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Blind deconvolution of sources from the transmission responses of one-dimensional inhomogeneous continuous and discrete layered absorbing media

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Abstract. Non-destructive testing and reconstruction of lossy non-uniform transmission lines require solving inverse scattering problems for inhomogeneous one-dimensional (1D) lossy media. Often the source waveform used to probe the medium is unknown, since it cannot be measured separately. This paper shows that the blind deconvolution problem of reconstructing both an unknown source and an unknown lossy 1D medium can be solved, provided the medium absorption is sufficiently large. We generalize the minimum-phase property of the transmission response to a more severe property for lossy media. We also show how partial knowledge of the source can be used to reconstruct low-loss and lossless 1D media. Both discrete and continuous 1D layered media are considered. Numerical examples are presented.

1. Introduction

Non-destructive testing often requires solving the inverse scattering problem for a one-dimensional (1D) inhomogeneous discrete- or continuous-layered medium in which the density and absorption both vary with depth. Electromagnetic wave propagation in a dielectric medium gives rise to inverse problems such as radar reflection from stratified dielectrics [1] in which the goal is to reconstruct a 1D absorbing non-dispersive medium in which the complex permittivity (imaginary part is absorption) varies with depth. Interchip communication and time-domain reflectometry can be modelled by lossy transmission lines [2], giving rise to an inverse problem in which the goal is to reconstruct the inductance, capacitance, and series and shunt resistances per unit length of a lossy transmission line to determine its propagation characteristics. All of these problems can be formulated as a 1D inverse scattering problem for an inhomogeneous absorbing medium in which both the local reflectivity and absorption vary with depth.

The solution of this inverse scattering problem from impulse reflection and transmission responses has been covered in [3–6]. However, in practice an impulsive source is usually unavailable. In fact, the source waveform used to probe the unknown medium is usually itself unknown. This leads to the blind deconvolution problem of reconstructing both an unknown source waveform and an unknown 1D inhomogeneous medium from the response of the latter to the former. It is not at all apparent that this problem can be solved at all.

This paper shows that this blind deconvolution problem can in fact be solved if the medium is sufficiently absorbing, i.e. the spatially varying absorption is bounded below

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by a minimum value. If the medium is not sufficiently absorbing (this is defined below) then the problem is ill-posed; however, partial knowledge of the source waveform can be used to solve the problem. A valuable feature of our approach is that no cost functionals or iterative algorithms are required; the unique solution is computed using only recursive (not iterative) layer-stripping algorithms which have excellent numerical stability properties. Interestingly, the same algorithms can be used for both the blind deconvolution and the subsequent reconstruction of the lossy scattering medium.

The blind deconvolution problem is solved by generalizing the minimum phase property of the transmission response of a lossless layered medium to a more stringent property for absorbing media. Provided the zeros of the source waveform are separated from the poles and zeros of the transmission response, the blind deconvolution problem can be solved. If this separation does not occur (as will be the case for lossless media) then partial knowledge of the source waveform is required in order to reconstruct the non-separated spectral factor of the source waveform. Such partial knowledge may well be available. For example, there may be a shallow homogeneous layer at the top of the medium, allowing observation of the initial part of the time waveform of the source, before reflections of the inhomogeneous part of the medium obscure the rest of the time waveform of the source.

This paper is organized as follows. Section 2 briefly reviews the asymmetric 1D inverse scattering problem in terms of which the inverse scattering problem for absorbing media can be formulated. It also reviews the asymmetric Levinson layer-stripping algorithm which solves *both* this inverse scattering problem *and* the blind deconvolution problem of deconvolving a signal into its minimum and maximum phase parts. Section 3 derives the new 'superminimum phase' property of the transmission response of 1D inhomogeneous absorbing media, and applies it to the blind deconvolution problem of reconstructing both an unknown source and medium. Section 4 shows how partial knowledge of the source waveform can be used to help solve the problem for lossless or low-loss media. Section 5 presents several numerical examples illustrating the results of sections 3 and 4. Section 6 derives analogous results for continuously varying 1D absorbing media as a limiting case of discrete medium results. Section 7 concludes with a summary.

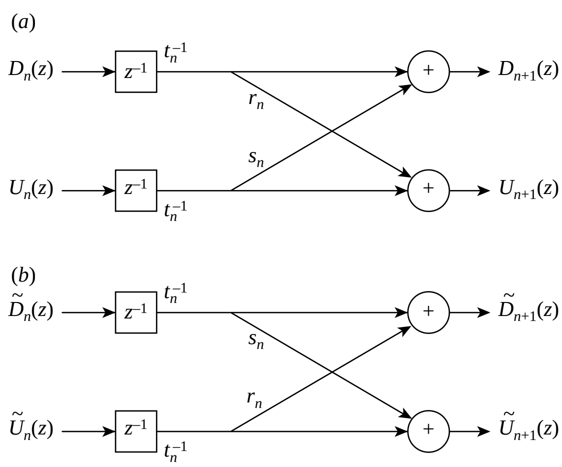

Figure 1. (*a*) A section of the actual scattering medium. (*b*) A section of the adjoint scattering medium.

2. Review of asymmetric inverse scattering and min/max deconvolution

2.1. The asymmetric two-component wave system

The inverse scattering problem for electromagnetic wave propagation in 1D lossy layered inhomogeneous dielectric media with a complex depth-varying permittivity and the inverse problem of reconstructing a lossy non-uniform transmission line can both be formulated [6] as the 1D asymmetric inverse scattering problem of reconstructing the two reflection coefficients r_n and s_n in the asymmetric two-component wave system illustrated by figure 1(a) and described by

$$\begin{bmatrix} D_{n+1}(z) \\ U_{n+1}(z) \end{bmatrix} = \frac{1}{t_n} \begin{bmatrix} z^{-1} & s_n \\ z^{-1}r_n & 1 \end{bmatrix} \begin{bmatrix} D_n(z) \\ U_n(z) \end{bmatrix} \tag{2.1}$$

from the impulse reflection and transmission responses of the cascade of many such systems. In (2.1) $D_n(z)$ and $U_n(z)$ are z-transforms of the downgoing and upgoing waves propagating in the medium just below the nth interface and $t_n = \sqrt{1 - r_n s_n}$ is the transmission loss. The z-transform of any discrete-time sequence $f(n)$ is $F(z) = \mathcal{Z}\{f(n)\} = \sum f(n) z^{-n}$.

The formulations of various inverse scattering problems such as (2.1) are made in [6] and will not be repeated here, save to note that for TE electromagnetic plane-wave propagation in an absorbing non-dispersive dielectric medium we have the following. Let $E_n(z)$ and $H_n(z)$ be the z-transforms of the tangential components of the electric and magnetic fields just below the nth interface, ϵ_i be the relative permittivity in layer i, a_i be the one-way absorption for propagation through layer i ($a_i = 0$ for lossless media; otherwise a_i is minus the imaginary part of the vertical wavenumber times layer thickness) and $\eta = \sqrt{\frac{\mu}{\epsilon}} = 377\Omega$ be the characteristic impedance of free space. Then we have [6] the waves

$$D_n(z) = \frac{e^{A_n}}{2}\left(\sqrt[4]{\epsilon_n} E_n - \frac{\eta}{\sqrt[4]{\epsilon_n}} H_n\right) \qquad U_n(z) = \frac{e^{-A_n}}{2}\left(\sqrt[4]{\epsilon_n} E_n + \frac{\eta}{\sqrt[4]{\epsilon_n}} H_n\right) \tag{2.2}$$

and the reflection coefficients (note $|r_n s_n| < 1$)

$$r_n = \frac{\sqrt{\epsilon_n} - \sqrt{\epsilon_{n+1}}}{\sqrt{\epsilon_n} + \sqrt{\epsilon_{n+1}}} e^{-2A_n} \qquad s_n = \frac{\sqrt{\epsilon_n} - \sqrt{\epsilon_{n+1}}}{\sqrt{\epsilon_n} + \sqrt{\epsilon_{n+1}}} e^{2A_n} \tag{2.3}$$

where cumulative absorption A_n is defined from absorption a_i in layer i as $A_n = \sum_{i=0}^{n} a_i$. For more details see [6].

For lossy transmission lines let $V_n(z)$ and $I_n(z)$ be the z-transforms of the voltage and the current just right of the nth interface, R_n, G_n, L_n and C_n be the series resistance, shunt conductance, series inductance and shunt capacitance for the nth segment (figure 2). Then

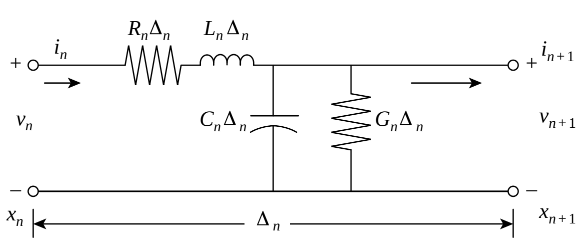

Figure 2. A section of the lossy transmission line.

we have [6] the waves

$$D_n(z) = \frac{e^{A_n}}{2}\left(\frac{V_n(z)}{\sqrt{Z_n}} + \sqrt{Z_n} I_n(z)\right) \qquad U_n(z) = \frac{e^{-A_n}}{2}\left(\frac{V_n(z)}{\sqrt{Z_n}} - \sqrt{Z_n} I_n(z)\right) \tag{2.4}$$

and the reflection coefficients (note $|r_n s_n| < 1$)

$$r_n = \frac{Z_n - Z_{n+1}}{Z_n + Z_{n+1}} e^{-2A_n} \qquad s_n = \frac{Z_n - Z_{n+1}}{Z_n + Z_{n+1}} e^{2A_n} \tag{2.5}$$

where $Z_n = \sqrt{\frac{R_n + j\omega L_n}{G_n + j\omega C_n}}$ is the characteristic impedance in the nth layer and losses a_i and A_n are defined as above. The line must be dispersionless, although in practice small amounts of dispersion can be handled [6].

2.2. Asymmetric inverse scattering problems

The inverse scattering problem for the asymmetric two-component wave system (2.1) can be stated as follows. A scattering medium described by (2.1) is probed by an impulse from one end and its reflection and transmission responses are measured. Then, in a separate experiment, the medium is probed by an impulse from the other end, and those reflection and transmission responses are measured. By reciprocity, the two transmission responses are identical, but the reflection responses differ. The goal is to compute the reflection coefficients r_n and s_n from the two sets of impulse reflection and transmission responses.

Procedures for solving this problem are discussed in [3–6]. A brief summary is as follows. Using both sets of reflection and transmission responses, the transmission and reflection responses of an adjoint medium (in which r_n and s_n have been interchanged) can be computed. The reflection responses of the real and adjoint media can be used in a layer-stripping algorithm [4–6] or in a coupled set of integral equations [3] (for the continuous-time problem) or matrix equations [6] (for the discrete-time problem as defined above) to recover r_n and s_n. More details are given in section 2.4 below.

For an absorbing non-dispersive dielectric medium it is clear from (2.3) that once r_n and s_n have been recovered we have immediately the cumulative absorption $A_n = \frac{1}{4}\log\frac{s_n}{r_n}$. Then the absorptions a_n in each layer can be recovered by differencing the A_n and the permittivity ϵ_n in each layer recovered from (2.3). For the lossy non-dispersive transmission line we can compute the absorption a_n and the characteristic impedance Z_n in each segment in the same way. Since the line is dispersionless we must have $R_n C_n = L_n G_n$; if we know any one of the line parameters or any other relation between them we can thus recover R_n, G_n, L_n, C_n individually in each segment.

2.3. The minimum phase/maximum phase deconvolution problem

The discrete-time min/max phase deconvolution problem is to deconvolve a signal $k(n)$ into its minimum phase and maximum phase components. That is, we factor $K(z) = A(z)B(1/z)$ where both $A(z)$ and $B(z)$ are minimum phase (have no zeros or poles outside the unit circle). The decomposition is only unique to a scale factor and $K(z)$ must have no poles or zeros on the unit circle.

It is curious that this deconvolution problem is in some ways better posed than the usual deconvolution problem of reconstructing a signal $x(n)$ from its convolution $y(n) = x(n) * h(n)$ with a known signal $h(n)$. This is because $Y(z) = H(z)X(z)$ must have zeros where the known $H(z)$ has zeros; any noise added to $y(n)$ will produce a deconvolution problem with no solution. Although noise added to $k(n)$ will move its zeros, *there will always be a solution* since even the shifted zeros will either be inside or outside the unit circle.

$a(n)$ and $b(n)$ being minimum phase means they are causal (zero for $n < 0$) and causally invertible (there exists a causal sequence $a^{-1}(n)$ such that $a(n) * a^{-1}(n) = \delta(n)$, and similarly for $b(n)$). Note that if $b(n)$ is minimum phase then $b(-n)$ is maximum phase (has all its poles and zeros outside the unit circle) since $F(1/z) = \mathcal{Z}\{f(-n)\}$.

In this paper we define a *superminimum phase* signal to be a signal whose z-transform has all of its poles and zeros inside a circle of radius $a < 1$. This is a more drastic condition than being minimum phase.

2.4. Solutions: the asymmetric Levinson algorithm

Both the asymmetric inverse scattering problem and the min/max deconvolution problem can be solved using the asymmetric Levinson algorithm. This $O(n^2)$ layer-stripping algorithm solves the $(n+1) \times (n+1)$ asymmetric Toeplitz system

$$\begin{bmatrix} 1 & k(1) & k(2) & \dots & k(n) \\ k(-1) & 1 & k(1) & \dots & k(n-1) \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ k(-n) & \dots & k(-2) & k(-1) & 1 \end{bmatrix} \begin{bmatrix} a_n(0) & b_n(0) \\ a_n(1) & b_n(1) \\ \vdots & \vdots \\ a_n(n) & b_n(n) \end{bmatrix} = \begin{bmatrix} q_n & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & q_n \end{bmatrix} \tag{2.6}$$

as follows:

$$A_0(z) = B_0(z) = 1 \qquad q_0 = 1 \tag{2.7a}$$

$$s_n = -\sum_{i=0}^{n} a_n(\mathrm{i}) k(n+1-\mathrm{i})/q_n \qquad r_n = -\sum_{i=0}^{n} b_n(\mathrm{i}) k(-\mathrm{i}-1)/q_n \tag{2.7b}$$

$$\begin{bmatrix} A_{n+1}(z) \\ B_{n+1}(z) \end{bmatrix} = \frac{1}{t_n} \begin{bmatrix} z^{-1} & s_n \\ z^{-1} r_n & 1 \end{bmatrix} \begin{bmatrix} A_n(z) \\ B_n(z) \end{bmatrix} \tag{2.7c}$$

$$q_n = \sqrt{1 - r_n s_n} q_{n-1}. \tag{2.7d}$$

Note that (2.7c) is the asymmetric two-component wave system (2.1) and q_n is the cumulative transmission loss. In fact, $A_n(z)$ and $B_n(z)$ are z-transforms of the components of the two-dimensional matrix Green's function for (2.1) (see [6]).

The asymmetric inverse scattering problem is solved using (2.7) as follows. Let $R_f(z)$ and $T_f(z)$ be the z-transforms of the half-space reflection and transmission impulse responses for probing from the top of the medium (or the left end of the transmission line) and $R_b(z)$ and $T_b(z)$ be the z-transforms of the half-space reflection and transmission impulse responses for probing from the bottom of the medium (or the right end of the transmission line). In fact $T_f(z) = T_b(z)$ by reciprocity. Form the scattering matrices

$$\mathcal{S}(z) = \begin{bmatrix} T_f(z) & R_b(z) \\ R_f(z) & T_b(z) \end{bmatrix} \qquad \mathcal{S}^{-T}(1/z) = \begin{bmatrix} T'_f(z) & R'_b(z) \\ R'_f(z) & T'_b(z) \end{bmatrix} \tag{2.8}$$

which define the primed responses from the given responses. It can be shown [6] that the primed responses are the reflection and transmission responses of the *adjoint* medium shown in figure 1(b) (note that r_n and s_n are exchanged from the actual medium in figure 1(a)). Now setting

$$k(n) = \mathcal{Z}^{-1}\{R_f(z)\} \qquad k(-n) = \mathcal{Z}^{-1}\{R'_f(1/z)\} \qquad n > 0 \tag{2.9}$$

in the asymmetric Levinson algorithm reconstructs the asymmetric scattering medium (2.1).

The min/max deconvolution problem can also be solved using (2.7). Rewrite $K(z) = A(z)B(1/z)$ as the two equations

$$K(z)^{-1}A(z) = B(1/z)^{-1} \qquad K(1/z)^{-1}B(z) = A(1/z)^{-1}. \tag{2.10}$$

Since $B(1/z)$ and $A(1/z)$ have all of their poles and zeros outside the unit circle, their reciprocals do also, and the inverse z-transforms are anticausal. Equating coefficients of powers of z gives the asymmetric Toeplitz system (2.6), which can be solved using the asymmetric Levinson algorithm (2.7). The min/max deconvolution problem can also be solved using cepstral techniques, but this is very unstable numerically. Using the same algorithm for both parts of the present problem seems to be preferable.

3. Blind deconvolution from superminimum phase transmission response

The rest of this paper consists of new results, although reductions to previous results will also be noted.

3.1. Superminimum phase transmission response of inhomogeneous absorbing media

First we generalize a result known for lossless media to lossy media.

Theorem 1. *Let $T(z)$ be the impulse transmission response of a 1D inhomogeneous discrete-layered absorbing medium in which the absorption varies from layer to layer but travel time through each layer is constant. Then $T(z)$ is minimum phase.*

Proof. Define $R_n(z) = \frac{U_n(z)}{D_n(z)}$ and $T_n(z) = \frac{T_f(z)}{D_n(z)}$ as the reflection response at depth n and the transmission response of the medium below depth n. From (2.1) it is straightforward to show $R_n(z)$ and $T_n(z)$ satisfy *layer-addition formulae for lossy media*

$$R_n(z) = \frac{1}{z}\frac{R_{n+1}(z) - r_n}{1 - s_n R_{n+1}(z)} \qquad T_n(z) = \frac{t_n}{z}\frac{T_{n+1}(z)}{1 - s_n R_{n+1}(z)}. \tag{3.1}$$

Now suppose momentarily that the medium is lossless. By conservation of energy $|R_n(z)| < 1$ on the unit circle. Applying the maximum modulus principle to $R_n(1/z)$ inside the unit circle shows that $|R(z)| < 1$ for $|z| \geqslant 1$ (this is well known in the lossless case). We also have $|s_n| < 1$ in (2.3*b*) and (2.5*b*).

Now consider the waves (2.2) and (2.4) for dielectrics and lossy transmission lines. Note that these definitions are the usual definitions for lossless media times $e^{\pm A_n}$; indeed $R_n(z) = R_{\text{lossless}}(z)e^{-2A_n}$. So $|R_n(z)| < e^{-2A_n}$ for $|z| \geqslant 1$. Physically this makes sense: the reflection response is clearly attenuated by the two-way cumulative absorption factor. Reflection coefficients s_n defined in (2.3*b*) and (2.5*b*) are the lossless reflection coefficients times e^{2A_n}, so $|s_n| < e^{2A_n}$. So we have $|1 - s_n R_n(z)| > 0$ for $|z| > 1$.

For a N-layer medium $T_N(z)$ is constant and so has no poles or zeros outside the unit circle. Using an induction argument, let $T_{n+1}(z)$ be minimum phase. Then since we have shown that the denominator in (3.1) is non-zero for $|z| > 1$, $T_n(z)$ is also minimum phase (has no zeros or poles in $|z| > 1$). □

Equations (3.1) have nice physical interpretations in terms of feedback. The effect of adding a layer *in the Born approximation* is as follows. For the reflection response, add reflection coefficient r_n at time $n = 0$ to the previous reflection response and then delay. For the transmission response, multiply by the transmission loss for the new interface and again delay. Now, the effect of multiple reflections is reverberation between the new interface (which has reflection coefficient s_n for waves coming from below!) and the previous reflection response. This accounts for the denominator in (3.1). The first formula (3.1) is well known for lossless media; the other is less familiar. Both are new for absorbing media.

Next we produce a more stringent result.

Theorem 2. *Let the absorption $a_i > a$ for each layer. Then $T(z)$ is superminimum phase: its z-transform has all its poles and zeros inside the disk $|z| < \mathrm{e}^{-a} < 1$. Note this is more drastic than being minimum phase.*

Proof. Suppose first that the medium absorption $a_i = a$ in each layer. Then *any* wave anywhere in the medium at time i is attenuated by e^{-ai} from what it would be in a lossless medium, since it has been attenuated by e^{-a} i times, no matter where it is in the medium or what else has happened to it. Using the z-transform relation $\mathcal{Z}\{a^n f(n)\} = F(z/a)$, we see that the effect of the constant loss is to replace z with z/e^{-a} in $D_n(z)$, $U_n(z)$, $R_n(z)$ and $T_n(z)$. Since $T_f(z)$ is minimum phase (has all its poles and zeros in $|z| < 1$), $T_f(z/\mathrm{e}^{-a})$ is superminimum phase (has all its poles and zeros in $|z| < \mathrm{e}^{-a} < 1$). If $a_i > a$, we now see that scaling z by e^{-a} transforms this problem into a problem with absorption $a_i - a > 0$ (attenuation $\mathrm{e}^{-a_i}\mathrm{e}^{a} = \mathrm{e}^{-(a_i-a)} < 1$), to which we can apply the above argument and the first theorem. □

3.2. Blind deconvolution for 1D inhomogeneous absorbing media

The above theorem leads to the following, which is one of our main results.

Theorem 3. *Let a 1D inhomogeneous absorbing discrete-layered medium be probed with an unknown source waveform $s(n)$ having z-transform $S(z)$. If the medium has absorption $a_i > a$ in each layer, and if the poles and zeros of $S(z)$ lie outside the circle of radius $\mathrm{e}^{-a} < 1$ (in $|z| > \mathrm{e}^{-a}$), then both the source and medium can be reconstructed from measurements of reflection and transmission responses of the medium to $s(n)$.*

Proof. The measured transmission response is $T_f(z)S(z)$. By the above theorem, the poles and zeros of $T_f(z)$ lie in $|z| < \mathrm{e}^{-a}$ and those of $S(z)$ lie in $|z| > \mathrm{e}^{-a}$, so $T_f(z)S(z)$ can be factored into $T_f(z)$ and $S(z)$ by computing $k(n) = \mathrm{e}^{an}\mathcal{Z}^{-1}\{T_f(z)S(z)\}$ and running the asymmetric Levinson algorithm (2.7). The multiplication by e^{an} moves all poles and zeros out in radius by a factor of e^{a} so that $T_f(z/\mathrm{e}^{a})$ is now the minimum phase factor and $S(z/\mathrm{e}^{a})$ is the maximum phase factor. $s(n)$ is then deconvolved from the other reflection responses and the medium is reconstructed, again using the asymmetric Levinson algorithm (2.7). □

Actually most of the additional deconvolutions are unnecessary. Suppose we do not perform the additional deconvolutions, but simply form the scattering matrices $\mathcal{S}(z)$ for the actual medium and $\mathcal{S}^{-T}(1/z)$ for the adjoint medium (see (2.8)). The elements of the former are multiplied by $S(z)$ and those of the latter are multiplied by $1/S(1/z)$, so $r'_f(n)$ etc can be obtained by *convolving* the given $\mathcal{Z}^{-1}\{\mathcal{S}^{-T}(1/z)\}$ with $s(-n)$. Only the deconvolution of $s(n)$ from $r_f(n) * s(n)$ is required. The asymmetric Levinson algorithm can be used for this by writing $r_f(n) * s(n)$ as the product of a known asymmetric Toeplitz matrix whose (i, j)th element is $r_f(i - j)$ and a vector of the unknown $s(n)$ (see (4.1) below).

Note that the blind deconvolution of $s(n)$ and $t_f(n)$ is unique only to a scale factor. This scale factor can be fixed if the energy $\sum_n s(n)^2$ is known, as it usually is (the sign is fixed by noting that the first non-zero value of $t_f(n)$ is the product of the transmission and absorption losses and is therefore positive). Alternatively, if the first layer of the medium is known (this is often the case in non-destructive testing) then r_1 is known, and this can be used to fix the scale factor.

Theorem 3 shows that a sufficiently absorbing (large a) layered medium can be reconstructed from its response to any unknown source $s(n)$ as long as $S(z)$ has no zeros or poles too near the origin. All that is required is an estimate of a and the energy of $s(n)$. This seems to be a new and significant result.

4. Special case: lossless layered media

In the special case of lossless media, theorem 1 is the well known result of [7–8] that the impulse transmission response of lossless layered medium is minimum phase. Theorem 3 only applies if the source $S(z)$ is known to be maximum phase ($a = 0$ so $e^{-a} = 1$), which is extremely unlikely. Some *a priori* knowledge about $s(n)$ must be used to obtain a unique reconstruction.

Theorem 4. *Let a 1D inhomogeneous lossless discrete-layered medium be probed with an unknown source waveform $s(n)$ having finite temporal support. Let B values, say without loss of generality $\{s(1), s(2) \dots s(B)\}$, be known, and let $S(z)$ have C zeros inside the unit circle. Then, if $B > C$, both the source and the medium can be reconstructed from measurements of the transmission response of the medium to $s(n)$.*

Proof. We are given $T_f(z)S(z)$. Use the asymmetric Levinson algorithm to factor $T_f(z)S(z)$ into $T_f(z)S_{\min}(z)$ and $S_{\max}(z)$, where $S(z) = S_{\min}(z)S_{\max}(z)$ is the min/max factorization of $S(z)$. Since $T_f(z)$ is minimum phase it will be included with $S_{\min}(z)$. Since $S(z)$ has C zeros inside the unit circle, $S_{\min}(z)$ is a polynomial of degree C with $C+1$ coefficients. Now equate coefficients of $\{z^{-i}, 1 \leqslant i \leqslant B\}$ in $S(z) = S_{\min}(z)S_{\max}(z)$. This yields the Toeplitz system of equations

$$\begin{bmatrix} s_{\max}(1) & s_{\max}(0) & \dots & 0 \\ s_{\max}(2) & s_{\max}(1) & \dots & 0 \\ \ddots & \ddots & 0 & \ddots \\ s_{\max}(B) & s_{\max}(B-1) & \dots & s_{\max}(B-C) \end{bmatrix} \begin{bmatrix} s_{\min}(0) \\ s_{\min}(1) \\ \vdots \\ s_{\min}(C) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(B) \end{bmatrix} \tag{4.1}$$

which can be solved using (again!) the asymmetric Levinson algorithm (2.7). Note since we have actual values of $s(n)$ there is no overall scale-factor ambiguity. The formulation for other sets of given values of $s(n)$ should be apparent. □

Such partial knowledge of $s(n)$ is often available. For example, there may be a shallow homogeneous layer at the top of the medium, allowing observation of the initial part of the time waveform of the source, before reflections of the inhomogeneous part of the medium obscure the rest of the source waveform.

Once the source $S(z)$ is found, it may be deconvolved from observations of $R_f(z)S(z)$ if the latter are also available (in addition to $T_f(z)S(z)$). Actually, the medium can be reconstructed directly from transmission response $T_f(z)$ using the lattice algorithm [9, 10] if the medium has a perfect reflector at the surface (as is often the case, especially in non-destructive testing). Using the lattice algorithm, a lossless layered medium can be reconstructed from just $T_f(z)S(z)$ and $\{s(1), s(2) \dots s(B)\}$.

The lattice algorithm is applied to the reconstruction from the transmission problem in [5, 9, 10] and is as follows (compare with (2.7)):

$$D_0(z) = U_0(z) = T(z) \qquad q_0 = 1 \tag{4.2a}$$

$$r_n = \sum_{i=0}^{\infty} d_n(\mathrm{i})t(\mathrm{i}+1)/q_n = \frac{\sum_{i=0}^{\infty} d_n(\mathrm{i})d_n(\mathrm{i}+1)}{\sum_{i=0}^{\infty} d_n(\mathrm{i})u_n(\mathrm{i})} \tag{4.2b}$$

$$\begin{bmatrix} D_{n+1}(z) \\ U_{n+1}(z) \end{bmatrix} = \frac{1}{t_n} \begin{bmatrix} z^{-1} & r_n \\ z^{-1}r_n & 1 \end{bmatrix} \begin{bmatrix} D_n(z) \\ U_n(z) \end{bmatrix} \tag{4.2c}$$

$$q_n = \sqrt{1 - r_n^2}\, q_{n-1}. \tag{4.2d}$$

The second of (4.2*b*) has two advantages: (1) the transmission response need not be stored and (2) (4.2*d*) can be dispensed with. However, it requires two inner-product computations (which can be performed in parallel).

5. Numerical examples

We present some simple numerical examples that illustrate the ideas presented in sections 3 and 4. In each example we use the lossy dielectric medium from [6]. This multilayer dielectric medium is depicted in figure 3. The r_n, s_n and A_n for this medium are given in figure 4 (as calculated in [6]). The actual and adjoint impulse reflection responses of the medium are shown in figure 5.

It should be remembered that in each example *both* the medium *and* the source $s(n)$ are unknown, except for the scale factor discussed in section 3.2.

5.1. Example 1

The medium is probed with the *time-reversal* of the signal shown in figure 6(*a*). The signal shown is minimum phase; its zero diagram is given in figure 6(*b*). Since theorem 3 requires that the poles and zeros of $S(z)$ lie *outside* the circle of radius e^{-a}, we *time-reverse* the signal in figure 6(*a*) to get a maximum phase signal (it is common practice in signal processing to use a zero diagram with all zeros inside, rather than outside, the unit circle,

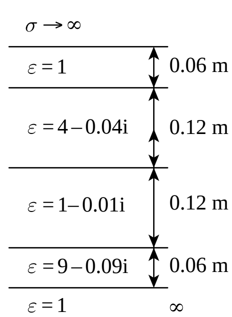

Figure 3. The dielectric medium used in the examples of section 5.

Interface $(n, n+1)$	R_n	e^{-2A_n}	r_n	s_n
0, 1	-.333	1.00	-.333	-.333
1, 2	+.333	.605	+.202	+.551
2, 3	-.500	.471	-.235	-1.063
3, 4	+.500	.323	+.161	+1.549

Figure 4. Parameters for the dielectric medium of figure 3.

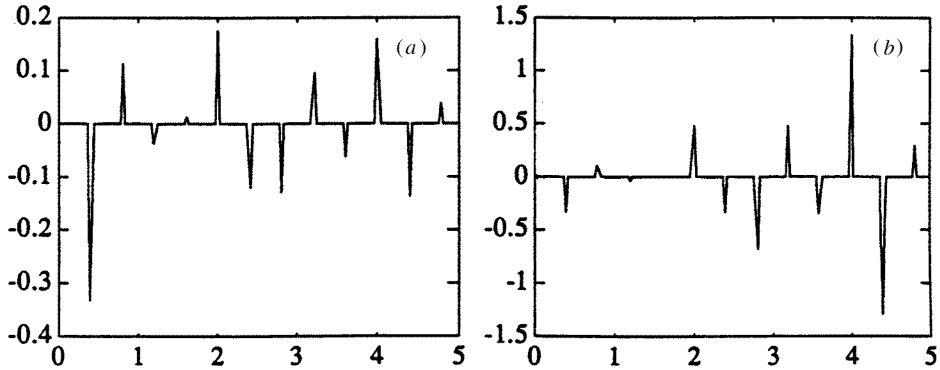

Figure 5. The impulse reflection responses of the actual and adjoint media. (a) Actual system response. (b) Adjoint system response.

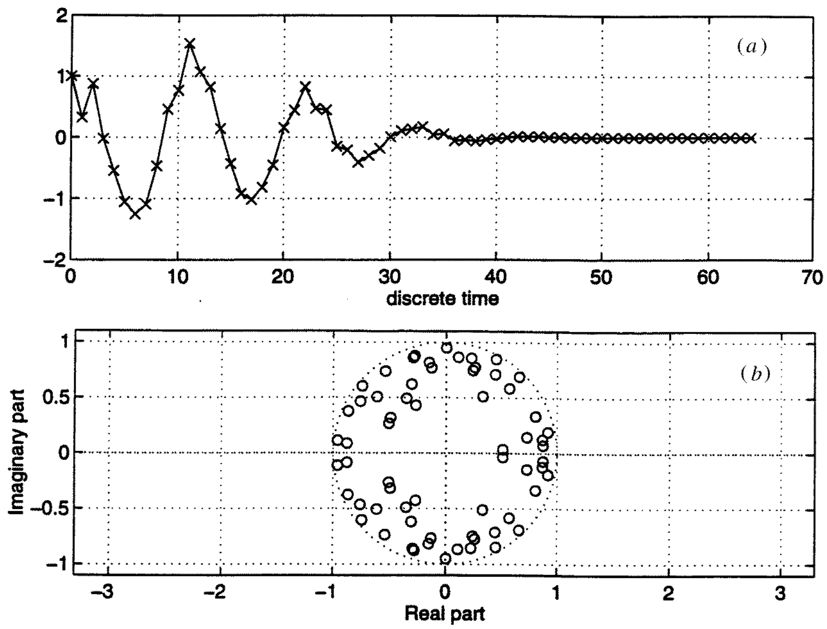

Figure 6. (a) Time-reversal of the (unknown) source signal for example 1. (b) Zero diagram for the signal shown in figure 6(a).

since this is neater). Since the signal has finite length there are no problems with causality (delaying the signal simply delays all of the responses to it).

Since $s(n)$ is maximum phase we know we can reconstruct both it and the medium, even if the medium is lossless. The reconstructed r_n and s_n are shown in figure 7. The reconstructions of both the medium and the source were perfect.

It is interesting to note how much more complicated the impulse reflection responses (shown in figure 5) are as compared with the reflection coefficients (shown in figure 7). This is due to the multiple reflections between the layers. The reader should try to distinguish primary reflections and reverberations in figure 5. The significance of this here is that our deconvolution results apply only if all multiple reflections are included—they do not apply

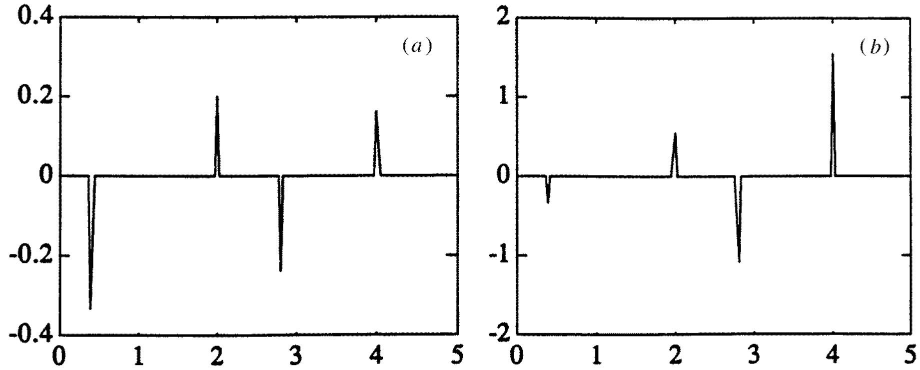

Figure 7. Reconstructed r_n and s_n; these match values given in figure 4. Actual system coefficients $\{r\}$. (b) Adjoint system coefficients $\{s\}$.

in the Born approximation (namely neglect all multiple reflections).

5.2. *Example 2*

Now the medium is probed with the (unknown) signal $s(n) = 0.9^n$, $0 \leqslant n \leqslant 19$. This signal has zeros at $0.9\mathrm{e}^{j2\pi k/20}$, $1 \leqslant k \leqslant 19$. Since these zeros are inside the unit circle, we need to know *a priori* that the medium is lossy with absorption factors a_n such that $\mathrm{e}^{-a_n} < 0.9$ (that this is true is evident from figure 4). Since this is true, we can reconstruct the medium and the source. Again the reconstructions were perfect and are not shown.

5.3. *Example 3*

Now the medium is probed with the (unknown) signal shown in figure 8(a). Its zero diagram is shown in figure 8(b). Since the innermost zeros have magnitude 0.5, the medium and source cannot be recovered. If the medium absorption factors were all increased so that $\mathrm{e}^{-a_n} < 0.5$, then we could reconstruct both the medium and the source.

6. Continuous layered media

We now derive analogous results for 1D inhomogeneous absorbing *continuous* layered media. These results can be obtained from the above results by scaling depth and time by Δ and letting $\Delta \to 0$. However, we present here rigorous derivations of the minimum phase and superminimum phase results, and we also discuss the use of the cepstrum to perform the min/max phase deconvolution.

6.1. *Continuous inverse scattering and min/max deconvolution*

The inverse scattering problem for electromagnetic wave propagation in 1D lossy continuous inhomogeneous dielectric media for complex depth-varying permittivity and the inverse problem of reconstructing a lossy non-uniform transmission line can both be formulated [5] as the 1D asymmetric inverse scattering problem of reconstructing the reflectivity functions

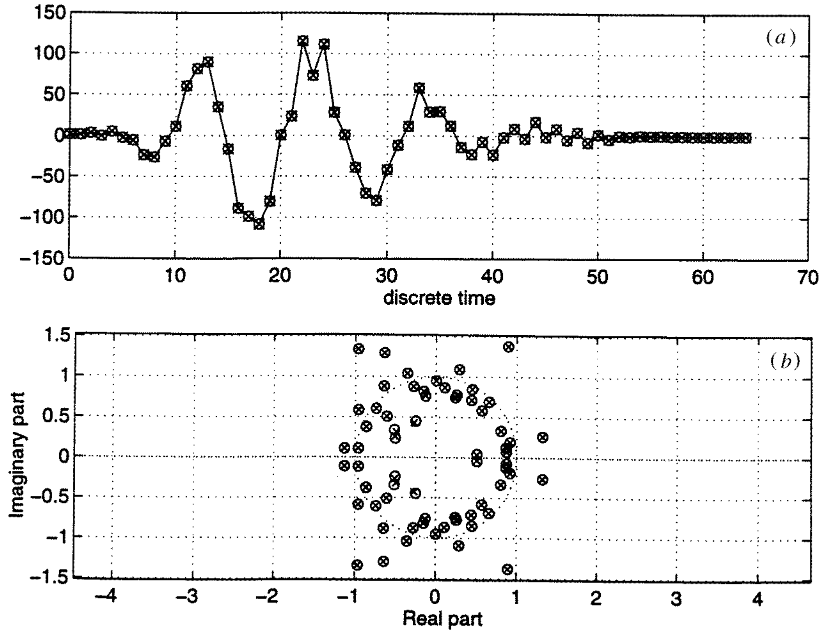

Figure 8. (*a*) The (unknown) source signal for example 3. (*b*) Zero diagram for the signal shown in figure 8(*a*).

$r(x)$ and $s(x)$ in the continuous asymmetric two-component wave system described by

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} D(x,\omega) \\ U(x,\omega) \end{bmatrix} = \begin{bmatrix} -j\omega & s(x) \\ r(x) & j\omega \end{bmatrix}\begin{bmatrix} D(x,\omega) \\ U(x,\omega) \end{bmatrix}. \tag{6.1}$$

The waves are the same as in (2.2) and (2.4), while the reflection coefficient definitions (2.3) and (2.5) are now [5]

$$r(x) = \frac{1}{Z(x)}\frac{\mathrm{d}Z}{\mathrm{d}x}\mathrm{e}^{-\int_0^x a(x')\,\mathrm{d}x'} \qquad s(x) = \frac{1}{Z(x)}\frac{\mathrm{d}Z}{\mathrm{d}x}\mathrm{e}^{\int_0^x a(x')\,\mathrm{d}x'} \tag{6.2}$$

where $Z(x) = \sqrt{\epsilon(x)}$ for dielectric media and A_n has become the integral in (6.2). Note all of these are obvious continuous-parameter limits of the discrete results.

The scattering matrix for adjoint media is formed from the scattering matrix $\mathcal{S}(\omega)$ for actual media by computing $\mathcal{S}^{-H}(\omega)$ (note the Hermitian is in lieu of $1/z$ in (2.8)). The continuous inverse scattering problem for (6.1) can be solved using the continuous version of the asymmetric Levinson algorithm (2.7) [5], which must be discretized to (2.7) in any case.

The continuous min/max deconvolution problem is to factor the Laplace transform (replacing the z-transform) $K(s)$ of $k(t)$ into $K(s) = K_{\min}(s)K_{\max}(s)$ where $K_{\min}(s)$ has all its poles and zeros in the left half of the complex plane and $K_{\max}(s)$ has all its poles and zeros in the right half-plane. Again $k_{\min}(t)$ is causal and causally invertible, and similarly for $k_{\max}(-t)$.

The continuous min/max deconvolution problem may be solved using the continuous asymmetric Levinson algorithm. It can also be solved using the cepstrum $\hat{x}(t)$ of $x(t)$, defined as [11]

$$\hat{x}(t) = \mathcal{F}^{-1}\{\log \mathcal{F}\{x(t)\}\} = \mathcal{F}^{-1}\{\log X(\omega)\}. \tag{6.3}$$

Provided $X(s)$ has no poles or zeros on the imaginary axis, the cepstrum $\hat{x}(t)$ is causal if and only if $x(t)$ is minimum phase (has all poles and zeros in the left half plane). Hence we may deconvolve $k(t) = k_{\min}(t) * k_{\max}(t)$ by taking the cepstrum, resulting in $\hat{k}(t) = \hat{k}_{\min}(t) + \hat{k}_{\max}(t)$. The first term is the causal part of $\hat{k}(t)$, and the second term is the anticausal part of $\hat{k}(t)$, so this effects the min/max deconvolution. The problem is that these operations are numerically unstable and noise sensitive.

6.2. Continuous superminimum phase transmission response

Theorem 1 generalizes to

Theorem 5. *Let $t_f(t)$ be the impulse transmission response of a 1D inhomogeneous continuous-layered absorbing medium in which the absorption varies with depth x. Then $t_f(t)$ is minimum phase.*

Proof. Define $R(x,\omega) = \frac{U(x,\omega)}{D(x,\omega)}$ and $T(x,\omega) = \frac{T_f(\omega)}{e^{j\omega x} D(x,\omega)}$ as the reflection response at x and transmission response below depth x. Note in $T(x,\omega)$ the downgoing wave $D(x,\omega)$ has been advanced in time using $e^{j\omega x}$ so that $T(x,\omega)$ is causal. From (6.1) it can be shown that $R(x,\omega)$ and $T(x,\omega)$ satisfy the *Riccati equations for absorbing media*

$$\frac{dR}{dx}(x,\omega) = 2j\omega R(x,\omega) + r(x) - s(x)R^2(x,\omega) \qquad \frac{dT}{dx}(x,\omega) = -s(x)R(x,\omega)T(x,\omega). \tag{6.4}$$

Riccati equations (6.4) can easily be obtained from the layer addition formulae (3.1) in the continuous limit. The first Riccati equation is well known for lossless media, and appeared in [5] for absorbing media. The second seems to be new for both lossless and absorbing media.

We now use the cepstrum (6.3). Substitute

$$\lim_{x\to\infty} D(x,\omega)e^{j\omega x} = T_f(\omega) \qquad \lim_{x\to\infty} T(x,\omega) = 1 \qquad \frac{1}{T}\frac{dT}{dx} = \frac{d\log T}{dx} \tag{6.5}$$

in (6.4) and integrate the result from x to ∞. This gives

$$\log T(x,\omega) = \int_x^\infty s(x')R(x',\omega)\,dx'. \tag{6.6}$$

Now take the inverse Fourier transform of (6.6). The right side of (6.6) is causal since the reflection response $R(x,\omega)$ must be causal in the time domain. Thus the cepstrum $\hat{t}(x,t) = \mathcal{F}^{-1}\{\log T(x,\omega)\}$ is causal and hence the transmission response $t(x,t)$ at x is minimum phase. In particular $t_f(t) = t(x=0,t)$ is minimum phase. □

Note the proof of theorem 5 differs completely from the proof of theorem 1.

Theorem 2 generalizes to

Theorem 6. *Let the absorption $a(x) > a$ for all x. Then $t_f(t)$ is superminimum phase: all its poles and zeros lie in the half-plane $RE[s] < -a < 0$.*

Proof. Suppose the medium attenuation is constant $a(x) = a$. Then *any* wave anywhere in the medium at time t is attenuated by e^{-at} since it travelled distance t at unity wave speed and was attenuated by $\int_0^t a(x)\,dx = at$. Using the Laplace transform relation $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$, we see the effect of the constant loss is to replace s with $s+a$ everywhere, so all poles and zeros of $t_f(t)$, already known to be in the left half-plane $RE[s] < 0$ for a lossless medium, are in fact in the half-plane $RE[s] < -a < 0$. If $a(x) > a$, shifting s by a maps the problem into a problem with absorption $a(x) - a > 0$, to which we can apply theorem 5. □

The following result for continuous-layered absorbing media is now evident.

Theorem 7. *A 1D inhomogeneous absorbing continuous-layered medium is probed with an unknown source waveform* $s(t)$ *with Laplace transform* $S(s)$*. If the medium has absorption* $a(x) > a$ *everywhere, and if the poles and zeros of* $S(s)$ *lie in the half-plane* $RE[s] > -a < 0$*, then both the source and medium can be reconstructed from measurements of reflection and transmission responses of the medium to* $s(t)$*.*

Proof. Analogous to the proof of theorem 3, using theorem 6. □

Comments analogous to those made in section 3.2 apply here and will not be repeated.

6.3. Special case: Continuous lossless layered media

As in the discrete case, the blind deconvolution problem of deconvolving the unknown transmission response $t_f(t)$ and source $s(t)$ can be solved using theorem 7 for lossless media only if $s(t)$ is known to be maximum phase, or if some values of $s(t)$ are known. This leads to

Theorem 8. *Let a 1D inhomogeneous lossless continuous-layered medium be probed with an unknown source waveform* $s(t)$*. Let* $s(t)$ *be known over the interval* $0 < t < B$*. Then both the source and the medium can be reconstructed from measurements of the transmission response to* $s(t)$ *if the integral equation*

$$\int_0^C s_{\max}(t-t')s_{\min}(t')\,\mathrm{d}t' = s(t) \qquad 0 < t < B \tag{6.7}$$

has a unique solution, where $s_{\min}(t)$ *is assumed to have support* $0 < t < C$*.*

Proof. Analogous to the proof of theorem 4. □

A continuous version of the lattice algorithm (4.2) can be used to reconstruct the medium from its deconvolved transmission response, if the medium has a perfect reflector at $x = 0$.

7. Conclusion

We have shown that a 1D absorbing continuous- or discrete-layered medium with depth-varying reflectivity and absorption can be reconstructed from its responses to an unknown source. That is, we can solve the blind deconvolution problem of reconstructing the transmission response of the medium and the source from the convolution of the two. The source is then deconvolved from the other responses and the asymmetric Levinson algorithm used to reconstruct the medium. It is interesting to note that the asymmetric Levinson algorithm solves *four* different problems: (1) reconstruction of the medium from its impulse responses; (2) the min/max deconvolution problem used to deconvolve the source from the transmission response; (3) the deconvolution of the source from other responses; and (4) the solution of (4.1) when some values of the source are known!

This is possible because the transmission response of an absorbing medium is superminimum phase. This result is not surprising, but it does not seem to have been recognized for absorbing media, nor has its implications for blind deconvolution of the source been recognized. The proofs of this result for discrete and continuous media are quite different, although results for the latter can be viewed as the continuous limit of results for the former. Since computations are necessarily discrete, we have concentrated on the discrete case, since the discrete-to-continuous transformation is easier than the continuous-to-discrete transformation. We have also assumed plane waves at normal incidence; if other excitations are allowed (point source) then more information can be recovered.

Acknowledgments

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