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Heterotic strings in two dimensions and new stringy phase transitions

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Abstract: We discuss heterotic string theories in two dimensions with gauge groups Spin(24) and Spin(8) $\times$ E_8. After compactification the theories exhibit a rich spectrum of states with both winding and momentum. At special points some of these stringy states become massless, leading to new first order phase transitions. For example, the thermal theories exhibit standard thermodynamics below the phase transition, but novel and peculiar behavior above it. In particular, when the radius of the euclidean circle is smaller than the phase transition point the torus partition function is not given by the thermal trace over the spacetime Hilbert space. The full moduli space of compactified theories is 13 dimensional, when Wilson lines are included; the Spin(24) and Spin(8) $\times$ E_8 theories correspond to distinct decompactification limits.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory, 2D Gravity.

Contents

1. Introduction and summary

There has recently been renewed interest in string theories with two-dimensional target space [1, 2] (for earlier work on string theory in two dimensions see e.g., [3, 4]). One aspect of this development is that several new theories have been proposed [5−11].

The goal of this paper is to study heterotic strings in two target space dimensions. These are theories that couple to $(1, 0)$ worldsheet supergravity. The supersymmetric side of the world-sheet theory has the same structure as the $N = 1$ non-critical superstring. The bosonic side matches the bosonic fields of the supersymmetric side and has, in addition, a $c_L = 12$ matter sector. This matter can be organized into either Spin(24) or Spin(8) $\times E_8$

current algebras, thus defining two consistent heterotic string theories. These theories were discussed in [12, 13]. Other theories with somewhat similar features were studied in [14 – 17].

The spectrum of the Spin(24) theory are 24 massless "tachyon" fields, as well as discrete states. The propagating modes of the $\mathrm{Spin}(8) \times E_8$ theory are 8_C massless fermions of one chirality, 8_S massless fermions of the other chirality, and 8_V massless "tachyons."

It is interesting to compactify the heterotic strings, with or without twisting by its discrete symmetries. Then each field theory degree of freedom gives rise to a tower of excitations *a la* Kaluza-Klein. An important novelty is that, unlike the bosonic, type 0 and type-II string theories, we find infinitely many states which have both momentum and winding. Thus there is a rich spectrum of "states" in the theory with compact time, with each level transforming as an increasingly complex representation of the gauge group. These modes can lead to interesting phenomena.

Some of our theories exhibit self-duality under inversion of the compactification radius R. At the self-dual points there are enhanced gauge symmetries such that T-duality is part of the gauge symmetry [18]. Due to the enhanced symmetry, there can be new massless particles which can give rise to phase transitions.

The most striking effect occurs when some of the string theory modes become massless (in the sense of one-dimensional Liouville theory [19]). In these cases the torus amplitude is non-analytic and the theory undergoes a phase transition. The mode that becomes massless can be either a complex boson Φ or a complex fermion χ. We can describe its one-dimensional Landau-Ginzburg mean field theory lagrangian as

$$\begin{aligned} \mathcal{L}_\Phi &= \frac{1}{2}|\partial_\phi \Phi|^2 + \frac{1}{2} m(R)^2 \Phi^2 \\ \mathcal{L}_\chi &= i\chi^\dagger \partial_\phi \chi + m(R)\chi^\dagger \chi \, . \end{aligned} \tag{1.1}$$

In our examples the mass $m(R)$ has a simple zero; specifically $m(R) = \frac{1}{2}\left(R - \frac{1}{R}\right)$. The one loop fluctuations of Φ and χ lead to finite, nonanalytic terms

$$\begin{aligned} Z_\Phi &= -\int \frac{V_L dp}{2\pi} \log(p^2 + m(R)^2) = -V_L |m(R)| + \text{const.} \\ Z_\chi &= \frac{1}{2}\int \frac{V_L dp}{2\pi} \log(p^2 + m(R)^2) = +\frac{1}{2} V_L |m(R)| + \text{const.} \end{aligned} \tag{1.2}$$

Here $V_L = \int d\phi \; 1$ is the size of the spatial direction ϕ. The (infinite) constants are independent of $m(R)$ and can be ignored. All our torus amplitudes are analytic functions of R plus possible terms arising from (1.2). Our results are summarized in table 2 and 3 in section 4.

The interpretation of the results raises conceptual issues that are not fully understood; they are discussed in section 4.4. For example, we will see that the torus amplitude of a theory compactified on a small thermal circle is not given by the standard thermodynamical trace over the spacetime Hilbert space. It is not clear whether there exists an alternate thermodynamical description of the physics with such small radius. The moduli space of other compactifications might have a boundary at finite radius beyond which the radius cannot be reduced.

We would like to clarify a few general points of potential confusion. The target space of the theories we consider have a linear dilaton along the spatial direction, ϕ, so that the string coupling constant varies as

$$g_s(\phi) = e^{\phi} \,. \tag{1.3}$$

We focus on the weakly coupled region of the target space where the string coupling is arbitrarily small $g_s(\phi) \to 0$, while the string scale M_s is finite. There, the infrared dynamics of the gauge theory can be ignored, because it is important only at energies below $g_s(\phi)M_s \to 0$. The typical energy scale we consider, including the scale $1/R$ set by our compactifications, is of order string scale M_s and, therefore, not affected by the infrared dynamics.

We will be interested in the string theory partition function written in the form

$$\int d\phi Z(\phi) = \int d\phi \left(e^{-2\phi} A_0 + A_1 + e^{2\phi} A_2 + \cdots \right) \tag{1.4}$$

where the ϕ dependence is associated with the powers of the string coupling (1.3) and, therefore, the coefficient A_n is the genus n contribution. The sphere term A_0 is proportional to the compactification radius R and is not interesting for our purpose. The torus amplitude A_1 is more interesting; for example, it receives the non-analytic contributions (1.2). Importantly, A_0 and A_1 depend only on physics in the weak coupling region, and on the weak coupling spectrum. A_n with larger n depend on the details of the interactions in the strong coupling region; but they are negligible for $\phi \to -\infty$. Usually, one turns on a tachyon background with coefficient μ to control the perturbative expansion (1.4) but this will not be needed here.

The theories we consider have discrete states formed from the gauge currents; but the ground ring and its associated towers of currents seems absent. In the bosonic and supersymmetric theories such currents are related to the symmetries of the dual matrix model description, specifically the symmetries expressing incompressibility of the free fermion representation. (Some discussions of discrete states and the ground ring in bosonic and superstring theories are [20 − 27].) The absence of this structure for the heterotic strings indicates that, if a dual matrix model description exists at all, it must have some significant new feature. Additionally, heterotic strings support no D-brane boundary states. Since the modern interpretation of matrix models identifies the matrix eigenvalues with D-brane coordinates [1, 2], this is another indication that a matrix model description cannot be simple. It would clearly be interesting to find a non-perturbative formulation of the heterotic strings discussed here.

An illuminating way to explore heterotic theories is to employ lattice technology. In two dimensions the uncompactified theories can be classified by even self-dual lattices in 16 euclidean dimensions, using the covariant lattice construction (which includes right moving fermions and superconformal ghosts). This confirms that there really are exactly two "fundamental" theories, with gauge groups Spin(24) and Spin(8) $\times$ E_8, respectively. This contrasts with ten dimensions where, in addition to the familiar supersymmetric Spin(32)/$\mathbb{Z}_2$ and $E_8 \times E_8$ theories, there are a number of non-supersymmetric theories.

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The lattice construction also shows that, after compactification, all theories are connected: there is a 13 dimensional moduli space, parametrized by the radius of compactification and 12 independent Wilson lines. This is much richer than for other strings in two dimensions. As illustrations, we show explicitly how the twisted lines of theories can be reinterpreted in terms of Wilson lines; and how T-duality relates the Spin(24) and the Spin(8) $\times E_8$ theories, after the introduction of suitable Wilson lines.

The remainder of the paper is organized as follows. In section 2 we define the basic Spin(24) and Spin(8) $\times E_8$ theories. We also discuss the discrete symmetries of the theories. In section 3 we consider compactifications of the two theories, with or without twisting of their discrete symmetries. T-duality and enhanced symmetry points are discussed as well. In section 4 we evaluate the torus partition function explicitly for the different lines of theories and discuss the phase transition in detail. Finally, we include an appendix where lattice constructions are used to classify the theories and reconsider their interconnections. Throughout the paper we use units in which $\alpha' = 2$.

2. Theories in noncompact space

The right movers are the $\hat{c} = 1$ noncritical string: a Liouville field ϕ, (euclidean) time x, and their fermionic superpartners ψ_ϕ and ψ_x. The slope of the Liouville field is $Q = 1$ which is such that it contributes $c_\phi = 13$ to the central charge. The left movers constitute a noncritical bosonic theory that includes the Liouville field, (euclidean) time and, in order to have total left moving central charge 26, a $c_L = 12$ bosonic CFT which we will take to be 24 free fermions $\bar{\lambda}^I$ with $I = 1, \ldots, 24$. In the remainder of this section we discuss the two natural theories constructed out of these building blocks.

2.1 Spin(24) theory

Here we correlate the spin structure of the free 24 fermions with that of the right movers. The physical vertex operators are

$$
\begin{aligned}
G &= \mathcal{J}\bar{\mathcal{J}} \\
A^{IJ} &= \mathcal{J}\bar{\lambda}^I\bar{\lambda}^J \\
T^I(k) &= e^{-\varphi}\bar{\lambda}^I V_k
\end{aligned}
\tag{2.1}
$$

where the operators

$$
\begin{aligned}
\mathcal{J} &= e^{-\varphi}\psi_x \\
\bar{\mathcal{J}} &= \bar{\partial}\bar{x}
\end{aligned}
\tag{2.2}
$$

are U(1) currents and the wave functions are

$$
V_k = e^{ik(x+\bar{x})+(1-|k|)(\phi+\bar{\phi})} \, . \tag{2.3}
$$

The absolute value in the coefficient of ϕ was explained in [19]. The discrete states G and A^{IJ} are the two dimensional graviton/dilaton and the Spin(24) gauge fields. $T^I(k)$ represent 24 massless scalars "tachyons." The Ramond sector does not lead to physical particles because the Spin(24) spin fields, $\bar{S}^\alpha$ and $\bar{S}^{\dot\alpha}$, have dimension $\bar\Delta = \frac{3}{2}$ and V_k has dimensions $(\Delta, \bar\Delta) = (\frac{1}{2}, \frac{1}{2})$ for all k. These fields, however, will play a role when we discuss the compactified theory. Clearly, the operators in (2.1) are mutually local. The partition function (in the notation of [28])

$$Z_F(\bar\tau) = \frac{1}{2}\left[Z_0^0(\bar\tau)^{12} - Z_1^0(\bar\tau)^{12} - Z_0^1(\bar\tau)^{12}\right] \tag{2.4}$$

is modular invariant. Also note that $Z_F(\bar\tau) = 24$ is independent of $\bar\tau$ [13], and hence it is manifestly modular invariant.

It is possible to turn on background tachyons which break the continuous symmetry $\mathrm{Spin}(24) \to \mathrm{Spin}(23)$.

2.2 Spin(8) $\times\, E_8$ theory

Here we divide the 24 fermions into two groups: $\bar\lambda^i$ with $i = 1, \ldots, 8$ and 16 other fermions. The latter lead to an E_8 left moving CFT. The spin structure of the $\bar\lambda^i$ is correlated with that of the right movers. In this theory the physical vertex operators are

$$\begin{aligned}
G &= \mathcal{J}\bar{\mathcal{J}}\\
A^{ij} &= \mathcal{J}\bar\lambda^i\bar\lambda^j\\
A^{ab} &= \mathcal{J}\bar{J}^{ab}\\
T^i(k) &= e^{-\varphi}\bar\lambda^i V_k\\
\Psi^\alpha &= e^{-\frac{1}{2}\varphi + i\frac{1}{2}H}\bar{S}^\alpha V_k\,, \qquad k \geq 0 \qquad \widetilde\Psi^{\dot\alpha} = e^{-\frac{1}{2}\varphi - i\frac{1}{2}H}\bar{S}^{\dot\alpha} V_k\,, \qquad k \leq 0\,.
\end{aligned} \tag{2.5}$$

Again, $\mathcal{J}$ and $\bar{\mathcal{J}}$ are the U(1) currents (2.2) which lead to discrete states: G is the graviton/dilaton, A^{ij} are the Spin(8) gauge fields and A^{ab} are the E_8 gauge fields constructed from the E_8 currents $\bar{J}^{ab}$. The other vertex operators represent propagating particles: T^i is a scalar in 8_V of Spin(8), Ψ^α is a left moving spacetime fermion in 8_S, and $\widetilde\Psi^{\dot\alpha}$ is a right moving spacetime fermion in 8_C. Unlike the Spin(24) theory, here, the dimension of the Spin(8) fields S^α and $S^{\dot\alpha}$ is $\Delta = \frac{1}{2}$, thus giving rise to physical fermions. The conditions on the momentum k arise from locality with respect to the world-sheet supercurrent (i.e. the Dirac equation). Note that the spectrum is anomaly free even though it is chiral. It is straightforward to check that the operators in (2.5) are mutually local. The partition function of the theory

$$\begin{aligned}
Z_F(\bar\tau) &= \frac{1}{2}\left[Z_0^0(\bar\tau)^4 - Z_1^0(\bar\tau)^4 - Z_0^1(\bar\tau)^4\right]\cdot\left(Z_0^0(\bar\tau)^8 + Z_0^1(\bar\tau)^8 + Z_1^0(\bar\tau)^8\right)\\
&= [(8-8) + (64-64)q + \cdots]\cdot[1 + 248q + \cdots] = 0
\end{aligned} \tag{2.6}$$

is modular invariant. Note that in this case $Z_F \equiv 0$ identically (this follows from the Jacobi identity, familiar from spacetime supersymmetry in 10 dimensions).

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It is possible to turn on background tachyons which break the continuous symmetry $\mathrm{Spin}(8) \to \mathrm{Spin}(7)$. The effective lagrangian can include a coupling of the form $T^i \Psi^\alpha \widetilde{\Psi}^{\dot\alpha} \gamma_{i\alpha\dot\alpha}$ with possible derivatives. It is amusing that this spectrum of particles is the same as in the worldsheet light cone description of the IIA critical string. However, unlike that theory, here, because of the linear dilaton, there is no two-dimensional Lorentz invariance and no $(8,8)$ two dimensional supersymmetry.

2.3 Discrete symmetries

We next discuss the discrete symmetries of the theories. We focus on transformations that do not break the gauge symmetry.

The spacetime fermion number F_R and the right moving world-sheet fermion number f_R are represented in the same way as in the superstring [11]

$$\begin{aligned} (-)^{F_R}: &\quad \varphi \to \varphi + 2\pi i \\ (-)^{f_R}: &\quad \varphi \to \varphi + \pi i \qquad H \to H + \pi\,. \end{aligned} \tag{2.7}$$

In the left moving sector we must proceed differently. The center of $\mathrm{Spin}(4n)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$. The generators of the center transform representations according to their $\mathrm{Spin}(4n)$ conjugacy class

$$\begin{aligned} \mathcal{Z}_1 = (-)^{F_L}: &\qquad \mathcal{O}_V \to \mathcal{O}_V\,; \qquad \mathcal{O}_S \to -\mathcal{O}_S\,; \quad \mathcal{O}_C \to -\mathcal{O}_C \\ \mathcal{Z}_2 = (-)^{f_L}: &\qquad \mathcal{O}_V \to -\mathcal{O}_V\,; \quad \mathcal{O}_S \to \mathcal{O}_S\,; \qquad \mathcal{O}_C \to -\mathcal{O}_C \end{aligned} \tag{2.8}$$

while the $\mathcal{O}_0$ are invariant. The transformation $\mathcal{Z}_1$ is a rotation by 2π around some axis in the internal space. It is therefore natural to define the left moving spacetime fermion number as $(-)^{F_L} \equiv \mathcal{Z}_1$. The transformation $\mathcal{Z}_2$ is world-sheet fermion number insofar as the $\mathrm{Spin}(2n)$ current algebra is realized in terms of $2n$ free fermions. It is therefore natural to define the left-moving world-sheet fermion number as $(-)^{f_L} \equiv \mathcal{Z}_2$. Note that this latter identification also makes sense for $\mathrm{Spin}(8) \times E_8$, because there are always an even number of E_8 fermions.

With these notations and conventions, the theories we consider are defined with diagonal GSO projections, i.e. the operators satisfy $(-)^{F_L+F_R} = (-)^{f_L+f_R} = 1$ on physical states. In contrast, the elements $(-)^{F_L}$ and $(-)^{f_L}$ in the center of $\mathrm{Spin}(4n)$ act as symmetries. In view of the GSO projection the symmetries can equally be characterized in terms of right moving quantities $(-)^{F_R}$ or $(-)^{f_R}$.

There are two more discrete transformations of interest. Spacetime parity $\mathcal{P}$ acts as

$$\mathcal{P}: \quad H \to -H\,; \qquad x \to -x\,; \qquad \bar{x} \to -\bar{x} \tag{2.9}$$

while charge conjugation $\mathcal{C}$ acts on the $\mathrm{Spin}(4n)$ lattice as

$$\mathcal{C}: S \leftrightarrow C\,. \tag{2.10}$$

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Neither of these transformations are symmetries of the Spin(8) $\times$ E_8 theory. For example, $\mathcal{P}$ would transform Ψ^α in (2.5) into a state with $k \leq 0$ and $H \to -H$, leaving $\bar{S}^\alpha$ intact; but there is no such state in the spectrum. However, the combined transformation $\mathcal{CP}$ *is* a symmetry of the theory: it simply interchanges the spinorial vertex operators Ψ^α and $\widetilde{\Psi}^{\dot\alpha}$. The $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{CP}$ are all symmetries of the Spin(24) theory.

The diagonal element in the center of Spin($4n$), generated by $\mathcal{Z}_1\mathcal{Z}_2 = (-)^{F_L+f_L}$, is related to $\mathcal{Z}_2$ through conjugation by $\mathcal{CP}$: we have $\mathcal{CP}(-)^{f_L}\mathcal{CP} = (-)^{F_L+f_L}$ when acting on any operator in the theories. Thus, $(-)^{F_L+f_L}$ acts in the same way as $(-)^{f_L}$, up to a change of conventions; so, later, it will be sufficient to consider orbifolds and twists by $(-)^{F_L}$ and $(-)^{f_L}$.

It is significant that the discrete symmetries are in fact elements of the center of the group, which can be continuously related to the identity. This means compactifications twisted by each of these symmetries are all connected to untwisted compactification. We will make this more explicit in the appendix.

As a final comment on discrete symmetries, recall that Spin(8) allows triality transformations, realized as outer automorphisms of the algebra. One element of the triality group acts on the weight lattice by cyclic permutation of the conjugacy classes $V \to S \to C \to V$. Concretely, this means we can replace the operators appearing in (2.5) according to

$$\bar\lambda^i \to \bar{S}^\alpha \to \bar{S}^{\dot\alpha} \to \bar\lambda^i \,. \tag{2.11}$$

It is important that no physical observable will be different in theories related by triality, because an automorphism just amounts to renaming of the representations.

2.4 Orbifolds

Starting from the Spin(24) and Spin(8) $\times$ E_8 theories discussed above it is natural to seek new theories by orbifolding with respect to the discrete symmetries. In the following we argue that this does not lead to interesting new possibilities.

First, let us orbifold by $(-)^{F_L}$. As indicated in (2.8) the untwisted sector is the V conjugacy class of the Spin($4n$) and also the discrete states, but the R-sector (if any) is projected out. An R-sector arises from the twisted states, but it has the opposite correlation between spacetime chirality and Spin($4n$) chirality. Explicitly, for the Spin(8) $\times$ E_8 theory in (2.5), the Ψ^α and $\widetilde{\Psi}^{\dot\alpha}$ are being replaced by $\widetilde{\Psi}^\alpha$ and $\Psi^{\dot\alpha}$. This does not lead to a genuinely new theory: it reduces to the transformation $\mathcal{C}$ introduced in (2.10). In the Spin(24) there are no R-states at all, so the orbifold leaves the theory invariant.

Next, let us consider orbifold by $(-)^{f_L}$. From (2.8) we see that the untwisted sector consists of the S conjugacy class along with the discrete states in the 0 conjugacy class. In the Spin(8) $\times$ E_8 theory this leaves the propagating states Ψ^α and so, adding the twisted states permitted by locality, we find the propagating states

$$\begin{aligned}
\Psi^\alpha &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\bar{S}^\alpha V_k \,, &\qquad k \geq 0 \\
T^{\dot\alpha} &= e^{-\varphi}\bar{S}^{\dot\alpha} V_k \\
\widetilde{\Psi}^i &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\bar\lambda^i V_k \,, &\qquad k \leq 0 \,.
\end{aligned} \tag{2.12}$$

This is related to the original spectrum (2.5) by triality.

The $(-)^{f_L}$ orbifold of the Spin(24) theory is formally expected to have the same structure as the Spin(8) $\times E_8$ theory but now, because the spin fields have dimensions $\bar{\Delta} = 3/2$, the only propagating states are the $\widetilde{\Psi}^i$, i.e. 24 chiral fermions. This spectrum is anomalous in spacetime and the theory is probably inconsistent.

3. Compactification

In this section we discuss the compactification of the Spin(24) and Spin(8) $\times E_8$ theories on a circle of radius R. We also consider theories that are twisted by discrete symmetries that commute with the gauge symmetry. As discussed in section 2.3, the non-trivial twists are $(-)^{F_L}$ and $(-)^{f_L}$.

3.1 Circle compactification: no twist

The spectrum at generic radius R includes the currents $\mathcal{J}$ and $\bar{\mathcal{J}}$ (2.2) and the discrete states G and A^{IJ} of (2.1) as well as

$$\begin{aligned} T_V &= e^{-\varphi}\bar{\mathcal{O}}_V\left(\frac{1}{2}+nw\right)V_{n,w} \\ \Psi_S &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\bar{\mathcal{O}}_S\left(\frac{1}{2}+nw\right)V_{n,w}\,, \qquad p_R \geq 0 \\ \widetilde{\Psi}_C &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\bar{\mathcal{O}}_C\left(\frac{1}{2}+nw\right)V_{n,w}\,, \qquad p_R \leq 0 \end{aligned} \tag{3.1}$$

where now the wave function is

$$V_{n,w} = e^{i\frac{n}{R}(x+\bar{x})+i\frac{w}{2}R(x-\bar{x})+(1-|p_R|)(\phi+\bar{\phi})} \tag{3.2}$$

with $p_R = \frac{n}{R} + \frac{wR}{2}$. The n, w are integers and $\bar{\mathcal{O}}_r(\bar{\Delta})$ are operators in the conjugacy class $r = V, S, C$ of Spin($4n$) with dimension $\bar{\Delta}$. The spectrum (3.1) is modular invariant (some details of this are discussed in section 4.1).

The theories are clearly invariant under $R \to \frac{2}{R}$. The right moving U(1) symmetry cannot be enhanced but, at the selfdual radius $R = \sqrt{2}$, the left moving U(1) symmetry whose current is $\bar{\partial}\bar{x}$ is enhanced to SU(2).

The list of operators in (3.1) represents schematically the spectrum of either the Spin(24) theory or the Spin(8) $\times E_8$ theory. The difference between the theories appears when constructing the operators $\bar{\mathcal{O}}_r$ explicitly. These must transform according to a representation in the appropriate conjugacy class, and with the correct conformal dimension. In the Spin(24) theory they are formed from the 24 $\bar{\lambda}^i$ ($\bar{\Delta} = \frac{1}{2}$; V representation) and the spin fields $\bar{S}^\alpha$ and $\bar{S}^{\dot{\alpha}}$ ($\bar{\Delta} = 3/2$; S or C). In the Spin(8) $\times E_8$ theory, there are only 8 $\bar{\lambda}^i$ and the spin fields have dimension $\bar{\Delta} = 1/2$; but then there are also operators from the E_8 part, including the adjoint operator $\bar{\mathcal{J}}^{ab}$ with $\bar{\Delta} = 1$. In either theory there are clearly numerous ways to construct operators with appropriate representations and dimensions. Thus, unlike the type 0 and type-II theories, here, because the central charge of the left movers, there is a large spectrum of physical operators obtained using the left moving oscillators.

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3.2 Compactification with $(-)^{F_L}$ twist (thermal theory)

We next consider the twisted theory where motion around the circle is accompanied by action with $(-1)^{F_L}$. The spectrum at generic radius R includes the currents $\mathcal{J}$ and $\bar{\mathcal{J}}$ and the discrete states G and A^{IJ} of (2.1) as well as

$$\begin{aligned}
T_V &= e^{-\varphi}\bar{\mathcal{O}}_V\left(\frac{1}{2}+2nw\right)V_{n,2w} \\
T_0 &= e^{-\varphi}\bar{\mathcal{O}}_0\left(\frac{1}{2}+\left(n+\frac{1}{2}\right)(2w+1)\right)V_{n+\frac{1}{2},2w+1} \\
\Psi_S &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\bar{\mathcal{O}}_S\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\,, \qquad p_R\geq 0 \\
\Psi_C &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\bar{\mathcal{O}}_C\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\,, \qquad p_R\geq 0 \\
\widetilde{\Psi}_S &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\bar{\mathcal{O}}_S\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\,, \qquad p_R\leq 0 \\
\widetilde{\Psi}_C &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\bar{\mathcal{O}}_C\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\,, \qquad p_R\leq 0
\end{aligned} \tag{3.3}$$

where again n and w are integers. The untwisted sector consists of all states with even winding; the spacetime bosons (fermions) have integer (half-integer) momentum, to compensate for the action with $(-1)^{F_L}$. The twisted sector (odd winding) has the opposite correlation. The spectrum (3.3) is modular invariant (some details of this are discussed in section 4.2).

The transformation $R\to 1/R$ leaves the set of operators (3.3) invariant, except for the trivial interchange of the S and C conjugacy classes. Hence the theory is self-dual.

Let us be explicit about the decompactification limits: as $R\to\infty$ only operators with no winding remain, such as Ψ_S and $\widetilde{\Psi}_C$. As $R\to 0$ it is the operators with vanishing momentum that remain, including Ψ_C and $\widetilde{\Psi}_S$. The spectra in the two limits are thus related by charge conjugation $\mathcal{C}$ (2.10) which, as discussed in section 2.3, amounts to a change of convention with no physical significance.

At the self-dual point $R=1$ there are additional discrete states

$$A^I_\pm = \mathcal{J}\bar{\lambda}^I e^{\pm i\bar{x}} \tag{3.4}$$

where $I=1,\cdots,N$ ($N=24$ for the Spin(24) theory and $N=8$ for Spin(8) $\times E_8$). Taken together with the operators A^{IJ} and G from (2.1) these form an Spin($N+2$) current algebra at level 1. This means the left moving symmetry is enhanced from Spin(N) $\times$ U(1) to Spin($N+2$) at the self-dual point.

The list (3.3) includes two states that become massless at $R=1$

$$T_0^\pm = e^{-\varphi}V_{\pm\frac{1}{2},\mp 1} = e^{-\varphi}e^{\pm i\bar{x}}e^{\phi+\bar{\phi}} \tag{3.5}$$

so that, in total, there are $N+2$ tachyons at $R=1$, transforming in the vector of Spin($N+2$). These modes are massless in the sense of one-dimensional Liouville theory [19], that is, they have Liouville dressing $e^{\phi+\bar{\phi}}$. In section 4.2 we will show that they signal a phase transition at $R=1$.

At $R=1$ the Ψ_S and Ψ_C combine into Spin($N+2$) spinors, as do $\widetilde{\Psi}_S$ and $\widetilde{\Psi}_C$.

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3.3 Compactification with $(-)^{f_L}$ twist

As the final compactification we consider the twisted theory where motion around the circle is accompanied by $(-)^{f_L}$. The propagating modes are

$$\begin{aligned}
T_V &= e^{-\varphi}\bar{\mathcal{O}}_V\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w} \\
T_C &= e^{-\varphi}\bar{\mathcal{O}}_C\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1} \\
\Psi_S &= e^{-\varphi/2+iH/2}\bar{\mathcal{O}}_S\left(\frac{1}{2}+2nw\right)V_{n,2w}\,, && p_R\geq 0 \\
\widetilde{\Psi}_C &= e^{-\varphi/2-iH/2}\bar{\mathcal{O}}_C\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\,, && p_R\leq 0 \\
\Psi_0 &= e^{-\varphi/2+iH/2}\bar{\mathcal{O}}_0\left(\frac{1}{2}+\left(n+\frac{1}{2}\right)(2w+1)\right)V_{n+\frac{1}{2},2w+1}\,, && p_R\geq 0 \\
\widetilde{\Psi}_V &= e^{-\varphi/2-iH/2}\bar{\mathcal{O}}_V\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\,, && p_R\leq 0\,.
\end{aligned}\tag{3.6}$$

The untwisted sector (even winding) has momentum shifted by half for odd world-sheet fermion number (V and C conjugacy classes). The twisted sectors have the opposite correlation. The theory is modular invariant (shown in detail in section 4.3).

The list (3.3) includes two fermionic states that become massless (in the one-dimensional Liouville sense described below (3.5)) at $R=1$

$$\Psi_0^{\pm}=e^{-\varphi/2+iH/2}e^{\pm i\bar{x}}e^{\phi+\bar{\phi}}\tag{3.7}$$

These will be responsible for a phase transition at $R=1$.

At $R=1$ the operator Ψ_0 with $n=w=0$ is the gravitino $S^+\bar{\mathcal{J}}$ where

$$\mathcal{S}^+=e^{-\varphi/2+iH/2}V_{1/2,1}=e^{-\varphi/2+iH/2+ix}\tag{3.8}$$

is a $(1,0)$ "supersymmetry" current. It is a special case of the construction of [29]. It exists only at precisely $R=1$ because only then does $V_{1/2,1}$ have conformal dimension $\Delta=1/2$.

In the $\mathrm{Spin}(8)\times E_8$ (but *not* the Spin(24) theory) there are also additional discrete states at $R=1$:

$$A_{\pm}^{\alpha}=\mathcal{J}\bar{S}^{\alpha}e^{\pm i\bar{x}}\,.\tag{3.9}$$

These combine with G and A^{IJ} to extend the gauge symmetry to $\mathrm{Spin}(10)\times E_8$. The Ψ_S and Ψ_0 combine into representations in the spinor conjugacy class of Spin(10). Similarly, T_V and T_C combine into Spin(10) spinors, as do $\widetilde{\Psi}_V$ and $\widetilde{\Psi}_C$.

Next we consider duality of the theory. Transforming $R\to 1/R$ on the operators (3.6) we find that the spectrum returns to its original form *except* that the modings of the operators in V and C conjugacy classes have been interchanged. In the $\mathrm{Spin}(8)\times E_8$ theory this is just triality, which just amounts to a change of conventions; so this theory is self-dual. Again the duality is an element of the enhanced gauge symmetry at $R=1$.

	Spin(24)	Spin(8) $\times E_8$
S_1	$R \to \frac{2}{R}$; Spin(24) $\times$ SU(2)	$R \to \frac{2}{R}$; Spin(8) $\times$ SU(2) $\times E_8$
$S_1/(-1)^{F_L}$	$R \to \frac{1}{R}$; Spin(26)	$R \to \frac{1}{R}$; Spin(10) $\times E_8$
$S_1/(-1)^{f_L}$	No duality/enhancement	$R \to \frac{1}{R}$; Spin(10) $\times E_8$

Table 1: Summary of T-duality symmetry and enhanced gauge symmetry at the self-dual point.

The Spin(24) theory is more confusing. Formally, the $R \to 1/R$ again interchanges the V and C conjugacy classes. However, for Spin(24) these representations are not equivalent, nor is there enhanced gauge symmetry at $R = 1$. We therefore find that there is a whole line of inequivalent theories. As $R \to 0$ the spectrum degenerates to the $(-)^{f_L}$ orbifold theory which, as discussed in the end of section 2.4, appears inconsistent.

As summary of this section, we tabulate for each line of theories the duality symmetry and the enhanced gauge symmetry at the self-dual point (see table 1).

4. The torus partition functions

In this section we analyze the torus partition function of the compactified Spin(24) and Spin(8) $\times E_8$ theories. We consider in turn the three theories discussed above: untwisted, thermal twist, and twist by world-sheet fermion number $(-)^{f_L}$. The result in each case takes the form

$$Z = aR + \frac{b}{R} \tag{4.1}$$

for some constants a and b. In the twisted theories there is a phase transition at $R = 1$; and so the constants a, b are different for $R > 1$ and $R < 1$. In each case we compute a, b and perform non-trivial checks on our results:

(i) We rewrite the string theory partition function in a form that isolates the field theory result (proportional to $1/R$) and the cosmological constant (proportional to R). The coefficients a, b are computed unambiguously this way.

(ii) The procedure in (i) uncovers a non-analytic contribution to the torus partition function of the twisted theories. This signals phase transitions at $R = 1$ for all the twisted theories. We trace the non-analytic term to modes that become massless at $R = 1$ and show how it arises in conventional field theory.

(iii) We compute the coefficient b independently in field theory. As explained in [3, 6, 11] this can be implemented efficiently by summing over momentum modes using ζ-function regularization $\sum_{n=1}^{\infty} n \to -\frac{1}{12}$ and $\sum_{n=0}^{\infty}(n+\frac{1}{2}) \to \frac{1}{24}$ for bosons (opposite sign for fermions). In the twisted theories these results are reliable for $R > 1$ only.

(iv) The coefficient a (the cosmological constant) is independent of the boundary condition (i.e. insensitive to the twists). This is a non-trivial check on the computations. Additionally, the coefficient a can be computed as in (iii), but now summing over winding modes. When there is phase transition the result obtained this way can be trusted only for $R < 1$.

Our final results are given in (4.8) and (4.9) for the untwisted theory and tables 2 and 3 in the twisted cases. The interpretation of the results is more tentative; it is discussed in section 4.4.

A final point to make before we move on concerns the calculation of the odd spin structure of the right-moving CFT. In higher dimensions the odd spin structure vanishes trivially, due to the presence of fermion zero modes; but, in two dimensions, these can be cancelled and a non-zero contribution arises in some cases [6]. For heterotic strings there are additional fermionic zero-modes due to the left moving fermions so, for diagonal GSO projection, the odd spin structure is again irrelevant. The only remaining question is for the compactifications with twist where, in general, the odd spin structure multiplies nonvanishing left movers. We now argue, following appendix A.1 of [6], that the odd spin structure in fact vanishes quite generally for heterotic strings.

The effect of the zero mode of the superconformal ghost, γ, is to cancel the zero mode of the fermionic partner of the Liouville field, ψ_ϕ, for both are associated with conformal Killing spinors on the torus. The zero mode of the supermodulus, β, leads to an insertion into the path integral on the torus of the supercharge

$$G(z) = \psi_x \partial x + \psi_\phi \partial\phi - 2\partial\psi_\phi \, . \tag{4.2}$$

This insertion absorbs the ψ_x zero mode leaving only $\langle \partial x(z) \rangle$ to be calculated, where only x is to be integrated over. This vanishes due to the $x \to -x$ symmetry of the worldsheet theory and so the odd spin structure also vanishes. In a theory with both left- and right-moving supercharges, the final result would be proportional to $\langle \partial x \bar{\partial} x \rangle$ and thus generically non-zero. So we see that it is a feature of the heterotic theories that this spin-structure vanishes even in $D = 2$.

4.1 Torus partition function: untwisted theory

First, consider the compactified theory with no twists. The partition function of the matter field alone is

$$\begin{aligned} Z_x(\tau) &= \frac{1}{|\eta(\tau)|^2} \sum_{n,w\in\mathbb{Z}} q^{\frac{1}{2}p_R^2} \bar{q}^{\frac{1}{2}p_L^2} \\ &= 2\pi R \cdot \frac{1}{\sqrt{8\pi^2\tau_2}} \frac{1}{|\eta(\tau)|^2} \sum_{m,w\in\mathbb{Z}} \exp\left(-\frac{\pi R^2 |m - w\tau|^2}{2\tau_2}\right) . \end{aligned} \tag{4.3}$$

In the first line the lattice sum is over $p_{R,L} = \frac{n}{R} \pm \frac{wR}{2}$. The symmetry under $R \to \frac{2}{R}$ is manifest in this form. The second line (obtained by Poisson resummation) is the instanton sum which is more convenient here. The corresponding partition function for the Liouville field is regulated by a volume V_L and there is no sum over instantons. The contribution from bosonic ghosts is $|\eta(\tau)|^4$. The complete torus partition function then takes the form

$$Z_{\text{circle}} = 2\pi R \cdot V_L \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{4\tau_2} \frac{1}{8\pi^2\tau_2} Z_F(\bar{\tau}) \sum_{m,w\in\mathbb{Z}} \exp\left(-\frac{\pi R^2 |m - w\tau|^2}{2\tau_2}\right) \tag{4.4}$$

where, in the present section, we write

$$Z_F(\bar\tau) = \frac{1}{2}\left(\chi_0^0(\bar\tau) - \chi_1^0(\bar\tau) - \chi_0^1(\bar\tau)\right) \tag{4.5}$$

as a convenient expression that combines (2.4) and (2.6) using the notation

$$\chi_i^j(\bar\tau) = \begin{cases} Z_i^j(\bar\tau)^{12} & \text{Spin}(24) \\ Z_i^j(\bar\tau)^4\left(Z_0^0(\bar\tau)^8 + Z_1^0(\bar\tau)^8 + Z_0^1(\bar\tau)^8\right) & \text{Spin}(8)\,. \end{cases} \tag{4.6}$$

The right-moving fermions and the super-conformal ghosts are included in Z_F; they cancel except for the relative signs of the various terms.

The modular integral can be carried out explicitly as follows [30 − 33]. First, divide the instanton sum over (m, w) into the zero-mode $m = w = 0$ and the non-zero-modes. Next, rewrite the non-zero-modes $m = kp$ and $w = kq$ with $k = \gcd(m, w)$; then p, q are mutually prime. For each mutually prime pair p, q there is a unique modular tranformation $(p, q) \to (1, 0)$ that maps the fundamental region $\mathcal{F}$ ($|\tau| > 1$, $|\tau_1| < \frac{1}{2}$, $\tau_2 > 0$) to a domain $E_{p,q} \subset E$, where E is the half-strip ($-\frac{1}{2} < \tau_1 < \frac{1}{2}$, $\tau_2 > 0$). The union $E = \cup_{p,q} E_{p,q}$ makes up the entire half-strip so the net result is to trade the sum over all non-zero-modes for a sum over only $(k, 0)$, while simultaneously extending the integration region from the fundamental region $\mathcal{F}$ to the entire half-strip E. In the present context this procedure gives

$$Z_{\text{circle}} = \frac{RV_L}{16\pi}\left[\int_{\mathcal{F}} \frac{d\tau d\bar\tau}{\tau_2^2} Z_F(\bar\tau) + 2\sum_{k=1}^{\infty}\int_E \frac{d\tau d\bar\tau}{\tau_2^2} Z_F(\bar\tau) e^{-\frac{\pi R^2 k^2}{2\tau_2}}\right]. \tag{4.7}$$

The first term (integration over $\mathcal{F}$) gives the cosmological constant, and the second term (integration over E) gives the standard quantum field theory result. In the field theory term, the integral over τ_1 simply implements level matching.

In the Spin(24) theory, carrying out the integral (note $d\tau d\bar\tau = 2d\tau_1 d\tau_2$) gives

$$Z_{\text{circle,Spin(24)}} = \frac{RV_L}{16\pi}\, 24\, \left(\frac{2\pi}{3} + 4\cdot\frac{2}{\pi R^2}\cdot\frac{\pi^2}{6}\right) = V_L\left(R + \frac{2}{R}\right) \tag{4.8}$$

where we have used $Z_F = 24$. As a check, note that (4.8) is consistent with the self-duality $R \to 2/R$. In the Spin(8) × E_8 theory the fermionic partition function (2.6) vanishes and so

$$Z_{\text{circle,Spin(8)}} = 0\,. \tag{4.9}$$

We can understand the results (4.8) and (4.9) independently from field theory: for Spin(24) we sum over momenta $\sum_n \frac{n}{R} \to -\frac{1}{12}\frac{1}{R}$ for each of the 24 spacetime bosons. Multiplication by $(-V_L)$ then gives the field theory term in (4.8). Similarly, summing over the winding $\sum_w \frac{wR}{2} \to -\frac{1}{24}R$ for each of the 24 bosons, we recover the term proportional to R. In the Spin(8) × E_8 theory the 8 bosons and 8 fermions cancel in each case; so the vanishing partition function (4.9) follows from field theory as well.

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4.2 Torus partition function: thermal theory

We next consider the theory with a $(-)^{F_L}$ twist. The twisting correlates the left-moving fermions and the lattice vectors non-trivially, as indicated in (3.3). The torus partition function for the thermal theory is

$$\begin{aligned} Z_{\text{thermal}} &= \int_{\mathcal{F}} \frac{d\tau d\bar\tau}{4\tau_2} \frac{V_L}{\sqrt{8\pi^2\tau_2}} \cdot \frac{1}{2} \times \\ &\times \sum_{n,w\in\mathbb{Z}} \Bigg\{ \left[\chi_0^0(\bar\tau) - \chi_1^0(\bar\tau)\right] q^{\frac12\left(\frac{n}{R}+\frac{2wR}{2}\right)^2} \bar q^{\frac12\left(\frac{n}{R}-\frac{2wR}{2}\right)^2} + \\ &\qquad + \left[\chi_0^0(\bar\tau) + \chi_1^0(\bar\tau)\right] q^{\frac12\left(\frac{n+\frac12}{R}+\frac{(2w+1)R}{2}\right)^2} \bar q^{\frac12\left(\frac{n+\frac12}{R}-\frac{(2w+1)R}{2}\right)^2} - \\ &\qquad - \chi_0^1(\bar\tau)\Bigg[q^{\frac12\left(\frac{n+\frac12}{R}+\frac{2wR}{2}\right)^2} \bar q^{\frac12\left(\frac{n+\frac12}{R}-\frac{2wR}{2}\right)^2} + \\ &\qquad\qquad q^{\frac12\left(\frac{n}{R}+\frac{(2w+1)R}{2}\right)^2} \bar q^{\frac12\left(\frac{n}{R}-\frac{(2w+1)R}{2}\right)^2} \Bigg]\Bigg\}. \end{aligned} \tag{4.10}$$

This expression is absolutely convergent for all R but, due to the additional tachyons $T_0^{\pm}$ (3.5) at $R=1$, it is *not* analytic in R. To see this explicitly, focus on the contribution from these states:

$$Z_{\text{thermal}} = \int \frac{d\tau d\bar\tau}{4\tau_2} \frac{V_L}{\sqrt{8\pi^2\tau_2}} \frac{1}{2} \left[\chi_0^0(\bar\tau) + \chi_1^0(\bar\tau)\right] \cdot 2\bar q^{\frac18(R+\frac1R)^2} q^{\frac18(R-\frac1R)^2} + \text{regular}\,. \tag{4.11}$$

Since $\frac12(\chi_0^0(\bar\tau) + \chi_1^0(\bar\tau)) \sim \bar q^{-1/2}$ for large τ_2 the exponential damping disappears for $R=1$; the integral remains convergent at $R=1$ only because of the powers of τ_2. However, the second derivative $\partial_R^2 Z_{\text{th}}$ diverges at $R=1$. This establishes a first order phase transition at $R=1$.

The expression (4.10) is symmetric under the duality $R\to 1/R$. However, because of the phase transition, we focus for now on $R>1$. Poisson resummation on the momenta gives

$$\begin{aligned} Z_{\text{thermal}} &= \int_{\mathcal{F}} \frac{d\tau d\bar\tau}{4\tau_2} \frac{2\pi R\cdot V_L}{8\pi^2\tau_2} \cdot \frac12 \times \\ &\times \sum_{m,w\in\mathbb{Z}} \Bigg\{ \left[\chi_0^0(\bar\tau) - \chi_1^0(\bar\tau) - (-)^m\chi_0^1(\bar\tau)\right] e^{-S(m,2w)} + \\ &\qquad + \left[(\chi_0^0(\bar\tau) + \chi_1^0(\bar\tau))(-)^m - \chi_0^1(\bar\tau)\right] e^{-S(m,2w+1)} \Bigg\} \end{aligned} \tag{4.12}$$

where $S(m,w) = \frac{\pi R^2}{2\tau_2}|m-w\tau|^2$. For sufficiently small R, the integral over individual terms in (4.12) diverges. The finite answer depends on first performing the sum over m,w and then integrating over τ. This lack of absolute convergence eventually leads to our phase transition.

It is useful to write (4.12) as

$$Z_{\text{thermal}} = Z_{\text{circle}} - 2Z_{\text{flip}} \tag{4.13}$$

where we assembled the "wrong sign" contributions into

$$Z_{\text{flip}} = \frac{RV_L}{16\pi}\int_{\mathcal{F}}\frac{d\tau d\bar{\tau}}{\tau_2^2}\,\frac{1}{2}\sum_{m,w\in\mathbb{Z}}\Big[\chi_0^0(\bar{\tau})e^{-S(2m+1,2w+1)} - \chi_1^0(\bar{\tau})e^{-S(2m,2w+1)} - \\ - \chi_0^1(\bar{\tau})e^{-S(2m+1,2w)}\Big]. \tag{4.14}$$

This expression is modular invariant: modular transformations permute the instanton factors $S(2m+1,2w+1)$, $S(2m,2w+1)$, and $S(2m+1,2w)$ nontrivially, but the fermion factors χ_0^0, $-\chi_1^0$, and $-\chi_0^1$ compensate for this, because they are permuted in exactly the same way. When mapping to the half-strip, the three instanton terms all map into $(2k+1)^2 S(1,0)$ (the greatest common divisor is odd in each case) and the fermion partition factors in each case maps to the $-\chi_0^1$. Between the three terms, all pairs of mutually primes are being covered; so the union of the integration regions after mapping is again the entire half-strip E. Therefore, the integral (4.14) can be written as

$$Z_{\text{flip}} = -\frac{RV_L}{16\pi}\int_E \frac{d\tau d\bar{\tau}}{\tau_2^2}\,\frac{1}{2}\chi_0^1(\bar{\tau})\sum_{k\in\mathbb{Z}} e^{-(2k+1)^2 S(1,0)}\,. \tag{4.15}$$

Collecting terms we find

$$Z_{\text{thermal}} = \frac{RV_L}{16\pi}\Bigg\{\int_{\mathcal{F}}\frac{d\tau d\bar{\tau}}{\tau_2^2}\,\frac{1}{2}\left[\chi_0^0(\bar{\tau}) - \chi_1^0(\bar{\tau}) - \chi_0^1(\bar{\tau})\right] + \\ + 2\sum_{k=1}^{\infty}\int_E\frac{d\tau d\bar{\tau}}{\tau_2^2}\,\frac{1}{2}\left[\chi_0^0(\bar{\tau}) - \chi_1^0(\bar{\tau}) - (-)^k\chi_0^1(\bar{\tau})\right]e^{-S(k,0)}\Bigg\}. \tag{4.16}$$

The significance of Z_{flip} in (4.12) is, therefore, to reverse the sign of the field theory contribution from spacetime fermions with odd momentum, as expected of thermal twisting.

In the form (4.16) the connection to standard thermodynamics is clear [30]. The first term is the vacuum energy and the second term is the trace over the spacetime Hilbert space of $e^{-\beta H}$. In the second term of (4.16) one has to first integrate over τ_1 thus implementing the level matching condition in the theory in noncompact space, and then integrates over τ_2. In our case this integral over τ_2 converges. However, recall the observation after (4.12) that, for sufficiently small R, the integral over τ of the individual terms in the sum diverges, and the correct, finite answer is obtained only if one first performs the sum and then integrate over τ. In the form (4.16) this translates to the statement that the result of the integral over the half strip E depends on the precise way it is performed. In particular, the correct answer might not correspond to first integrating over τ_1 and then over τ_2. We will now see that for small R this naive result of (4.16) is in fact wrong.

In the Spin(24) theory there are no spacetime fermions; $Z_{\text{flip}} = 0$ by level matching. Accordingly, the torus partition function (4.16) for the thermal theory agrees exactly with the result (4.8) from the circle theory

$$Z_{\text{thermal,Spin(24)}} = V_L\left(R + \frac{2}{R}\right), \qquad R > 1 \tag{4.17}$$

This makes sense physically: the cosmological constant term (proportional to R) should not be sensitive to boundary conditions; and the field theory contribution (proportional to $1/R$) of the 24 spacetime bosons is also unaffected by the twist.

The result for $R < 1$ is easily determined as

$$Z_{\text{thermal,Spin}(24)} = V_L \left(2R + \frac{1}{R}\right), \qquad R < 1 \tag{4.18}$$

by imposing the symmetry under $R \to 1/R$ of the original expression (4.10). In more detail, this can be obtained by Poisson resummation on w, rather than n. This gives alternate expressions similar to (4.12), (4.14), and (4.16); but now with good convergence for small R.

The two expressions (4.17) and (4.18) agree at $R = 1$ but, as expected from (4.11), the derivative is discontinuous there. This result can be understood from effective field theory, as explained in the introduction. Indeed, combining the string theory results (4.17) and (4.18) into

$$Z_{\text{thermal,Spin}(24)} = \frac{3}{2} V_L \left(R + \frac{1}{R}\right) - \frac{1}{2} V_L \left|R - \frac{1}{R}\right|, \qquad \forall R \tag{4.19}$$

we see that the field theory result (1.2) for a complex boson with $m(R) = \frac{1}{2}\left|R - \frac{1}{R}\right|$ accounts precisely for the non-analyticity.

For the $\text{Spin}(8) \times E_8$ theory the integral (4.16) gives

$$Z_{\text{thermal,Spin}(8)} = \frac{V_L}{R}, \qquad R > 1\,. \tag{4.20}$$

The cosmological constant (proportional to R) vanishes as it did for the circle theory; so it is independent of the twist as it should be. The field theory contribution is nonvanishing because the spacetime fermions are sensitive to the $(-)^k$ weight of the instanton sectors in (4.16); they no longer cancel the bosons. The result (4.20) also follows from field theory, by summing up the momenta $\sum_n n/R$ of 8 bosons and the shifted momenta $\sum_n (n + \frac{1}{2})/R$ of 8 fermions.

The partition function in the small R phase follows from duality:

$$Z_{\text{thermal,Spin}(8)} = V_L R\,, \qquad R < 1\,. \tag{4.21}$$

Combining the results (4.20) and (4.21) into

$$Z_{\text{thermal,Spin}(8)} = \frac{1}{2} V_L \left(R + \frac{1}{R}\right) - \frac{1}{2} V_L \left|R - \frac{1}{R}\right|, \qquad \forall R \tag{4.22}$$

we see that the non-analyticity in the $\text{Spin}(8) \times E_8$ theory is identical to that found in the $\text{Spin}(24)$ theory; it is again accounted for by the complex boson with $m(R) = \frac{1}{2}\left|R - \frac{1}{R}\right|$. For easy reference we summarize in table 2 the results for the thermal theory.

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	$R>1$	$R<1$
$Z_{\text{thermal, Spin}(24)}$	$V_L\left(R+\frac{2}{R}\right)$	$V_L\left(2R+\frac{1}{R}\right)$
$Z_{\text{thermal, Spin}(8)}$	$V_L\frac{1}{R}$	$V_L R$

Table 2: Torus partition functions after the thermal twist $(-)^{F_L}$.

4.3 Torus partition function: theory with $(-)^{f_L}$ twist

The theory with a $(-)^{f_L}$ twist has the partition function

$$\begin{aligned} Z_{\text{twist}} &= \int_{\mathcal{F}} \frac{d\tau d\bar\tau}{4\tau_2}\,\frac{V_L}{\sqrt{8\pi^2\tau_2}}\times \\ &\times\sum_{n,w\in\mathbb{Z}}\Bigg\{\frac{1}{2}(\chi_0^0(\bar\tau)-\chi_1^0(\bar\tau))q^{\frac{1}{2}\left(\frac{n+\frac{1}{2}}{R}+wR\right)^2}\bar q^{\frac{1}{2}\left(\frac{n+\frac{1}{2}}{R}-wR\right)^2} - \\ &\qquad -\frac{1}{4}\chi_0^1(\bar\tau)\left(q^{\frac{1}{2}\left(\frac{n}{R}+wR\right)^2}\bar q^{\frac{1}{2}\left(\frac{n}{R}-wR\right)^2}+q^{\frac{1}{2}\left(\frac{n+\frac{1}{2}}{R}+wR\right)^2}\bar q^{\frac{1}{2}\left(\frac{n+\frac{1}{2}}{R}-wR\right)^2}\right)+ \\ &\qquad +\frac{1}{2}\chi_0^1(\bar\tau)q^{\frac{1}{2}\left(\frac{n}{R}+\frac{2w+1}{2}R\right)^2}\bar q^{\frac{1}{2}\left(\frac{n}{R}-\frac{2w+1}{2}R\right)^2} - \\ &\qquad -\frac{1}{4}(\chi_0^0(\bar\tau)+\chi_1^0(\bar\tau))q^{\frac{1}{2}\left(\frac{n+\frac{1}{2}}{R}+\frac{2w+1}{2}R\right)^2}\bar q^{\frac{1}{2}\left(\frac{n+\frac{1}{2}}{R}-\frac{2w+1}{2}R\right)^2} - \\ &\qquad -\frac{1}{4}(\chi_0^0(\bar\tau)-\chi_1^0(\bar\tau))q^{\frac{1}{2}\left(\frac{n}{R}+\frac{2w+1}{2}R\right)^2}\bar q^{\frac{1}{2}\left(\frac{n}{R}-\frac{2w+1}{2}R\right)^2}\Bigg\} \end{aligned} \tag{4.23}$$

As in the thermal theory, there is a non-analytic feature at $R=1$, interpreted as a phase transition. The origin of the phase transition is the two fermions (3.7) that become massless at $R=1$. Their contribution to (4.23) is

$$Z_{\text{twist}} = \int_{\mathcal{F}} \frac{d\tau d\bar\tau}{4\tau_2}\,\frac{V_L}{\sqrt{8\pi^2\tau_2}}\left(-\frac{1}{4}\right)(\chi_0^0(\bar\tau)+\chi_1^0(\bar\tau))\;2\;q^{\frac{1}{8}(\frac{1}{R}+R)^2}\bar q^{\frac{1}{8}(\frac{1}{R}-R)^2}+\text{regular} \tag{4.24}$$

which clearly has a divergent second derivative.

After Poisson resummation, the partition function can be written as

$$\begin{aligned} Z_{\text{twist}} &= \frac{RV_L}{16\pi}\int_{\mathcal{F}}\frac{d\tau d\bar\tau}{\tau_2^2}\times \\ &\times\sum_{m,w\in\mathbb{Z}}\Bigg\{\frac{1}{2}\chi_0^0(\bar\tau)(e^{-S(2m,2w)}-e^{-S(2m+1,2w)}-e^{-S(2m,2w+1)}) - \\ &\qquad -\frac{1}{2}\chi_1^0(\bar\tau)(e^{-S(2m,2w)}-e^{-S(2m+1,2w+1)}-e^{-S(2m+1,2w)}) \\ &\qquad -\frac{1}{2}\chi_0^1(\bar\tau)(e^{-S(2m,2w)}-e^{-S(2m,2w+1)}-e^{-S(2m+1,2w+1)})\Bigg\} \end{aligned} \tag{4.25}$$

This expression is manifestly modular invariant. Comparing with quantities defined in the previous sections, we find

$$Z_{\text{twist}}(R) = Z_{\text{circle}}(2R)-\frac{1}{2}Z_{\text{circle}}(R)-\frac{1}{2}Z_{\text{thermal}}(R) \tag{4.26}$$

for all R. This gives table 3.

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	$R > 1$	$R < 1$
$Z_{\text{twist, Spin}(24)}$	$V_L\left(R - \frac{1}{R}\right)$	$\frac{V_L}{2}\left(R - \frac{1}{R}\right)$
$Z_{\text{twist, Spin}(8)}$	-$V_L \frac{1}{2R}$	-$V_L \frac{R}{2}$

Table 3: Torus partition functions after twist by $(-)^{f_L}$.

For $R > 1$ the entries can be understood from general principles: the cosmological constant (proportional to R) is the same as in the untwisted theories (4.8), (4.9). For Spin(24) the field theory term (proportional to $1/R$) is due to 24 bosons with momenta $(n + \frac{1}{2})/R$; and, for Spin(8) $\times E_8$ it is due to 8 bosons with momenta $(n + \frac{1}{2})/R$, 8 real fermions with momenta n/R, and 8 real fermions with momenta $(n + \frac{1}{2})/R$.

The results in the table can be combined as

$$\begin{aligned} Z_{\text{twist, Spin}(24)} &= \frac{3}{4}V_L\left(R - \frac{1}{R}\right) + \frac{1}{4}V_L\left|R - \frac{1}{R}\right|, \qquad \forall R \\ Z_{\text{twist, Spin}(8)} &= -\frac{1}{4}V_L\left(R + \frac{1}{R}\right) + \frac{1}{4}V_L\left|R - \frac{1}{R}\right|, \qquad \forall R\,. \end{aligned} \tag{4.27}$$

We see that, as expected, the singularity in each theory takes the form predicted by field theory (1.2) for one complex fermion with mass $m(R) = \frac{1}{2}\left|R - \frac{1}{R}\right|$. It is also manifest in (4.27) that only the Spin(8) $\times E_8$ theory satisfies the duality symmetry $R \to 1/R$.

4.4 Discussion

Up to this point we have discussed the torus partition function as a rather abstract object. Our determination of this object is unambiguous, but the interpretation is not necessarily straightforward. Here we discuss these issues.

Thermodynamic interpretation of theories twisted by $(-)^{F_L}$.

In the thermal theory it is natural to try to identify $2\pi R = \beta = T^{-1}$. Recalling that the total partition function, including disconnected amplitudes, is related to the torus amplitude through $Z_{\text{tot}} = e^{Z_{\text{sphere}} + Z_{\text{torus}} + \cdots}$, we then find the energy density $\epsilon = -\frac{1}{V_L}\frac{\partial \ln Z_{\text{tot}}}{\partial \beta} = -\frac{1}{V_L}\frac{\partial Z_{\text{torus}}}{\partial \beta} + \cdots$, and the free energy density $f = \epsilon - Ts = -\frac{1}{V_L \beta}\ln Z_{\text{tot}} = -\frac{1}{V_L \beta} Z_{\text{torus}} + \cdots$. The results of this procedure are recorded in table 4.

	$T < \frac{1}{2\pi}$	$T > \frac{1}{2\pi}$
f_{24}	$-4\pi T^2 - \frac{1}{2\pi}$	$-2\pi T^2 - \frac{1}{\pi}$
ϵ_{24}	$4\pi T^2 - \frac{1}{2\pi}$	$2\pi T^2 - \frac{1}{\pi}$
s_{24}	$8\pi T$	$4\pi T$
f_8	$-2\pi T^2$	$-\frac{1}{2\pi}$
ϵ_8	$2\pi T^2$	$-\frac{1}{2\pi}$
s_8	$4\pi T$	0

Table 4: Thermodynamics of the theories on a thermal circle.

In either theory the transition is characterized by negative latent heat per volume $\ell = T\Delta s = -\frac{1}{\pi}$. This means the high temperature phase is *more* ordered than the low temperature phase. This situation is usually considered unacceptable in thermodynamics. Let us make some comments on the possible interpretation of the result.

A clear benchmark for the interpretation is the evaluation of the trace Tr $e^{-\beta H}$ over the spacetime Hilbert space. This is represented by the half-strip in our computations and gives the result indicated in the table for $T < 1/2\pi$, but now at all T. The reason this cannot be the correct answer at high T is that it is inconsistent with the duality $R \to 1/R$; and also it does not take into account the presence of additional light modes at $T = 1/2\pi$. However, it would be surprising if this result was invalid for $T < 1/2\pi$.

The table above does not reflect standard thermodynamics. Usually, when considering a first order phase transition, one would reason that near the transition there are two candidate phases which have free energies taking the forms given in the table, with the range for *each* phase analytically continued to other values of T. Then one determines the stable phase as the one with the lowest free energy at each T. We were guided instead by the field theory interpretation at low temperature, and then applied duality. Our procedure apparently amounts to taking the *highest* free energy in each phase. If taken at face value this means that both phases are unstable. However, as explained above such an instability would be very surprising, at least in the low temperature phase.

We are thus lead to a picture where our results and their thermodynamic interpretation can be trusted at $T < 1/2\pi$. For $R < 1$ the theories exist and we can compute reliably; but it seems that the proper interpretation of the string calculation in this regime cannot be standard thermodynamics. One indication of this is that the torus partition function has no obvious field theory interpretation for $R < 1$.

Finally, we should clarify that, because of T-duality, the theory with $R < 1$ can, of course, be interpreted in terms of a system with $1/R > 1$ and, in these variables, there exist standard thermodynamics with temperature $R/2\pi < 1/2\pi$. The question discussed here is whether, in addition, this regime permits an interpretation as a genuinely new phase with $T > 1/2\pi$.

Interpretation of theories twisted by $(-)^{f_L}$.

We next discuss the interpretation of the phase transition in the theories twisted by $(-)^{f_L}$. Again, the results for $R > 1$ have clear field theory interpretations. As for the regime $R < 1$, a distinction must be made between the Spin(24) theory and the Spin(8) $\times$ E_8 theory. In the latter, the duality $R \to 1/R$ is part of the gauge symmetry at the self-dual point. This means that the $R < 1$ phase exists in the Spin(8) $\times$ E_8 theory. In contrast, in the Spin(24) theory, there is no duality $R \to 1/R$, and also no enhanced gauge symmetry at $R = 1$. In this case it is possible that the theory simply does not exist at $R < 1$. One appealing consequence of this possibility is that it would exclude the apparently inconsistent limit $R \to 0$ from moduli space.

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A. Lattice Constructions

In this appendix we first classify the possible uncompactified theories using covariant lattices, finding exactly the Spin(24) and Spin(8) $\times$ E_8 theories. Next, we consider the compactified theory and show that there is a single contiguous moduli space with 13 dimensions. The relation between the Spin(24) and Spin(8) $\times$ E_8 theories is found explicitly.

A.1 Classification of uncompactified theories

We will use the covariant lattice approach following [34] . In this formalism the superconformal ghosts are represented as three bosons $\vec{x}_{\rm gh}$ with canonical normalization and signature. The correspondence is

$$e^{q\phi} \leftrightarrow e^{i\vec{v}\cdot\vec{x}_{\rm gh}}\,. \tag{A.1}$$

The canonical qhost pictures map as

$$\begin{aligned} R: q_0 &= -\frac{1}{2} \leftrightarrow \vec{v}_0 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \\ NS: q_0 &= -1 \;\leftrightarrow \vec{v}_0 = (0, 0, -1)\,. \end{aligned} \tag{A.2}$$

Under this identification the levels of simple operators map as

$$\Delta(e^{q\phi}) = -q - \frac{1}{2}q^2 = \frac{1}{2}\vec{v}^2 = \Delta(e^{i\vec{v}\cdot\vec{x}_{\rm gh}}) \tag{A.3}$$

and locality conditions are preserved because

$$-q_1 q_2 = \vec{v}_1 \cdot \vec{v}_2 \qquad \mathrm{mod}\ 1 \tag{A.4}$$

for any pairs q_1 and q_2; and the corresponding $\vec{v}_1$ and $\vec{v}_2$. We can now write vertex operators of propagating states as

$$V_{\vec{w}_R,\vec{w}_L,k} = e^{i\vec{w}_R\cdot\vec{H}_R} e^{i\vec{w}_L\cdot\vec{H}_L} \mathcal{O}_{R,L} V_k\,. \tag{A.5}$$

Here the $\vec{H}_R = (H, \vec{x}_{\rm gh})$ denote the 4 right-moving bosons and the $\vec{H}_L$ are the corresponding 12 bosons, making up the lattice on the left side. The V_k is the $(\frac{1}{2}, \frac{1}{2})$ operator (2.3) associated with the bosonic matter fields x and ϕ. Finally, the $\mathcal{O}_{R,L}$ are operators constructed from the towers of bosonic oscillators. In the present context physical conditions will not leave any operators of this kind before compactification. Discrete states take a similar form, with V_k replaced by the identity operator.

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The consistency conditions on the string theory are satisfied in the covariant lattice construction by demanding that $w = (\vec{w}_R, \vec{w}_L)$ forms an *even, self-dual lattice of signature* $(4, 12)$. This comes as usual from the level matching condition

$$\frac{1}{2}\vec{w}_R^2 - \frac{1}{2}\vec{w}_L^2 \in \mathbb{Z} \tag{A.6}$$

which requires the lattice to be even, and invariance under the modular transformation $\tau \to -1/\tau$ imposes self-duality. The locality condition

$$\vec{w}_{R1} \cdot \vec{w}_{R2} - \vec{w}_{L1} \cdot \vec{w}_{L2} \in \mathbb{Z} \tag{A.7}$$

is automatic after level matching. The only subtlety in these statements is that, in this formalism, the description of the superconformal ghosts has more redundancy than is familiar, a feature that can be factored out [34]. In summary, the uncompactified theories are classified by the even self-dual lattices of signature $(4, 12)$.

Such lattices are in one-to-one correspondance with even self-dual lattices in 16 *euclidean* dimensions. It is well-known that there are precisely two such lattices, $\mathrm{Spin}(32)/\mathbb{Z}_2$ and $E_8 \times E_8$. Physical string theories then follow from the decomposition[1]

$$\Gamma_{16} \supset \mathrm{Spin}(8) \otimes \Gamma_{12} \tag{A.8}$$

where the first factor encodes the right-moving fermions and the super-conformal ghosts, while the second factor is the 12-dimensional lattice of left-moving bosons. For $\mathrm{Spin}(32)/\mathbb{Z}_2$ this decomposition leaves as stabilizer the lattice $\mathrm{Spin}(24)$ and so the first of the 2D heterotic theories. For $E_8 \times E_8$ the embedding goes into one E_8-factor as $\mathrm{Spin}(8) \subset \mathrm{Spin}(16) \subset E_8$. The stabilizer of this embedding is $\mathrm{Spin}(8) \times E_8$, leading to the other 2D heterotic string.

In both cases the lattice embeddings align conjugacy classes of $\mathrm{Spin}(2n) \times \mathrm{Spin}(2m) \subset \mathrm{Spin}(2n + 2m)$ in the obvious diagonal fashion[2] $(0, 0) \oplus (V, V) \oplus (S, C) \oplus (C, S) = 0 \oplus C$. This is the lattice analogue of the diagonal GSO in the CFT language.

The covariant lattice construction similarly classifies heterotic string theories in 10 dimensions. In this case the relevant lattices have signature $(8, 16)$, with the first factor $8 = 5 + 3$ from bosonized fermions and superconformal ghosts. Such lattices are classified by the even self-dual lattices in 24 euclidean dimensions, i.e. the Niemeyer lattices. Decomposing these lattices as $\Gamma_{24} \supset \mathrm{Spin}(16) \otimes \Gamma_{16}$ identifies 8 distinct heterotic string theories associated with different Γ_{16}.[3] Among these, two correspond to simple factorization $\Gamma_{24} = \mathrm{Spin}(16) \otimes \Gamma_{16}$. These are the usual supersymmetric heterotic string theories with gauge groups $\mathrm{Spin}(32)/\mathbb{Z}_2$ and $E_8 \times E_8$, constructed using a chiral GSO projection. In two

[1] We do not make any distinctions between the $\mathrm{Spin}(2n)$ lattice and the lattice of the Lie algebra D_n.

[2] The covariant lattice $0 \oplus S$ gives a more symmetric decomposition. This corresponds to the theory obtained from ours using the $\mathcal{C}$ transformation (2.10) which amounts to a different convention.

[3] The covariant lattice construction misses one 10D heterotic string theory, the one with gauge group E_8. The failure in this case is that the fermions cannot be bosonized using lattices. There seems to be no analogous possibilities in two dimensions.

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dimensions there are no analogues of these theories. The remaining heterotic string theories in ten dimensions involve nontrivial embeddings and correspond to non-chiral GSO projections. The closest analogue to the 2D heterotic string theories we study are the non-supersymmetric Spin(32) theory (based on the Niemeyer lattice $\mathrm{Spin}(48)/\mathbb{Z}_2$) and the $E_8\times\mathrm{Spin}(16)$ theory (based on the Niemeyer lattice $E_8\times\mathrm{Spin}(32)$) which both correspond to simple diagonal embeddings [35, 36]. The famous tachyon-free $O(16)\times O(16)$ string theory [35, 37] is based on the Niemeyer lattice $\mathrm{Spin}(16)^3$ which has no analogue in 16 euclidean dimensions so, from this perspective, there can be no analogous construction in two dimensions.

A.2 Compactification

We next discuss toroidal compactifications using lattices. Thus, the matter field X is assumed periodic with period R. Additionally, a general compactification has Wilson lines. These can can be introduced as usual through the shifted momenta

$$\begin{aligned} p_R &= \left(\frac{n}{R} - \vec{w}_L\cdot\vec{A} + \frac{wR}{2}\vec{A}^2\right) + \frac{wR}{2} \\ p_L &= \left(\frac{n}{R} - \vec{w}_L\cdot\vec{A} + \frac{wR}{2}\vec{A}^2\right) - \frac{wR}{2} \\ \vec{k_L} &= \vec{w_L} - wR\vec{A}\,. \end{aligned} \tag{A.9}$$

Here $\vec{w}_L$ refer to the vectors of the 12 dimensional left moving lattice prior to compactification. Since we are using a non-standard GSO projection, the bosonic lattice does not decouple completely from the right moving fermions the way it usually does in compactifications of 10D heterotic strings. The reason that the usual procedure works anyway is that

$$\vec{k}_L^2 + p_L^2 - p_R^2 = \vec{w}_L^2 - 2nw \in 2\mathbb{Z} \tag{A.10}$$

so, if the original set of lattice vectors were even, then the deformed set is even as well as well. Additionally, if we keep the original conjugacy classes, the covariant lattice remains self-dual. Thus the theory must be consistent also after deformation.

In the covariant lattice approach, the theory is consistent exactly when the full lattice vector $(\vec{k}_R, p_R; \vec{k}_L, p_L)$ belongs to an even, self-dual lattice of signature $(5,13)$. The sublattice obtained by restricting $\vec{k}_R$ to the canonical ghost pictures (the $\vec{k}_R\equiv\vec{w}_R$ is unchanged by the Wilson lines) has signature $(1,13)$ and its moduli space is

$$\mathcal{H}\backslash O(1,13,\mathbb{R})/O(13,\mathbb{R})\,. \tag{A.11}$$

This 13 dimensional moduli space of compactifications is parametrized locally by the radius of compactification R and the 12 Wilson lines $\vec{A}$. The global identifications indicated by $\mathcal{H}$ would be $\mathcal{H}=O(1,13,\mathbb{Z})$ in standard heterotic theory but the situation is not clear here.

Interestingly, this discussion is independent of which theory is taken as starting point: Spin(24) or $\mathrm{Spin}(8)\times E_8$; whether twisted or not. This means all these theories must belong to the same moduli space; they must be continuously related. In the following we verify this by explicit comparison.

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$\widetilde{n}$	w	Spin(8, 24)
$\mathbb{Z}$	$2\mathbb{Z}$	$(0,0),\ (V,V)$
$\mathbb{Z}+\frac{1}{2}$	$2\mathbb{Z}$	$(C,S),\ (S,C)$
$\mathbb{Z}+\frac{1}{2}$	$2\mathbb{Z}+1$	$(0,V),\ (V,0)$
$\mathbb{Z}$	$2\mathbb{Z}+1$	$(S,S),\ (C,C)$

Table 5: Covariant lattice representation of the thermal Spin(24) theory.

Let us first show how the theory twisted by $(-)^{F_L}$, i.e. the thermal theory, can be obtained from the untwisted theories by turning on a suitable Wilson line. The computation works the same way for the Spin(24) and the Spin(8) $\times$ E_8 theories so we just consider the former.

Before twisting the covariant lattice Spin(8, 24) can be in $0 \oplus C$ which, in canonical ghost picture, decomposes under Spin(8) $\times$ Spin(24) as $(0,0) \oplus (V,V) \oplus (C,S) \oplus (S,C)$. Each sector allows $n, w \in \mathbb{Z}$ along the thermal direction. The thermal twist corresponds to the Wilson line $R\vec{A} = (1, 0^{11}) \in V$ of Spin(24). The shift $\vec{w}_L \to \vec{k}_L = \vec{w}_L - w\vec{A}R$ from (A.9) means we still have the conjugacy classes $(0,0) \oplus (V,V) \oplus (C,S) \oplus (S,C)$ for $w \in 2\mathbb{Z}$ (untwisted sector) but now the conjugacy classes are $(0,V)\oplus(V,0)\oplus(S,S)\oplus(C,C)$ for $w \in 2\mathbb{Z}+1$ (twisted sector). The shifts (A.9) of p_L, p_R due to the Wilson line introduces the shifted momentum

$$\widetilde{n} = n - \vec{w}_L \cdot \vec{A}R + \frac{w}{2}\vec{A}^2R^2 = n - \frac{w}{2} - \vec{k}_L \cdot \vec{A}R\,. \tag{A.12}$$

The $\vec{k}_L \cdot \vec{A}R$ is integer (half-integer) for $\vec{k}_L \in 0 \oplus V$ $(S \oplus C)$ so the $\widetilde{n}$ is shifted to half-integer values for $\vec{k}_L \in S \oplus C$ in the untwisted sector and for $\vec{k}_L \in 0 \oplus V$ in the twisted sector. In summary, the spectrum after twisting is as in table 5.

This spectrum agrees precisely with the one given already in (3.3).

The spectrum of the theory compactified with $(-)^{f_L}$ twist can be obtained in an entirely analogous manner, by including a Wilson-line $R\vec{A} = (\frac{1}{2}^{12}) \in S$ of Spin(24). The untwisted sector (even winding) is given by the decomposition of the covariant lattice $0 \oplus C = (0,0) \oplus (V,V) \oplus (C,S) \oplus (S,C)$ while the twisted sector (odd winding) is the shifted lattice $(V,C) \oplus (0,S) \oplus (C,0) \oplus (S,V)$. The $\vec{k}_L \cdot \vec{A}R$ is half-integral for $\vec{k}_L \in V \oplus C$, so these are the conjugacy classes that have half-integral momentum in the untwisted sectors. Since $\vec{A}^2R^2 = 3$ the changes of modings are the opposite in the twisted sector, i.e. the momentum is half-integral for $\vec{k}_L \in 0 \oplus S$. This gives the spectrum in table 6 which is identical to that given already in (3.6).

The lattice implementation of the $(-)^{F_L}$ and $(-)^{f_L}$ twists simply recasts the discussion of the center of Spin($4n$) (section 2.3) in terms of Wilson lines. The identification is that $\mathcal{Z}_1 \sim (\vec{A}R \in V)$ and $\mathcal{Z}_2 \sim (\vec{A}R \in S)$. The advantage of the lattice technology is that it automates the procedure; and this is helpful when considering Wilson lines that are not in the center of the gauge groups. As an example of this, let us show that the Spin(24) and the Spin(8) $\times$ E_8 theories are related by T-duality $R \to 1/R$ after suitable Wilson lines are turned on. The computation is similar to the standard comparison between the Spin(32)/$\mathbb{Z}_2$ and $E_8 \times E_8$ heterotic theories in ten dimensions [38]. The computation (and

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$\widetilde{n}$	w	$\mathrm{Spin}(8,24)$
$\mathbb{Z}$	$2\mathbb{Z}$	$(0,0)$, (C,S)
$\mathbb{Z}+\frac{1}{2}$	$2\mathbb{Z}$	(V,V), (S,C)
$\mathbb{Z}$	$2\mathbb{Z}+1$	(V,C), (S,V)
$\mathbb{Z}+\frac{1}{2}$	$2\mathbb{Z}+1$	$(C,0)$, $(0,S)$

Table 6: Covariant lattice representation of the Spin(24) theory with $(-)^{f_L}$ twist.

$\widetilde{n}$	w	$\mathrm{Spin}(8,8)\times\mathrm{Spin}(16)$
$\mathbb{Z}$	$2\mathbb{Z}$	$(0,0)$, (S,S)
$\mathbb{Z}+\frac{1}{2}$	$2\mathbb{Z}$	(V,V), (C,C)
$\mathbb{Z}$	$2\mathbb{Z}+1$	$(0,S)$, $(S,0)$
$\mathbb{Z}+\frac{1}{2}$	$2\mathbb{Z}+1$	(C,V), (V,C)

Table 7: Spectrum of the Spin(24) theory with Wilson line $R\vec{A}=(0^4;\frac{1}{2}^8)$.

the relation to 10 dimensions) is clearest if we start from the covariant lattice $0\oplus S$.[4] The strategy is to decompose the covariant lattice $\mathrm{Spin}(8,24)$ into $\mathrm{Spin}(8,8)\times\mathrm{Spin}(16)$ and then add Wilson lines.

The covariant lattice of the Spin(24) theory with conjugacy classes $0\oplus S$ decomposes to $\vec{w}_L\in(0,0)\oplus(V,V)\oplus(S,S)\oplus(C,C)$ under the $\mathrm{Spin}(8,8)\times\mathrm{Spin}(16)$ subgroup. The Wilson line $R\vec{A}=(0^4;\frac{1}{2}^8)$ is in S of Spin(16) and leaves the $\mathrm{Spin}(8,8)$ intact. The shifted lattice-vector $\vec{k}_L=\vec{w}_L-wR\vec{A}$ has the spectrum

$$\begin{aligned}&(0,0)\oplus(V,V)\oplus(S,S)\oplus(C,C) \quad ;\ w\in 2\mathbb{Z}\\&(0,S)\oplus(S,0)\oplus(V,C)\oplus(C,V) \quad ;\ w\in 2\mathbb{Z}+1\,.\end{aligned} \tag{A.13}$$

The lattice vectors in the X-direction take the form $p_{R,L}=\frac{\widetilde{n}}{R}\pm\frac{wR}{2}$ where

$$\widetilde{n}=n-\vec{w}_L\cdot\vec{A}R+\frac{w}{2}\vec{A}^2R^2=n-w-\vec{k}_L\cdot\vec{A}R \tag{A.14}$$

since $\vec{A}^2R^2=2$. The $\vec{k}_L\cdot\vec{A}R$ is integer (half-integer) when the Spin(16) part of $\vec{k}_L\in\mathrm{Spin}(8)\times\mathrm{Spin}(16)$ is in $0\oplus S$ $(C\oplus V)$. The complete spectrum of the Spin(24) theory is then specified as in table 7.

Next, we consider the $\mathrm{Spin}(8)\times E_8$ theory. Then the covariant lattice decomposes to $(0,0)\oplus(0,S)\oplus(S,0)\oplus(S,S)$ under the $\mathrm{Spin}(8,8)\times\mathrm{Spin}(16)$ subgroup. The Wilson line $R\vec{A}=(1,0^3;1,0^7)$ belongs to the (V,V) conjugacy class of the full $\mathrm{Spin}(8,8)\times\mathrm{Spin}(16)$. It shifts the lattice-vector $\vec{k}_L=\vec{w}_L-wR\vec{A}$ has the spectrum

$$\begin{aligned}&(0,0)\oplus(0,S)\oplus(S,0)\oplus(S,S)\ ; \qquad w\in 2\mathbb{Z}\\&(V,V)\oplus(C,C)\oplus(V,C)\oplus(C,V)\ ; \qquad w\in 2\mathbb{Z}+1\end{aligned} \tag{A.15}$$

[4] As noted in a previous footnote, this is related to the convention used in the rest of the paper through conjugaction by $\mathcal{C}$ defined in (2.10).

$\widetilde{n}$	w	$\mathrm{Spin}(8,8) \times \mathrm{Spin}(16)$
$\mathbb{Z}$	$2\mathbb{Z}$	$(0,0)$, (S,S)
$\mathbb{Z}$	$2\mathbb{Z}+1$	(V,V), (C,C)
$\mathbb{Z}+\frac{1}{2}$	$2\mathbb{Z}$	$(0,S)$, $(S,0)$
$\mathbb{Z}+\frac{1}{2}$	$2\mathbb{Z}+1$	(C,V), (V,C)

Table 8: Spectrum of the $\mathrm{Spin}(8) \times E_8$ theory with Wilson line $R\vec{A} = (1, 0^3; 1, 0^7)$.

In this case $\vec{k}_L \cdot \vec{A}R$ is integer (half-integer) for the diagonal (off-diagonal) conjugacy classes. This shifts the allowed values of $\widetilde{n}$ (defined in (A.14)) so that the spectrum becomes as in table 8

The duality between the two heterotic theories can now be established by comparing table 7 and table 8; they agree after taking $R \to 1/R$ and $\widetilde{n} \leftrightarrow w/2$ (i.e. $n' = w/2$ and $w' = 2\widetilde{n}$). This concludes the explicit verification that the two theories are on the same moduli space.

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