Class. Quantum Grav. **14** (1997) 1899–1909. Printed in the UK PII: S0264-9381(97)80807-6

Taub numbers at future null infinity: III. The Bondi mass

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Received 31 December 1996, in final form 7 April 1997

Abstract. This work extends the ideas developed in two previous papers by the authors. First- and second-order perturbation solutions of Einstein's equations (in Newman–Penrose form) for the Bondi–Sachs metric are found on a background Minkowski manifold. These solutions allow a tensorial calculation of the Bondi mass using the Taub superpotential.

PACS numbers: 0425N, 0440N

1. Introduction

The goal of this work is to present a relativistic tensorial method for computing the global mass of isolated astrophysical systems. We use the Bondi–Sachs metric, which describes asymptotically flat radiative systems, and present the Taub method for computing the Bondi mass. The Taub method has been developed in [1, 2] and extended to Einstein–Maxwell spaces [3].

An isolated system has a well defined global mass. If the system *is not* radiating, the mass at spatial infinity agrees with the mass at future null infinity $\mathcal{I}^+$ [4–6]. The Komar superpotential [7] provides a tensorial method for such a calculation (up to the well known factor of two anomaly [8]) and so does the Penrose–Goldberg superpotential [9, 10], when it exists, which seems to be only for asymptotically flat type D solutions [11].

When the system *is* radiating, i.e. when the matter fields produce a flux of energy–momentum, we focus exclusively upon null infinity, since there is no field which can produce a flux of any sort at spatial infinity. With *non-zero news*, the first calculation which yielded the Bondi mass as a 2-surface integral over a spherical cut of $\mathcal{I}^+$ was done by Goldberg [12] in an heroic work, which used the Einstein pseudotensor and an associated superpotential, and a transformation from asymptotically rectangular coordinates to Bondi coordinates.

Winicour and Tamburino [13] constructed a tensorial calculation by modifying the Komar superpotential. For the null surface $u =$ constant they added a term which eliminated off-surface derivatives. Using an asymptotic symmetry, the integral of the modified Komar superpotential, called a 'linkage', yields the Bondi mass at $\mathcal{I}^+$. Unfortunately, the linkage construction does not arise from a variational principle.

This work presents a calculation of the Bondi mass which is tensorial, arises from a variational principle, and yields the Bondi mass as a sum of perturbations from a background

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manifold. Each of the Taub numbers in the sequence $\tau_1, \tau_2, \ldots, \tau_n$ is derived from a variational derivative of $\int \sqrt{-g}\, g^{\alpha\beta} R_{\alpha\beta}\, \mathrm{d}^4x$, the nth number from the $(n+1)$th variation. We show below that the Bondi mass is the sum of τ_1 and τ_2, where τ_1 gives the curvature part of the mass and τ_2 contains the news function.

This paper is organized as follows: in the second section Taub numbers and the Taub superpotential are reviewed. The space of Lorentz metrics containing the curve of solutions of Einstein's equations is discussed and the Taub mass is defined as a sum of Taub numbers for a timelike Killing translation. The Minkowski metric is chosen as the background metric with background mass $M_0 = 0$. In section 3 the Bondi–Sachs metric and associated Newman–Penrose tetrad are discussed. The Bondi mass is therein defined as the 2-surface integral of the mass aspect over a topological 2-sphere at future null infinity. Initial data are given for the Newman–Penrose form of Einstein's equations whose Bondi–Sachs solution is the background Minkowski metric. In section 4 the Bondi–Sachs perturbations are presented. Minkowski null tetrad components of the Taub superpotential are evaluated for the perturbation tensor $h_{\mu\nu}$. Solutions of the linearized Newman–Penrose equations for the Bondi–Sachs metric along with the details of the tetrad constraints which precede linearization are given in section 5 where the first-order Taub mass is obtained. Similarly, second-order solutions are given in section 6 and the second-order Taub mass is obtained. The Bondi mass and quadrupole mass loss are presented in section 7. Following the discussion, the linearized Newman–Penrose field equations are given in appendix A, and the second-order equations appear in appendix B.

In this work, Greek indices range over $0, 1, 2, 3$ and upper case Latin indices range over $2, 3$. Our sign conventions are $2A_{\nu;[\alpha\beta]} = A_\mu R^\mu{}_{\nu\alpha\beta}$, and $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$. We use $\eth$ to symbolize the differential operator edth acting on 2-spheres in Minkowski space.

2. Taub numbers and superpotential

Here we review the basic ideas of the Taub method for computing mass. A Taub number τ_n is defined with respect to tensors $h^{(n)}_{\mu\nu}$ on a curve of asymptotically flat Einstein solutions $\hat{g}_{\mu\nu}$ where

$$\begin{aligned} \hat{g}_{\mu\nu}(\lambda) &= g_{\mu\nu} + h_{\mu\nu}(\lambda), \\ h_{\mu\nu}(\lambda) &= \lambda h^{(1)}_{\mu\nu} + \lambda^2 h^{(2)}_{\mu\nu} + \cdots, \end{aligned} \tag{1}$$

and a linearized Einstein operator $D_g G_{\alpha\beta} \cdot h^{(n)}$ (a directional derivative on the space of Lorentz metrics in the direction $h^{(n)}_{\mu\nu}$ evaluated at $g_{\mu\nu}$). Taub's theorem, namely $\nabla^\alpha (D_g G_{\alpha\beta} \cdot h^{(n)}) = 0$, must hold in order for all Taub numbers to be well defined. Taub's theorem is true when $G_{\alpha\beta}(g) = 0$. Since all curves $\hat{g}_{\mu\nu}(\lambda)$ pass through the background $g_{\mu\nu}$, the background metrics are required to satisfy $G_{\alpha\beta}(g) = 0$. In this work the background is chosen to be the flat Minkowski metric $\eta_{\mu\nu}$. A sequence of field equations is determined by the coefficients of $G_{\alpha\beta}(\hat{g})$ expanded as a series in λ along the curve of solutions $\hat{g}_{\mu\nu}(\lambda)$

$$G_{\alpha\beta}(g) = 0, \tag{2a}$$

$$D_g G_{\alpha\beta} \cdot h^{(1)} = 0, \tag{2b}$$

$$D^2_g G_{\alpha\beta} \cdot (h^{(1)}, h^{(1)}) + D_g G_{\alpha\beta} \cdot h^{(2)} = 0, \tag{2c}$$

etc.

The first equation of the sequence is the vacuum Einstein equation for the background metric. The next equation is the linearized Einstein equation for $h^{(1)}_{\mu\nu}$, and the subsequent equations each determine an $h^{(n)}_{\mu\nu}$.

Taub numbers result from integrating a vector density $t^{\alpha}_{(n)} = (-g)^{1/2}(D_g G^{\alpha}{}_{\beta} \cdot h^{(n)})k^{\beta}$, conserved by virtue of Taub's theorem and Killing's equation, over a 3-surface.

$$\tau_n := \int_{\Sigma\to\mathcal{N}} t^{\alpha}_{(n)}\, \mathrm{d}S_{\alpha} \tag{3}$$

$t^{\alpha}_{(n)}$ is integrated over a four-dimensional region D given in [1]. D is bounded by two 3-surfaces Σ_1 and Σ_2, which meet in the same S^2 cut of $\mathcal{I}^+$. Σ_1 is a null surface in the vacuum region (becoming smoothly spacelike in the interior source region), and Σ_2 lies to the future of Σ_1; it is spacelike in the source and vacuum regions and becomes null asymptotically where both Σ_1 and $\Sigma_2 \to \mathcal{N}$ and intersect $\mathcal{I}^+$ in the same cut.

A superpotential $U^{\alpha\beta}_{Taub}(h^{(n)})$ for all τ_n, $n \geqslant 1$ has been found [2]. The superpotential has the same functional form for all the $h^{(n)}$ where

$$h^{(n)}_{\mu\nu} := \frac{1}{n!}\left[\frac{\mathrm{d}^n \hat{g}_{\mu\nu}(\lambda)}{\mathrm{d}\lambda^n}\right]_{\lambda=0}. \tag{4}$$

When $h_{\mu\nu}$ is known, one can compute the entire sum of Taub numbers by using $h_{\mu\nu}$ in the superpotential:

$$U^{\alpha\beta}_{Taub} = (-g)^{1/2}\left(k^{[\alpha}h^{\beta]}{}_{\mu}{}^{;\mu} - k^{[\alpha}h^{;\beta]} + \tfrac{1}{2}hk^{[\alpha;\beta]} + k^{\mu}h_{\mu}{}^{[\alpha;\beta]} + k^{\mu;[\alpha}h^{\beta]}{}_{\mu}\right), \tag{5}$$

where

$$\nabla_{\beta} U^{\alpha\beta}_{Taub} = (-g)^{1/2}(D_g G^{\alpha}{}_{\beta} \cdot h)k^{\beta}. \tag{6}$$

For Einstein solutions which admit Kerr–Schild form, such as the Kerr metric (given in appendix B of [1]) $\hat{g}_{\mu\nu} = \eta_{\mu\nu} - 2mN_{\mu}N_{\nu}$, one knows $h_{\mu\nu}$ since $2mN_{\mu}N_{\nu}$ is a solution of the linearized Einstein equations. The sequence of solutions $h^{(n)}_{\mu\nu}$ terminates with $h^{(1)}_{\mu\nu} = h_{\mu\nu}$.

Any point on the curve of solutions can be chosen as a background metric provided it has a Killing symmetry for the physical quantity one wishes to compute. The superpotential is a function of the background metric, a Killing vector k^{μ} on the background (*timelike* for mass), and $h_{\mu\nu}$. The covariant derivatives in (5) are with respect to the background $g_{\mu\nu} = \eta_{\mu\nu}$ and $h := g^{\mu\nu}h_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu}$. The sum of all Taub numbers is

$$\tau_{\Sigma} = \sum_{n=1}^{\infty} \tau_n. \tag{7}$$

(Note that (7) differs by $n!$ from equations (3.4) and (3.5) in [2]. Here we include the $n!$ within the definition of $h^{(n)}_{\mu\nu}$.) Global Taub numbers are evaluated on an S^2 cut of $\mathcal{I}^+$ as one goes out on a null surface $\mathcal{N}$ to future null infinity. The global Taub mass is

$$\tau_{\Sigma}(k_t, h) = -\frac{1}{8\pi}\oint_{\partial\mathcal{N}} U^{\alpha\beta}_{Taub}\, \mathrm{d}S_{\alpha\beta}. \tag{8}$$

The Taub mass calculation is done in the coordinate system of the background metric. We require that the metric components of the physical Einstein solution $\hat{g}_{\mu\nu}$ have a valid Taylor expansion over the domain of their coordinates. This insures that when all of the source parameters are set to zero the resulting metric is globally diffeomorphic to the Minkowski metric. The mass of solution $\hat{g}$ is

$$\mathrm{Mass}\,(\hat{g}) = M_0 + \tau_{\Sigma}(k_t, h). \tag{9}$$

The background $\eta_{\mu\nu}$ is flat with mass $M_0 = 0$.

There are an infinite number of solution curves that go between the background solution $\eta_{\mu\nu}$ at $\lambda = 0$ and the asymptotically flat solution $\hat{g}_{\mu\nu}$ at $\lambda = 1$. We can select a family of curves (a 1-jet) by requiring the tangent to the curve, $[d\hat{g}/d\lambda]_{\lambda=0}$, to be the linearized solution which gives the monopole moment of the source. (Asymptotically flat systems are linearization stable [14].) Janis and Newman [15] have defined the multipole structure of gravitational sources in terms of initial data for the asymptotic solutions. We can use their data for the monopole to fix $h^{(1)}$:

$$\begin{aligned} &\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \qquad \sigma^0 = 0, \\ &\Psi_2 = a_0/r^3, \qquad a_0 \text{ real, constant.} \end{aligned} \tag{10}$$

Of course a unique curve will be known only when all the $h^{(n)}$ and their respective initial data are given.

It can be useful to define the reverse trace metric $\gamma_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2}hg_{\alpha\beta}$ since it allows $D_g G_{\alpha\beta} \cdot h$ to be written as

$$D_g G_{\alpha\beta} = \tfrac{1}{2}\nabla_\mu \nabla_\nu (\gamma_{\alpha\beta} g^{\mu\nu} + \gamma^{\mu\nu} g_{\alpha\beta} - 2\gamma^{\mu}{}_{(\alpha}\delta^{\nu}{}_{\beta)}). \tag{11}$$

The Taub superpotential then has the simpler form

$$U^{\alpha\beta}_{Taub} = (-g)^{1/2}(k^{\mu;[\alpha}\gamma^{\beta]}_{\ \mu} + k^\mu \gamma_\mu{}^{[\alpha;\beta]} + k^{[\alpha}\gamma^{\beta];\mu}_{\ \mu}). \tag{12}$$

The de Donder/Hilbert/harmonic gauge is $\nabla_\beta \gamma^{\alpha\beta} = 0$, analogous to the Lorentz gauge $\nabla_\beta A^\beta = 0$. This gauge would simplify the Taub superpotential above but will not be used here since it is difficult to preserve the gauge to all orders.

3. The Bondi–Sachs metric

The radiative systems considered here are described by

$$\hat{g}^{\text{B–S}}_{\mu\nu}\, dx^\mu\, dx^\nu = \frac{V e^{2b}}{r}\, du^2 + 2e^{2b}\, du\, dr - r^2 H_{AB}(dx^A - U^A\, du)(dx^B - U^B\, du), \tag{13}$$

where outgoing null hypersurfaces are labelled by $x^0 = u =$ constant. The Bondi–Sachs metric [16] extends Bondi's original metric [17] to include φ dependence and has six independent functions V, b, U^A, y and q, of u, r, ϑ, φ. The rays of each $u =$ constant null surface are null geodesics $x^\alpha(r)$ with tangent dx^α/dr where $x^1 = r$ is a luminosity parameter. Coordinates $x^2 = \vartheta$ and $x^3 = \varphi$ are constant along each ray. The luminosity parameter is defined by $r^4 \sin^2\vartheta = \det(g_{AB}) = \det(r^2 H_{AB})$, where

$$H_{AB} = \begin{bmatrix} e^{2y}\cosh(2q) & \sinh(2q)\sin\vartheta \\ \sinh(2q)\sin\vartheta & e^{-2y}\cosh(2q)\sin^2\vartheta \end{bmatrix}.$$

The boundary conditions on the metric functions in the limit of $\mathcal{I}^+$ are

$$rU^A \to 0, \qquad b \to 0, \qquad y \to 0, \qquad q \to 0, \qquad V/r \to 1.$$

We use the tetrad choice and asymptotic solution given in Glass and Goldberg [18] with the functions y and q contained in ξ_A, where $r^2 H_{AB} = \xi_A\bar{\xi}_B + \bar{\xi}_A\xi_B$. The notation in [18] was chosen to avoid confusion between Sachs metric functions and Newman–Penrose spin coefficients:

$$2y = \gamma + \delta \text{ (Sachs)}, \qquad 2q = \gamma - \delta \text{ (Sachs)}, \qquad b = \beta \text{ (Sachs)}.$$

The u = constant hypersurfaces have null geodesic tangent $\hat{l}^\alpha\partial_\alpha$, which is also hypersurface orthogonal as $\hat{l}_\alpha\,\mathrm{d}x^\alpha = \mathrm{d}u$. The twist of $\hat{l}^\alpha$ is zero and its expansion and shear are given by

$$\rho = -\mathrm{e}^{-2b}/r, \qquad \sigma = -\mathrm{e}^{-2b}[(\partial_r y)\cosh(2q) + \mathrm{i}(\partial_r q)] \tag{14}$$

which follow from taking r as a luminosity parameter, and where the phase of σ is determined by the choice of tetrad orientation. The asymptotic solution for the metric function V is found in equation (B16) of [18]:

$$V = r - 2M + \mathrm{O}(1/r),$$

where $M(u, \vartheta, \varphi)$ is the Bondi mass aspect given by

$$-2M = \Psi_2^0 + \bar{\Psi}_2^0 + \partial_u(\sigma^0\bar{\sigma}^0). \tag{15}$$

The Bondi mass is the 2-surface integral of the mass aspect over a topological 2-sphere at $\mathcal{I}^+$

$$M_{Bondi} = -\frac{1}{8\pi}\oint_{S^2}[\Psi_2^0 + \bar{\Psi}_2^0 + \partial_u(\sigma^0\bar{\sigma}^0)]\,\mathrm{d}\Omega. \tag{16}$$

There is characteristic initial data for the Newman–Penrose equations

$$\Psi_0(u_0, r, \vartheta, \varphi) = \Psi_1^0(u_0, \vartheta, \varphi) = \Psi_2^0(u_0, \vartheta, \varphi) + \bar{\Psi}_2^0(u_0, \vartheta, \varphi) = \sigma^0(u, \vartheta, \varphi) = 0,$$

which yields Bondi–Sachs metric functions $U^A = b = y = q = 0$, $V = r$. The Bondi–Sachs solution is then the flat Minkowski metric

$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r - r^2 H_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B. \tag{17}$$

This is the background metric for perturbation calculations. The Bondi–Sachs metric covers the vacuum region outside the sources and in that region the coordinates of the Minkowski metric (17) and the Bondi–Sachs metric (13) coincide. In the following, carets over tetrad letters distinguish the Bondi–Sachs tetrad from the zero-order Minkowski tetrad.

4. The Bondi–Sachs metric and perturbations

In order to understand the individual terms in the Bondi mass, we use a direct perturbation method to find the $h^{(n)}_{\mu\nu}$. In the background Minkowski frame $h^{(n)}_{\mu\nu}$ is computed by integrating the nth-order Newman–Penrose field equations. The general form of $h^{(n)}_{\mu\nu}$ is

$$\begin{aligned} h^{(n)}_{\mu\nu} = {}& h_0 l_\mu l_\nu + h_1(l_\mu n_\nu + n_\mu l_\nu) + h_2(l_\mu m_\nu + m_\mu l_\nu) + \bar{h}_2(l_\mu \bar{m}_\nu + \bar{m}_\mu l_\nu) \\ & + h_3 m_\mu m_\nu + \bar{h}_3 \bar{m}_\mu \bar{m}_\nu + h_4(m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu). \end{aligned} \tag{18}$$

Here the components of $h^{(n)}_{\mu\nu}$ are given in the coordinates and tetrad of the background Minkowski frame:

$$\begin{aligned} &\mathrm{d}u = l_\alpha\,\mathrm{d}x^\alpha, \qquad \mathrm{d}r = (n_\alpha - \tfrac{1}{2}l_\alpha)\,\mathrm{d}x^\alpha, \\ &\mathrm{d}\vartheta = -(1/\sqrt{2}r)(m_\alpha + \bar{m}_\alpha)\,\mathrm{d}x^\alpha, \\ &\mathrm{d}\varphi = (\mathrm{i}/\sqrt{2}r\sin\vartheta)(m_\alpha - \bar{m}_\alpha)\,\mathrm{d}x^\alpha. \end{aligned} \tag{19}$$

The Minkowski spin coefficients are

$$\rho = -1/r, \qquad \mu = -1/(2r), \qquad \alpha = -\cot\vartheta/(2\sqrt{2}r), \qquad \beta = -\alpha. \tag{20}$$

The background timelike Killing vector $k_t = (n^\alpha + \frac{1}{2}l^\alpha)\partial_\alpha = \partial_u$ is used to compute the mass in the Taub superpotential and is covariantly constant. Thus (5) reduces to

$$U^{\alpha\beta}_{Taub}(k_t, h) = (-g)^{1/2}k^\nu(\delta_\nu{}^{[\alpha}h^{\beta]}{}_\mu{}^{;\mu} - \delta_\nu{}^{[\alpha}h^{;\beta]} + h_\nu{}^{[\alpha;\beta]}). \tag{21}$$

The global Taub mass will be computed from (8), (18) and (21). The superpotential, evaluated from (21), has bivector components

$$
\begin{aligned}
(-g)^{-1/2}U^{\alpha\beta}_{Taub} = {}& l^{[\alpha}n^{\beta]}[2\rho h_0 - (\rho+2\mu)h_1 - (\delta-2\alpha)h_2 - (\bar\delta-2\alpha)\bar h_2 \\
& + (D-\rho)h_4 - 2(\Delta+\mu)h_4] \\
& + l^{[\alpha}m^{\beta]}\tfrac{1}{2}[-2\bar\delta h_0 + (D-3\rho)h_2 - 2(\Delta+\mu)h_2 + (\delta-4\alpha)h_3 - \bar\delta h_4] \\
& + l^{[\alpha}\bar m^{\beta]}\tfrac{1}{2}[-2\delta h_0 + (D-3\rho)\bar h_2 - 2(\Delta+\mu)\bar h_2 + (\bar\delta-4\alpha)\bar h_3 - \delta h_4] \\
& + n^{[\alpha}m^{\beta]}[\bar\delta h_1 - 2\rho h_2 + (\delta-4\alpha)h_3 - \bar\delta h_4] \\
& + n^{[\alpha}\bar m^{\beta]}[\delta h_1 - 2\rho\bar h_2 + (\bar\delta-4\alpha)\bar h_3 - \delta h_4] \\
& + m^{[\alpha}\bar m^{\beta]}[(\bar\delta-2\alpha)\bar h_2 - (\delta-2\alpha)h_2]. \qquad (22)
\end{aligned}
$$

Here $D = \partial_r$, $\Delta = \partial_u - \frac{1}{2}\partial_r$, and $(\delta + 2s\alpha)\eta = -(\eth\eta)/r$ for η a spin-weight s scalar. The tetrad components given in (18) have the following spin-weights:

Component	h_0	h_1	h_2	h_3	h_4
Spin-weight	0	0	−1	−2	0

5. The linearized asymptotic solution

The linearized Newman–Penrose equations were developed by Torrence and Janis [19] for a slightly different tetrad choice and so the linearized spin coefficients given here differ from theirs because of our different tetrad constraints. Choosing $\mathrm{d}u = \hat l_\mu\,\mathrm{d}x^\mu$ requires

$$\kappa = 0, \qquad \rho = \bar\rho, \qquad \tau = \bar\alpha + \beta, \qquad \epsilon + \bar\epsilon = 0.$$

The imaginary part of ϵ is set to zero by propagating $\hat m^\mu$ along $\hat l^\mu$ according to $D\hat m^\mu = \bar\pi \hat l^\mu$. $\hat m^\mu$ and c.c.($\hat m^\mu$) are required to be surface forming (carets are omitted in the following equation):

$$m^\mu_{;\nu}\bar m^\nu - \bar m^\mu_{;\nu}m^\nu = a_1 m^\mu + a_2\bar m^\mu$$

which in turn requires $\mu = \bar\mu$. Finally, requiring r to be a luminosity distance fixes the expansion of $\hat l^\mu$ to be $\rho = -\mathrm{e}^{-2b}/r$. The tetrad used in [18], which obeys the constraints above, is

$$
\begin{aligned}
&\hat l_\mu\,\mathrm{d}x^\mu = \mathrm{d}u, && \hat l^\mu\partial_\mu = \mathrm{e}^{-2b}\partial_r\,, \\
&\hat n_\mu\,\mathrm{d}x^\mu = (V\mathrm{e}^{2b}/2r)\,\mathrm{d}u + \mathrm{e}^{2b}\,\mathrm{d}r, && \hat n^\mu\partial_\mu = \partial_u - (V/2r)\partial_r + U^A\partial_A\,, \qquad (23)\\
&\hat m_\mu\,\mathrm{d}x^\mu = \xi_A\,\mathrm{d}x^A - (U^A\xi_A)\,\mathrm{d}u, && \hat m^\mu\partial_\mu = \xi^A\partial_A\,.
\end{aligned}
$$

The linearization method developed in [19] starts with the complete set of Newman–Penrose equations for the tetrad above. The equations are then linearized on the Minkowski background, with the Minkowski tetrad (19), and zero-order spin coefficients (20). The linearized equations are given in appendix A. The necessary initial data [15] are

$$\Psi_0(u_0, r, \vartheta, \varphi), \qquad \Psi_1^0(u_0, \vartheta, \varphi), \qquad \Psi_2^0(u_0, \vartheta, \varphi) + \bar\Psi_2^0(u_0, \vartheta, \varphi), \qquad \sigma^0(u, \vartheta, \varphi).$$

In the following $O(1/r^n)$ is abbreviated by O_n and the perturbation order is omitted from Ψ_2^0 and σ^0

$$\rho_{(1)} = 0, \qquad \sigma_{(1)} = \sigma^0/r^2 + O_4, \tag{24a}$$

$$\alpha_{(1)} = \bar{\alpha}^0\bar{\sigma}^0/r^2 - \eth\bar{\sigma}^0/r^2 + O_3, \qquad \beta_{(1)} = -\alpha^0\sigma^0/r^2 + O_3, \tag{24b}$$

$$\tau_{(1)} = -\bar{\eth}\sigma^0/r^2 + O_3, \qquad \pi_{(1)} = \bar{\tau}_{(1)}, \tag{24c}$$

$$\gamma_{(1)} = -(\Psi_2^0/2 + \bar{\alpha}^0\eth\bar{\sigma}^0 - \alpha^0\bar{\eth}\sigma^0)/r^2 + O_3, \tag{24d}$$

$$\mu_{(1)} = -(\Psi_2^0 + \bar{\Psi}_2^0 + \eth^2\bar{\sigma}^0 + \bar{\eth}^2\sigma^0)/(2r^2) + O_3, \tag{24e}$$

$$\lambda_{(1)} = \partial_u\bar{\sigma}^0/r + (\bar{\sigma}^0 - 2\bar{\eth}\eth\bar{\sigma}^0)/(2r^2) + O_3, \tag{24f}$$

$$\nu_{(1)} = -\bar{\eth}(\Psi_2^0 + \bar{\Psi}_2^0)/(2r^2) + O_3, \tag{24g}$$

where $\alpha^0 = -\cot\vartheta/(2\sqrt{2})$. The linearized metric components are

$$b_{(1)} = 0, \tag{25a}$$

$$V_{(1)} = \Psi_2^0 + \bar{\Psi}_2^0 + O_2, \tag{25b}$$

$$H_{AB}^{(1)} = (2\bar{\sigma}^0/r)\xi_A^0\xi_B^0 + (2\sigma^0/r)\bar{\xi}_A^0\bar{\xi}_B^0 + O_3, \tag{25c}$$

$$U_{(1)}^A = (\eth\bar{\sigma}^0/r^2)\xi^{A0} + (\bar{\eth}\sigma^0/r^2)\bar{\xi}^{A0} + O_4, \tag{25d}$$

where $\xi_A^0 = (1/r)m_\alpha\delta_A^\alpha$, $\xi^{A0} = r\,m^\alpha\delta_\alpha^A$. The linearized contribution to $h_{\mu\nu}$ is given by

$$h_{\mu\nu}^{(1)} = [(\Psi_2^0 + \bar{\Psi}_2^0)/r + O_3]l_\mu l_\nu - (2\bar{\sigma}^0/r + O_3)m_\mu m_\nu - (2\sigma^0/r + O_3)\bar{m}_\mu\bar{m}_\nu$$
$$- (\eth\bar{\sigma}^0/r + O_3)(l_\mu m_\nu + m_\mu l_\nu) - (\bar{\eth}\sigma^0/r + O_3)(l_\mu\bar{m}_\nu + \bar{m}_\mu l_\nu). \tag{26}$$

We integrate $U_{Taub}^{\alpha\beta}$ over a u = constant, r = constant 2-surface at $\mathcal{I}^+$ with $\mathrm{d}S_{\alpha\beta} = l_{[\alpha}n_{\beta]}\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$

$$U_{Taub}^{\alpha\beta}l_{[\alpha}n_{\beta]} = r\sin\vartheta[h_0 - h_1 - \partial_r(rh_4) + r\partial_u h_4 - \tfrac{1}{2}\eth h_2 - \tfrac{1}{2}\bar{\eth}\bar{h}_2]. \tag{27}$$

Substituting the components of (26) in (27) and taking the limit to $\mathcal{I}^+$, we find the first-order Taub mass

$$m_{(1)} = \tau_1(k_t, h^{(1)})$$
$$= -\frac{1}{8\pi}\oint_{S^2}(\Psi_2^0 + \bar{\Psi}_2^0 + \tfrac{1}{2}\eth^2\bar{\sigma}^0 + \tfrac{1}{2}\bar{\eth}^2\sigma^0)\,\mathrm{d}\Omega$$
$$= -\frac{1}{8\pi}\oint_{S^2}(\Psi_2^0 + \bar{\Psi}_2^0)\,\mathrm{d}\Omega, \tag{28}$$

which is the monopole mass moment, the Bondi mass when the news $\partial_u\sigma^0$ is zero. It was not necessary to use the Janis–Newman monopole data (10) to restrict (26), since all the σ^0 terms integrated out on the 2-sphere. The Komar superpotential also gives this mass [6] (up to the factor of two mass anomaly [8]). For the Schwarzschild solution, where $\sigma = 0$ and $\Psi_2^0 = -m$, we note that $\eta_{\mu\nu} + h_{\mu\nu}^{(1)}$ is a Kerr–Schild representation of the Schwarzschild metric with Bondi mass m.

6. The second-order asymptotic solution

The second-order Newman–Penrose equations are found in [20] and are the null tetrad expansion of (2*c*). The second-order Bianchi identities in Newman–Penrose form are given in appendix B, where one can see terms from $D_g^2 G_{\alpha\beta} \cdot (h^{(1)}, h^{(1)})$ in (2*c*) appearing as sources. The metric functions at second order are

$$b_{(2)} = -\sigma^0\bar{\sigma}^0/4r^2 + \mathrm{O}_4, \tag{29a}$$

$$V_{(2)} = \partial_u(\sigma^0\bar{\sigma}^0) + \mathrm{O}_1, \tag{29b}$$

$$H_{AB}^{(2)} = (2\sigma^0\bar{\sigma}^0/r^2)(\xi_A^0\bar{\xi}_B^0 + \bar{\xi}_A^0\xi_B^0) + \mathrm{O}_3, \tag{29c}$$

$$U_{(2)}^A = \mathrm{O}_3. \tag{29d}$$

The difference between r as an affine parameter and r as a luminosity distance is first observed at second order in $b_{(2)}$ above. The second-order contribution to $h_{\mu\nu}$ is given by

$$h_0^{(2)} = \partial_u(\sigma^0\bar{\sigma}^0)/r + \mathrm{O}_2, \tag{30a}$$

$$h_1^{(2)} = -\sigma^0\bar{\sigma}^0/2r^2 + \mathrm{O}_4, \tag{30b}$$

$$h_2^{(2)} = \mathrm{O}_2, \tag{30c}$$

$$h_3^{(2)} = \mathrm{O}_3, \tag{30d}$$

$$h_4^{(2)} = -2\sigma^0\bar{\sigma}^0/r^2 + \mathrm{O}_3. \tag{30e}$$

We substitute (6) into (27) and obtain

$$U_{Taub}^{\alpha\beta} l_{[\alpha}n_{\beta]} = \sin\vartheta[\partial_u(\sigma^0\bar{\sigma}^0) + \mathrm{O}_1].$$

Again we integrate $U_{Taub}^{\alpha\beta}$ over a 2-surface at $\mathcal{I}^+$ with $\mathrm{d}S_{\alpha\beta} = l_{[\alpha}n_{\beta]}\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$. This yields

$$m_{(2)} = \tau_2(k_t, h^{(2)}) = -\frac{1}{8\pi}\oint_{S^2}[\partial_u(\sigma^0\bar{\sigma}^0)]\,\mathrm{d}\Omega. \tag{31}$$

7. The Bondi mass and quadrupole mass loss

$h_{\mu\nu}^{(3)}$ and higher orders yield terms which cause the coefficient of $l^{[\alpha}n^{\beta]}$ in (22) to be O_3 or greater, therefore only two orders contribute to the result at $\mathcal{I}^+$. Adding the first- and second-order Taub masses yields the Bondi mass

$$\begin{aligned} M_{Bondi} &= m_{(1)} + m_{(2)} \\ &= -\frac{1}{8\pi}\oint_{S^2}[\Psi_2^0 + \bar{\Psi}_2^0 + \partial_u(\sigma^0\bar{\sigma}^0)]\,\mathrm{d}\Omega. \end{aligned} \tag{32}$$

The order-by-order computation reveals where each part of the Bondi mass arises.

Multipole mass loss expressions can be found by differentiating the Bondi mass

$$\dot{M}_{Bondi} = -\frac{1}{8\pi}\oint_{S^2}[\partial_u\Psi_2^0 + \partial_u\bar{\Psi}_2^0 + \partial_u^2(\sigma^0\bar{\sigma}^0)]\,\mathrm{d}\Omega. \tag{33}$$

The asymptotic solutions in equations (B28), (B29), and (B33) in [18] provide

$$\partial_u\Psi_2^0 + \sigma^0\partial_u^2\bar{\sigma}^0 + \eth^2\partial_u\bar{\sigma}^0 + \text{c.c.} = 0. \tag{34}$$

Substituting (34) in (33) yields the Bondi mass loss

$$\dot{M}_{Bondi} = -\frac{1}{8\pi}\oint_{S^2} 2\,\partial_u\sigma^0\partial_u\bar{\sigma}^0\,\mathrm{d}\Omega. \tag{35}$$

We use first-order solutions from [20], where weak, 2^l-pole, gravitational radiation exploding from a Schwarzschild mass is studied. With $q(u)$ as the first-order quadrupole moment, we find σ^0 in appendix C of [20]:

$$\sigma^0 = \text{constant} \times (\partial_u^2\bar{q})\,{}_2Y_{20}$$

and so

$$\dot{M}_{Bondi} = -A(\partial_u^3 q)\,(\partial_u^3\bar{q}), \qquad A = \text{constant} > 0. \tag{36}$$

This is one of a number of 'quadrupole equations' discussed by Damour [21].

8. Discussion

The early history of Taub numbers showed their usefulness in the straightforward establishment of linearization instability for a class of cosmologies. It was shown that closed cosmologies with isometries and compact Cauchy surfaces had constant Taub numbers with zero values and these constants over-constrained a well posed initial value problem, leading directly to linearization instability [22]. On the other hand, asymptotically flat systems have been shown to be linearization stable [14] and for those systems Taub numbers organize a set of physical parameters in a logical tensorial manner through Noether's theorem. In particular, the global mass and angular momentum of non-radiative systems have been computed as conserved Taub numbers generated by time translations and axial symmetries, respectively.

The Bondi mass of radiative systems has previously been computed [12, 13] by using time translations of the asymptotic BMS group. Here we have used the exact time symmetry of a Minkowski background and first- and second-order perturbations from that background manifold. Using the Taub superpotential, the Bondi mass then appeared directly as the 2-surface integral of the mass aspect over a spherical cut of $\mathcal{I}^+$. The second and third variational derivatives of the Hilbert action provide Taub numbers τ_1 and τ_2, which comprise the Bondi mass.

Acknowledgments

We thank the Physics Department of the University of Michigan and Professor Jean Krisch for their hospitality. This work has been partially supported by an NSERC of Canada grant.

Appendix A. Linearized equations

The linearized Bianchi identities are (for $A = 0, 1, 2, 3$)

$$(r\partial_r + 4 - A)\,\Psi^{(1)}_{A+1} + \bar{\eth}\,\Psi^{(1)}_A = 0, \tag{A1a}$$

$$(2r\partial_u - r\partial_r - A - 1)\,\Psi^{(1)}_A + 2\,\eth\,\Psi^{(1)}_{A+1} = 0, \tag{A1b}$$

with solutions given in [20]. The definition of edth is given in [18]. The linearized spin coefficient equations are

$$\partial_r(r^2\rho_{(1)}) = 0, \tag{A2a}$$

$$\partial_r(r^2\sigma_{(1)}) = r^2\,\Psi_0^{(1)}, \tag{A2b}$$

$$\partial_r(r\tau_{(1)}) = -\bar{\pi}_{(1)} + r\,\Psi_1^{(1)}, \tag{A2c}$$

$$\partial_r(r\alpha_{(1)}) = -\bar{\alpha}^0\bar{\sigma}_{(1)} - \pi_{(1)}, \tag{A2d}$$

$$\partial_r(r\beta_{(1)}) = \alpha^0\sigma_{(1)} + r\,\Psi_1^{(1)}, \tag{A2e}$$

$$r\partial_r\gamma_{(1)} = \alpha^0(\tau_{(1)} + \bar{\pi}_{(1)}) - \bar{\alpha}^0(\bar{\tau}_{(1)} + \pi_{(1)}) + r\,\Psi_2^{(1)}, \tag{A2f}$$

$$\partial_r(r\lambda_{(1)}) = -\bar{\eth}\pi_{(1)} - \bar{\sigma}_{(1)}/2, \tag{A2g}$$

$$\partial_r(r\mu_{(1)}) = -\eth\pi_{(1)} + r\,\Psi_2^{(1)}, \tag{A2h}$$

$$2\partial_r\nu_{(1)} = (2\partial_u - \partial_r - r^{-1})\pi_{(1)} - r^{-1}\bar{\tau}_{(1)} + 2\,\Psi_3^{(1)}. \tag{A2i}$$

The choice of r as a luminosity distance relates b to ρ through $\mathrm{e}^{2b}\rho = -1/r$ and so $b_{(1)} = 0$. The remaining linearized metric equations are

$$\partial_r(V_{(1)}/2r) = \gamma_{(1)} + \bar{\gamma}_{(1)}, \tag{A3a}$$

$$r\partial_r U_{(1)}^A = (\tau_{(1)} + \bar{\pi}_{(1)})\bar{\xi}^{A0} + (\bar{\tau}_{(1)} + \pi_{(1)})\xi^{A0}, \tag{A3b}$$

$$\partial_r(r\xi_{(1)}^A) = \sigma_{(1)}\bar{\xi}^{A0}, \tag{A3c}$$

where $\xi^{A0} = (1/\sqrt{2})(\delta_\vartheta^A + \mathrm{i}\,\mathrm{cosec}\,\vartheta\,\delta_\varphi^A) = r\,m^\alpha\delta_\alpha^A$. To complete the solution of the first-order equations, we require

$$\lambda_{(1)}^0 = \partial_u\bar{\sigma}^0, \tag{A4a}$$

$$\Psi_{2(1)}^0 - \bar{\Psi}_{2(1)}^0 = \bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0, \tag{A4b}$$

$$\Psi_{3(1)}^0 = \eth(\partial_u\bar{\sigma}^0), \tag{A4c}$$

$$\Psi_{4(1)}^0 = -\partial_u^2\bar{\sigma}^0. \tag{A4d}$$

Appendix B. Second-order equations

The second-order Bianchi identities are (for $A = 0, 1, 2, 3$)

$$(r\partial_r + 4 - A)\,\Psi_{A+1}^{(2)} + \bar{\eth}\,\Psi_A^{(2)} = rR_{A+1}, \tag{B1a}$$

$$(2r\partial_u - r\partial_r - A - 1)\,\Psi_A^{(2)} + 2\,\eth\,\Psi_{A+1}^{(2)} = rD_A, \tag{B1b}$$

where R_{A+1} and D_A are products of first-order terms, with solutions given in [20]. The equations for second-order spin coefficients and metric components are similarly iterated.

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