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COMMENT

Reply to a Comment by R Newton

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Abstract. We point out that: (1) the major contribution of [1] is the generalised fast Cholesky algorithm, not the integral equations; (2) the Marchenko integral equation in [1] is correct as written, but does *not* determine the scattered field; and (3) in [1] it is implicitly assumed that the regular solution exists. The last two points suggest that the generalised fast Cholesky algorithm may be the *only* viable solution to the non-local problem.

The purpose of [1] was twofold: (1) to derive fast, differential, 'layer stripping' algorithms for solving inverse scattering problems with non-local potentials diagonal in the radius; and (2) to present integral equation counterparts to these differential algorithms.

Two different types of solution are involved. The scattering solution $\psi(x, t, e)$ is used in the generalised fast Cholesky differential algorithm, and in the Marchenko integral equation. The regular solution $\varphi(x, t, e)$ is used in the generalised Levinson differential algorithm, and in the Gel'fand–Levitan integral equation. The object is to generalise the one-dimensional results of [2].

Hence [1] proposes *four*, not two, methods for solving the inverse scattering problem with a non-local potential (although one of them is in fact incorrect). In real applications, the generalised fast Cholesky algorithm seems to be the simplest procedure, since it requires the least amount of computation. The other three methods may be primarily of academic interest, showing how the integral equations generalise to the non-local potential case.

There seems to be some confusion over equations reducing to other equations. In interpreting the reduction of (3.5) and (3.6) to the local potential case, it should be noted that the limit of time t in both equations, for a given x, is the instant at which the scattered field $u(x, t, e_i)$ becomes non-zero, as the wavefront passes x. In the non-local case this is $t = -|x|$, while in the local case this is $t = e_i \cdot x$. The reduction of (3.5) and (3.6) to the local case must be made with this in mind. Note that if $t = -|x|$ in (3.5) and (3.6) is replaced with $t = t_{\mathrm{LIM}}$, where $u(x, t, e_i)$ has support $[t_{\mathrm{LIM}}, \infty]$, then the reduction is as noted in the paper. I think this would be overly cumbersome to the reader; nonetheless the reductions should indeed have been explained in more detail.

I disagree with the last comment. The regular solution $\varphi(x, t, e_i)$ has the support indicated in (2.7), and solves the Schrödinger equation. This leads directly to (3.17) and (3.19). $\varphi(x, t, e_i)$ also satisfies the orthogonality condition (3.7), and this leads to the Gel'fand–Levitan equation. Assuming that a given $A(k, e_s, e_i)$ is uniquely associated with a $V(x, e)$, the proposed Gel'fand–Levitan procedure must lead to the correct answer. This uniqueness *is* an assumption, and this issue is still open even in the local case. However, (2.6) does provide some justification.

The Marchenko equation (3.5) is correct. An alternative derivation amplifying the comment in the line below it is as follows. Using a representation theorem argument, for *all* t we have [3, 4]

$$u(x,t,e_i)-u(x,-t,-e_i)$$
$$=\int_{S^2} G(t-e_s\cdot x, e_s, e_i)\,\mathrm{d}e_s+\int_{S^2}\int_{-\infty}^{\infty} G(t+\tau,-e',e_i)u(x,\tau,e')\,\mathrm{d}e'.$$

In the *local* case $u(x,t,e_i)=0$ for $t<e_i\cdot x$ implies $u(x,-t,-e_i)=0$ for $t>e_i\cdot x$, so that the $u(x,-t,-e_i)$ term in the above equation disappears for $t>e_i\cdot x$. This is the Marchenko *integral* equation for the local case. However, for the *non-local* case $u(x,t,e_i)=0$ for $t<-|x|$ implies $u(x,-t,-e_i)=0$ for $t>|x|$, so that the $u(x,-t,-e_i)$ term in the above equation disappears *only* for $t>|x|$. This is (3.5).

However, (3.5) does *not* determine the scattered field $u(x,t,e_i)$, for the reasons noted in the Comment. Hence the proposed Marchenko procedure does indeed collapse. This is of course an error, but it does not invalidate the rest of the paper. And (3.5) is interesting in that it shows that the Marchenko *form* does generalise to the non-local case—it just cannot be used to solve an inverse scattering problem.

Also, the Comment makes the important point that the regular solution $\varphi(x,t,e_i)$ is *assumed* to exist in the non-local case, as it does (generically) in the local case. This assumption certainly should have been stated in the paper, although there is some evidence in its favour. Note that if $\varphi(x,t,e)$ has support in $[-|x|,|x|]$, and $J(t,e_1,e_2)$ is causal in t, then their convolution (the inverse Fourier transform of the integrand in (2.8*b*) of [1]) will have support in $[-|x|,\infty]$, which is the support of the *non-local* $\psi(x,t,e)$. $\varphi(x,t,e)$ and $J(t,e_1,e_2)$ must have additional properties for the $\psi(x,t,e)$ associated with a local potential to have support $[|x|,\infty]$. Of course this does not show $\varphi(x,t,e)$ exists, but it does provide some indirect evidence.

References

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