Hidden beauty in multiloop amplitudes

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ABSTRACT: Planar L-loop maximally helicity violating amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory are believed to possess the remarkable property of satisfying iteration relations in L. We propose a simple new method for studying iteration relations for four-particle amplitudes which involves the use of certain linear differential operators and eliminates the need to fully evaluate any loop integrals. We carry out this procedure in explicit detail for the two-loop amplitude and prove that this method can be applied to any multiloop integral, allowing a conjectured iteration relation for any given amplitude to be tested up to polynomials in logarithms.

Contents

1. Introduction

Maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills (SYM) theory possesses remarkably rich mathematical structure which has been the subject of intense investigation over the past several years. One motivation for much of this work is the AdS/CFT correspondence [1], which asserts that the strongly-coupled SYM theory admits an equivalent formulation as gravity in AdS$_5$, thereby opening a new window for studying quantum gravity. A complementary motivation, which has seen a dramatic resurgence following [2] (see [3] for a review), is the desire to explore the mathematical structure of Yang-Mills perturbation theory and to exploit that structure to aid the calculation of scattering amplitudes. Scattering amplitudes are the basic building blocks which enter into the calculation of experimentally measured processes. As the LHC comes on line in the next couple of years there will be increasing pressure to bring the theoretical uncertainties in QCD calculations, especially at higher loops, under control.

Optimistically, we hope that the rich structure of Yang-Mills perturbation theory and the simplicity of the strongly-coupled theory expected from AdS/CFT are two sides of the same coin, and that we might in some cases be able to see some hint of the structure which enables perturbation theory to be resummed to match onto AdS/CFT.

One intriguing step in this direction has been the study of iterative relations amongst planar maximally helicity violating (MHV) loop amplitudes in dimensionally regulated $(d = 4 - 2\epsilon)$ $\mathcal{N} = 4$ SYM [4, 5]. In [4] Anastasiou, Bern, Dixon and Kosower suggested that two-loop MHV amplitudes obey the iteration

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)M_n^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon), \tag{1.1}$$

where we use $M_n^{(L)}(\epsilon) = A_n^{(L)}(\epsilon)/A_n^{(0)}$ to denote the ratio of the L-loop n-particle color-stripped partial amplitude to the corresponding tree-level amplitude, and

$$f^{(2)}(\epsilon) \equiv \frac{1}{\epsilon}(\psi(1-\epsilon) - \psi(1)) = -(\zeta(2) + \zeta(3)\epsilon + \zeta(4)\epsilon^2 + \cdots), \qquad \psi(x) \equiv \frac{d}{dx}\ln\Gamma(x). \quad (1.2)$$

We refer to (1.1) for general n as the ABDK conjecture. In section 4 we review an exponentiated ansatz conjectured in [5] which generalizes the iterative structure (1.1) for MHV amplitudes to all loops. These conjectures were motivated by studies of the infrared [6–8] and collinear [4, 9–11]-[12] behavior of multiloop amplitudes, where similar basic iterative structures appear.

The problem of calculating $L > 1$ amplitudes and then verifying these iterative structures involves numerous extremely difficult technical challenges. So far the only iteration relations which have been explicitly verified are the two-loop ABDK relation (1.1) for $n = 4$ particles [4] (the amplitude $M_4^{(2)}(\epsilon)$ was originally calculated in [13] using unitarity based methods [14–20], and its three-loop generalization [5], again just for $n = 4$, where the calculation of $M_4^{(3)}(\epsilon)$ by Bern, Dixon and Smirnov through $\mathcal{O}(\epsilon^0)$ is an impressive tour de force. Even at two loops the $n > 4$ amplitudes are not yet known. For some progress in this direction see [13, 21].

In this paper we introduce a new method for studying these iteration relations for $n = 4$ particle amplitudes at L loops, beginning with a direct check of the form of the ABDK relation at two loops. We first express each amplitude appearing in (1.1) in a familiar integral form. We then apply a linear differential operator $\mathcal{L}^{(2)}$ which is chosen in such a way that the terms in (1.1) can be nicely combined into a single integral which is valid in a neighborhood of $\epsilon = 0$. The result (1.1) then follows by simply expanding in ϵ under the integral sign, eliminating the need to fully evaluate any loop integrals. Consequently, in our calculation we never need to use the complicated explicit formulas for the ϵ series expansions of $M_4^{(1)}(\epsilon)$ and $M_4^{(2)}(\epsilon)$ in terms of polylogarithm functions. The only ambiguity introduced by acting with $\mathcal{L}^{(2)}$ is just a polynomial in $\ln^2(t/s)$, which we show can be fixed up to an additive numerical constant by the known infrared and collinear behavior of the two-loop amplitude.

We then prove that this method can be generalized to higher L. Specifically, we use general properties of four-particle Feynman integrals to prove that for any given polynomial in multiloop integrals, it is always possible to construct a differential operator $\mathcal{L}^{(L)}$ which generalizes the nice functionality of $\mathcal{L}^{(2)}$. It is important to note, however, that it is not yet known which particular integrals contribute to the $L > 3$ amplitudes. Once the participating integrals have been catalogued, our method should allow the conjectured iteration relations to be tested without evaluating the integrals.

In order to provide some motivation and context for our calculation we begin in section 2 with a more technical review of the difficulties involved in checking (1.1) and an outline of how we propose to overcome them. We present our calculation of the ABDK relation for $n = 4$ in section 3. section 4 contains a general argument which proves that at any loop order L it is always possible to construct a differential operator $\mathcal{L}^{(L)}$ which generalizes the nice functionality of $\mathcal{L}^{(2)}$.

2. Advanced introduction and motivation

The ABDK conjecture is highly nontrivial because infrared divergences in $M_n^{(1)}(\epsilon)$, which start at $\mathcal{O}(1/\epsilon^2)$, imply that it is necessary to know $M_n^{(1)}(\epsilon)$ through $\mathcal{O}(\epsilon^2)$ in order to check (1.1) through $\mathcal{O}(\epsilon^0)$. The proof of (1.1) for $n = 4$ given in [4] proceeds by evaluating both sides explicitly in terms of polylogs and finding that they agree up to terms of $\mathcal{O}(\epsilon)$.

Even for $n = 4$, the direct evaluation of the ABDK relation is difficult for two reasons. First of all, both $M_4^{(2)}(\epsilon)$ and $M_4^{(1)}(\epsilon)$ are very complicated functions of the kinematic ratio $x = -t/s$. Even using state-of-the-art techniques, such as Mellin-Barnes representations, the computation of the ϵ expansions of these two amplitudes requires the resummation of an infinite number of poles that give rise to various polylogarithms. To the order required to verify (1.1), these amplitudes involve polylogarithms of degree up to and including 4. Secondly, even after $M_4^{(2)}(\epsilon)$ and $M_4^{(1)}(\epsilon)$ have both been evaluated to the required order, it is necessary to use nontrivial polylog identities to check (1.1) explicitly order by order.

The goal of this paper is to provide a far simpler method for studying iterative relations which we hope may shed some light on the general structure of these relations and may be useful to test the conjectures in cases where their status is not currently known. Our direct proof relies on three basic ingredients.

First of all, we use the fact that loop amplitudes can be expressed in terms of Mellin-Barnes integral representations (a detailed introductory treatment can be found in the book [22]). These have the nice feature that the x dependence can be isolated in a simple manner. In particular, any four-particle loop amplitude can be written as an integral of the form $\int_{-i\infty}^{+i\infty} dy\, (-x)^y F(\epsilon, y)$ for some function $F(\epsilon, y)$. Often it is not very hard to find the function $F(\epsilon, y)$, rather it is the final integral over y which is exceedingly difficult to evaluate. It is clearly tempting to try to collect all of the terms in (1.1) under a single y integral and check the ABDK conjecture by expanding in ϵ 'under the integral sign.' In other words, we might like to look at the *inverse* Mellin transform[1] of (1.1) with respect to the variable $-x$.

However, this is usually not possible. The reason is that Mellin-Barnes representations of the form $\int_{-i\infty}^{+i\infty} dy\, (-x)^y F(\epsilon, y)$ are generally only valid for ϵ in an open set which does not contain $\epsilon = 0$. In order to make sense of statements such as (1.1), which involve a series expansion around $\epsilon = 0$, it is necessary to analytically continue the amplitudes in ϵ. Unfortunately, as one analytically continues in ϵ towards $\epsilon = 0$ one frequently finds that along the way the y integration contour will hit poles in $F(\epsilon, y)$. In crossing these poles one picks up residue terms which no longer have y integrals. In this paper we will refer to any such term which spoils our ability to collect everything under a single y integral and then expand around $\epsilon = 0$ as an *obstruction*.

We overcome this difficulty with a second ingredient: linear differential operators which can be used to eliminate all obstructions, or at least push them off so that they are $\mathcal{O}(\epsilon)$ and can be ignored. At two loops, the simplest example of an operator which accomplishes

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[1]Actually, since the contour for y is taken from $-i\infty$ to $+i\infty$, it is more appropriately viewed as a Fourier transform with respect to $\ln(-x)$ rather than a Mellin transform with respect to $-x$.

this goal is $(x\frac{d}{dx})^5$. However we prefer to use the more elegant choice

$$\mathcal{L}^{(2)} \equiv \left(x\frac{d}{dx} - \epsilon \right)^3 \left(x\frac{d}{dx} + \epsilon \right)^3 \tag{2.1}$$

which has several attractive features, even though it is of higher degree.[2] Most importantly, we will see that $\mathcal{L}^{(2)}$ kills all obstructions *exactly*, to all orders in ϵ. In fact we will see that the choice of $\mathcal{L}^{(2)}$ is completely natural from the Mellin-Barnes integral representation of the amplitudes: acting with $\mathcal{L}^{(2)}$ explicitly removes a number of poles in $F(\epsilon, y)$. In contrast, acting with $(x\frac{d}{dx})^5$ does not kill any of the poles; the residues of all of the original poles must still be calculated and shown to be $\mathcal{O}(\epsilon)$ before one can be sure that they can be ignored. Also as a minor bonus we note that $\mathcal{L}^{(2)}$ conveniently preserves the manifest $x \to 1/x$ symmetry of four-particle amplitudes.

Armed with $\mathcal{L}^{(2)}$ we present an elementary proof of the identity

$$\mathcal{L}^{(2)} \left[M_4^{(2)}(\epsilon) - \frac{1}{2} \left(M_4^{(1)}(\epsilon) \right)^2 \right] = \mathcal{O}(\epsilon), \tag{2.2}$$

without needing to evaluate the y integral. The proof involves only the evaluation of a finite (and small!) number of residues and the use of a single Barnes lemma integral.[3] Clearly (2.2) is a weaker statement than (1.1) because we can add to $M_4^{(2)}(\epsilon) - \frac{1}{2}(M_4^{(1)}(\epsilon))^2$ any function $K(\epsilon, x)$ in the kernel of $\mathcal{L}^{(2)}$. It is clear upon inspection that this kernel consists of just a simple function of $\ln^2(-x)$ — therefore, all polylogarithms in the ABDK relation are unambiguously fixed already by (2.2). This involves nontrivial cancellation between polylogarithms appearing in the two terms, but we promise the reader that not a single polylogarithm will appear in this paper. For us all of the cancellation in (2.2) occurs under the y integral.

The final ingredient in our proof is the known infrared and collinear behavior of the two-loop amplitude $M_4^{(2)}$. We will show that these constraints fix the ambiguity discussed above to be of the form $K = f^{(2)}(\epsilon)M_4^{(1)}(2\epsilon) + C + \mathcal{O}(\epsilon)$, where C is a numerical constant which our method does not determine. In this particular case we happen to know from the work of [4] that $C = -\pi^4/72$, but in general it would probably not be terribly difficult to determine C — once it is known to be independent of x — by performing the Mellin-Barnes integrals numerically at any convenient value of x (see for example [23, 24]).

3. The two-loop calculation

In this section we begin by proving the identity

$$\mathcal{L}^{(2)} \left[M_4^{(2)}(\epsilon) - \frac{1}{2} \left(M_4^{(1)}(\epsilon) \right)^2 \right] = \mathcal{O}(\epsilon), \tag{3.1}$$

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[2]In fact there is even a degree 4 operator $\left(x\frac{d}{dx} - \epsilon \right)^2 \left(x\frac{d}{dx} + \epsilon \right)^2$ which kills many, but not all, of the obstructions, leaving a remainder which is just an innocent number plus $\mathcal{O}(\epsilon)$. This operator would be sufficient for verifying the ABDK conjecture, although it complicates the analysis slightly because one has to keep track of the residues which are not completely killed but only injured.

[3]Barnes lemmas are integrals of products of Γ functions that give rise to more Γ functions and their derivatives.

and then use the known infrared and collinear behavior of $M_4^{(2)}$ to fix the ambiguity arising from the differential operator $\mathcal{L}^{(2)}$. Of course, one could verify this identity by directly substituting the known expressions for $M_4^{(2)}(\epsilon)$ and $\frac{1}{2}(M_4^{(1)}(\epsilon))^2$ through $\mathcal{O}(\epsilon)$ and hitting them with $\mathcal{L}^{(2)}$. Our goal in this section is to demonstrate that it is straightforward to prove (3.1) directly at the level of the Mellin-Barnes *integrand*, without having to first fully evaluate the Mellin-Barnes integral in terms of polylogarithms. Along the way we will see the special feature of the particular choice of the differential operator $\mathcal{L}^{(2)}$ which makes this possible.

3.1 The $\frac{1}{2}(M_4^{(1)})^2$ term

Let us first look at the second term in (3.1). Taking a convenient expression for the one-loop box integral from (7) of [25] (see also [26, 27]), we find the following simple Mellin-Barnes representation for the one-loop amplitude:

$$M_4^{(1)}(\epsilon) = -\frac{1}{2} \frac{e^{\epsilon\gamma}}{(st)^{\epsilon/2}\Gamma(-2\epsilon)} \int \frac{dz}{2\pi i} \; (-x)^{z+\epsilon/2} \Gamma(1+\epsilon+z)\Gamma^2(z)\Gamma^2(-\epsilon-z)\Gamma(1-z). \quad (3.2)$$

The contour for the z integral runs from $-i\infty$ to $+i\infty$ and passes to the right of all poles of the two $\Gamma(\cdots + z)$ functions and to the left of all poles of the two $\Gamma(\cdots - z)$ functions. The contour can be taken to be a straight line parallel to the imaginary axis as long as the arguments of all Γ functions have a positive real part[4]. This is only possible if $\mathrm{Re}(\epsilon) < 0$, in which case we can take $0 < \mathrm{Re}(z) < -\mathrm{Re}(\epsilon)$. Note that $M_4^{(1)}(\epsilon)$ is symmetric under $s \leftrightarrow t$, or equivalently $x \leftrightarrow 1/x$, as can be seen by making the change of variable $z \to -z - \epsilon$.

Using (3.2) we can write

$$\begin{aligned}
\frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 = \frac{1}{8}\frac{e^{2\epsilon\gamma}}{(st)^\epsilon\Gamma^2(-2\epsilon)} \int \frac{dz}{2\pi i}\frac{dw}{2\pi i}(-x)^{w-z} \\
\times \; \Gamma(1+\epsilon+z)\Gamma^2(z)\Gamma^2(-\epsilon-z)\Gamma(1-z) \\
\times \; \Gamma(1+\epsilon+w)\Gamma^2(w)\Gamma^2(-\epsilon-w)\Gamma(1-w). \quad (3.3)
\end{aligned}$$

Now make the change of variables

$$z = u - \frac{1}{2}y, \qquad w = u + \frac{1}{2}y \quad (3.4)$$

to consolidate the x dependence into the factor $(-x)^y$. This is convenient because acting with the differential operator $\mathcal{L}^{(2)}$ simply inserts a factor of $(y^2 - \epsilon^2)^3$ into the integral. Note that we hold the product st fixed while differentiating with respect to $x = -t/s$. Therefore

$$\mathcal{L}^{(2)}\left[\frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2\right] = \frac{1}{8}\frac{e^{2\epsilon\gamma}}{(st)^\epsilon\Gamma^2(-2\epsilon)} \int \frac{du}{2\pi i}\frac{dy}{2\pi i}F(u,y) \quad (3.5)$$

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[4]We remind the reader that this requirement comes about in the following way. Many of the Γ functions appearing in Mellin-Barnes integral representations originate from integrals of the form $\int_0^1 d\alpha \; \alpha^{a-1}(1-\alpha)^{b-1} = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ over Feynman parameters α. This integral only converges for $\mathrm{Re}(a), \mathrm{Re}(b) > 0$. Other Γ functions come directly from the Mellin-Barnes representation of quantities such as $(X + Y)^{-\nu}$; the arguments of the Γ functions arising in this way must also have arguments with positive real parts.

where

$$F(u,y) = (-x)^y \left(y^2 - \epsilon^2\right)^3 \Gamma\left(1 + \epsilon + u - \tfrac{1}{2}y\right) \Gamma\left(u - \tfrac{1}{2}y\right)^2 \Gamma\left(-\epsilon - u + \tfrac{1}{2}y\right)^2$$

$$\times \Gamma\left(1 - u + \tfrac{1}{2}y\right) \Gamma\left(1 + \epsilon + u + \tfrac{1}{2}y\right) \Gamma\left(u + \tfrac{1}{2}y\right)^2$$

$$\times \Gamma\left(-\epsilon - u - \tfrac{1}{2}y\right)^2 \Gamma\left(1 - u - \tfrac{1}{2}y\right). \tag{3.6}$$

Let us now review[5] the procedure for manipulating Mellin-Barnes integrals such as (3.5). The formula (3.5) defines a meromorphic function of ϵ for ϵ in an open set which does not contain $\epsilon = 0$. As we take $\epsilon \to 0$, the integration contours become pinched until at $\epsilon = 0$ they are forced to run through some poles of the Γ functions. In order to construct a formula for $(M_4^{(1)}(\epsilon))^2$ which is valid in an open neighborhood of $\epsilon = 0$ we must analytically continue (3.5) in ϵ.

For definiteness, let us choose $\mathrm{Re}(u - \tfrac{1}{2}y) > \mathrm{Re}(u + \tfrac{1}{2}y) > 0$. The contour $\mathcal{C}$ for the u and y variables must be such that the argument of each Γ function has a positive real part. This can be satisfied as long as[6] $\mathrm{Re}(\epsilon) < \mathrm{Re}(-u - \tfrac{1}{2}y) < 0$. With this choice, the first pole that hits one of the integration contours as we try to take $\epsilon \to 0$ is the first pole of $\Gamma(-\epsilon - u + \tfrac{1}{2}y)$. Passing the pole through this contour produces two terms: the first term is the same integral as in (3.5), but with a contour $\mathcal{C}'$ that now passes to the *right* of the first pole of $\Gamma(-\epsilon - u + \tfrac{1}{2}y)$, and the second term is the residue of $F(u,y)$ at $u = -\epsilon + \tfrac{1}{2}y$. The residue comes with a minus sign because the contour passes across the pole from left to right, leaving a clockwise integral around the pole. Therefore we have

$$\int_{\mathcal{C}} \frac{du}{2\pi i} \frac{dy}{2\pi i} F(u,y) = \int_{\mathcal{C}'} \frac{du}{2\pi i} \frac{dy}{2\pi i} F(u,y) - \int_{\mathcal{C}} \frac{dy}{2\pi i} \mathrm{Res}_{u=-\epsilon+\frac{1}{2}y} F(u,y). \tag{3.7}$$

Now when $\mathrm{Re}(\epsilon)$ becomes larger than $\mathrm{Re}(-u + \tfrac{1}{2}y)$, the contour $\mathcal{C}'$ can be taken to be the straight line $\mathcal{C}$. Therefore we replace $\mathcal{C}'$ by $\mathcal{C}$ in (3.7) to write

$$\int_{\mathcal{C}} \frac{du}{2\pi i} \frac{dy}{2\pi i} F(u,y) \Rightarrow \int_{\mathcal{C}} \frac{du}{2\pi i} \frac{dy}{2\pi i} F(u,y) - \int_{\mathcal{C}} \frac{dy}{2\pi i} \mathrm{Res}_{u=-\epsilon+\frac{1}{2}y} F(u,y). \tag{3.8}$$

The notation $\Rightarrow$ is a reminder of the logic here: it would be incorrect to write $=$, since that would require the second term on the right-hand side to be zero. The left- and right-hand sides of (3.8) represent the same meromorphic function of ϵ. The left-hand side is valid in a neighborhood of $\mathrm{Re}(\epsilon) < \mathrm{Re}(-u + \tfrac{1}{2}y)$ while the right-hand side defines the analytic continuation of the left-hand side for a neighborhood $\mathrm{Re}(\epsilon) > \mathrm{Re}(-u + \tfrac{1}{2}y)$.

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[5]See [26, 28] for pioneering work on how this is implemented at two loops. A very convenient program which automates these kinds of manipulations has recently been made available by M. Czakon [24].

[6]There is also a lower bound on $\mathrm{Re}(\epsilon)$. We do not worry about this since we are always interested in pushing ϵ towards 0, not away from it. We can imagine starting out with $\mathrm{Re}(\epsilon)$ infinitesimally less than $\mathrm{Re}(-u - \tfrac{1}{2}y)$.

The only remaining pole which we encounter as we take ϵ to zero is the first pole of $\Gamma(-\epsilon - u - \frac{1}{2}y)$, at $u = -\epsilon - \frac{1}{2}y$. This gives

$$
\int_{\mathcal{C}} \frac{du}{2\pi i} \frac{dy}{2\pi i} F(u,y) \Rightarrow \int_{\mathcal{C}} \frac{du}{2\pi i} \frac{dy}{2\pi i} F(u,y)
$$
$$
- \int_{\mathcal{C}} \frac{dy}{2\pi i} \text{Res}_{u=-\epsilon-\frac{1}{2}y} F(u,y) - \int_{\mathcal{C}} \frac{dy}{2\pi i} \text{Res}_{u=-\epsilon+\frac{1}{2}y} F(u,y), \quad (3.9)
$$

with the left-hand side defined for $\text{Re}(\epsilon) < \text{Re}(-u + \frac{1}{2}y)$ and the right-hand side defined for $\text{Re}(\epsilon) > \text{Re}(-u - \frac{1}{2}y)$. There are no more contours standing in the way of taking $\epsilon \to 0$, so the three terms on the right-hand side of (3.9) can therefore be evaluated in an open neighborhood of $\epsilon = 0$ by simply making a power series expansion in ϵ under the y integral. The first term in (3.9) (the double integral term) is manifestly $\mathcal{O}(1)$ because $F(u,y)$ itself is. When we remember the prefactor in front of the integral (3.5), which contains an explicit $1/\Gamma(-2\epsilon)$, we see that this term only contributes to (3.5) at $\mathcal{O}(\epsilon)$, so we can ignore it. All we have to do is evaluate the two residues on the second line of (3.9) and expand the result (including the prefactor in (3.5)) through $\mathcal{O}(\epsilon^0)$, which gives

$$
(st)^\epsilon \mathcal{L}^{(2)} \left[\frac{1}{2} \left(M_4^{(1)}(\epsilon) \right)^2 \right] = \int \frac{dy}{2\pi i} \, (-x)^y y^6 \Gamma(1-y) \Gamma(-y)^2 \Gamma(y)^{*2} \Gamma(1+y)
$$
$$
\times \left[-\frac{2}{\epsilon} + y \left(f^{(2)}(y) - f^{(2)}(-y) \right) \right] + \mathcal{O}(\epsilon), \quad (3.10)
$$

where $f^{(2)}$ is the function defined in (1.2). It is very intriguing to see this function, which is related to the infrared and collinear behavior of two-loop amplitudes, emerge in an interesting way from a one-loop amplitude squared. We follow the notation of [22] in using Γ^* as a reminder that the contour for the y integral passes the first pole of the indicated Γ function $\Gamma^2(y)$ on the wrong side. The appropriate contour for (3.10) is $-1 < \text{Re}(y) < 0$, reflecting a choice we made in setting up the original u and y contours in (3.5).

It may appear that the application of the differential operator $\mathcal{L}^{(2)}$, and the corresponding insertion of the factor $(y^2 - \epsilon^2)^3$ into the integral, was not important in this analysis. In fact it plays a crucial role in removing a third-order obstruction. Without including this factor in F we would have found

$$
-\text{Res}_{u=-\epsilon+\frac{1}{2}y} F(u,y) = -\frac{2(-x)^\epsilon \Gamma(-\epsilon)^4 \Gamma(1+\epsilon)^4}{(\epsilon-y)^3} + \mathcal{O}((\epsilon-y)^{-2}), \quad (3.11)
$$

which contains an obstruction to taking $\epsilon \to 0$ in the form of a triple pole at $y = \epsilon$. A factor of $(y-\epsilon)^3$ is needed to kill this obstruction. The necessity of the other factor $(y+\epsilon)^3$ only becomes apparent in the analysis of $M_4^{(2)}(\epsilon)$, to which we now turn our attention.

3.2 The $M_4^{(2)}$ term

In the previous subsection we reviewed in detail the procedure for manipulating Mellin-Barnes integrals. In this section we will calculate $\mathcal{L}^{(2)} M^{(2)}(\epsilon)$ by the same procedure, although we will not show each step in as much detail. The full two-loop amplitude is

given by the sum of the two terms [13]

$$M_4^{(2)}(\epsilon) = \frac{1}{4}s^2 t I_4^{(2)}(s,t) + \frac{1}{4}st^2 I_4^{(2)}(t,s), \qquad (3.12)$$

where $I_4^{(2)}(s,t)$ is the two-loop massless scalar box function. We start with a convenient Mellin-Barnes representation for the two-loop box function given by (11) of [29][7]:

$$\frac{1}{4}s^2 t I_4^{(2)}(s,t) =$$
$$= \frac{1}{2}\frac{e^{2\epsilon\gamma}}{(st)^\epsilon \Gamma(-2\epsilon)} \int \frac{d\sigma}{2\pi i}\frac{dz}{2\pi i}\frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\ (-x)^{1-\sigma+\epsilon}$$
$$\times \frac{\Gamma(-z_1-z_2)\Gamma(-z_1+z_2)\Gamma(2+\epsilon+z+z_1)}{\Gamma(1-z_1-z_2)\Gamma(1-z_1+z_2)}\frac{\Gamma(\sigma)\Gamma(1+\epsilon-z-z_1-\sigma)}{\Gamma(1-2\epsilon+z+z_1)}$$
$$\times \Gamma(-1-\epsilon-z-z_2)\Gamma(-1-\epsilon-z+z_2)\Gamma(1+z-z_1)\Gamma(1+z+z_1)$$
$$\times \Gamma(-\epsilon+z+z_2+\sigma)\Gamma(-z+z_1-\sigma)\Gamma(-\epsilon+z-z_2+\sigma)\Gamma(1-\sigma). \qquad (3.13)$$

The full two-loop amplitude (3.12) is obtained by adding together this quantity and the same expression with x replaced by $1/x$. Our goal is to reduce this to a single integral over the variable σ, since that is the only variable which sets the x dependence through the factor $(-x)^{1-\sigma+\epsilon}$. The contour is fixed by requiring that the argument of each function has a positive real part. The final result does not depend on the particular choice, but intermediate steps can look a little different. For definiteness, we imagine taking the contour $\text{Re}(z) = -2/3$, $\text{Re}(z_1) = -1/8$, $\text{Re}(z_2) = -1/16$ and $\text{Re}(\sigma) = 1/2$.

A simple analysis of (3.13) shows that all of the poles we hit as we analytically continue ϵ to 0 sit at only four different possible values of z_2,

$$z_2 = 1+\epsilon+z, \qquad z_2 = -1-\epsilon-z, \qquad z_2 = \epsilon-\sigma-z, \qquad z_2 = -\epsilon+\sigma+z. \quad (3.14)$$

Our first step is therefore to deform the contour past these poles. This leaves us with a sum of four residue terms, each of which is only a triple integral, plus the original four-fold integral (3.13), which can now be evaluated in a neighborhood of $\epsilon = 0$. In fact the four-fold integral is manifestly $\mathcal{O}(\epsilon)$ due to the explicit $1/\Gamma(-2\epsilon)$ factor sitting in front of (3.13). Therefore we only need to follow the four residues (3.14). Let's look at the first residue term,

$$\frac{1}{4}s^2 t I_4^{(2)}(s,t) =$$
$$= \frac{1}{2}\frac{e^{2\epsilon\gamma}}{(st)^\epsilon \Gamma(-2\epsilon)} \int \frac{d\sigma}{2\pi i}\frac{dz_1}{2\pi i}\frac{dz}{2\pi i}\ (-x)^{1-\sigma+\epsilon}$$
$$\times \frac{\Gamma(-1-\epsilon-z-z_1)\Gamma(1+\epsilon+z-z_1)\Gamma(2+\epsilon+z+z_1)}{\Gamma(-\epsilon-z-z_1)\Gamma(2+\epsilon+z-z_1)}\frac{\Gamma(\sigma)\Gamma(1+\epsilon-z-z_1-\sigma)}{\Gamma(1-2\epsilon+z+z_1)}$$
$$\times \Gamma(-2-2\epsilon-2z)\Gamma(1+z-z_1)\Gamma(1+z+z_1)$$
$$\times \Gamma(1+2z+\sigma)\Gamma(-z+z_1-\sigma)\Gamma(-1-2\epsilon+\sigma)\Gamma(1-\sigma)$$
$$+ \text{ three more residues} + \mathcal{O}(\epsilon). \qquad (3.15)$$

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[7]Our formula follows from theirs after the change of variables $\alpha = z_1 + z_2$, $\beta = z_1 - z_2$, $\tau = z - z_1$.

This expression is now valid for $-11/24 < \text{Re}(\epsilon) < -1/3$ (given our specific choice for the contours). As we continue taking ϵ closer to zero we encounter more poles. The ones we are concerned about are poles which force us to take a residue in σ. For example, when $\text{Re}(\epsilon)$ reaches $-1/4$ we hit a pole at $\sigma = 1 + 2\epsilon$. The residue at this pole is a double integral $\int dz_1 dz$. The presence of this term spoils our goal of trying to perform all manipulations under the σ integral.

The dangerous pole at $\sigma = 1 + 2\epsilon$ comes from the $\Gamma(-1 - 2\epsilon + \sigma)$ factor in (3.15). We could get rid of this pole by inserting a factor of $-1 - 2\epsilon + \sigma$ into the integral. But this is precisely the factor we would get if we acted on (3.15) with the differential operator $-x\frac{d}{dx} - \epsilon$! Applying this differential operator kills the pole completely; the corresponding residue is exactly zero, not just zero to $\mathcal{O}(\epsilon)$.

It is a straightforward exercise to continue following the Mellin-Barnes procedure all the way to $\epsilon = 0$. It turns out to be necessary to cube the factor $(-1 - 2\epsilon + \sigma)$ inside the integrand as this kills other obstructions which appear at later steps. This analysis motivates consideration of the differential operator $(-x\frac{d}{dx} - \epsilon)^3$. However, we have only looked at the term $\frac{1}{4}s^2 t I_4^{(2)}(s,t)$. The complete two-loop amplitude (3.12) is the sum of this and $\frac{1}{4}s t^2 I_4^{(2)}(t,s)$, which is the same up to $x \to 1/x$. The same analysis applied to this other term suggests that we should also hit $M_4^{(2)}(\epsilon)$ with the differential operator $(x\frac{d}{dx} - \epsilon)^3$. Taken together, these observations motivate the choice of $\mathcal{L}^{(2)}$ in (2.1).

Let us now display the final result for $\mathcal{L}^{(2)}$ acting on the Mellin-Barnes integral (3.15). Each of the triple integral terms in (3.15) eventually branches into the same triple integral, now valid in a neighborhood of $\epsilon = 0$, plus a sum of residue terms which involve double or single integrals. All four triple integral terms, and several double integral terms which appear later on, turn out to be $\mathcal{O}(\epsilon)$, again because of the explicit factor $1/\Gamma(-2\epsilon)$ in (3.15). However, this explicit factor $1/\Gamma(-2\epsilon)$ certainly does not mean that everything is $\mathcal{O}(\epsilon)$ since taking residues can produce explicit singularities which cancel this factor. Ultimately we find that only 6 residues contribute through $\mathcal{O}(\epsilon)$,

$$
\begin{aligned}
\mathcal{L}^{(2)}\left[\frac{1}{4}s^2 t I_4^{(2)}(s,t)\right] = \frac{1}{2}\frac{e^{2\epsilon\gamma}}{(st)^\epsilon \Gamma(-2\epsilon)}\Bigg[&-\int \frac{d\sigma}{2\pi i}\, \text{Res}_{z_1=\epsilon}\text{Res}_{z=-1-\epsilon}\text{Res}_{z_2=1+\epsilon+z} \\
&+\int \frac{d\sigma}{2\pi i}\, \text{Res}_{z_1=-\frac{1}{2}-\epsilon+\frac{1}{2}\sigma}\text{Res}_{z=-1-\epsilon-z_1}\text{Res}_{z_2=1+\epsilon+z} \\
&-\int \frac{d\sigma}{2\pi i}\, \text{Res}_{z_1=-\frac{1}{2}-\epsilon+\frac{1}{2}\sigma}\text{Res}_{z=-1-\epsilon-z_1}\text{Res}_{z_2=-1-\epsilon-z} \\
&+\int \frac{d\sigma}{2\pi i}\, \text{Res}_{z_1=\epsilon}\text{Res}_{z=\epsilon-\sigma}\text{Res}_{z_2=\epsilon-z-\sigma} \\
&+\int \frac{d\sigma}{2\pi i}\int \frac{dz_1}{2\pi i}\, \text{Res}_{z=\epsilon+z_1-\sigma}\text{Res}_{z_2=\epsilon-z-\sigma} \\
&-\int \frac{d\sigma}{2\pi i}\int \frac{dz_1}{2\pi i}\, \text{Res}_{z=\epsilon+z_1-\sigma}\text{Res}_{z_2=-\epsilon+\sigma+z}\Bigg]G + \mathcal{O}(\epsilon), \quad (3.16)
\end{aligned}
$$

where G is the quantity appearing inside the integral in (3.13). The signs are easily fixed by keeping in mind that a residue of the form $\text{Res}_{v=\cdots\pm\epsilon}$ comes with a sign of ± 1.

A simple calculation reveals that the second and third line in (3.16) are equal to each other, as are the fifth and sixth lines. Let us look at these last two terms first since they still involve an extra z_1 integral. Expanding to $\mathcal{O}(\epsilon)$ we find that they contribute

$$\frac{1}{(st)^\epsilon} \int \frac{d\sigma}{2\pi i} \, (-x)^{1-\sigma} (1-\sigma)^6 \Gamma(\sigma) \Gamma(\sigma-1) \Gamma^{*2}(1-\sigma)$$
$$\times \int \frac{dz_1}{2\pi i} \, \Gamma(2-\sigma+2z_1) \Gamma^*(2z_1) \Gamma(-2z_1) \Gamma^*(\sigma-1-2z_1) + \mathcal{O}(\epsilon), \qquad (3.17)$$

where as before we keep track of the integration contour by using a $*$ to denote a Γ function whose first pole is crossed in the 'wrong' direction. The remaining z_1 integral can be performed with the help of the generalized Mellin-Barnes identity

$$\int \frac{dz}{2\pi i} \, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma^*(\lambda_3-z) = \qquad (3.18)$$
$$= \Gamma(\lambda_2+\lambda_3)\Big(\Gamma(\lambda_1-\lambda_2)\left[\psi(\lambda_1-\lambda_2)-\psi(\lambda_1+\lambda_3)\right] - \Gamma(-\lambda_2-\lambda_3)\Gamma(\lambda_1+\lambda_3)\Big),$$

which follows straightforwardly from (D.2) of [22].

Combining (3.17) with the first four terms in (3.16), expanded through $\mathcal{O}(\epsilon^0)$, gives

$$\mathcal{L}^{(2)}\left[\frac{1}{4}s^2 t I_4^{(2)}(s,t)\right] = \frac{1}{(st)^\epsilon} \int \frac{d\sigma}{2\pi i} \, (-x)^{1-\sigma} (1-\sigma)^6 \Gamma(\sigma) \Gamma^2(\sigma-1) \Gamma^{*2}(1-\sigma) \Gamma(2-\sigma)$$
$$\times \left[-\frac{1}{\epsilon} + 4\pi \cot(\pi(1-\sigma)) - \ln(-x) + \psi(2-\sigma) - \psi(1)\right] + \mathcal{O}(\epsilon).$$
$$(3.19)$$

To get the full two-loop amplitude (3.12) we should add (3.19) to the same quantity with x replaced by $1/x$. This looks equivalent to leaving $(-x)^{1-\sigma}$ alone and replacing $\sigma \to \sigma' = 2 - \sigma$ (and $\ln(-x) \to -\ln(-x)$) inside the integral, but there is a subtlety. The contour for σ was chosen to run along $\mathrm{Re}(\sigma) = \frac{1}{2}$, which would place the contour for σ' at $\frac{3}{2}$. We would like to rename σ' back to σ and combine both terms under a single σ integral. In order to do this we must check that we don't cross any poles in taking σ' from $\frac{3}{2} \to \frac{1}{2}$. Fortunately the high power $(1-\sigma)^6$ is more than adequate to kill the pole at $\sigma = 1$, so there is no problem.

Finally we conclude that

$$\mathcal{L}^{(2)}\left[M_4^{(2)}(\epsilon)\right] = \frac{1}{(st)^\epsilon} \int \frac{d\sigma}{2\pi i} \, (-x)^{1-\sigma} (1-\sigma)^6 \Gamma(\sigma) \Gamma^2(\sigma-1) \Gamma^{*2}(1-\sigma) \Gamma(2-\sigma)$$
$$\times \left[-\frac{2}{\epsilon} + \psi(2-\sigma) - 2\psi(1) + \psi(\sigma)\right] + \mathcal{O}(\epsilon). \qquad (3.20)$$

The change of variable $\sigma \to 1 - y$ and the identity

$$\psi(1-y) + \psi(1+y) = \psi(y) + \psi(-y) \qquad (3.21)$$

precisely transform (3.20) into (3.10). This concludes the elementary proof that

$$\mathcal{L}^{(2)}\left[M_4^{(2)}(\epsilon) - \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2\right] = \mathcal{O}(\epsilon). \qquad (3.22)$$

Note that while $M_4^{(2)}(\epsilon)$ and $\frac{1}{2}(M_4^{(1)}(\epsilon))^2$ are both individually $\mathcal{O}(\epsilon^{-4})$, (3.10) and (3.20) indicate that they become $\mathcal{O}(\epsilon^{-1})$ after being hit with the differential operator $\mathcal{L}^{(2)}$. This fact is evident when one looks at the explicit formulas

$$(st)^\epsilon M_4^{(2)}(\epsilon) = \frac{2}{\epsilon^4} - \frac{1}{\epsilon^2}\left[\frac{1}{2}\ln^2(-x) + \frac{5}{4}\pi^2\right] + \mathcal{O}(\epsilon^{-1}),$$

$$(st)^\epsilon \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 = \frac{2}{\epsilon^4} - \frac{1}{\epsilon^2}\left[\frac{1}{2}\ln^2(-x) + \frac{4}{3}\pi^2\right] + \mathcal{O}(\epsilon^{-1}). \tag{3.23}$$

There is at least one slightly intriguing aspect of the calculation in this subsection. In the calculation of [4] it was remarked that there is a non-trivial cancellation of terms, involving the use of polylogarithm identities, between the two integrals $I_4^{(2)}(s,t)$ and $I_4^{(2)}(t,s)$ which contribute to the two-loop amplitude. In our calculation this cancellation manifests itself under the σ integral as

$$\cot(\pi(1-\sigma)) + \cot(\pi(\sigma-1)) = 0. \tag{3.24}$$

3.3 Fixing the ambiguity

The result (3.22) implies that

$$M_4^{(2)}(\epsilon) - \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 = \frac{1}{(st)^\epsilon}K(x,\epsilon) + \mathcal{O}(\epsilon), \tag{3.25}$$

where $K(x,\epsilon)$ is a conveniently normalized undetermined function in the kernel of $\mathcal{L}^{(2)}$. To conclude the check of the ABDK relation (1.1) we have to study this kernel. Now $K(x,\epsilon)$ has to be invariant under $x \to 1/x$ (since the left-hand side of (3.25) is), and it is easy to see that the most general function annihilated by $\mathcal{L}^{(2)}$ which respects this symmetry is

$$\begin{aligned}
K(x,\epsilon) = &\, A(\epsilon)\left((-x)^\epsilon + (-x)^{-\epsilon}\right) \\
&+ B(\epsilon)\ln(-x)\left((-x)^\epsilon - (-x)^{-\epsilon}\right) \\
&+ C(\epsilon)\ln^2(-x)\left((-x)^\epsilon + (-x)^{-\epsilon}\right),
\end{aligned} \tag{3.26}$$

where $A(\epsilon)$, $B(\epsilon)$ and $C(\epsilon)$ are arbitrary functions of ϵ but independent of x. Note that (3.26) is a function of $\ln^2(-x)$ only.

By inspecting the original relation (1.1) it is clear that an important consistency check of our method is to find that $(st)^\epsilon M_4^{(1)}(2\epsilon)$ can be expressed in the form (3.26) up to terms of order ϵ. Indeed, a little algebra shows that

$$(st)^\epsilon M_4^{(1)}(2\epsilon) = \left[-\frac{1}{4\epsilon^2} + \frac{\pi^2}{3}\right]\left((-x)^\epsilon + (-x)^{-\epsilon}\right) + \frac{\ln(-x)}{4\epsilon}\left((-x)^\epsilon - (-x)^{-\epsilon}\right) + \mathcal{O}(\epsilon). \tag{3.27}$$

Equivalently, it is of course also possible to check directly that $\mathcal{L}^{(2)}M_4^{(1)}(2\epsilon) = \mathcal{O}(\epsilon)$. Since we have verified that the quantity $(st)^\epsilon M_4^{(1)}(2\epsilon)$ lies in the kernel of $\mathcal{L}^{(2)}$ (to $\mathcal{O}(\epsilon)$), we can freely absorb this into the ambiguity in $K(x,\epsilon)$ to rewrite (3.25) in the equivalent form

$$M_4^{(2)}(\epsilon) - \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 - f^{(2)}M_4^{(1)}(2\epsilon) = \frac{1}{(st)^\epsilon}K(x,\epsilon) + \mathcal{O}(\epsilon). \tag{3.28}$$

Let us now take a look at what we can say about the remaining ambiguity $K(x, \epsilon)$ from the infrared singularity structure of two-loop amplitudes. This has been studied in QCD [7, 6] and can be applied to $\mathcal{N} = 4$ SYM as well. The known behavior implies that the quantity appearing on the left-hand side of (3.28) is free of IR divergences. This implies that $K(x, \epsilon)$ must not have any poles in ϵ. Looking at the most general form (3.26) of $K(x, \epsilon)$, it is easy to see that the only function compatible with the infrared behavior is

$$\frac{1}{(st)^\epsilon} K(x, \epsilon) = C + E \ln^2(-x) + F \ln^4(-x) + \mathcal{O}(\epsilon), \quad (3.29)$$

where C, E and F are numerical constants.

In fact it is easy to reduce the ambiguity slightly (3.29) by using a lower-order differential operator instead of $\mathcal{L}^{(2)}$. For example, the operator $(x\frac{d}{dx} + \epsilon)^2 (x\frac{d}{dx} - \epsilon)^2$, which we mentioned in section 2, kills most, but not all, of the obstructions, leaving as remainders just numerical constants (which can be absorbed into C in (3.29)). One could even consider just the operator $(x\frac{d}{dx})^4$, which does not completely kill any obstruction, but reduces all of them to numbers plus $\mathcal{O}(\epsilon)$. The use of these operators would have complicated the analysis of section 2, since we would have to keep track of all of these unkilled obstructions. Their only advantage is that these operators have smaller kernels than $\mathcal{L}^{(2)}$, so one can use them to argue that F in (3.29) must be zero.

An independent argument which reduces the ambiguity (3.29) even further involves considering the collinear limit of the two-loop amplitude $M_4^{(2)}(\epsilon)$. The known behavior [4, 9 – 11]-[12] implies that the quantity on the left-hand side of (3.28) must be finite as $x \to 0$ and $x \to \infty$, which immediately fixes $E = F = 0$. In conclusion, we have verified the form of the ABDK relation

$$M_4^{(2)}(\epsilon) = \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon) M_4^{(1)}(2\epsilon) + C + \mathcal{O}(\epsilon) \quad (3.30)$$

up to an overall additive numerical constant.

4. Higher loops

In this section we would like to tie together the threads which have been weaving around throughout the analysis of sections 2 and 3 into a coherent picture. We show that the method used in this paper can be generalized for $n = 4$ particle amplitudes at $L > 2$ loops, by proving that it is always possible to find a generalization of the differential operator $\mathcal{L}^{(2)}$ which naturally removes all obstructions to combining all of the terms in the L-loop four-particle iteration relation inside a single integral.

The multiloop iteration relations proposed in [5] take the form

$$M_n^{(L)}(\epsilon) = X_n^{(L)}[M_n^{(l)}(\epsilon)] + f^{(L)}(\epsilon) M_n^{(1)}(L\epsilon) + C^{(L)} + \mathcal{O}(\epsilon), \quad (4.1)$$

where $C^{(L)}$ are numerical constants, $f^{(L)}$ are functions of ϵ only, and $X_n^{(L)}$ is a polynomial in the lower loop amplitudes which is conveniently summarized in the expression

$$X_n^{(L)}[M_n^{(l)}] = M_n^{(L)} - \ln\left(1 + \sum_{l=1}^{\infty} a^l M_n^{(l)}\right)\Bigg|_{a^L \text{ term}}. \quad (4.2)$$

For $L = 2$ we recover (1.1) while for $L = 3$ we have for example

$$M_n^{(3)}(\epsilon) = -\frac{1}{3}\left(M_n^{(1)}(\epsilon)\right)^3 + M_n^{(1)}(\epsilon)M_n^{(2)}(\epsilon) + f^{(3)}(\epsilon)M_n^{(1)}(3\epsilon) + C^{(3)} + \mathcal{O}(\epsilon). \quad (4.3)$$

This relation has been shown to be correct for $n = 4$ by explicit evaluation of the participating amplitudes [5].

We would like to collect all of the terms in (4.3) or its higher L generalizations, for $n = 4$ particles, under a single Mellin integral of the form

$$\int dy \ (-x)^y F(\epsilon, y) \quad (4.4)$$

and then to prove the iteration relations by expanding in ϵ under the integral sign. We found that in general this was not possible due to obstructions which prevent us from taking $\epsilon \to 0$ inside the integral. The most general possible obstruction is a residue of the form

$$\text{Res}_{y=g(\epsilon)}[(-x)^y F(\epsilon, y)] \quad (4.5)$$

where $g(\epsilon)$ is determined by the precise arguments of the Γ functions appearing in $F(\epsilon, y)$. In general it will be a linear function of ϵ. (For example, the obstruction we discussed below (3.15) corresponds to $g(\epsilon) = -\epsilon$ when the σ variable is transcribed to the y variable used in this section.) The crucial feature of the expression (4.5) is that while the resulting ϵ dependence can be very complicated, the x dependence is simple. Let us suppose that the pole where we are taking the residue is a pole of order k. Then the residue (4.5) will evaluate to

$$(-x)^{g(\epsilon)} P_k(\ln(-x), \epsilon), \quad (4.6)$$

where P_k is some polynomial in $\ln(-x)$ of degree k. The coefficients of that polynomial might have very complicated ϵ dependence. A differential operator which exactly kills the residue (4.6) is

$$\left(x\frac{d}{dx} - g(\epsilon)\right)^{k+1}. \quad (4.7)$$

In (4.5) we analyzed the contribution from a single obstruction. In a general amplitude (or product of amplitudes, such as appear in the iteration relations) there will be many such obstructions. Obviously it is always possible to construct a differential operator which removes all of these obstructions by taking the product of all of the individual differential operators of the form (4.7)—one for each obstruction.

Now suppose that we would like to test a conjectured iteration relation for $M_4^{(L)}(\epsilon)$ using some differential operator $\mathcal{L}^{(L)}$ constructed in the manner we have just described. By construction, applying $\mathcal{L}^{(L)}$ to the iteration relation will allow us to combine all of the terms under a single y integral, and then to verify the iteration relation by expanding the integrand through $\mathcal{O}(\epsilon^0)$ inside the integral. The ambiguity introduced by acting with $\mathcal{L}^{(L)}$ consists precisely of terms of the form (4.6), which span the kernel of $\mathcal{L}^{(L)}$ (by construction). Since $g(\epsilon)$ is always a linear function of ϵ, we can expand (4.6) in ϵ to see that the only things which can appear are powers of $\ln(-x)$ (actually, the ambiguity must be a polynomial in $\ln^2(-x)$, due to the $x \to 1/x$ symmetry). There can never be anything as complicated as a polylogarithm function — those are uniquely fixed even after $\mathcal{L}^{(L)}$ is applied.

In summary, we have demonstrated in this subsection that it is always possible to construct a differential operator $\mathcal{L}^{(L)}$ which, when applied to a conjectured iterative relation for $M_4^{(L)}(\epsilon)$, would allow the relation to be tested without explicitly evaluating the y integral. Moreover all of the complicated generalized polylogarithm functions are completely fixed by this procedure; the only ambiguity introduced by $\mathcal{L}^{(L)}$ is a polynomial in $\ln^2(-x)$. It might be possible to argue away some of this ambiguity by a more careful general argument.

5. Summary and outlook

In this paper we have proposed to use certain simple linear differential operators as effective tools for studying the iterative structure of planar four-particle L-loop MHV amplitudes in $\mathcal{N} = 4$ SYM. A key ingredient in our analysis is played by Mellin-Barnes integral representations of loop amplitudes, and by what we call 'obstructions.' Obstructions are terms which appear as we analytically continue Mellin-Barnes integrals towards the region around $\epsilon = 0$; their presence implies that it is impossible to collect everything under a single integral which simultaneously preserves the x-dependence in the simple factor $(-x)^y$ and also admits a series expansion in ϵ under the y integral. We showed that obstructions are always functions of $\ln^2(-x)$ and may be killed by the application of simple differential operators. The differential operator may be chosen to be simply $x\frac{d}{dx}$ to a sufficiently high power if one is only interested in pushing the appearance of obstructions to a sufficiently high order in ϵ, or a slightly more complicated operator if one wishes to kill them exactly. The operator $\mathcal{L}^{(2)}$ defined in (2.1) was proven to kill all obstructions appearing in the two-loop ABDK iterative relation (1.1).

The advantage of removing obstructions is obvious: by getting rid of them, we can directly study the *inverse* Mellin transform of an iterative relation in an expansion around $\epsilon = 0$. In other words we have no need to evaluate the final, and often exceedingly complicated, y integral explicitly in terms of generalized polylogarithms. The resulting simplicity is nicely exhibited by our elementary verification of the structure of the two-loop ABDK relation (1.1) for $n = 4$. The proof of (3.22) for required nothing more than evaluating a small number of residues and using the single Mellin-Barnes integral (3.18).

Fortunately, the price we pay for this beautiful simplification is not too high: by construction, the only ambiguities introduced by the differential operators we study are polynomials in $\ln^2(-x)$. The complicated generalized polylogarithms are unambiguously fixed. It is likely that the remaining $\ln^2(-x)$ ambiguity can in general be fixed by the known infrared and collinear behavior of L-loop amplitudes.

Although we have proven that for any given polynomial in L-loop four-particle Feynman integrals it is always possible to construct a differential operator which kills all obstructions, the important problem of determining which integrals contribute to the L-loop amplitude remains open. A candidate proposal put forward in [13] (the 'rung rule') has the feature that it encapsulates the correct two-particle cuts to all loops. Although the rung rule works at two and three loops, it is not clear that it correctly generates all of the contributions at higher loops [5].

It is natural to suppose that these techniques can also be generalized to $n > 4$ particle amplitudes. In general these admit Mellin-Barnes representations where we isolate the dependence on the various independent kinematic variables,

$$\int \frac{dy_1}{2\pi i}(k_1 \cdot k_2)^{y_1} \int \frac{dy_2}{2\pi i}(k_1 \cdot k_3)^{y_2} \cdots F(\epsilon; y_1, y_2, \ldots). \tag{5.1}$$

There will now be a more complicated picture involving complete obstructions and partial obstructions in some subset of the variables, but it should nevertheless be possible to construct partial differential operators in several variables which might prove useful in studying the iterative structure of these amplitudes.

Another interesting direction might be to use the converse mapping theorem to compute the asymptotic behavior of Feynman integrals [30] as $t/s \to 0$ and as $t/s \to \infty$ in order to study possible iterative relations in cases where proving the full structure is not within reach[8]. Particularly interesting would be to study the $n = 4$ four-loop case.

Acknowledgments

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