underline{\epsilon}$ is defined by Eq. A. 3, may be replaced by determining the largest value of $\|\underline{\epsilon}^{(v)}(\underline{x}_v^*)\|_\infty$ from the set $\{\|\underline{\epsilon}^{(v)}(\underline{x}_v^*)\|_\infty : v = 1, 2, \ldots, \binom{q}{n+1}\}$, namely,

$$\|\underline{\epsilon}^*\|_\infty = \max_{1 \leq v \leq \binom{q}{n+1}} \{\|\underline{\epsilon}^{(v)}(\underline{x}_v^*)\|_\infty\} \tag{A. 10}$$

where the vector $\underline{\epsilon}^{(v)}(\underline{x}_v^*)$ is given by Eq. A. 8 when $\underline{x} = \underline{x}_v^*$, and where $\underline{x}_v^*$ defines the best approximating vector, $\underline{f}^{*(v)} \triangleq [A^{(v)}]\ \underline{x}_v^*$ to $\underline{f}^{(v)}$ in E_v^{n+1}, and is determined by using Theorem 2. 3. Hence, the method of solution of Eq. A. 2 proposed by Stiefel is based on the method defined by Eq. A. 10.

Basically Stiefel's algorithm is as follows: We begin by arbitrarily picking an (n+1)-dimensional reference subspace, say, E_k^{n+1}, where $k \in \{v = 1, 2, \ldots, \binom{q}{n+1}\}$. This yields the relation

$$\underline{f}^{(k)} = [A^{(k)}]\ \underline{x} + \underline{\epsilon}^{(k)}, \quad \text{in } E_k^{n+1} \tag{A. 11}$$

Then, by applying Theorem 2. 3 to Eq. A. 11, the vector $\underline{x}_k{}^*$ in E^n which minimizes $\|\underline{\epsilon}^{(k)}\|_\infty$ and the vector $\underline{\epsilon}^{(k)}(\underline{x}_k^*)$ in E_k^{n+1} are determined. Knowing the value of $\underline{x}_k{}^*$, the value of the vector $\underline{\epsilon}(\underline{x}_k{}^*)$ in E^q is determined from Eq. A.3, when $\underline{x} = \underline{x}_k{}^*$. At this point we compare the ℓ_∞-norms of the two error vectors $\underline{\epsilon}^{(k)}(\underline{x}_k{}^*)$ and $\underline{\epsilon}(\underline{x}_k^*)$, and see if they satisfy either

$$\|\underline{\epsilon}(\underline{x}_k{}^*)\|_\infty = \|\underline{\epsilon}^{(k)}(\underline{x}_k{}^*)\|_\infty \tag{A. 12}$$

or

$$\|\underline{\epsilon}(\underline{x}_k^*)\|_\infty > \|\underline{\epsilon}^{(k)}(\underline{x}_k^*)\|_\infty \qquad (A.13)$$

If Eq. A.12 is satisfied, then the vector $\underline{x}_k^* = \underline{x}^*$, where $\underline{x}^*$ is the vector $\underline{x}$ which prescribes the solution of the Chebyshev approximation problem defined by Eq. A.2. If, on the other hand, Eq. A.13 is satisfied, then a new estimate of $\underline{x}^*$ must be sought. Since we are seeking an estimate of $\underline{x}^*$, which will increase the value of $\|\underline{\epsilon}^{(k)}(\underline{x}_k^*)\|_\infty$ so that eventually Eq. A.12 will be satisfied, then the new estimate of $\underline{x}^*$, say, $\underline{x}_m^*$, must be chosen so that the new value of $\|\underline{\epsilon}(\underline{x}_m^*)\|_\infty$, where $\underline{\epsilon}(\underline{x}_m^*)$ is given by Eq. A.3 when $\underline{x} = \underline{x}_m^*$, will satisfy

$$\|\underline{\epsilon}(\underline{x}_k^*)\|_\infty > \|\underline{\epsilon}(\underline{x}_m^*)\|_\infty > \|\underline{\epsilon}^{(k)}(\underline{x}_k^*)\|_\infty$$

or alternately,

$$\|\underline{\epsilon}(\underline{x}_k^*)\|_\infty > \|\underline{\epsilon}^{(m)}(\underline{x}_m^*)\|_\infty > \|\underline{\epsilon}^{(k)}(\underline{x}_k^*)\|_\infty \qquad (A.14)$$

Hence, we have the problem of finding a new (n+1)-dimensional reference subspace E_m^{n+1} which will yield an $\underline{x}_m^*$ so that Eq. A.14 will be satisfied. The method of solution of this problem represents the contribution of Stiefel, and is summarized in his Exchange Theorem (Ref. 22).

Now, before considering Stiefel's Exchange Theorem, let us make the following observation: When the initial estimate of $\underline{x}^*$, i.e., $\underline{x}_k^*$ yields an error vector $\underline{\epsilon}(\underline{x}_k^*)$ whose ℓ_∞-norm satisfies Eq. A.13, there exists at least one component of $\underline{\epsilon}(\underline{x}_k{}^*)$ in E^q whose absolute value equals to $\|\underline{\epsilon}(\underline{x}_k{}^*)\|_\infty$. Let us denote this component by $\epsilon_b(\underline{x}_k{}^*)$, where $\epsilon_b(\underline{x}_k{}^*)$ satisfies

$$|\epsilon_b(\underline{x}_k{}^*)| = \|\underline{\epsilon}(\underline{x}_k{}^*)\|_\infty \tag{A. 15}$$

and where the component ϵ_b is not one of the components of the vector $\underline{\epsilon}$ in E^q which formed the vector $\underline{\epsilon}^{(k)}$ in E_k^{n+1}. Then, the subspace which contains the subspace E_k^{n+1} plus the b-th component of the vector $\underline{\epsilon}$ in E^q is an (n+2)-dimensional subspace, E^{n+2}. This can be best illustrated as follows:

Let the vector relation in E_k^{n+1} be given by[1]

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \epsilon_1(\underline{x}) \\ \epsilon_2(\underline{x}) \\ \vdots \\ \epsilon_{n+1}(\underline{x}) \end{bmatrix} \tag{A. 16}$$

where $\underline{f}$ in E^{n+1} and the (n+1) x n matrix [A] are prescribed. Applying Theorem 2.3 to Eq. A.16 yields the values of the vectors $\underline{x}^*$ and $\underline{\epsilon}(\underline{x}^*)$ where the components of the vector $\underline{\epsilon}(\underline{x}^*)$ are given by

---

[1]To simplify the notation, we shall drop the superscript and subscript k in Eqs. A.16 through A.20 and Theorem A.1, since there is no danger of ambiguity.

$$\epsilon_i(\underline{x}^*) = \rho \text{ sgn } \epsilon_i , \quad i = 1, 2, \ldots, n+1 \tag{A.17}$$

Let us assume that when substituting the value of $\underline{x}^*$ into Eq. A.3, the component $\epsilon_{n+2}(\underline{x}^*)$, of the vector $\underline{\epsilon}(\underline{x}^*) \in E^q$, satisfies[2]

$$|\epsilon_{n+2}(\underline{x}^*)| > |\rho| \tag{A. 18}$$

We now form the following vector relation which defines the (n+2)-dimensional subspace E^{n+2}, namely,

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n+1} \\ f_{n+2} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1,n} \\ a_{21} & \cdots & a_{2,n} \\ \vdots & & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,n} \\ a_{n+2,1} & \cdots & a_{n+2,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \rho \text{ sgn } \epsilon_1 \\ \rho \text{ sgn } \epsilon_2 \\ \vdots \\ \rho \text{ sgn } \epsilon_{n+1} \\ \epsilon_{n+2} \end{bmatrix} \tag{A. 19}$$

where the vectors $\underline{f}$, $[A]\ \underline{x}$ and $\underline{\epsilon}$ in E^{n+2} are all known, and where the value of $|\epsilon_{n+2}| > |\rho|$. We now have the problem of finding the (n+1)-dimensional reference subspace E_m^{n+1} of E^{n+2} so that the parameter vector $\underline{x}_m^*$, which defines the best Chebyshev approximating vector in E_m^{n+1}, according to Theorem 2.3, yields an error vector $\underline{\epsilon}(\underline{x}_m^*) \in E^{n+2}$, whose $\|\underline{\epsilon}(\underline{x}_m^*)\|_\infty$ satisfies

---

[2]Note that we are also assuming that $|\epsilon_{n+2}(\underline{x}^*)| = \|\underline{\epsilon}(\underline{x}^*)\|_\infty$, where $\underline{\epsilon}(\underline{x}^*) \in E^q$, i.e., that Eq. A. 15 holds when b = n+2.

$$|\epsilon_{n+2}| > \|\underline{\epsilon}(\underline{x}_m^*)\|_\infty > |\rho| \tag{A.20}$$

The solution to this problem is given by Stiefel in the following theorem:

Theorem A. 1: (Exchange Theorem). Let the vector [A] $\underline{x}$, where [A] is an (n+2) x n matrix, of rank n, and $\underline{x}$ is in E^n, represent the approximation vector to the prescribed vector $\underline{f}$ in E^{n+2} such that the resulting approximation error vector $\underline{\epsilon}$ has the form

$$\underline{\epsilon} \triangleq \underline{f} - [A]\,\underline{x} = \begin{bmatrix} \rho \operatorname{sgn} \epsilon_1 \\ \rho \operatorname{sgn} \epsilon_2 \\ \vdots \\ \rho \operatorname{sgn} \epsilon_{n+1} \\ \epsilon_{n+2} \end{bmatrix} \tag{A.21}$$

where

$$|\epsilon_{n+2}| > |\rho| \tag{A.22}$$

Then, there exists an n-dimensional vector $\underline{y} \neq \underline{x}$ in E^n, such that if [A] $\underline{y}$ is another approximation vector of $\underline{f}$ in E^{n+2}, then the approximation error vector $\underline{\eta}$ has the form

$$\underline{\eta} \triangleq \underline{f} - [A]\,\underline{y} = \begin{bmatrix} \delta \operatorname{sgn} \eta_1 \\ \vdots \\ \delta \operatorname{sgn} \eta_{j-1} \\ \eta_j \\ \delta \operatorname{sgn} \eta_{j+1} \\ \vdots \\ \delta \operatorname{sgn} \eta_{n+2} \end{bmatrix} \tag{A.23}$$

where $j \in \{1, 2, \ldots, n+1\}$, and where δ satisfies

$$|\rho| < |\delta| < |\epsilon_{n+2}| \tag{A.24}$$

__Proof:__ Let us begin by considering Eq. A. 19. Assume that the two independent vectors in E^{n+2}, which span the orthogonal complement subspace of the n-dimensional subspace $C_n(A)$, are given by[3]

$$\underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n+1} \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n+1} \\ 1 \end{bmatrix} \tag{A.25}$$

---

[3] It should be noted that the vector $\underline{\lambda}$ is also orthogonal to the projection of $C_n(A)$ on the (n+1)-dimensional subspace E^{n+1} formed by the first (n+1) components of Eq. A. 19.

Then, applying the orthogonality conditions $([A]\underline{x}, \underline{\lambda}) = 0$, and $([A]\underline{x}, \underline{\mu}) = 0$, to Eq. A.19, we obtain the following two equations:

$$(\underline{f} - \underline{\epsilon}, \underline{\lambda}) = \sum_{i=1}^{n+1} \lambda_i(f_i - \epsilon_i) = 0 \tag{A.26}$$

$$(\underline{f} - \underline{\epsilon}, \underline{\mu}) = \sum_{i=1}^{n+1} \mu_i(f_i - \epsilon_i) + (f_{n+2} + \epsilon_{n+2}) = 0 \tag{A.27}$$

Multiplying Eq. A.26 by $-\mu_j$ and Eq. A.27 by λ_j and adding the resulting equations, yields

$$\sum_{\substack{i=1 \\ i \neq j}}^{n+1} (\mu_i\lambda_j - \mu_j\lambda_i)(f_i - \epsilon_i) + \lambda_j(f_{n+2} - \epsilon_{n+2}) = 0 \tag{A.28}$$

This equation may be written as

$$\sum_{\substack{i=1 \\ i \neq j}}^{n+1} \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right) \lambda_i \epsilon_i + \epsilon_{n+2} = \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right) \lambda_i f_i + f_{n+2} \tag{A.29}$$

Let us denote the coefficients of ϵ_i's and f_i's by ν_i, i.e., let

$$\nu_i \triangleq \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right) \lambda_i \tag{A.30}$$

Then, Eq. A.29 becomes

$$\sum_{\substack{i=1 \\ i \neq j}}^{n+1} \nu_i \epsilon_i + \epsilon_{n+2} = \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \nu_i f_i + f_{n+2} \tag{A.31}$$

Applying the lemma of de la Vallee Poussin (i. e., Lemma 2. 1, Chapter II), to A. 31, we find that the minimal value of the expression

$$\max_{\substack{i=1, \ldots, n+2 \\ i \neq j}} \{|\epsilon_i|\}$$

is given by

$$\delta = \frac{\sum_{\substack{i=1 \\ i \neq j}}^{n+1} \nu_i f_i + f_{n+2}}{\sum_{\substack{i=1 \\ i \neq j}}^{n+1} |\nu_i| + 1} \tag{A.32}$$

Observe that Eq. A. 32 gives an alternate representation to Eq. A. 31, namely,

$$\sum_{\substack{i=1 \\ i \neq j}}^{n+1} \nu_i f_i + f_{n+2} = \delta \left[\sum_{\substack{i=1 \\ i \neq j}}^{n+1} |\nu_i| + 1 \right] \tag{A.33}$$

Equating Eq. A. 31 to Eq. A. 33, yields

$$\sum_{\substack{i=1 \\ i \neq j}}^{n+1} \nu_i \epsilon_i + \epsilon_{n+2} = \delta \left[\sum_{\substack{i=1 \\ i \neq j}}^{n+1} |\nu_i| + 1 \right] \tag{A.34}$$

or

$$\left[\sum_{\substack{i=1 \\ i \neq j}}^{n+1} \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right) \lambda_i \epsilon_i + \epsilon_{n+2} \right] = \delta \left[\sum_{\substack{i=1 \\ i \neq j}}^{n+1} \left| \frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right| \cdot |\lambda_i| + 1 \right] \tag{A.35}$$

Recall from A.21 and Lemma 2.1 that

$$\lambda_i \epsilon_i = |\lambda_i| \rho , \quad \text{for } i = 1, \ldots, n+1 \tag{A.36}$$

then, by taking the absolute value of both sides of Eq. A.35, one obtains

$$|\rho| \cdot \left| \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right) \cdot |\lambda_i| + \frac{\epsilon_{n+2}}{\rho} \right| = |\delta| \cdot \left| \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \left| \frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right| \cdot |\lambda_i| + 1 \right| \tag{A.37}$$

Since $|\epsilon_{n+2}| > |\rho|$ by definition (i.e., Eq. A.22), we find that if the relation

$$|\rho| < |\delta| \tag{A.38}$$

is to be guaranteed, then the sign of ϵ_{n+2} must equal the sign of

$$\left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right)$$

that is,

$$\text{(a) if } \epsilon_{n+2} > 0, \text{ then } \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j}\right) > 0, \text{ or} \tag{A.39}$$

$$\text{(b) if } \epsilon_{n+2} < 0, \text{ then } \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j}\right) < 0 \tag{A.40}$$

To show that $|\epsilon_{n+2}| > |\delta|$, let us consider Eq. A.37 under either the condition of (A.39), or (A.40), depending on the sign of ϵ_{n+2}. This yields

$$|\epsilon_{n+2}| - |\delta| = [\,|\delta| - |\rho|\,] \left[\sum_{\substack{i=1 \\ i \neq j}}^{n+1} \left| \frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right| \cdot |\lambda_i| \right] \tag{A.41}$$

Since the right hand side of Eq. A.41 is greater than zero, then

$$|\epsilon_{n+2}| > |\delta| \tag{A.42}$$

and the theorem is proved.

It is noted that this method does not necessarily increase the deviation $|\delta|$ at the fastest possible rate since Theorem A.1 does not guarantee that

$$|\delta| \geq |\eta_j|, \quad j = 1, 2, \ldots, n+1 \tag{A.43}$$

However, by considering all possible n+1 values of $|\delta_j|$, given by Eq. A.32 for each j, that is,

$$|\delta_j| = \frac{\left| \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right) \lambda_i f_i + f_{n+2} \right|}{\sum_{\substack{i=1 \\ i \neq j}} \left| \frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j} \right| \cdot |\lambda_i| + 1}, \quad j = 1, 2, \ldots, n+1 \tag{A.44}$$

and choosing the j which maximizes $|\delta_j|$, will result in the fastest possible increase in $|\delta|$ for each replacement step. Furthermore, the resulting error vector $\underline{\eta}$, defined in Eq. A.23, becomes the best Chebyshev error vector $\underline{\eta}^*$ in E^{n+2}, that is,

$$\|\underline{\eta}^*\|_\infty = |\delta|$$

Hence, the resulting vector $[A]\,\underline{y}^*$ is the best Chebyshev approximation vector of $\underline{f}$ in E^{n+2}.

Now before presenting Stiefel's algorithm in detail, it should be mentioned that there are other algorithms which converge faster than Stiefel's point-by-point exchange algorithm. One of these is the method proposed by Remez (Ref. 17) which exchanges all the (n+1) points at each step of the algorithm, i.e., it selects a completely new (n+1)-dimensional reference subspace E_m^{n+1}. We shall not consider this method here and the reader interested in this method is referred to the treatment of this subject by Rice (Ref. 19, pp. 176-180).

Let us now present the step-by-step procedure proposed by Stiefel for the solution of the Chebyshev approximation problem of Eq. A.2.

A.2 Stiefel's Iterative Procedure

(1) Write out Eq. A.3, substituting for $\underline{f}$ and [A] their respective prescribed values.

(2) Select any (n+1)-dimensional subspace out of the set of $\left\{E_v^{n+1}\right\}$, $v = 1, 2, \ldots, \binom{q}{n+1}$, in E^q. Let us denote it by E_k^{n+1}, where $k \in \left\{v = 1, 2, \ldots, \binom{q}{n+1}\right\}$.

Write Eq. A.8 in the form

$$\underline{f}^{(k)} = [A^{(k)}]\,\underline{x} + \underline{\epsilon}^{(k)}, \quad \text{in } E_k^{n+1} \tag{A.45}$$

where $\underline{f}^{(k)}$ is an (n+1)-dimensional vector representing the projection of the prescribed q-dimensional vector $\underline{f}$ onto E_k^{n+1} and similarly, $[A^{(k)}]$ is an (n+1) x n matrix formed by selecting the appropriate rows from the prescribed q x n matrix [A].

(3) Find the (n+1)-dimensional vector $\underline{\lambda}^{(k)}$ in E_k^{n+1} which is orthogonal to the column space of $[A^{(k)}]$, $C_n(A^{(k)})$, from

$$[A^{(k)}]^T \underline{\lambda}^{(k)} = 0 \tag{A.46}$$

where

$$\underline{\lambda}^{(k)} = \begin{bmatrix} \lambda_1^{(k)} \\ \vdots \\ \lambda_n^{(k)} \\ 1 \end{bmatrix}$$

Note that if the $(n+1) \times n$ matrix $[A^{(k)}]$ is of rank n, then the vector $\underline{\lambda}^{(k)}$ can be determined directly.

(4) From Eq. A.45 and from the fact that $([A^{(k)}]\, \underline{x}, \underline{\lambda}^{(k)}) = 0$ for all $\underline{x}$ in $C_n(A^{(k)})$, form the equation[4]

$$(\underline{\epsilon}^{(k)}, \underline{\lambda}^{(k)}) = (\underline{f}^{(k)}, \underline{\lambda}^{(k)}) \tag{A.47}$$

$$= \sum_{i=1}^{n+1} f_i^{(k)} \lambda_i^{(k)} = c \tag{A.48}$$

where $\underline{f}^{(k)}$ and $\underline{\lambda}^{(k)}$ are given in Steps 2 and 3, respectively.

(5) From Eq. A.48, determine the vector $\underline{\epsilon}^{(k)}$, whose ℓ_∞-norm is minimal. This vector (denoted by $\underline{\epsilon}^{*(k)}$) is given by Lemma 2.1 as follows:

$$\underline{\epsilon}^{*(k)} = \rho_k \begin{bmatrix} \operatorname{sgn} \epsilon_1^{*(k)} \\ \operatorname{sgn} \epsilon_2^{*(k)} \\ \vdots \\ \operatorname{sgn} \epsilon_{n+1}^{*(k)} \end{bmatrix}$$

[4] This equation implies that the orthogonal projection of $\underline{f}^{(k)}$ and $\underline{\epsilon}^{(k)}$ on $\lambda^{(k)}$ are equal.

where

$$\rho_k = \frac{c}{1 + \sum_{i=1}^{n} |\lambda_i^{(k)}|}$$

and

$$\text{sgn}\, \epsilon_i^{*(k)} = \text{sgn}\, \lambda_i^{(k)}, \qquad i = 1, \ldots, n+1$$

(6) Determine $\underline{x}_k^*$, from any n rows of the vector relation

$$[A^{(k)}]\, \underline{x}_k^* = \underline{f}^{(k)} - \underline{\epsilon}^{*(k)}$$

where $\underline{\epsilon}^{*(k)}$ takes the value found in Step 5.

(7) Solve for $\underline{\epsilon}$ in E^q from Eq. A.3 when $\underline{x} = \underline{x}_k^*$, where $\underline{x}_k^*$ was found in Step 6, that is,

$$\underline{\epsilon}(\underline{x}_k^*) = \underline{f} - [A]\, \underline{x}_k^*$$

(8) (a) If $\|\underline{\epsilon}(\underline{x}_k^*)\|_\infty = |\rho_k|$ then $\underline{x}_k^* = \underline{x}^*$, the best Chebyshev solution vector of Eq. A.3.

(b) If $\|\underline{\epsilon}(\underline{x}_k^*)\|_\infty > |\rho_k|$, go to Step 9.

(9) Let the b-th component of $\underline{\epsilon}(\underline{x}_k^*)$, determined in Step 7, be the component whose absolute value equals to $\|\underline{\epsilon}(\underline{x}_k^*)\|_\infty$, i.e.,

$$|\epsilon_b(\underline{x}_k^*)| = \|\underline{\epsilon}(\underline{x}_k^*)\|_\infty$$

Then, form a new n+1-dimensional subspace (denoted by E_m^{n+1}) by using Theorem A. 1. This will be done in Steps 10 through 12.

(10) Calculate the (n+2)-dimensional vector $\underline{\mu}$ defined by

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n+1} \\ 1 \end{bmatrix}$$

from the equation[5]

$$\left[\begin{array}{c:c} [A^{(k)}] & \underline{\lambda}^{(1)} \\ \hdashline A_b & 0 \end{array}\right]^T \mu = 0$$

where (n+1) x n submatrix $[\underline{A}^{(k)}]$ is defined in Step 2, and the 1 x n submatrix A_b is the b-th row of the q x n matrix [A] equivalent to the component ϵ_b.

(11) Form a set of (n+1) values of the ratios of the components of the vector $\underline{\mu}$ to those of the vector $\underline{\lambda}^{(k)}$ found in Step 2, that is,

---

[5]Note that this equation will yield a vector $\underline{\mu}$ which is orthogonal to the (n+1)-dimensional subspace, spanned by $C_n(A)$ and the vector $\underline{\lambda}$. Although this simplifies the calculation of $\underline{\mu}$, it suffices to find a vector $\underline{\mu}$ which simply is not in the (n+1)-dimensional subspace.

$$\left\{ \frac{\mu_i}{\lambda_i^{(k)}} \right\} \quad \text{where } i = 1, 2, \ldots, n+1 \qquad \text{(A. 49)}$$

(12) Determine the coordinate j, of the subspace E_k^{n+1}, to be replaced by the coordinate b, as follows:

(i) If $\epsilon_b > 0$, then j is the subscript i which corresponds to the <u>smallest</u> element of the set $\left\{ \frac{\mu_i}{\lambda_i^{(k)}} \right\}$ defined by Eq. A. 49, that is,

$$\left(\frac{\mu_j}{\lambda_j^{(k)}} \right) = \min_{1 \leq i \leq n+1} \left\{ \frac{\mu_i}{\lambda_i^{(k)}} \right\}$$

(ii) If $\epsilon_b < 0$, then j is the subscript i which corresponds to the <u>largest</u> element of the set $\left\{ \frac{\mu_i}{\lambda_i^{(k)}} \right\}$ defined by Eq. A. 49, that is,

$$\left(\frac{\mu_j}{\lambda_j^{(k)}} \right) = \max_{1 \leq i \leq n+1} \left\{ \frac{\mu_i}{\lambda_i^{(k)}} \right\}$$

(13) Form the new vector equation

$$\underline{f}^{(m)} = [A^{(m)}] \, \underline{x} + \underline{\epsilon}^{(m)}, \quad \text{in } E_m^{n+1}$$

where the (n+1) vector, $\underline{f}^{(m)}$, and the (n+1) x n matrix, $[A^{(m)}]$, correspond to $\underline{f}^{(k)}$ and $[A^{(k)}]$, respectively, when the j-th row has been replaced by the b-th row of the vector relation given in Step 1.

(14) Determine $\underline{\epsilon}^{*(m)}$ and $\underline{x}_m{}^*$ by repeating the procedure starting with Step 3.

(15) This procedure is repeated until the relation in Step 8 is satisfied.

A. 3 Illustrative Example

Given:

$$\underline{f} = \begin{bmatrix} 4 \\ 2 \\ 7 \\ -3 \end{bmatrix} \quad \text{and} \quad [A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

Find the vector $\underline{x}^*$ in E^2, such that if $\underline{f}^* \triangleq [A]\, \underline{x}^*$, then,

$$\|\underline{\epsilon}^*\| \triangleq \|\underline{f} - \underline{f}^*\|_\infty < \|\underline{f} - [A]\, \underline{x}\|_\infty$$

for all $\underline{x} \neq x^*$ in E^2.

Note: $q = 4$, $n = 2$.

(1) Equation A. 2 becomes

$$\begin{bmatrix} 4 \\ 2 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix} \qquad \text{(A. 50)}$$

(2) Since $n = 2$, then $E_k^{n+1} = E_k^3$. Hence, select any 3 component of Eq. A. 50 to represent the vector relation A. 45. Let $k = 1$, then

Eq. A. 45 becomes

$$\underline{f}^{(1)} = [A^{(1)}]\,\underline{x} + \underline{\epsilon}^{(1)} \quad \text{in } E_1^3$$

where

$$\underline{f}^{(1)} = \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} \qquad \underline{\epsilon}^{(1)} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

$$[A^{(1)}] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 3 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(3) Equation A. 46 becomes

$$[A^{(1)}]^T \underline{\lambda}^{(1)} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1^{(1)} \\ \lambda_2^{(1)} \\ 1 \end{bmatrix} = 0$$

Solving for $\left\{\lambda_i^{(1)}\right\}$ from this equation yields

$$\underline{\lambda}^{(1)} = \begin{bmatrix} -0.5 \\ -0.5 \\ 1 \end{bmatrix}$$

(4) Equation A. 47 becomes

$$c = (\underline{\epsilon}^{(1)}, \underline{\lambda}^{(1)}) = (\underline{f}^{(1)}, \underline{\lambda}^{(1)})$$

$$= 4(-0.5) + 2(-0.5) + 7(1) = 4$$

(5) The value of

$$\rho_1 = \frac{c}{\sum_{i=1}^{3} |\lambda_i^{(1)}|} = \frac{4}{2} = 2$$

Therefore,

$$\underline{\epsilon}^{*(1)} = \rho_1 \begin{bmatrix} \text{sgn}\,\lambda_1^{(1)} \\ \text{sgn}\,\lambda_2^{(1)} \\ \text{sgn}\,\lambda_3^{(1)} \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

(6) Determine $\underline{x}_1^*$ from

$$[A^{(1)}]\, \underline{x}_1^* = \underline{f}^{(1)} - \underline{\epsilon}^{*(1)}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$$

Solving this equation yields

$$\underline{x}_1^* = \begin{bmatrix} -8 \\ 7 \end{bmatrix}$$

(7) Substituting $\underline{x}_1{}^*$ into Eq. A. 50 and solving for $\underline{\epsilon}$ yields,

$$\underline{\epsilon}(\underline{x}_1{}^*) = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 7 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 6 \end{bmatrix}$$

(8) Since $\|\underline{\epsilon}(\underline{x}_1{}^*)\|_\infty = 6 > |\rho_1| = 2$, go to Step 9 of procedure.

(9) From Step 7 of procedure, b = 4 since

$$|\epsilon_4(\underline{x}_1{}^*)| = 6 = \|\underline{\epsilon}(\underline{x}_1{}^*)\|_\infty$$

(10) Determine the vector $\underline{\mu}$ by solving the following equation:

$$\begin{bmatrix} [A^{(1)}] & \underline{\lambda}^{(1)} \\ A_b & 0 \end{bmatrix}^T \underline{\mu} = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 1 \\ -0.5 & -0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ 1 \end{bmatrix} = 0$$

Hence,

$$\underline{\mu} = \begin{bmatrix} 7/3 \\ -5/3 \\ 1/3 \\ 1 \end{bmatrix}$$

(11) Form the set defined by Eq. A. 49

$$\left\{\frac{\mu_i}{\lambda_i^{(1)}}\right\} = \left\{\frac{-14}{3}, \frac{10}{3}, \frac{1}{3}\right\}$$

(12) Since $\epsilon_4 = 6 > 0$, found in Step 7, then $j = 1$ because

$$\frac{-14}{3} = \min_{1 \leq i \leq 3} \left\{\frac{\mu_i}{\lambda_i^{(1)}}\right\} \quad \text{of Step 11}$$

(13) Form

$$\underline{f}^{(2)} = [A^{(2)}]\,\underline{x} + \underline{\epsilon}^{(2)} \quad \text{in } E_2^{\,3}$$

where

$$\underline{f}^{(2)} = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}; \qquad \underline{\epsilon}^{(2)} = \begin{bmatrix} \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix}$$

$$[A^{(2)}] = \begin{bmatrix} 3 & 4 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}; \quad \text{and} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(14) From

$$[A^{(2)}]^T \underline{\lambda}^{(2)} = \begin{bmatrix} 3 & 2 & 2 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{(2)} \\ \lambda_2^{(2)} \\ 1 \end{bmatrix} = 0$$

one obtains

$$\underline{\lambda}^{(2)} = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$

$$(\underline{\epsilon}^{(2)}, \underline{\lambda}^{(2)}) = (\underline{f}^{(2)}, \underline{\lambda}^{(2)}) = 24$$

$$\rho_2 = \frac{24}{\sum_{i=1}^{3} |\lambda_i^{(2)}|} = \frac{24}{10} = 2.4$$

Therefore,

$$\underline{\epsilon}^{(2)} = \begin{bmatrix} -2.4 \\ 2.4 \\ 2.4 \end{bmatrix} \quad , \quad \text{and}$$

$$\underline{x}_2^* = \begin{bmatrix} -5.2 \\ 5.0 \end{bmatrix}$$

Substituting $\underline{x}_2^*$ for $\underline{x}$ in Eq. A. 50, yields,

$$\underline{\epsilon}(\underline{x}_2^*) = \begin{bmatrix} -0.8 \\ -2.4 \\ 2.4 \\ 2.4 \end{bmatrix} \qquad \text{(A. 51)}$$

Since $\|\underline{\epsilon}(\underline{x}_2^*)\|_\infty = 2.4 = \rho$, then $\underline{x}_2^* = \underline{x}^*$, i.e., the best Chebyshev solution vector. To summarize, the vector $\underline{x}^*$ of Eq. A. 50 is

$$\underline{x}^* = \begin{bmatrix} -5.2 \\ 5.0 \end{bmatrix}$$

Hence,

$$\underline{f}^* = [A]\,\underline{x}^* = \begin{bmatrix} 4.8 \\ 4.4 \\ 4.6 \\ -5.4 \end{bmatrix} \quad \text{and}$$

$\|\underline{\epsilon}^*\|_\infty = 2.4$, where $\underline{\epsilon}^*$ is given by Eq. A. 51.

APPENDIX B

PROOF OF THE RELATIONS OF LEMMA 4.1 GIVEN BY EQ. 4.9 THROUGH EQ. 4.14

In this section we shall prove the following relations:

(1) If $\underline{z}_o \neq 0$ and if $\|\underline{z} - \underline{z}_o\|_1 \leq \frac{1}{2} \|\underline{z}_o\|_\infty$, then

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z}_o\|_\infty \right| \leq c_1(\underline{z}_o) \|\underline{z} - \underline{z}_o\|_1 \quad \text{(B. 1)}$$

where

$$c_1(\underline{z}_o) = 2\|\underline{f}\|_\infty \|\underline{z}_o\|_\infty^{-1} \sum_{j=1}^{q-1} \binom{q-1}{j} \quad \text{(B. 2)}$$

and

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z}_o)\|_\infty \right| \leq c_2(\underline{z}_o) \|\underline{z} - \underline{z}_o\|_1 \quad \text{(B. 3)}$$

where

$$c_2(\underline{z}_o) = 2(n+1) \|\underline{f}\|_\infty \|\underline{z}_o\|_1^{-1} \sum_{j=1}^{q-1} \binom{q-1}{j} \quad \text{(B. 4)}$$

(2) If $\underline{z}_o = 0$ and if $\|\underline{z}\|_1 \leq 1$, then

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z}_o)\|_\infty \right| \leq 2\|\underline{f}\|_\infty \quad \text{(B. 5)}$$

and

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z}_o)\|_\infty \right| \leq 2\|\underline{f}\|_\infty \quad \text{(B. 6)}$$

First, we note that since

$$\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z})\|_\infty \triangleq \|\underline{f} - [Z(\underline{z})]\,\underline{\beta}^*(\underline{z})\|_\infty \geq \|[Z(\underline{z})]\,\underline{\beta}^*(\underline{z})\|_\infty - \|\underline{f}\|_\infty \tag{B.7}$$

and

$$\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z})\|_\infty \leq \|\underline{f}\|_\infty \tag{B.8}$$

then

$$\|[Z(\underline{z})]\,\underline{\beta}^*(\underline{z})\|_\infty \triangleq \max_{1\leq i\leq q} \left| \sum_{k=1}^{n} \beta_k^*(\underline{z})\, z_k^{i-1} \right| \leq 2\|\underline{f}\|_\infty \tag{B.9}$$

Furthermore, we note that the binomial expansion yields

$$\begin{aligned}[z_o^m - z^m] &= [z_o^m - (z_o - [z_o - z])^m] \\ &= -\sum_{j=1}^{m} \binom{m}{j} z_o^{m-j}\,[z - z_o]^j \end{aligned} \tag{B.10}$$

(1) Using the relation given by Eqs. B. 9 and B. 10, let us now obtain the relations given by Eqs. B. 1 through B. 4 when $\underline{z} \neq 0$ as follows

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z}_o)\|_\infty \right|$$

$$\leq \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z}) - \underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z}_o)\|_\infty$$

$$= \| [Z(\underline{z}_o) - Z(\underline{z})]\, \underline{\beta}^*(\underline{z}_o)\|_\infty$$

$$= \max_i \left| \sum_{k=1}^{n} \beta_k^*(\underline{z}_o) \left[z_{o_k}^{i-1} - z_k^{i-1} \right] \right|$$

$$= \max_i \left| \sum_{k=1}^{n} \beta_k^*(\underline{z}_o)\, z_{o_k}^{i-1} \sum_{j=1}^{i-1} \binom{i-1}{j} z_{o_k}^{-j} \left[z_k - z_{o_k} \right]^j \right|$$

$$\leq \left\{ \max_i \left| \sum_{k=1}^{n} \beta_k^*(\underline{z}_o)\, z_{o_k}^{i-1} \right| \right\} \left\{ \max_i \sum_{j=1}^{i-1} \binom{i-1}{j} \sum_{k=1}^{n} \left| z_{o_k} \right|^{-j} \left| z_k - z_{o_k} \right|^j \right\}$$

$$\leq 2\|\underline{f}\|_\infty \max_i \sum_{j=1}^{i-1} \binom{i-1}{j} \|\underline{z}_o\|_\infty^{-j} \sum_{k=1}^{n} \left| z_k - z_{o_k} \right|^j \qquad \text{(B. 11)}$$

where

$$\|\underline{z}_o\|_\infty^{-j} = \max_{1 \leq k \leq n} \left| z_{o_k} \right|^{-j}$$

Since

$$\sum_{k=1}^{n} \left| z_k - z_{o_k} \right|^j \leq \left[\sum_{k=1}^{n} \left| z_o - z_{o_k} \right| \right]^j \triangleq \|\underline{z} - \underline{z}_o\|_1^j \qquad \text{(B. 12)}$$

then Eq. B. 11 yields

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z}_o)\|_\infty \right| \leq 2\|\underline{f}\|_\infty \max_i \sum_{j=1}^{i-1} \binom{i-1}{j} \|\underline{z}_o\|_\infty^{-j} \cdot \|\underline{z} - \underline{z}_o\|_1^j$$

(B. 13)

Let us now note that

$$\sum_{j=1}^{i-1} \binom{i-1}{j} \|\underline{z}_o\|_\infty^{-j} \|\underline{z}-\underline{z}_o\|_1^{j} = 0 \qquad , \text{ when } i = 1 \tag{B. 14}$$

$$= \|\underline{z}_o\|_\infty^{-1} \|\underline{z}-\underline{z}_o\|_1. \qquad , \text{ when } i = 2 \tag{B. 15}$$

$$= \sum_{j=1}^{q-1} \binom{q-1}{j} \|\underline{z}_o\|_\infty^{-j} \|\underline{z}-\underline{z}_o\|_1^{j} , \text{ when } i = q \tag{B. 16}$$

Furthermore, if $\|\underline{z} - \underline{z}_o\|_1 \leq \|\underline{z}_o\|_\infty$, then Eq. B. 16 becomes

$$\sum_{j=1}^{q-1} \binom{q-1}{j} \|\underline{z}_o\|_\infty^{-j} \|\underline{z} - \underline{z}_o\|_1^{j} \leq \|\underline{z}_o\|_\infty^{-1} \sum_{j=1}^{q-1} \binom{q-1}{j} \|\underline{z} - \underline{z}_o\|_1 \tag{B. 17}$$

Hence, from Eqs. B. 13 through B. 17, we obtain Eq. B. 1, namely,

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o),\underline{z}_o)\|_\infty \right| \leq c_1(\underline{z}_o) \|\underline{z} - \underline{z}_o\|_1 \tag{B. 18}$$

where

$$c_1(\underline{z}_o) = 2\|\underline{f}\|_\infty \|\underline{z}_o\|_\infty^{-1} \sum_{j=1}^{q-1} \binom{q-1}{j}$$

under the constraint $\|\underline{z} - \underline{z}_o\|_1 \leq \|\underline{z}_o\|_\infty \neq 0$. Clearly this constraint contains the assumed constraint, i. e., $\|\underline{z} - \underline{z}_o\|_1 \leq \frac{1}{2} \|\underline{z}_o\|_\infty$.

To obtain the relation given by Eq. B. 3 we simply interchange the variables $\underline{z}$ and $\underline{z}_o$ in Eqs. B. 11 through B. 18. This yields

$$\|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}),\underline{z}_o)\|_\infty \leq c_1(\underline{z}) \|\underline{z} - \underline{z}_o\|_1 \tag{B. 19}$$

where

$$c_1(\underline{z}) = 2\|\underline{f}\|_\infty \|\underline{z}\|_\infty^{-1} \sum_{j=1}^{q-1} \binom{q-1}{j} \qquad \text{(B.20)}$$

under the constraint $\|\underline{z} - \underline{z}_o\|_1 \leq \|\underline{z}\|_\infty$. Clearly, if $\|\underline{z} - \underline{z}_o\|_1 < \|\underline{z}\|_\infty$ when $\|\underline{z}_o\|_\infty \neq 0$; and since $\|\underline{z} - \underline{z}_o\|_\infty \leq \|\underline{z} - \underline{z}_o\|_1$ then

$$0 < \|\underline{z}_o\|_\infty - \|\underline{z} - \underline{z}_o\|_\infty \leq \|\underline{z}\|_\infty$$

or in other words $\|\underline{z}\|_\infty \neq 0$.

Since $\underline{z} \neq 0$ and $\underline{z}_o \neq 0$ then from the constraint $\|\underline{z}\|_\infty \geq \|\underline{z} - \underline{z}_o\|_1 \geq \|\underline{z}_o\|_1 - \|\underline{z}\|_1$, and from the relation $\|\underline{z}\|_1 \leq n\|\underline{z}\|_\infty$, we obtain

$$\|\underline{z}\|_\infty \geq \|\underline{z}_o\|_1 - n\|\underline{z}\|_\infty$$

or alternately,

$$\|\underline{z}\|_\infty^{-1} \leq (n+1)\|\underline{z}_o\|_1^{-1} \qquad \text{(B.21)}$$

Substituting Eq. B.21 into B.20 yields

$$c_2(\underline{z}_o) = 2(n+1)\|\underline{f}\|_\infty \|\underline{z}_o\|_1^{-1} \sum_{j=1}^{q-1} \binom{q-1}{j} \geq c_1(z) \qquad \text{(B.22)}$$

Hence Eq. B. 19 yields Eq. B. 3, namely,

$$\Big| \|\underline{\epsilon}(\beta^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\beta^*(\underline{z}), \underline{z}_o)\|_\infty \Big| \leq c_2(\underline{z}_o)\|\underline{z} - \underline{z}_o\|_1 \qquad \text{(B.23)}$$

under the constraint $\|\underline{z} - \underline{z}_o\| \leq \|\underline{z}\|_\infty$. To show that this constraint contains the assumed constraint $\|\underline{z} - \underline{z}_o\|_1 \leq \frac{1}{2} \|\underline{z}_o\|_\infty$, we note that

$$\|\underline{z}_o\|_\infty - \|\underline{z}\|_\infty \leq \|\underline{z}_o - \underline{z}\|_\infty = \|\underline{z} - \underline{z}_o\|_\infty \leq \|\underline{z} - \underline{z}_o\|_1 \leq \|\underline{z}\|_\infty$$

that is

$$\|\underline{z}\|_\infty \geq \frac{1}{2} \|\underline{z}_o\|_\infty$$

(2) When $\underline{z}_o = 0$, Eq. (B.9) becomes

$$\|[Z(0)]\, \underline{\beta}^*(0)\|_\infty = \left| \sum_{k=1}^{n} \beta_k^*(0) \right| \leq 2\|\underline{f}\|_\infty \qquad \text{(B.24)}$$

Hence, when $\underline{z}_o = 0$, then

$$\Big| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}_o), \underline{z}_o)\|_\infty \Big|$$

$$\leq \|[Z(\underline{z}) - Z(0)]\, \underline{\beta}^*(0)\|_\infty = \max_{1 \leq i \leq q-1} \left| \sum_{k=1}^{n} \beta_k^*(0)\, z_k^i \right|$$

$$\leq \left(\left| \sum_{k=1}^{n} \beta_k^*(0) \right| \right) \cdot \left(\max_{1 \leq i \leq q-1} \sum_{k=1}^{n} |z_k|^i \right)$$

$$\leq 2\|\underline{f}\|_\infty \max_{1 \leq i \leq q-1} \|\underline{z}\|_1^i$$

$$\leq 2\|\underline{f}\|_\infty \|\underline{z}\|_1, \quad \text{if} \quad \|\underline{z}\|_1 \leq 1 \qquad \text{(B.25)}$$

Furthermore, when $\underline{z} = 0$, then

$$\left| \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z})\|_\infty - \|\underline{\epsilon}(\underline{\beta}^*(\underline{z}), \underline{z}_0)\|_\infty \right|$$

$$\leq \| [Z(\underline{z}) - Z(0)]\, \underline{\beta}^*(\underline{z})\|_\infty = \max_{1 \leq i \leq q-1} \left| \sum_{k=1}^{n} \beta_k^*(\underline{z})\, z_k^i \right|$$

$$\leq \left| \sum_{k=1}^{n} \beta_k^*(\underline{z}) \right| \cdot \max_{1 \leq i \leq q-1} \sum_{k=1}^{n} |z_k|^i$$

$$\leq 2 \|\underline{f}\|_\infty \; \|\underline{z}\|_1 , \quad \text{if} \quad \|\underline{z}\|_1 \leq 1 \qquad \text{(B.26)}$$

since $\left| \sum_{k=1}^{n} \beta_k^*(\underline{z}) \right| \leq 2 \|\underline{f}\|_\infty$ from Eq. B.9 when $i = 1$.

APPENDIX C

PRONY'S EXTENDED METHOD IN AN m-DIMENSIONAL SUBSPACE, $m \geq n+1$

In this section we shall derive the relation given by Eq. 4.58, namely

$$[\Lambda^{(w)}(\underline{r})]^T \underline{f}^{(w)} = [\Lambda^{(w)}(\underline{r})]^T \underline{\epsilon}^{(w)}, \qquad w = 1, \ldots, \binom{q}{2n+1} \qquad \text{(C. 1)}$$

where $\underline{f}^{(w)}$ and $\underline{\epsilon}^{(w)}$ represent the projections of the vectors $\underline{f}$ and $\underline{\epsilon}$ in U^q onto U_w^{2n+1}, respectively, and where $[\Lambda^{(w)}(\underline{r})]$ is a $(2n+1) \times (n+1)$ matrix as a function of $\underline{r} \in \mathscr{R}$. Since one can obtain an analogous relation in any m-dimensional reference subspace of U^q, where $m \geq n+1$, we shall formulate "Prony's Extended Method" in a general m-dimensional reference subspace U_k^m which is defined by Definition 2.1, when $m \geq n+1$.

In the m-dimensional k-th reference subspace U_k^m, where[1] $k \in \{\mu = 1, \ldots, \binom{q}{m}\}$, the m-dimensional projections of the vectors $\underline{f}$, $[Z(\underline{z})] \underline{\beta}$, and $\underline{\epsilon}$ in U^q are given by

$$\underline{f}^{(k)} = [I_k]^T \underline{f} \quad \epsilon U_k^m \qquad \text{(C. 2)}$$

---

[1] Note that one can form $\binom{q}{m}$ distinct m-dimensional reference subspaces from U^q.

$$[Z^{(k)}(\underline{z})]\,\underline{\beta} = [I_k]^T [Z(\underline{z})]\,\underline{\beta} \quad \epsilon\ U_k^m, \quad \text{and} \tag{C.3}$$

$$\underline{\epsilon}^{(k)} = [I_k]^T\,\underline{\epsilon} \quad \epsilon\ U_k^m \tag{C.4}$$

where $[I_k]$ is a q x m matrix defined by Eq. 2.25. The relation between these vectors in U_k^m is given by

$$\underline{f}^{(k)} = [Z^{(k)}(\underline{z})]\,\underline{\beta} + \underline{\epsilon}^{(k)} \tag{C.5}$$

Note that $[Z^{(k)}(\underline{z})]$ is an (m x n)-matrix, where $m \geq n+1$. To formulate the relation of Eq. C.5 in terms of "Prony's Extended Method" we must define a matrix which is a function of[2] $\underline{r} \in \mathscr{R}$ and whose column space is the orthogonal complement of the column space of the m x n matrix $[Z^{(k)}(\underline{z})]$ in U_k^m. Observe that since the matrix $[Z^{(k)}(\underline{z})]$ is of rank n then its column space, $C_n(Z^{(k)})$, defines an n-dimensional subspace in U_k^m. Then, the orthogonal complement subspace in U_k^m must be an (m-n)-dimensional subspace. This (m-n)-dimensional subspace can be represented by the column space of an m x (m-n) matrix which will be denoted by $[\Lambda^{(k)}(\underline{r})]$. We shall now show that the matrix $[\Lambda^{(k)}(\underline{r})]$ can be obtained directly from the matrix $[R(\underline{r})]$ by appropriate column and row operations.

[2] Note that when the vector $\underline{r}$ is in $\mathscr{R}$, the relationship between the q x n matrix $[Z(\underline{z})]$ and the q x (q-n) matrix $[R(\underline{r})]$, given by Lemma 3.1, is preserved. This point will be discussed further.

Consider the relation of Eq. 3.16, namely

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_n & 0 & & \cdots & 0 \\ 0 & r_0 & r_1 & \cdots & r_n & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & r_0 & r_1 & \cdots & r_n & & 0 \\ 0 & \cdots & 0 & r_0 & r_1 & \cdots & & r_n \end{bmatrix} \begin{bmatrix} f_1 - \epsilon_1 \\ f_2 - \epsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ f_q - \epsilon_q \end{bmatrix} \qquad \text{(C. 6)}$$

or alternately

$$\sum_{i=0}^{n} r_i f_{i+\nu} = \sum_{i=0}^{n} r_i \epsilon_{i+\nu}, \qquad \nu = 1, \ldots, q-n \qquad \text{(C. 7)}$$

Observe that any one equation out of the set given by Eq. C. 7 relates (n+1) consecutive components of $\underline{f}$ and $\underline{\epsilon} \in U^q$. Hence, it suffices to say that the ν-th equation of Eq. C. 7 represents the relation between the projections of $\underline{f}$ and $\underline{\epsilon}$ in U^q onto a particular (n+1)-dimensional reference subspace, U_ν^{n+1}, of U^q, where U_ν^{n+1} contains only the (n+1) components of the vectors which related by the ν-th equation of of Eq. C. 7.

It stands to reason that if $m \geq n+1$, then any (m-n) consecutive equations out of the set given by Eq. C. 7 represent the relation between the projection of the vectors $\underline{f}$ and $\underline{\epsilon}$ in some particular m-dimensional reference subspace U_k^m. To illustrate this let us consider the m-dimensional subspaces U_μ^m, $\mu = 1,\ldots,q-m$, defined by mapping $I_\mu : U^q \rightarrow U_\mu^m$, where

$$[I_\mu] = [\underline{\xi}_\mu \;\; \underline{\xi}_{\mu+1} \;\; \cdots \;\; \underline{\xi}_{\mu+m-1}], \qquad \mu = 1,\ldots,q-m \tag{C.8}$$

and where $\underline{\xi}_j$ denotes a q-dimensional vector whose j-th component equals one and all the other components are equal to zero. Hence, in each U_μ^m the projections of $\underline{f}$ and $\underline{\epsilon}$ in U^q are given by

$$\underline{f}^{(\mu)} = [I_\mu]^T \underline{f} = \begin{bmatrix} f_\mu \\ \vdots \\ f_{\mu+m-1} \end{bmatrix}, \quad \text{and } \underline{\epsilon}^{(\mu)} = [I_\mu]^T \underline{\epsilon} = \begin{bmatrix} \epsilon_\mu \\ \vdots \\ \epsilon_{\mu+m-1} \end{bmatrix} \tag{C.9}$$

From Eq. C. 7 the relation between the vectors $\underline{f}^{(\mu)}$ and $\underline{\epsilon}^{(\mu)}$ in U_μ^m becomes

$$[\Lambda^{(\mu)}(\underline{r})]^T \underline{f}^{(\mu)} = [\Lambda^{(\mu)}(\underline{r})]^T \underline{\epsilon}^{(\mu)} \tag{C.10}$$

where $[\Lambda^{(\mu)}(\underline{r})]$ is an m x (m-n) matrix given by

$$[\Lambda^{(\mu)}(\underline{r})] = \underbrace{\begin{bmatrix} r_0 & 0 & \cdots & 0 \\ r_1 & r_0 & & \vdots \\ \vdots & r_1 & & 0 \\ & \vdots & & r_0 \\ r_n & & & r_1 \\ 0 & r_n & & \vdots \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}}_{(m-n)} \Bigg\} m \tag{C. 11}$$

It should be noted that Eq. C. 10 represents Prony's formulation of the relation

$$\underline{f}^{(\mu)} = [Z^{(\mu)}(\underline{z})]\,\underline{\beta} + \underline{\epsilon}^{(\mu)}, \quad \mu = 1,\ldots,q-m \tag{C. 12}$$

where $[Z^{(\mu)}(\underline{z})]$ is an m x n matrix given by

$$[Z^{(\mu)}(\underline{z})] = \begin{bmatrix} z_1^{\mu-1} & \cdots & z_n^{\mu-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ z_1^{\mu-m-1} & \cdots & z_n^{\mu-m-1} \end{bmatrix}$$

Furthermore, by using the relation between the vectors $\underline{r}$ and $\underline{z}$ given by Definition 3.1, it can be shown,[3] that the column spaces of $[Z^{(\mu)}(\underline{z})]$ and $[\Lambda^{(\mu)}(\underline{r})]$ are orthogonal complements in U_{μ}^{m}.

Up to this point we have considered the m-dimensional subspaces $\{U_{\mu}^{m} : \mu = 1, \ldots, q-m\}$ which contain m-consecutive components of the vectors in U^q. Let us now obtain the matrix $[\Lambda^{(\mu)}(\underline{r})]$ in the subspaces $\{U_{\mu}^{m} : \mu = q-m+1, \ldots, \binom{q}{m}\}$, which do not contain m-consecutive components of the vectors in U^q. Here the matrix $[\Lambda^{(k)}(\underline{r})]$ is obtained by taking the linear combinations of the columns of the matrix $[R(\underline{r})]$. To show this let us observe that the matrix $[\Lambda^{(\mu)}(\underline{r})]$ given by Eq. C.11 has been obtained from Eq. C.6 (or Eq. C.7) by selecting a set of (m-n)-independent rows of the matrix $[R(\underline{r})]^T$ which relate only[4] the m-components of the vectors in U^q that are in U_{μ}^{m}. Hence, to determine the general m x (m-n) matrix $[\Lambda^{(k)}(\underline{r})]$ one must obtain a set of (m-n) independent rows of $[R(\underline{r})]^T$ (or columns of $[R(\underline{r})]$) which relate only the m-components of the vectors in U^q that are in the subspace U_k^m.

The following example illustrates the method of forming the m x (m-n) matrix $[\Lambda^{(k)}(\underline{r})]$ from the matrix $[R(\underline{r})]$.

---

[3]This will be done in Lemma C.1.

[4]Note that the elements of these (m-n) rows of the matrix $[R(\underline{r})]^T$, which correspond to the (q-m) components of the vectors in $\overline{U}^q$ that are not in U_{μ}^{m}, are equal to zero.

Example: Let q = 9, n = 3, and m = 6. Equation C. 6 becomes

$$[R(\underline{r})]^T (\underline{f} - \underline{\epsilon}) = \begin{bmatrix} r_0 & r_1 & r_2 & r_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_0 & r_1 & r_2 & r_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_0 & r_1 & r_2 & r_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_0 & r_1 & r_2 & r_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_0 & r_1 & r_2 & r_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_0 & r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} f_1 - \epsilon_1 \\ f_2 - \epsilon_2 \\ f_3 - \epsilon_3 \\ f_4 - \epsilon_4 \\ f_5 - \epsilon_5 \\ f_6 - \epsilon_6 \\ f_7 - \epsilon_7 \\ f_8 - \epsilon_8 \\ f_9 - \epsilon_9 \end{bmatrix} = 0 \qquad \text{(C. 13)}$$

Consider the subspace U_1^6 which contains the components of the vectors $\underline{f}$ and $\underline{\epsilon} \in U^9$, given by the vectors

$$\underline{f}^{(1)} = \begin{bmatrix} f_1 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_8 \end{bmatrix} \quad \text{and} \quad \underline{\epsilon}^{(1)} = \begin{bmatrix} \epsilon_1 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_8 \end{bmatrix} \qquad \text{(C. 14)}$$

Find the 6 x 3 matrix $[\Lambda^{(1)}(\underline{r})]$ which relates the vectors $\underline{f}^{(1)}$ and $\underline{\epsilon}^{(1)}$ by

$$[\Lambda^{(1)}(\underline{r})]^T \underline{f}^{(1)} = [\Lambda^{(1)}(\underline{r})]^T \underline{\epsilon}^{(1)} \qquad \text{(C. 15)}$$

Let us post-multiply the matrix $[R(\underline{r})]$, given in Eq. C. 13, by the 6 x 6 matrix $[Y]$, given by

$$[Y] = \begin{bmatrix} r_0 & 0 & 0 & 0 & 0 & 0 \\ -r_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -r_2 & 0 \\ 0 & 0 & 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This yields

$$[R(\underline{r})]\,[Y] = \begin{bmatrix} r_0^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_0 & 0 & 0 & 0 & 0 \\ (r_0r_2 - r_1^2) & r_1 & r_0 & 0 & 0 & 0 \\ (r_0r_3 - r_1r_2) & r_2 & r_1 & r_0 & -r_0r_2 & 0 \\ -r_1r_3 & r_3 & r_2 & r_1 & (r_0r_3 - r_1r_2) & 0 \\ 0 & 0 & r_3 & r_2 & (r_1r_3 - r_2^2) & r_0 \\ 0 & 0 & 0 & r_3 & 0 & r_1 \\ 0 & 0 & 0 & 0 & r_3^2 & r_2 \\ 0 & 0 & 0 & 0 & 0 & r_3 \end{bmatrix}$$

Note that the 1st, 3rd, and 5th columns of the above matrix $[R(\underline{r})]\,[Y]$ relate only the components of $\underline{f}$ and $\underline{\epsilon} \in U^9$ given by Eq. C. 14. Using these three columns, the matrix $[\Lambda^{(1)}(\underline{r})]$ of Eq. C. 15 becomes

$$[\Lambda^{(1)}(\underline{r})] = \begin{bmatrix} r_0^2 & 0 & 0 \\ (r_0r_2 - r_1^2) & r_0 & 0 \\ (r_0r_3 - r_1r_2) & r_1 & -r_0r_2 \\ -r_1r_3 & r_2 & (r_0r_3 - r_1r_2) \\ 0 & r_3 & (r_1r_3 - r_2^2) \\ 0 & 0 & r_3^2 \end{bmatrix}$$

Now that the preliminary ideas have been discussed let us show that in the m-dimensional subspace U_k^m, where $k \in \{\mu = 1,\ldots,\binom{q}{m}\}$, the relation

$$[\Lambda^{(k)}(\underline{r})]^T \underline{f}^{(k)} = [\Lambda^{(k)}(\underline{r})]^T \underline{\epsilon}^{(k)} \qquad \text{(C. 16)}$$

represents Eq. C. 5 in terms of "Prony's Extended Method." We begin by making the following definitions:

<u>Definition C. 1</u>: Let $[Z(\underline{z})]$ be the q x n matrix defined by Eq. 1. 18 (or Eq. 1. 18a), and let $[Z_k(\underline{z})]$ by a q x n matrix defined by[5]

$$[Z_k(\underline{z})] \triangleq \begin{bmatrix} Z^{(k)}(\underline{z}) \\ \text{------} \\ A^{(k)}(\underline{z}) \end{bmatrix} \begin{matrix} \}\, m \\ \}\, q-m \end{matrix} \qquad \text{(C. 17)}$$

[5]Note that the matrix $[Z_k(\underline{z})]$ represents the rearrangement of the rows of the matrix $[Z(z)]$.

where $[Z^{(k)}(\underline{z})]$ is the (m x n) matrix defined by[6] Eq. C.3, $m \geq n+1$. Then the q x q nonsingular matrix representing the elementary row operations, which takes $[Z(\underline{z})]$ into $[Z_k(\underline{z})]$, is denoted by $[X_k]$ and defined by[7]

$$[Z_k(\underline{z})] = [X_k]\,[Z(\underline{z})] \tag{C.18}$$

Definition C.2: Let $[R(\underline{r})]$ be the q x (q-n) matrix defined by Eq. 3.2, and let $[X_k]$ be the q x q nonsingular matrix defined by Eq. C.18. Then we denote by $[R_k(\underline{r})]$ the q x (q-n) matrix which is equivalent to $[R(\underline{r})]$ by the transformation

$$[R_k(\underline{r})] = [X_k]\,[R(\underline{r})]\,[Y_k] \tag{C.19}$$

where $[Y_k]$ is a (q-n) x (q-n) nonsingular matrix, representing column operations. The matrix $[Y_k]$ is chosen so that when $[R_k(\underline{r})]$ is represented by

$$[R_k(\underline{r})] \triangleq \begin{array}{c} \overbrace{\phantom{R_{11}^{(k)}(\underline{r})}}^{m-n} \quad \overbrace{\phantom{R_{12}^{(k)}(\underline{r})}}^{q-m} \\ \left[\begin{array}{c|c} R_{11}^{(k)}(\underline{r}) & R_{12}^{(k)}(\underline{r}) \\ \hline R_{21}^{(k)}(\underline{r}) & R_{22}^{(k)}(\underline{r}) \end{array}\right] \begin{array}{l} \}\ m \\ \}\ q-m \end{array} \end{array} \tag{C.20}$$

then the (q-m) x (m-n) submatrix $[R_{21}^{(k)}]$ is a zero matrix, i.e.,

$$[R_{21}^{(k)}(\underline{r})] = [0] \tag{C.21}$$

---

[6] Note, from Eq. C.3, that $[Z^{(k)}(\underline{z})] = [I_k]^T\,[Z(\underline{z})]$.

[7] The matrix $[X_k]$ is equivalent to the product of the elementary matrices $[F_{i_k}]$ defined by Thrall (Ref. 24, p. 94).

Definition C. 3: Let $[R_{11}^{(k)}(\underline{r})]$ be the (m x m-n) submatrix, where $m \geq n+1$, of the q x (q-n) matrix $[R_k(\underline{r})]$ defined by Eq. C. 20. Then we denote by $[\Lambda^{(k)}(\underline{r})]$ the m x (m-n) matrix which is column equivalent to $[R_{11}^{(k)}(\underline{r})]$ by the transformation

$$[\Lambda^{(k)}(\underline{r})] = [R_{11}^{(k)}(\underline{r})]\,[K_k] \tag{C. 22}$$

where $[K_k]$ is an (m-n) x (m-n) nonsingular matrix representing the column operations. The matrix $[K_k]$ is chosen so that the matrix $[\Lambda^{(k)}(\underline{r})]$ will have the form

$$[\Lambda^{(k)}(\underline{r})] = \begin{bmatrix} \lambda_{1,1}^{(k)}(\underline{r}) & 0 & \cdots & 0 \\ \lambda_{2,1}^{(k)}(\underline{r}) & \lambda_{2,2}^{(k)}(\underline{r}) & & \vdots \\ \vdots & \lambda_{3,2}^{(k)}(\underline{r}) & & 0 \\ \lambda_{n+1,1}^{(k)}(\underline{r}) & \vdots & & \lambda_{m-n,m-n}^{(k)}(\underline{r}) \\ 0 & \lambda_{n+2,2}^{(k)}(\underline{r}) & & \lambda_{m-n+1,m-n}^{(k)}(\underline{r}) \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_{m,m-n}^{(k)}(\underline{r}) \end{bmatrix} \tag{C. 23}$$

where[8]

$$\lambda_{i,j}^{(k)}(\underline{r}) = \sum_{\nu} \cdots \sum_{\mu} a_{\nu \ldots u}^{(i,j)} \, r_0^{\nu} \cdots r_n^{\mu}$$

Let us now state the following lemma:

Lemma C.1: Let $[Z(\underline{z})]$ be a $q \times n$ matrix, defined by Eq. 1.18 (or Eq. 1.18a), of maximal rank n ($q > 2n$), and let $[R(\underline{r})]$ be a $q \times (q-n)$ matrix, defined by Eq. 3.2, of rank $(q-n)$. Furthermore, let $[Z^{(k)}(\underline{z})]$ be a $(m \times n)$ matrix, of rank n ($m \geq n+1$), which is related to the matrix $[Z(\underline{z})]$ by Eq. C.3; and let $[\Lambda^{(k)}(\underline{r})]$ be a $m \times (m-n)$ matrix, of rank $(m-n)$ which is related to the matrix $[R(\underline{r})]$ according to Definition C.3. Then, $C_n(Z^{(k)})$, the n-dimensional subspace of U_k^m, defined by the column space of $[Z^{(k)}(\underline{z})]$, and $C_{m-n}(\Lambda^{(k)})$, the $(m-n)$-dimensional subspace complement of U_k^m, defined by the column space of $[\Lambda^{(k)}(\underline{r})]$ are orthogonal complement subspaces in U_k^m, that is,

$$C_n(Z^{(k)}) \oplus C_{m-n}(\Lambda^{(k)}) = U_k^m \qquad \text{(C.24)}$$

---

[8]The nonzero elements, $\lambda_{ij}^{(k)}(\underline{r})$, of the matrix $[\Lambda^{(k)}(\underline{r})]$, represent some polynomial in the n+1-variables $\{r_0, r_1, \ldots, r_n\}$ denoting the components of the vector $\underline{r} \in \mathscr{R}$.

where

$$C_n(Z^{(k)}) \perp C_{m-n}(\Lambda^{(k)})$$

if,

$$C_n(Z) \oplus C_{q-n}(R) = U^q \qquad \text{(C. 25)}$$

where

$$C_n(Z) \perp C_{q-n}(R)$$

and where $C_n(Z)$ is the n-dimensional subspace of U^q, defined by the column space of $[Z(\underline{z})]$, and $C_{q-n}(R)$ is the (q-n)-dimensional subspace of U^q, defined by the column space of $[R(\underline{r})]$.

Proof: Let us consider the q x n matrix[9] $[Z_k]$ which is row equivalent to the q x n matrix $[Z]$ and defined by Eq. C. 17. Furthermore, let us consider the q x (q-n) matrix $[R_k]$ which is equivalent to the q x (q-n) matrix $[R]$ and defined by Eq. C. 19. Taking the matrix product $[R_k]^T [Z_k]$, yields

$$\begin{aligned} [R_k]^T [Z_k] &= [Y_k]^T [R]^T [X_k]^T [X_k] [Z] \\ &= [Y_k]^T [R]^T [Z] \end{aligned} \qquad \text{(C. 26)}$$

[9] To simplify our notation, we shall drop the parameter vector $\underline{z}$ or $\underline{r}$ when writing the matrices $[Z(\underline{z})]$ or $[R(\underline{r})]$.

since the matrix product $[X_k]^T [X_k] = [I]$ by definition (Ref. 24, p. 94), where $[I]$ is a q x q identity matrix. Recall from Lemma 3.1 that Eq. C.25 is satisfied, when the matrix product $[R]^T [Z] = [0]$. Since, by definition, the matrix $[Y_k]$ is nonsingular, then Eq. C.25 is satisfied, if the matrix product $[R_k]^T [Z_k]$, of Eq. C.26, satisfies

$$[R_k]^T [Z_k] = [0] \tag{C.27}$$

By using the partitionings of $[R_k]$ and $[Z_k]$, which are defined in Eq. C.20 and Eq. C.17, respectively, one obtains the following relations from Eq. C.27:

$$[R_{11}^{(k)}]^T [Z^{(k)}] + [R_{21}^{(k)}]^T [A^{(k)}] = 0 \tag{C.28}$$

$$[R_{12}^{(k)}]^T [Z^{(k)}] + [R_{22}^{(k)}]^T [A^{(k)}] = 0 \tag{C.29}$$

Since the matrix $[Y_k]$ was chosen so that the matrix $[R_{21}^{(k)}]$ is a zero matrix, then Eq. C.28 yields,

$$[R_{11}^{(k)}]^T [Z^{(k)}] = [0] \tag{C.30}$$

where $[R_{11}^{(k)}]$ is a m x (m-n) matrix defined by Eq. C.20 and $[Z^{(k)}]$ is a m x n matrix defined by Eq. C.3. Furthermore, using the relation

$$[R_{11}^{(k)}] = [\Lambda^{(k)}] [K_k]^{-1} \tag{C.31}$$

where $[\Lambda^{(k)}]$ is a m x (m-n) matrix, defined by Eq. C. 23; then Eq. C. 30 yields

$$[K_k^T]^{-1} [\Lambda^{(k)}(\underline{r})]^T [Z^{(k)}(\underline{z})] = [0]$$

or alternately

$$[\Lambda^{(k)}(\underline{r})]^T [Z^{(k)}(\underline{z})] = [0] \tag{C. 32}$$

since $[K_k^T]$ is invertible. Clearly, Eq. C. 32 states that the subspaces $C_n(Z^{(k)})$ and $C_{m-n}(\Lambda^{(k)})$ are orthogonal in U_k^m. Since the dimensionality of $C_n(Z^{(k)})$ is n and the dimensionality of $C_{m-n}(\Lambda^{(k)})$ is (m-n), then $C_n(Z^{(k)})$ and $C_{m-n}(\Lambda^{(k)})$ are orthogonal complement subspaces in U_k^m and so Eq. C. 24 is satisfied. Thus, the lemma is proved.

It should be mentioned that the hypothesis of Lemma C. 1, which is given by Eq. C. 25, can be replaced by the polynomial equation $P(z) = 0$, defined by Eq. 3.4 of Chapter III. This leads to the following corollary:

Corollary C. 1: Equation C. 24 of Lemma C. 1 is satisfied if the components $\{z_1, z_2, \ldots, z_n\}$ of the vector $\underline{z} \in \mathcal{Z}$ are the roots of the n-th order real polynomial equation

$$P(z) = \sum_{i=0}^{n} r_i z^i = 0 \tag{C. 33}$$

where the ordered set $\{r_0, r_1, \ldots, r_n\}$ defines the vector $\underline{r} \in \mathcal{R}$.

Proof: All of this corollary is contained in Lemma 3.1, Chapter III, and Lemma C.2. Clearly, Eq. C.33 simply replaces the condition given by Eq. C.24 of Lemma C.1. The validity of such a replacement has been established by Lemma 3.1.

At this point let us examine the converse problem, namely, if Eq. C.24 is true, is there a unique n-th order polynomial equation, P(z) = 0, which relates the vectors $\underline{r}$ and $\underline{z}$? Recall that if Eq. C.33 is true, i.e., the subspaces $C_n(Z^{(k)})$ and $C_{m-n}(\Lambda^{(k)})$ are orthogonal complements in U_k^m, then the matrices $[Z^{(k)}(\underline{z})]$ and $[\Lambda^{(k)}(\underline{r})]$ are related by Eq. C.32, or equivalently by Eq. C.28 (since the matrix $[R_{21}{}^{(k)}] = [0]$ by definition). Using Eqs. C.28, C.30, C.31, and C.32, let us relate the matrix product $[\Lambda^{(k)}]^T[Z^{(k)}]$ to the matrix product $[R]^T[Z]$ as follows:

$$[K_k^T]^{-1}[\Lambda^{(k)}]^T[Z^{(k)}] = \left[[R_{11}{}^{(k)}]^T \,\vdots\, [R_{21}{}^{(k)}]^T \right] \begin{bmatrix} [Z^{(k)}] \\ \text{-----} \\ [A^{(k)}] \end{bmatrix}$$

$$= [I_y]^T[R_k]^T[Z_k]$$

$$= [I_y]^T[Y_k]^T[R]^T[Z] = 0 \qquad \text{(C.34)}$$

where $[I_y]$ is a (q-n) x (m-n) elementary matrix which picks the first (m-n) rows of the matrix $[R_k]^T$. Premultiplying both sides of Eq. C.34 by the matrix $[K_k^T]$, yields

$$[\Lambda^{(k)}]^T[Z^{(k)}] = [K_k]^T[I_y]^T[Y_k]^T[R]^T[Z] = 0 \qquad \text{(C.35)}$$

Since the matrix product $[K_k]^T [I_y]^T [Y_k]^T$ represents the elementary row operations on the matrix product $[R]^T [Z]$, which is defined by Eq. 3.8, then in carrying out the matrix multiplication $[K_k]^T [I_y]^T [Y_k]^T [R]^T [Z]$, one obtains

$$[\Lambda^{(k)}]^T [Z^{(k)}] = \begin{bmatrix} Q_1^{(k)}(z_1)P(z_1) & \cdots & Q_1^{(k)}(z_n)P(z_n) \\ Q_2^{(k)}(z_1)P(z_1) & \cdots & Q_2^{(k)}(z_n)P(z_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ Q_{m-n}^{(k)}(z_1)P(z_1) & \cdots & Q_{m-n}^{(k)}(z_n)P(z_n) \end{bmatrix} = [0] \qquad \text{(C. 36)}$$

where $P(z_j)$ is defined by Eq. 3.9 and $\{Q_i^{(k)}(z_j) : i = 1,2,\ldots,m-n\}$ represents a set of N_i-th order polynomial, where $N_i < q-n$. Since the ij-th element of the matrix product $[\Lambda^{(k)}]^T [Z^{(k)}]$ can be represented by $Q_i^{(k)}(z_j)\, P(z_j)$, then the n-roots, $\{z_1, z_2, \ldots, z_n\}$, which are common to all the rows of Eq. C. 36, are given by the polynomial equation

$$P(z) = 0 \qquad \text{(C. 37)}$$

Note that Eq. C. 37 gives a relation between the vectors $\underline{r}$ and $\underline{z}$ which is independent of the superscript k (i. e., invariant of the m -dimensional reference subspace) and also the subscript i where $i = 1,2,\ldots,m-n$. These results are summarized by the following lemma:

Lemma C.2: Let $[Z(\underline{z})]$ be a q x n matrix, defined by Eq. 1.18 (or Eq. 1.18a), of maximal rank n ($q > 2n$), and let $[R(\underline{r})]$ be a q x (q-n) matrix, defined by Eq. 3.2, of rank (q-n). Furthermore, let $[Z^{(k)}(\underline{z})]$ be a m x n matrix ($m \geq n+1$), of rank n, which is related to the matrix $[Z(\underline{z})]$ by Eq. C.3, and let $[\Lambda^{(k)}(\underline{r})]$ be a m x (m-n) matrix, of rank (m-n), which is related to the matrix $[R(\underline{r})]$ according to Definition C.3. Then,

$$C_n(Z) \oplus C_{q-n}(R) = U^q \qquad \text{(C.38)}$$

where

$$C_n(Z) \perp C_{q-n}(R)$$

if

(1) $$C_n(Z^{(k)}) \oplus C_{m-n}(\Lambda^{(k)}) = U_k^m \qquad \text{(C.39)}$$

where

$$C_n(Z^{(k)}) \perp C_{m-n}(\Lambda^{(k)}), \quad \text{and if}$$

(2) $$\{Q_i^{(k)}(z) \neq 0 : i = 1,2,\ldots,m-n\}$$

when

$$Q_i^{(k)}(z) \neq P(z) \tag{C.40}$$

where $\{Q_i^{(k)} : i = 1,2,\ldots,m-n\}$ is defined by Eq. C.36.

The proof of the lemma follows directly from the discussion that was presented above.

We are now in a position to formulate "Prony's Extended Method" in each m-dimensional reference subspace U_k^m, where $k \in \{\mu = 1,2,\ldots,\binom{q}{m}\}$. Recall that Lemma C.1 tells us that for each $\underline{r} \in \mathscr{R}$ we can define a vector $[\Lambda^{(k)}(\underline{r})]\underline{\xi} \in U_k^m$ which is orthogonal to the vector $[Z^{(k)}(\underline{z})]\underline{\beta} \in U_k^m$, that is

$$\left([\Lambda^{(k)}(\underline{r})]\underline{\xi},\ [Z^{(k)}(z)]\underline{\beta}\right) = 0 \tag{C.41}$$

for all $\underline{\beta} \in \mathscr{B}_z$ and $\underline{\xi} \in E^{m-n}$. Let us take the inner product of both sides of Eq. C.5 with respect to the vector $[\Lambda^{(k)}(\underline{r})]\underline{\xi}$. This yields

$$\left([\Lambda^{(k)}(\underline{r})]\underline{\xi},\ \underline{f}^{(k)}\right) = \left([\Lambda^{(k)}(\underline{r})]\underline{\xi},\ \underline{\epsilon}^{(k)}\right) \tag{C.42}$$

when using the relation of Eq. C.41. For Eq. C.42 to hold for all $\underline{\xi} \in E^{m-n}$, we obtain the relation

$$[\Lambda^{(k)}(\underline{r})]^T\,\underline{f}^{(k)} = [\Lambda^{(k)}(\underline{r})]^T\,\underline{\epsilon}^{(k)}(\underline{r}) \tag{C.43}$$

where $\underline{f}^{(k)}$ is a prescribed real vector in U_k^m, and where $\underline{r} \in \mathscr{R}$ and[10] $\underline{\epsilon}^{(k)}(\underline{r}) \in U_k^m$ are unknown vectors, $k \in \{\mu = 1,2,\ldots,\binom{q}{m}\}$. This equation defines the basic relation which must be satisfied when using "Prony's Extended Method" to solve the Chebyshev approximation problem in U_k^m, $k \in \{\mu = 1,2,\ldots,\binom{q}{m}\}$, defined by Eq. C.5.

The following theorem shows that for every $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}^{(k)} \in U_k^m$ satisfying Eq. C.43, one can determine a unique vector $\underline{\epsilon} \in U^q$ which satisfies the relation $[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon}$:

Theorem C.1: Let $\underline{f}$ be a prescribed real vector in U^q. Then for each $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}^{(k)} \in U_k^m$ satisfying

$$[\Lambda^{(k)}(\underline{r})]^T \underline{f}^{(k)} = [\Lambda^{(k)}(\underline{r})]^T \underline{\epsilon}^{(k)} \tag{C.44}$$

$k \in \{\mu = 1,2,\ldots,\binom{q}{m}\}$, where $\underline{f}^{(k)} \in U_k^m$ is related to $\underline{f} \in U^q$ by Eq. C.2, and where $[\Lambda^{(k)}(\underline{r})]$ is defined by Eq. C.23; there exists a unique vector $\underline{\epsilon} \in U^q$ which satisfies the relation

$$[R(\underline{r})]^T \underline{f} = [R(\underline{r})]^T \underline{\epsilon} \tag{C.45}$$

where $[R(\underline{r})]$ is defined by Eq. 3.2.

---

[10] Strictly speaking the vector $\underline{\epsilon}^{(k)}(\underline{r})$ should be denoted by $\underline{\epsilon}^{(k)}(\underline{\beta},\underline{r})$ since it represents the vector $\underline{\epsilon}^{(k)}(\underline{\beta},\underline{z})$ of Eq. C.5.

Proof: Consider any index $k \in \{\mu = 1,2,\ldots,\binom{q}{m}\}$. Recall from Definition C. 3 that the (m) x (m-n) matrix $[\Lambda^{(k)}(\underline{r})]$ is column equivalent to the m x (m-n) matrix $[R_{11}^{(k)}(\underline{r})]$, that is,

$$[\Lambda^{(k)}(\underline{r})] = [R_{11}^{(k)}(\underline{r})]\,[K_k] \tag{C.46}$$

Furthermore, from Definition C. 2 the matrix $[R_{11}^{(k)}(\underline{r})]$ is a submatrix of the q x (q-n) matrix $[R_k(\underline{r})]$ defined by Eq. C. 20. Let us now represent the relation of Eq. C. 45 in terms of the matrix $[R_k(\underline{r})]$ by using Eq. C. 19. This yields

$$[R_k(\underline{r})]^T [X_k]\, \underline{f} = [R_k(\underline{r})]^T [X_k]\, \underline{\epsilon} \tag{C.47}$$

By using Definition C. 1, the vectors $[X_k]\underline{f}$ and $[X_k]\underline{\epsilon}$ may be represented by

$$[X_k]\,\underline{f} = \begin{bmatrix} \underline{f}^{(k)} \\ ---- \\ \underline{f}_1^{(k)} \end{bmatrix}, \quad [X_k]\,\underline{\epsilon} = \begin{bmatrix} \underline{\epsilon}^{(k)} \\ ---- \\ \underline{\epsilon}_1^{(k)} \end{bmatrix}$$

where $\underline{f}^{(k)}$ and $\underline{\epsilon}^{(k)}$ represent the projections of $\underline{f}$ and $\underline{\epsilon}$ in U^q onto U_k^m, respectively. Hence, Eq. C. 47 becomes

$$[R_k(\underline{r})]^T \begin{bmatrix} \underline{f}^{(k)} \\ ---- \\ \underline{f}_1^{(k)} \end{bmatrix} = [R_k(\underline{r})]^T \begin{bmatrix} \underline{\epsilon}^{(k)} \\ ---- \\ \underline{\epsilon}_1^{(k)} \end{bmatrix}$$

which yields the following two equations, when using the partitioning of the matrix $[R_k(\underline{r})]$, defined by Eq. C. 20:

$$[R_{11}^{(k)}(\underline{r})]^T \underline{f}^{(k)} = [R_{11}^{(k)}(\underline{r})]^T \underline{\epsilon}^{(k)} \tag{C.48}$$

and

$$[R_{12}(\underline{r})]^T \underline{f}^{(k)} + [R_{22}(\underline{r})]^T \underline{f}_1^{(k)} = [R_{12}(\underline{r})]^T \underline{\epsilon}^{(k)} + [R_{22}(\underline{r})]^T \underline{\epsilon}_1^{(k)} \tag{C.49}$$

First we note that the vectors $\underline{r}$ and $\underline{\epsilon}^{(k)}$ which satisfy Eq. C.44 will also satisfy Eq. C.48 since the matrices $[\Lambda^{(k)}(\underline{r})]$ and $[R_{11}^{(k)}(\underline{r})]$ are equivalent (see Eq. C.46). Next we note that when the vectors $\underline{r}$ and $\underline{\epsilon}^{(k)}$ are known then the only unknowns in Eq. C.49 are given by the vector $\underline{\epsilon}_1^{(k)}$, since the vectors $\underline{f}^{(k)}$ and $\underline{f}_1^{(k)}$ are prescribed. Since the (q-m) x (q-m) matrix $[R_{22}^{(k)}(\underline{r})]$ is nonsingular for all $\underline{r} \neq 0 \in \mathscr{R}$, then from Eq. C.49 the (q-m)-dimensional unknown vector $\underline{\epsilon}_1^{(k)}$ is given by

$$\underline{\epsilon}_1^{(k)} = \underline{f}_1^{(k)} + \left[[R_{22}^{(k)}(\underline{r})]^T\right]^{-1} [R_{12}^{(k)}(\underline{r})]^T \left(\underline{f}^{(k)} - \underline{\epsilon}^{(k)}\right) \tag{C.50}$$

Using the relation

$$\underline{\epsilon} = [X_k]^{-1} \begin{bmatrix} \underline{\epsilon}^{(k)} \\ \hdashline \underline{\epsilon}_1^{(k)} \end{bmatrix}$$

it is seen that the vector $\underline{\epsilon} \in U^q$ which satisfies Eq. C.45 is fully determined when the vectors $\underline{r}$ and $\underline{\epsilon}^{(k)}$ are known and satisfy Eq. C.44. Thus the theorem is proved.

An alternate way of proving the above result is to note that for a given pair of vectors $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}^{(k)} \in U_k^m$, which satisfy Eq. C. 44, one can first obtain the vector pair[11] $(\underline{\beta}, \underline{z})$ from Eqs. C. 33 and C. 5. Then obtain the vector $\underline{\epsilon} \in U^q$ by using Eq. 4. 47 of Chapter IV. Specifically, knowing the vector $\underline{r} \in \mathscr{R}$, one can obtain the vector $\underline{z} \in \mathscr{Z}$ from the roots of the polynomial equation of Eq. C. 33. Then the vector $\underline{\beta}$ can be obtained from the relation

$$\underline{f}^{(k)} = [Z^{(k)}(\underline{z})]\,\underline{\beta} + \underline{\epsilon}^{(k)} \quad \text{in} \quad U_k^m$$

since the vectors $\underline{f}^{(k)}$ and $\underline{\epsilon}^{(k)}$ are known and the matrix $[Z^{(k)}(\underline{z})]$ is fully determined by using the above calculated value of $\underline{z}$. Once the vector pair $(\underline{\beta}, \underline{z}) \in \mathscr{B}_z \times \mathscr{Z}$ is known, then the error vector $\underline{\epsilon} \in U^q$ can be determined from the relation of Eq. 4. 47 of Chapter IV. Clearly the advantage of using the method, outlined in the proof of Theorem C.1, to obtain the vector $\underline{\epsilon} \in U^q$ from each estimate of $\underline{r} \in \mathscr{R}$ and $\underline{\epsilon}^{(k)} \in U_k^m$, is that one need not determined the value of $\underline{\beta} \in \mathscr{B}_z$.

---

[11]That such a vector pair exists has been shown in Lemma C. 2.

LIST OF REFERENCES

1. P. R. Aigrain and E. M. Williams, "Synthesis of n-Reactance Networks for Desired Transient Response," J. of Applied Physics, Vol. 20. 6, June 1949, pp. 597-600.

2. N. I. Akhiezer, Theory of Approximation, Frederick Ungar Publishing Co., New York, 1956.

3. J. D. Brule, "Improving the Approximation to a Prescribed Time Response," IRE Trans. on Circuit Theory, Vol. CT-6, December 1959, pp. 355-361.

4. B. M. Brown, The Mathematical Theory of Linear Systems, John Wiley and Sons, Inc., New York, 1961, pp. 149-152.

5. J. W. Carr, III, "An Analytic Investigation of Transient Synthesis by Exponentials," M. S. Thesis, M. I. T., 1949.

6. B. Friedman, Principles and Techniques of Applied Mathematics, John Wiley and Sons, Inc., New York, 1956, p. 18.

7. T. N. E. Greville, "The Pseudoinverse of a Rectangular or Singular Matrix and Its Application to the Solution of Systems of Linear Equations," SIAM Review, Vol. 1, No. 1, January 1959, pp. 38-43.

8. W. H. Huggins, "Signal Theory," IRE Trans. on Circuit Theory, Vol. CT-3, No. 4, December 1956, pp. 210-216.

9. E. I. Jury, "A Simplified Stability Criterion for Linear Discrete Systems," Proc. IRE, Vol. 50, No. 6, June 1962, pp. 1493-1500.

10. W. H. Kautz, "Transients Synthesis in the Time Domain," IRE Trans. on Circuit Theory, Vol. CT-1, No. 3, September 1954, pp. 29-39.

REFERENCES (Cont.)

11. A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis: Vol. 1, Metric and Normed Spaces, Graylock Press, Rochester, N. Y., 1957.

12. R. M. Lerner, "The Representation of Signals," Trans. of the 1959 International Symposium on Circuit and Information Theory, June 16-18, 1959, pp. 197-216.

13. R. N. McDonough, "Representation and Analysis of Signals: Part XV, Matched Exponents for the Representation of Signals," Dept. of Electrical Engineering, The Johns Hopkins University, April 30, 1963.

14. C. de la Vallée Poussin, "Sur la Méthode de l'approximation Minimum," Soc. Sci. de Bruxelles, Annales, Seconde Partie, Memoires 35, 1911, pp. 1-16.

15. R. Prony, "Essai Experimental et Analytique sur les lois de la Dilatabilite des Fluides Élastiques, et sur Celles de la Force Expansive de la Vapeur de L'eau et de la Vapeur de L'alkool, à Differentes Temperatures," J. de l'Ecole Polytechnique Vol. 1, Cahier 2, An. III, Paris, 1795, pp. 24-76.

16. H. Radamacher and I. J. Schoenberg, "Helley's Theorem on Convex Domains and Tchebyscheff's Approximation Problem," Canadian J. of Math., 2, 1950, pp. 245-256.

17. E. Ya Reméz, General Computational Method of Chebyshev Approximations. The Problem with Linear Real Parameters. Izdatel'stvo Akademia Nauk Ukrainskoii, S. S. R., Kiev, 1957. Translation: Document No. AEC-tr-4491, Office of Technical Services, Department of Commerce, Washington 25, D. C.

18. T. J. Rivlin, "Overdetermined System of Linear Equations," SIAM Review, Vol. 5, No. 1, January 1963, pp. 52-66.

19. J. R. Rice, The Approximation of Functions: Vol. 1, Linear Theory, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1964.

REFERENCES (Cont.)

20. H. Ruston, "Synthesis of R-L-C Networks by Discrete Tschebyscheff Approximations in the Time Domain," Technical Report No. 107, Electronic Defense Group, Dept. of Electrical Engineering, University of Michigan, Ann Arbor, Michigan, April 1960.

21. G. F. Simmons, Introduction to Topology and Modern Analysis, McGraw-Hill Book Co., New York, 1963.

22. E. Stiefel, "Über Discrete and Lineare Tschebyscheff Approximationen," Numerishe Mathematik, Band 1, Springer Verlag 1959, pp. 1-28.

23. D. T. Tang, "Tschebyscheff Approximation of a Prescribed Impulse Response with RC Network Realization," 1961 IRE International Convention Record, Vol. 9, Part 4, pp. 214-220.

24. R. M. Thrall and L. Tornkeim, Vector Spaces and Matrices, John Wiley and Sons, Inc., London, 1957.

25. D. F. Tuttle, Jr., "Network Synthesis for Prescribed Transient Response," D. Sc. Dissertation, M.I.T., 1949.

26. W. C. Yengst, "Approximation to a Specified Time Response," IRE Trans. on Circuit Theory, Vol. CT-9, No. 2, June 1962, pp. 152-162.

27. T. Y. Young and W. H. Huggins, "Complementary Signals and Orthogonalized Exponentials," IRE Trans. on Circuit Theory, Vol. CT-9, No. 4, December 1962, pp. 362-370.

28. L. A. Zadeh and C. A. Desoer, Linear System Theory, The State Space Approach, McGraw-Hill Book Co., New York, 1963, pp. 577-581.

DISTRIBUTION LIST

No. of Copies	
23	Defense Documentation Center Attn: DDC-IRS Cameron Station (Bldg. 5) Alexandria, Virginia 22314
1	Office of Assistant Secretary of Defense (Research and Engineering) Attn: Technical Library, RM 3E1065 Washington, D. C. 20301
1	Bureau of Ships Technical Library Attn: Code 312 Main Navy Building, Room 1528 Washington, D. C. 20325
1	Director U. S. Naval Research Laboratory Attn: Code 2027 Washington, D. C. 20390
1	Commanding Officer and Director U. S. Navy Electronics Laboratory Attn: Library San Diego, California 92101
2	Headquarters, U. S. Air Force Attn: AFCIN Washington, D. C. 20325
1	Rome Air Development Center Attn: EMLAL-1 Griffiss Air Force Base New York 13442
1	Headquarters Strategic Air Command Attn: DOCE Offutt Air Force Base, Nebraska 68113

DISTRIBUTION LIST (Cont.)

No. of Copies	
1	Systems Engineering Group (SEPIR) Wright-Patterson Air Force Base, Ohio 45433
2	Electronic Systems Division (AFSC) Scientific and Technical Info Div (ESTI) L. G. Hanscom Field Bedford, Massachusetts 01731
2	Air Force Cambridge Research Laboratories Attn: CRXL-R L. G. Hanscom Field Bedford, Massachusetts 01731
2	Chief of Research and Development Department of the Army Washington, D. C. 20315
1	Ofc of the Chief of Communications-Electronics Attn: OCC-E Department of the Army Washington, D. C. 20315
2	Commanding General U. S. Army Materiel Command Attn: R & D Directorate Washington, D. C. 20315
2	Redstone Scientific Information Center Redstone Arsenal, Alabama 35809
1	Commanding Officer 52d USASASOC Fort Huachuca, Arizona 85613
1	Commanding General U. S. Army Combat Developments Command Attn: CDCMR-E Fort Belvoir, Virginia 22060

DISTRIBUTION LIST (Cont.)

No. of Copies	
1	Deputy Commander U. S. Army Combat Developments Command Communications-Electronics Agency Fort Huachuca, Arizona 85613
1	Commanding Officer Harry Diamond Laboratories Connecticut Ave. and Van Ness St., N.W. Washington, D. C. 20438
1	USAECOM Commodity Management Office AMSEL-EW Fort Monmouth, New Jersey
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-RD-LNR Fort Monmouth, New Jersey 07703
1	Commanding Officer U. S. Army Electronics Command R&D Activity White Sands Missile Range, New Mexico 88002
1	Chief, Mountain View Office Electronic Warfare Lab., USAECOM P. O. Box 205 Mountain View, California 94042
2	Commanding General U. S. Army Security Agency Attn: IADEV Arlington Hall Station Arlington, Virginia 22212
1	Commanding General U. S. Army Security Agency Attn: IACON Arlington Hall Station Arlington, Virginia 22212

DISTRIBUTION LIST (Cont.)

No. of Copies	
1	Commanding General U. S. Army Security Agency Combat Development Activity Arlington Hall Station Arlington, Virginia 22212
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-WL-D Fort Monmouth, New Jersey 07703
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-WL-C Fort Monmouth, New Jersey 07703
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-WL-S Fort Monmouth, New Jersey 07703
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-WL-N Fort Monmouth, New Jersey 07703
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-WL-E Fort Monmouth, New Jersey 07703
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-RD-GFR Fort Monmouth, New Jersey 07703
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-NL-E (Mr. Reindl) Fort Monmouth, New Jersey 07703

DISTRIBUTION LIST (Cont.)

No. of Copies	
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-NL-E (Mr. Hecker) Fort Monmouth, New Jersey 07703
1	Commanding General U. S. Army Electronics Command Attn: AMSEL-NL-E (Mr. Tepper) Fort Monmouth, New Jersey 07703
10	Cooley Electronics Laboratory The University of Michigan Ann Arbor, Michigan
1	Stanford Electronics Laboratories Stanford University Stanford, California

ERRATA

Page xvi: Definition of "$\mathscr{R}$" should be "the set of all parameter vectors $\underline{r} \in E^{n+1}$ with $\|\underline{r}\|_1 = 1$ (See Definition 3.3) 102."

Page xxvi, line 3: Replace word "optimum" by the word "unknown."

Page 3, line 8: add "dt" in the integral.

Page 5, line 10: $(\underline{\alpha}, s)$ should be $(\underline{\alpha}, \underline{s})$.

Page 6, line 4 from bottom: replace word "order" by the word "ordered."

Page 9, line 3 from bottom: replace the word "repeated" by "repeated real."

Page 13, line 15: "Eq. 1.19" should be "Eq. 1.9."

Page 20, line 1: The elements of the second column should be multiplied by Δt. The elements of the third column should be multiplied by $(\Delta t)^{j-1}$.

Page 20, line 4: Replace the word "identical" by the word "identical and real."

Page 53, line 4: Replace the word "fector" by the word "vector."

Page 99, bottom line: delete the words "and only if."

Page 121, Footnote: add sentence "Clearly, if there is no point where $\nu_n \neq 0$, then there is no finite $\underline{z}$."

Page 139, line 1: $\mathscr{R}$ should be $\mathscr{R}^1$.

ERRATA - Continued

Page 175, Footnote: $\mathcal{R}$ should be $\mathcal{R}^1$.

Page 307, Footnote: $\mathcal{R}$ should be $\mathcal{R}^1$.

Page 328, Footnote: Replace footnote by "Note that such a vector pair exists if $\underline{\nu} \in \mathcal{R}^1$ (See Lemma C. 2)."

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) The University of Michigan Cooley Electronics Laboratory Ann Arbor, Michigan 48105	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

On the Best Chebyshev Approximation of an Impulse Response Function at a Finite Set of Equally-Spaced Points

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report (Doctoral Thesis)

5. AUTHOR(S) (Last name, first name, initial)

Fischl, Robert

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1966	351	28

8a. CONTRACT OR GRANT NO. DA 28-043 AMC-01870(E) b. PROJECT NO. 280-M6-00287(E) c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 7695-2-T 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) TR-172

10. AVAILABILITY/LIMITATION NOTICES This document is subject to special export controls and each transmittal to foreign governments or nationals may be made only with prior approval of CG, U. S. Army Electronics Command, Fort Monmouth, N. J., Attn: AMSEL-WL-N.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U. S. Army Electronics Command Fort Monmouth, New Jersey 07703

13. ABSTRACT This study considers the approximation of a prescribed impulse response function, h(t), at a finite set of equally-spaced points of t, by a linear combination of exponential functions, so that the resulting error is minimum in the Chebyshev sense. Specifically, given a set of q value $\{h(t_i)\}$ of h(t), where $t_i = t_1 + (i-1)\Delta t$, $i = 1, \ldots, q$; find the 2n complex constants $\{\alpha_k, s_k\}$, $k = 1, \ldots, n$, of the function $\hat{h}(t) = (\alpha_1 \exp[s_1 t] + \ldots + \alpha_n \exp[s_n t])$ so that the Chebyshev error, $\|\epsilon(t_i)\|_\infty = \max_i |h(t_i) - \hat{h}(t_i)|$, is minimum when $q > 2n$. Since the complex α_k's and s_k's must occur in conjugate pairs, the Laplace transform of $\hat{h}(t)$ will be a rational function of s, i.e., a network function. The time domain approximation problem is formulated in terms of an approximation in the finite dimensional vector space. Prony's method, for determining the unknown $\{\alpha_k, s_k\}$ when $q = 2n$, is formally extended to the case when $q > 2n$. The existence of the best Chebyshev approximation is proved and the bounds within which lies the minimal value of $\|\epsilon(t_i)\|_\infty$ is found. A computational algorithm, which determines the unknown α_k's and s_k's simultaneously, is presented.

DD FORM 1 JAN 64 1473

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
(1) Discrete Chebyshev Approximation						
(2) Exponential Approximation						
(3) Time Domain Network Synthesis						
(4) Prony's Extended Method						
(5) Computational Algorithm						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ______________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ______________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ______________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.