Class. Quantum Grav. **13** (1996) L19–L22. Printed in the UK

LETTER TO THE EDITOR

String fluid dynamics

Larry L Smalley† and Jean P Krisch‡

† Department of Physics, University of Alabama in Huntsville, Huntsville, AL 35899, USA
‡ Department of Physics, University of Michigan, Ann Arbor, MI 48109, USA

Received 26 October 1995

Abstract. We present the general energy–momentum tensor for a dynamical string fluid and show that it predicts that strings are very strong during early cosmological times, weak after the inflationary period, and negligible during the late-time evolution of the universe.

PACS numbers: 0440N, 1127, 9880C

The discussion of the cosmological implications of strings goes back nearly twenty years to the work of Zeldovich [1] and Vilenkin [2]. A string dust model was introduced by Letelier [3] from the description of the string function as a *spacetime*, surface forming bivector

$$\Sigma^{\mu\nu} = \epsilon^{AB} \frac{\partial x^\mu}{\partial \lambda^A} \frac{\partial x^\nu}{\partial \lambda^B} \tag{1}$$

where

$$\epsilon^{AB} = \begin{cases} 1 & \text{if } A = 0, B = 1 \\ -1 & \text{if } A = 1, B = 0 \\ 0 & \text{if } A = B \end{cases} \tag{2}$$

The form of equation (1) follows closely the form of the spin bivector developed by Halbwachs [4], and therefore the description of a string fluid energy density will use the parallel description of the velocity matrix used in discussions of fluid dynamics in continuum mechanics. (For a more complete discussion see, for example, the discussions of fluids with spin and twist in metric affine geometry [5].) This means that in a fluid context we can develop a Ray–Hilbert variational principle [6] for a string fluid by introducing the string in terms of a set of tetrads

$$\Sigma^{ij} = \rho\lambda(x)(a^{4i}a^{3j} - a^{4j}a^{3i}) \tag{3}$$

where ρ is the string density, $a^{\mu i}$ are the tetrads, where the latin indices, $i = 0, 1, 2, 3$, are the holonomic coordinates, greek indices, $\mu = 1, 2, 3, 4$, $\lambda(x)$ is the string (module) function (considered as a parametric function) which will not be varied directly. The holonomic metric has signature $g_{ij} = (-1, 1, 1, 1)$, and the anholonomic metric, $\eta_{\mu\nu} = (1, 1, 1, -1)$. We also identifiy the tetrad $a^{4i} \equiv u^i$. Using equation (3), the equivalent 'angular' velocity matrix takes the form

$$\omega^{ij} = \dot{a}^{\alpha i}\ a_\alpha{}^j \tag{4}$$

so that the string energy-density is given by

$$T_{ST} = \tfrac{1}{2}\Sigma^{ij}\omega_{ij} = \rho\lambda a^{4}{}_{i}\dot{a}^{3i}. \tag{5}$$

Some additional justification for this formulation is given by Nieto [7]. It is interesting to note the similarity of the string energy-density in the fluid framework to that which occurs in general spin/twist formulations. In this work, we do not restrict the generality of the spacetime manifold to the Riemannian form. One is tempted to argue from the standpoint of a theorem [8] on the uniqueness of a symmetric connection in a Riemannian manifold (which is both torsion and non metricity free) that this is equivalent to the 'gauging' of the torsion and non-metricity away. If one compares this to electromagnetic theory with the invariance of the electric and magnetic fields under the gauging of the vector potential, one quickly notes that the gravitational field is not invariant because of the *different* possible monifolds in the metric affine geometry. For example, if we investigate the effects of intrinsic spin in the cosmological arena, we discover that in the Riemann–Cartan (RC) spacetime, the spin energy enters with spin-squared terms, whereas in general relativity (GR), the spin-squared terms are missing [9]. This means that an expanded class of meaningful cosmologies is possible in RC spacetime compared with general relativity [10]. In addition, there is the theorem by Krisch and Smalley [11] that shows that every static perfect fluid solution in general relativity (i.e. the Riemannian spacetime) with metric g_{ij} is equivalent to a stationary spin fluid solution in a spacetime with torsion in a Einstein–Cartan theory (i.e. RC spacetime) with the same metric but with zero RC vorticity. However, the matter contents of these two spacetimes are very different as can be seen by either solving the field equations or by writing the field equations in a psuedo-Riemannian form [9] and then comparing the matter content to the static general relativistic case.

From the above arguments and the framework leading to equation (*5*), we justify that extending our manifold to Riemann–Cartan spacetime (which will also contain the GR limit) can lead to further understanding of the full implications of strings. We further note recent investigations of strings which have considered the relevance of torsion to spacetime defects [12, 13], and an exact solution of a family of strings in RC spacetime [14]. Thus in this work we investigate our string fluids in the RC spacetime of the Einstein–Cartan theory where $S_{ij}{}^{k} = \Gamma_{[ij]}{}^{k}$, square brackets indicate antisymmetrization of the indices ij and $\Gamma_{ij}{}^{k}$ is the connection in RC spacetime. We have, however, chosen to sort out the details of the dynamics of string fluids without the additional complications of spinning strings, although the RC geometry is the natural framework for spinning fluids [15]. We leave this to a later work.

The Ray–Hilbert Lagrangian for a string fluid in RC spacetime becomes

$$\mathcal{L}_{\mathcal{G}} = \mathcal{L}_{\mathcal{M}} + \mathcal{L}_{\mathcal{G}} \tag{6}$$

where

$$\mathcal{L}_{\mathcal{M}} = e\Big\{ -\rho\,[1+\epsilon\,(\rho,s)] + \lambda_{\mu\nu}\left(a^{\mu i}a^{\nu j}g_{ij} - \eta^{\mu\nu}\right) + \lambda_2 \overset{*}{\nabla}_i \left(\rho u^i\right) + \lambda_3 u^i X_{,i} + \lambda_4 u^i s_{,i} - \rho\lambda a_i^4 \dot{a}^{3i}\Big\} \tag{7}$$

and

$$\mathcal{L}_{\mathcal{G}} = \frac{eR}{2\kappa} \tag{8}$$

where $e = \sqrt{-g}$, $\kappa = 4\pi G$, R is the Riemann scalar in RC spacetime, $\lambda_{\mu\nu}$ are the Lagrange multipliers, and we write $\lambda_{44} = \lambda_1$ in the variation. Note that in this initial model, only the third and fourth component of the tetrads occur in the Lagrangian, and therefore it is not necessary to vary those that do not appear, since the variational equations are trivial. Also in this formulation, we do not include the string variable in the extended thermodynamics, i.e. the internal energy is not considered a function of the string as in the improved energy–momentum formulations [15]. Thus

$$d\epsilon = T\,ds + \frac{p}{\rho^2}\,d\rho. \tag{9}$$

The variational variables are then g_{ij}, ${S_{ij}}^k$, ρ, s, X, u^i, a^{3i}, and the various Lagrange multipliers. After varying equation (6), we obtain the metric field equation

$$G^{(ij)} - \overset{*}{\nabla}_k\,(T^{kij} + T^{kji}) = \kappa T^{ij}_{SF} \tag{10}$$

where the string fluid energy–momentum tensor is given by

$$T^{ij}_{SF} = T^{ij}_F + T^{ij}_{ST} \tag{11}$$

where

$$T^{ij}_F = [\rho(1+\epsilon)]u^i u^i + p g^{ij} \tag{12}$$

is the perfect fluid energy–momentum tensor of the string fluid, and

$$T^{ij}_{ST} = \overset{*}{\nabla}_k\,[u^{(i}\Sigma^{j)k}] + \overset{*}{\nabla}_\ell\,[u^\ell \Sigma^{k(j}]u^{i)}u_k \tag{13}$$

is the string energy–momentum tensor. In reducing the metric equation to its final form given by equations (10–13), we use the relationship for the Lagrange multiplier for the fluid continuity

$$\rho\dot{\lambda}_2 = -\left[\rho\,(1+\epsilon) + p\right] - T_{ST}. \tag{14}$$

As a result, it is quite remarkable the that the string energy density T_{ST} does not occur in the fluid energy–momentum tensor given by equation (12). This occurs because of the identity (which can be shown directly)

$$u^{(i}\Sigma^{j)k}\dot{u}_k = -T_{ST}u^i u^j \tag{15}$$

where the first term, which would have occurred in the string energy–momentum tensor, cancels with the right-hand term, which would have occurred in the fluid energy–momentum tensor.

The torsion equation takes the form

$$T^{kij} = \tfrac{1}{2}\kappa\,\Sigma^{ki}u^j \tag{16}$$

which has interesting consequences for the the metric field equation (10) when symmetized on ij. However the string function can be shown to have the much simpler form

$$\kappa\,\Sigma^{ij} = -8u^{[i}S^{j]} \tag{17}$$

which shows that the string depends only on the torsion vector. Taking the trace of equation (16) on ij shows that

$$S^k = \frac{\kappa}{4}u_j\Sigma^{jk} \tag{18}$$

whereas contracting equation (17) gives

$$\frac{\kappa}{4}u_j\Sigma^{jk} = S^k + \frac{1}{4}u^k u_i S^i \tag{19}$$

which shows that $u_i S^i = 0$, so that the torsion vector is a spacelike vector (i.e. not timelike). If we use equation (3), we can expand the right-hand side of equation (18), so that the torsion vector is proportional to a^{3i}. However if we had used the fluid constraint $\nabla_k[\rho u^k] = 0$ instead of the one with the 'star' derivative used in the Lagrangian equation (7), then the term $\kappa\lambda_2\rho u^{[i}g^{k]j}$ would occur on the right-hand side of equation (16). Eventually one finds that $3\kappa\rho\lambda_2 = u_i S^i$, and in general

$$S^i = 3\kappa\rho\lambda_2 u^i - \frac{\kappa}{4}\kappa\rho\lambda a^{3i}. \tag{20}$$

Thus the fluid constraint with the star derivative keeps the torsion vector from having a timelike piece.

The variational equation δa^{3i} has two interesting consequences. First antisymmetrizing with a^{3j} shows that $\dot{u}^{[i}a^{3j]} = 0$ which implies that $\dot{u}^i \propto a^{3i}$; and secondly, antisymmetrizing with $\dot{u}^i$ and using the previous results, one finds that the string module function satisfies the equation

$$\dot{\lambda} + \lambda = 0. \tag{21}$$

Integrating gives the parametric representation

$$\lambda = C\mathrm{e}^{-\tau} \tag{22}$$

where C is a constant and τ is a proper time parameter in the co-moving frame.

The above discussion describes a universe populated by a simple string fluid (in contrast to the dust model described by Letelier [3, 16]). In the far past, the strings are extremely strong as can be seen from equation (22). Thus in a more compreshensive model, it would be easy to see how they could have been the seed perturbations for matter density fluctuations that lead to the formation of super clusters and galaxies during early cosmological times; however, relatively soon after their creation, the strings become extremely weak and therefore do not play a major role in the late-time dynamics of the universe.

In a future publication, we will present details of the present calculation and show how the the basic model considered here can be extended to spinning strings with and without improved energy–momentum tensors.

References

[1] Zeldovich Ya B 1980 *Mon. Not. R. Astron. Soc.* **192** 663
[2] Vilenkin A 1981 *Phys. Rev. Lett.* **46** 1169; 1496 (erratum)
[3] Letelier P S 1979 *Phys. Rev.* D **20** 1294
[4] Halbwachs F 1960 *Theorie Relativiste des Fluide a Spin* (Paris: Gathier-Villar)
[5] Smalley L L and Krisch J P 1995 *J. Math. Phys.* **36** 778
[6] Ray J R 1972 *J. Math. Phys.* **13** 1451
[7] Nieto J A 1995 From a relativistic point particle to string theory *Preprint* Dept of Physics and Mathematics, University of Michoacana, 58000 Morelia Michoacan, Mexico, August, 1995
[8] Choquet-Bruhat Y, Dewitt-Morette C and Dillard-Bleich M 1982 *Analysis, Manifolds, and Physics* 2nd edn (Amsterdam: North-Holland) pp 300–1
[9] Smalley L L and Krisch J P 1994 *Class. Quantum Grav.* **11** 2517
[10] Smalley L L and Krisch J P 1994 *Class. Quantum Grav.* **11** 2375
[11] Krisch J P and Smalley L L 1993 *Class. Quantum Grav.* **10** 995
[12] Letelier P S 1995 Spacetime defects: torsion loops *Preprint* Dept de Matemática Aplicada-IMECC, Unive. Estadualde Campinas, 13018-970 Campinas, SP, Brazil
[13] Tod K P 1994 *Class. Quantum. Grav.* **11** 1331
[14] Krisch J P 1995 Family of strings with spin *Preprint* Dept of Physics, University of Michigan
[15] Ray J R and Smalley L L 1982 *Phys. Rev. Lett.* **49** 1059; **50** 525 (erratum)
[16] Letelier P S 1983 *Phys. Rev.* D **28** 2414