Two-fluid atmosphere for relativistic stars

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Received 6 July 1998, in final form 21 December 1998

Abstract. We have extended the Vaidya radiating metric to include both a radiation fluid and a string fluid. This paper expands our brief introduction to extensions of the Schwarzschild vacuum which appeared in 1998 *Phys. Rev.* D **57** R5945. Assuming diffusive transport for the string fluid, we find new analytic solutions of Einstein's field equations.

PACS numbers: 0420J, 0440D, 9760

1. Introduction

Strings have become a very important ingredient in many physical theories. They may have been present in the early universe and played a role in the seeding of density inhomogeneities [1]. The idea of strings is fundamental to superstring theories [2]. String fluids have been suggested as dark-matter candidates. The lensing properties of cosmic strings and string systems have been discussed [3] and the apparent relationship between counting string states and the entropy of the Schwarzschild horizon [4–6] suggests an association of strings with black holes. Recently, Glass and Krisch [7] have pointed out that allowing the Schwarzschild mass parameter to be a function of radial position creates an atmosphere with a string fluid stress–energy around a static, spherically symmetric, object. If the mass is also a function of retarded time a Vaidya radiation fluid is present in addition to the string fluid. Since the metric has one arbitrary function, $m(u, r)$, invariantly defined by the sectional curvature of the u, r 2-surfaces, a given mass distribution determines the stress–energy.

The string fluid stress–energy is a macroscopic, statistically averaged, description of a microscopic distribution of Planck length string bits. The string fluid lies on a two-dimensional timelike worldsheet (u, r) and its stress–energy is parametrized by radial derivatives of the mass function. The radial stress, $p_r < 0$, is the expected string tension. Since the string fluid is constrained to stay in the (u, r)-plane, the Planck scale string bits have only radial motions. Interactions of the string bits along the radial directions are modelled by the macroscopic string tension p_r and create, on average, a long string stretching radially away from the core or horizon of the mass distribution. The Glass–Krisch atmosphere allows transverse stresses. These stresses, if present, can be modelled by the presence of a pressureless dust fluid. The transverse pressures arise from the different velocities of the dust and string fluids. The most general atmosphere has three components: a Vaidya radiation fluid, a string fluid with radial tension and a dust fluid. This can be interpreted as a two-fluid atmosphere with a transverse pressure. We also discuss other, less general, possibilities.

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This atmosphere can model a variety of physical situations at different distance scales. It can describe the atmosphere around a black hole with a distance scale of multiples of Schwarzschild radii. It could also describe a globular cluster with a dark-matter component and a distance scale of the order of parsecs.

The contracted Bianchi identities are satisfied for arbitrary $m(u, r)$ and so the mode of mass propagation is a modelling choice from the many possible propagation equations of classical and quantum physics. Each choice allows the generation of new analytic solutions to the field equations. In this work we develop new solutions by considering the diffusive transport of the string fluid elements. With this choice we find that the string fluid diffuses inward as the Vaidya photons carry energy outward. The overall effect is to slow down the time scale of the Vaidya energy loss.

In the next section we present our extension of Vaidya's metric [8] as a further extension of the Schwarzschild vacuum solution. The arbitrary mass function, which appears in the metric, is identified as the sectional curvature of nested 2-spheres. The string fluid and an associated static two-fluid model are described in sections 3 and 4. We are able to provide a two-fluid kinematic interpretation of the transverse stresses in this model. In section 5 diffusive transport is developed and analytic solutions for the energy density are found. In section 6 analytic mass solutions are presented and interpreted. The last two sections contain the analysis of the horizon structure of some of our solutions and a discussion of the general results.

In this work Greek indices range over $(0, 1, 2, 3) = (u, r, \vartheta, \varphi)$. Our sign conventions are $2A_{\nu;[\alpha\beta]} = A_\mu R^\mu{}_{\nu\alpha\beta}$, and $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$. Overdots denote $\partial/\partial u$ and primes denote $\partial/\partial r$. Overhead carets denote unit vectors. We use units where $G = c = 1$. Einstein's field equations are $G_{\mu\nu} = -8\pi T_{\mu\nu}$, and the metric signature is $(+, -, -, -)$.

2. Extending the Schwarzschild vacuum

The spacetime metric covering the region exterior to a spherical star is given by

$$\mathrm{d}s^2 = A\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \tag{1}$$

where $A = 1 - 2m(u, r)/r$. Initially $m(u, r) = m_0$ provides the vacuum Schwarzschild solution in the region $r > 2m_0$. At later times that region admits a two-fluid description of radial strings and outward flowing short-wavelength photons (sometimes called a 'null fluid'). Metric (1) is spherically symmetric and is given in terms of the retarded time coordinate u. With the use of a Newman–Penrose null tetrad the Einstein tensor is computed from (1) and given by

$$G_{\mu\nu} = -2\Phi_{11}(l_\mu n_\nu + n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu) - 2\Phi_{22} l_\mu l_\nu - 6\Lambda g_{\mu\nu}. \tag{2}$$

Here the null tetrad components of the Ricci tensor are

$$\Phi_{11} = (2m' - rm'')/(4r^2), \tag{3a}$$

$$\Phi_{22} = -\dot{m}/r^2, \tag{3b}$$

$$\Lambda = R/24 = (rm'' + 2m')/(12r^2). \tag{3c}$$

The only non-zero component of the Weyl tensor is

$$\Psi_2 = -m/r^3 + (4m' - rm'')/(6r^2). \tag{4}$$

The metric is Petrov type **D** with l_μ and n_μ principal null geodesic vectors

$$l_\mu\,\mathrm{d}x^\mu = \mathrm{d}u, \tag{5a}$$

$$n_\mu\,\mathrm{d}x^\mu = (A/2)\,\mathrm{d}u + \mathrm{d}r, \tag{5b}$$

$$m_\mu\,\mathrm{d}x^\mu = -(r/\sqrt{2})(\mathrm{d}\vartheta + \mathrm{i}\sin\vartheta\,\mathrm{d}\varphi), \tag{5c}$$

where

$$l_{\mu;\nu} = (A'/2)l_\mu l_\nu - (1/r)(m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu), \tag{6a}$$

$$n_{\mu;\nu} = -(A'/2)n_\mu l_\nu + (A/2r)(m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu), \tag{6b}$$

$$m_{\mu;\nu} = (A/2r)l_\mu m_\nu - (1/r)n_\mu m_\nu + (\cot\vartheta/\sqrt{2}r)(m_\mu m_\nu - m_\mu \bar{m}_\nu). \tag{6c}$$

In order to clearly see the two-fluid description we introduce a timelike unit velocity vector $\hat{v}^\mu$ and three unit spacelike vectors $\hat{r}^\mu$, $\hat{\vartheta}^\mu$, $\hat{\varphi}^\mu$ such that

$$g_{\mu\nu} = \hat{v}_\mu \hat{v}_\nu - \hat{r}_\mu \hat{r}_\nu - \hat{\vartheta}_\mu \hat{\vartheta}_\nu - \hat{\varphi}_\mu \hat{\varphi}_\nu.$$

The unit vectors are defined by

$$\hat{v}_\mu \, dx^\mu = A^{1/2} \, du + A^{-1/2} \, dr, \qquad \hat{v}^\mu \partial_\mu = A^{-1/2} \partial_u, \tag{7a}$$

$$\hat{r}_\mu \, dx^\mu = A^{-1/2} \, dr, \qquad \hat{r}^\mu \partial_\mu = A^{-1/2} \partial_u - A^{1/2} \partial_r, \tag{7b}$$

$$\hat{\vartheta}_\mu \, dx^\mu = r \, d\vartheta, \qquad \hat{\vartheta}^\mu \partial_\mu = -r^{-1} \partial_\vartheta, \tag{7c}$$

$$\hat{\varphi}_\mu \, dx^\mu = r \sin\vartheta \, d\varphi, \qquad \hat{\varphi}^\mu \partial_\mu = -(r \sin\vartheta)^{-1} \partial_\varphi. \tag{7d}$$

$\hat{v}^\mu$ is hypersurface-orthogonal, i.e. $\hat{v}_{[\mu;\nu}\hat{v}_{\alpha]} = 0$, with $h_{\mu\nu}$ the first fundamental form of the hypersurface. Since $\hat{v}_\mu \, dx^\mu = f(u, r) \, dt$, the components of $h_{\mu\nu}$ show explicitly that $\hat{v}^\mu$ lies along $t =$ constant time lines:

$$\begin{aligned} h_{\mu\nu} \, dx^\mu \, dx^\nu &= (g_{\mu\nu} - \hat{v}_\mu \hat{v}_\nu) \, dx^\mu \, dx^\nu \\ &= -A^{-1} \, dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2). \end{aligned} \tag{8}$$

The kinematics of the $\hat{v}^\mu$ flow are described by

$$\hat{v}^\mu{}_{;\nu} = a^\mu \hat{v}_\nu + \sigma^\mu_\nu - (\Theta/3)(\hat{r}^\mu \hat{r}_\nu + \hat{\vartheta}^\mu \hat{\vartheta}_\nu + \hat{\varphi}^\mu \hat{\varphi}_\nu), \tag{9}$$

where

$$a^\mu = [\dot{m}/r + A\partial_r(m/r)]A^{-3/2}\hat{r}^\mu, \tag{10a}$$

$$\sigma^\mu{}_\nu = (\Theta/3)(-2\hat{r}^\mu \hat{r}_\nu + \hat{\vartheta}^\mu \hat{\vartheta}_\nu + \hat{\varphi}^\mu \hat{\varphi}_\nu), \tag{10b}$$

$$\Theta = (\dot{m}/r)A^{-3/2}. \tag{10c}$$

The Einstein tensor can now be written as a two-fluid system:

$$G_{\mu\nu} = (2\dot{m}/r^2)l_\mu l_\nu - (2m'/r^2)(\hat{v}_\mu \hat{v}_\nu - \hat{r}_\mu \hat{r}_\nu) + (m''/r)(\hat{\vartheta}_\mu \hat{\vartheta}_\nu + \hat{\varphi}_\mu \hat{\varphi}_\nu). \tag{11}$$

Spherical symmetry allows the function $m(u, r)$ to be identified as the mass within 2-surfaces of constant u and r, and invariantly defined from the sectional curvature [24] of those surfaces:

$$-2m/r^3 = R_{\alpha\beta\mu\nu} \hat{\vartheta}^\alpha \hat{\varphi}^\beta \hat{\vartheta}^\mu \hat{\varphi}^\nu. \tag{12}$$

3. String fluid

The string bivector is defined by

$$\Sigma^{\mu\nu} = \epsilon^{BC} \frac{\partial x^\mu}{\partial x^B} \frac{\partial x^\nu}{\partial x^C}, \qquad (B, C) = (0, 1) \text{ or } (2, 3).$$

Spherical symmetry demands that the averaged string bivector will have a worldsheet in either the (u, r)- or (ϑ, φ)-plane. The condition that the worldsheets are timelike, i.e. $\gamma := \frac{1}{2}\Sigma^{\mu\nu}\Sigma_{\mu\nu} < 0$, implies that only the Σ_{ur} component is non-zero. It is useful to write $\Sigma^{\mu\nu}$ in terms of unit vectors

$$\Sigma^{\mu\nu} = \hat{r}^\mu \hat{v}^\nu - \hat{v}^\mu \hat{r}^\nu. \tag{13}$$

It is now clear that $\Sigma^{\mu\alpha}\Sigma_{\alpha}{}^{\nu} = \hat{v}^{\mu}\hat{v}^{\nu} - \hat{r}^{\mu}\hat{r}^{\nu}$. We follow Letelier [9, 10] and write a string energy–momentum tensor in analogy with one for a perfect fluid

$$T^{fluid}_{\mu\nu} = \rho u_{\mu}u_{\nu} - ph_{\mu\nu},$$

where $h^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - u^{\mu}u_{\nu}$, $h^{\mu}{}_{\nu}u^{\nu} = 0$. The string energy–momentum is given by

$$T^{string}_{\mu\nu} = \rho(-\gamma)^{1/2}\hat{\Sigma}_{\mu}{}^{\alpha}\hat{\Sigma}_{\alpha\nu} - p_{\perp}H_{\mu\nu}, \tag{14}$$

where $H^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \hat{\Sigma}^{\mu\alpha}\hat{\Sigma}_{\alpha\nu}$, $H^{\mu}{}_{\nu}\hat{\Sigma}^{\nu\beta} = 0$. Although here $\gamma = -1$, we have kept γ explicit in (14) and written $\hat{\Sigma}^{\mu\nu} := (-\gamma)^{-1/2}\Sigma^{\mu\nu}$ to show how $\hat{\Sigma}^{\mu\nu}$ is made invariant to reparametrizations of the worldsheets [9].

Einstein's field equations allow the matter portion of $G_{\mu\nu}$ in equation (11) to be identified as a string fluid:

$$\begin{aligned} T_{\mu\nu} &= T^{rad}_{\mu\nu} + T^{string}_{\mu\nu}, \\ &= \psi l_{\mu}l_{\nu} + \rho\hat{v}_{\mu}\hat{v}_{\nu} + p_r\hat{r}_{\mu}\hat{r}_{\nu} + p_{\perp}(\hat{\vartheta}_{\mu}\hat{\vartheta}_{\nu} + \hat{\varphi}_{\mu}\hat{\varphi}_{\nu}). \end{aligned} \tag{15}$$

Thus

$$4\pi\psi = -\dot{m}/r^2, \tag{16a}$$

$$4\pi\rho = -4\pi p_r = m'/r^2, \tag{16b}$$

$$8\pi p_{\perp} = -m''/r. \tag{16c}$$

Since the contracted Bianchi identities are satisfied for arbitrary $m(u, r)$, it follows that the equations of motion $T^{\mu\nu}{}_{;\nu} = 0$ are identically satisfied for the components of $T_{\mu\nu}$ given in equation (15).

The components of the contracted Bianchi identities which do not vanish because of spherical symmetry, but rather because of the explicit form $A = 1 - 2m(u, r)/r$, are

$$l_{\mu}G^{\mu\nu}{}_{;\nu} = -\nabla_{\nu}[(2\Phi_{11} + R/4)l^{\nu}] - G^{\mu\nu}l_{\mu;\nu}$$

and

$$n_{\mu}G^{\mu\nu}{}_{;\nu} = -\nabla_{\nu}[(2\Phi_{11} + R/4)n^{\nu} + \Phi_{22}l^{\nu}] - G^{\mu\nu}n_{\mu;\nu}.$$

4. Static fluid models

Static models are useful in developing insights about time-dependent fluids. We consider two static models that are equivalent to the general stress–energy tensor described in equation (15) when it is time independent. The first is a static isotropic string fluid, and the second provides a two-fluid interpretation of (15) and an interpretation of the transverse stress.

4.1. Isotropic string fluid

Consider the stress–energy tensor in equation (15) with static mass function $m(r)$ and with $p_r = p_{\perp}$:

$$T^{iso}_{\mu\nu} = -p_r(\hat{v}_{\mu}\hat{v}_{\nu} - \hat{r}_{\mu}\hat{r}_{\nu} - \hat{\vartheta}_{\mu}\hat{\vartheta}_{\nu} - \hat{\varphi}_{\mu}\hat{\varphi}_{\nu}). \tag{17}$$

This is clearly an isotropic cloud of strings with equation of state $\rho + p_r = 0$. The mass is determined by the constraint $p_r = p_{\perp}$ or

$$\frac{m''}{2r} = \frac{m'}{r^2}. \tag{18}$$

The solution of equation (18), in the more general time-dependent case with null radiation fluid, is

$$m(u, r) = r^3 c_1(u) + c_2(u). \tag{19}$$

For the static case we have c_1 and c_2 constant. As can be seen from the energy–momentum equation (17), the isotropic string cloud is in an Einstein spacetime.

4.2. Static two-fluid model

One can use two different 4-velocities, $\hat{u}_\mu$ and $\hat{w}_\mu$, to write a two-fluid model:

$$T^{2fluid}_{\mu\nu} = \rho_2 \hat{w}_\mu \hat{w}_\nu + \rho_1 \hat{u}_\mu \hat{u}_\nu + p_r \hat{r}_\mu \hat{r}_\nu. \tag{20}$$

The fluid with ρ_2 is dust and the other is a fluid with a radial stress ($\dot{m} = 0$ and there is no Vaidya radiation fluid). Letelier [11] has described a procedure for casting two-fluid stress–energies into the form of an anisotropic fluid. His method is not adapted for the string fluid equation of state. As a variation of his method, we transform the fluid velocities to create two unnormalized vectors: V_μ timelike and Y_μ spacelike,

$$\sqrt{\rho_1} V_\mu = \sqrt{\rho_1} \cos\alpha\, \hat{u}_\mu + \sqrt{\rho_2} \sin\alpha\, \hat{w}_\mu \tag{21}$$

$$\sqrt{\rho_2} Y_\mu = -\sqrt{\rho_1} \sin\alpha\, \hat{u}_\mu + \sqrt{\rho_2} \cos\alpha\, \hat{w}_\mu. \tag{22}$$

This transformation obeys

$$\rho_1 \hat{u}_\mu \hat{u}_\nu + \rho_2 \hat{w}_\mu \hat{w}_\nu = \rho_1 V_\mu V_\nu + \rho_2 Y_\mu Y_\nu. \tag{23}$$

Since Y_μ is spacelike, equation (22) is only valid for $\hat{u}_\mu \neq \hat{w}_\mu$ and $\alpha \neq n(\pi/2)$. The parameter α is defined by $V_\mu Y^\mu = 0$:

$$\cot 2\alpha = \frac{\rho_1 - \rho_2}{2\sqrt{\rho_1 \rho_2} \hat{u}_\mu \hat{w}^\mu}.$$

The stress–energy $T^{2fluid}_{\mu\nu}$ can be written as

$$T^{transf}_{\mu\nu} = \rho \hat{V}_\mu \hat{V}_\nu + p_r \hat{r}_\mu \hat{r}_\nu + p_\perp \hat{Y}_\mu \hat{Y}_\nu \tag{24}$$

with

$$\rho := V_\alpha V^\alpha\, \rho_1, \qquad p_\perp := -Y_\alpha Y^\alpha\, \rho_2. \tag{25}$$

The transformation in equation (23) constrains the new density and transverse pressure to obey

$$\rho_1 + \rho_2 = \rho - p_\perp. \tag{26}$$

$T^{transf}_{\mu\nu}$ has the same form as $T^{string}_{\mu\nu}$ in equation (15).

If the two fluids have the same velocity, $\hat{u}_\mu = \hat{w}_\mu$, then the two-fluid stress–energy becomes a single fluid with a radial stress.

$$T^{2fluid}_{\mu\nu} \to T^{1fluid}_{\mu\nu} = (\rho_1 + \rho_2) \hat{u}_\mu \hat{u}_\nu + p_r \hat{r}_\mu \hat{r}_\nu. \tag{27}$$

The density of this string fluid is just the sum of the two densities. In the case $\hat{u}_\mu = \hat{w}_\mu$ the transverse pressure is zero. The transverse pressure reflects the different velocities of the fluids in the two-fluid model. When the transverse pressure is zero then $m = c_1 r + c_2$.

Using a multi-fluid model, one can also describe the radial stress as reflecting a velocity difference between two dusts so that the complete stress–energy tensor (15) could be described by a three-fluid dust model with all of the dusts moving with different velocities. Which of the interpretations discussed in this section is most likely depends on the actual physical situation being modelled.

5. Diffusive transport

Diffusion has been a seminal process for the development of our understanding of many modern systems. Since the work of Einstein [12], Smoluchowski [13] and Langevin [14], the ideas implicit in diffusion have found new application in areas as diverse as the behaviour of stock option values to cosmic strings. Vilenkin [15] has introduced diffusion into the description of cosmic strings by characterizing string evolution as the formation of Brownian trajectories. This has also been discussed by Bennett [16]. Diffusion is playing an important part in the growth of our understanding of fluctuations in quantum gravity [17, 18] and in the very early life of our universe. For example, Watabiki [19] has used the classical diffusion equation to characterize diffusion times in fractal quantum gravity [20, 21]. Diffusion may also play a role in eternal inflation models where inflation field fluctuations can be modelled as random walks [22]. The path integral technique, developed by Norbert Weiner [23] to describe diffusive processes, has become an essential part of the modern view of quantum mechanics.

String collisions, unlike point particle collisions, do not occur at a single spacetime point (interaction vertex). The 2-surface picture of strings requires the collision (interaction) to be a curve on a worldsheet. Observers in different Lorentz frames will see the interaction occurring at different points along the curve. Statistically, the coarse-grained picture of phase space for a set of collisions hides the lack of Lorentz invariance of a single collision. We assume string diffusion is like point particle diffusion in that the number density diffuses from higher numbers to lower according to

$$\partial_u n = \mathcal{D}\,\nabla^2 n \tag{28}$$

where $\nabla^2 = r^{-2}(\partial/\partial r)\,r^2(\partial/\partial r)$ and $\mathcal{D}$ is the positive coefficient of self-diffusion (which we henceforth take to be constant). Classical transport theory derives the diffusion equation by starting with Fick's law

$$\vec{J}_{(n)} = -\mathcal{D}\vec{\nabla} n \tag{29}$$

where $\vec{\nabla}$ is a purely spatial gradient. Then 4-current conservation $J^{\mu}_{(n);\mu} = 0$, where

$$\begin{aligned} J^{\mu}_{(n)}\partial_\mu &= (n, \vec{J}_{(n)}) \\ &= n\partial_u - \mathcal{D}(\partial n/\partial r)\partial_r, \end{aligned} \tag{30}$$

yields the diffusion equation (28). We label the 4-current $J_{(n)}$ to indicate n diffusion but we could have also written $J_{(\rho)}$ since the string number density and string fluid density must be related by $\rho = M_s n$ where M_s is the constant mass of the string species. M_s must be a multiple of the Planck mass since it is only over Planck length scales that point particles resolve into strings.

By rewriting the $T_{\mu\nu}$ components (16*a*) and (16*b*) as $\dot{m} = -4\pi r^2\psi$ and $m' = 4\pi r^2\rho$, we can write the integrability condition for m as

$$\dot{\rho} + r^{-2}\partial_r(r^2\psi) = 0. \tag{31}$$

If we compare the diffusion equation (28) (n replaced by ρ)

$$\dot{\rho} = \mathcal{D}\,r^{-2}\partial_r(r^2\partial\rho/\partial r) \tag{32}$$

with $\dot{\rho}$ in equation (31) we obtain

$$\dot{m} = 4\pi\mathcal{D}\,r^2\partial\rho/\partial r. \tag{33}$$

Thus solving the diffusion equation for ρ and then integrating those solutions to obtain m provides exact Einstein solutions which can be interpreted as either anisotropic fluids or diffusing string fluids.

There are some analytic solutions of equation (32):

$$\rho = \rho_0 + k_1/r, \tag{34}$$

$$\rho = \rho_0 + (k_2/6)r^2 + k_2\mathcal{D}u, \tag{35}$$

$$\rho = \rho_0 + k_3(\mathcal{D}u)^{-3/2}\exp[-r^2/(4\mathcal{D}u)], \tag{36}$$

$$\rho = \rho_0 + (k_6/r)\exp(-k_4^2\mathcal{D}u)[\sin(k_4 r) + k_5\cos(k_4 r)]. \tag{37}$$

Solutions (34) and (36) appear in [7]. In addition to the solutions above, we have obtained a number of analytic solutions using well known similarity techniques. Those solutions will be presented elsewhere. The physical behaviour of the density solutions provides a variety of atmospheric models.

Solution (34) describes an atmosphere with a simple drop off in radius and no time dependence. k_1 must be positive to avoid negative densities. ρ_0 is the density at spatial infinity.

The density described by equation (35) has two very different behaviours. For $k_2 > 0$, the density is not physically realistic. It increases with radius and grows with time. However, for $k_2 < 0$, the density decreases with radius and has a zero indicating a bounded string atmosphere. Since we are working with a string fluid whose equation of state is $\rho = -p_r$, the boundary also has zero radial pressure. The position of the boundary moves inward as $\mathcal{D}u$ increases; the extent of the atmospheric shell decreases with time.

The third density solution, equation (36), requires $k_3 > 0$ for positive densities. The solution begins with a high central density which falls off with radius. As time progresses the density decreases to the constant value ρ_0, the density at spatial infinity.

The fourth density solution, equation (37), models a complete array of atmospheric behaviours. For example, the parameter choices $k_6/\rho_0 = 10$, $k_4 = k_5 = 1$, describe an atmosphere that starts with a string boundary at $\mathcal{D}u = 1$ and as time progresses becomes an unbounded string cloud. Other parameter choices model atmospheric shells which are always unbounded. Parameters k_5 and k_6 must be positive for positive densities.

6. Diffusive mass solutions

6.1. Analytic solutions

Upon integrating $m' = 4\pi r^2\rho$ and $\dot{m} = 4\pi\mathcal{D}\, r^2\partial\rho/\partial r$ we obtain, in the same order as the densities above, the following:

$$m(u,r) = m_0 + (4\pi/3)r^3\rho_0 + 2\pi k_1(r^2 - 2\mathcal{D}u), \tag{38}$$

$$m(u,r) = m_0 + (4\pi/3)r^3\rho_0 + (4\pi/3)r^3k_2(\mathcal{D}u + r^2/10), \tag{39}$$

$$m(u,r) = m_0 + (4\pi/3)r^3\rho_0 + 16\pi k_3[-\eta\exp(-\eta^2) + (\sqrt{\pi}/2)\,\mathrm{erf}(\eta)], \tag{40}$$

$$m(u,r) = m_0 + (4\pi/3)r^3\rho_0 + (4\pi k_6/k_4^2)\exp(-k_4^2\mathcal{D}u)B, \tag{41}$$

where $\eta := r\,(4\mathcal{D}u)^{-1/2}$ and $B = \sin(k_4 r) - k_4 r\cos(k_4 r) + k_5[\cos(k_4 r) + k_4 r\sin(k_4 r)]$. Solutions (38) and (40) appear in [7].

If parameters k_1, k_2, k_3 and k_6 are zero then all the mass solutions above become the static solution $m_0 + (4\pi/3)r^3\rho_0$. This is the mass for the isotropic string fluid described by equation (19).

6.2. Interpreting the mass

Diffusive transport has a conserved 4-current for the density

$$J^\mu\partial_\mu = \rho\partial_u - \mathcal{D}(\partial\rho/\partial r)\partial_r$$

which can be written in terms of the null tetrad

$$J^\mu = \rho n^\mu + [(A/2)\rho - \mathcal{D}(\partial\rho/\partial r)]l^\mu. \tag{42}$$

When $J^\mu_{;\mu} = 0$ is integrated over a 4-volume, Stokes theorem casts the integral onto the bounding 3-surfaces:

$$\int_{R_4} J^\mu_{,\mu}\sqrt{-g}\,\mathrm{d}^4x = \oint_{\partial R_4} J^\mu\sqrt{-g}\,\mathrm{d}S_\mu \tag{43}$$

where we integrate over $u =$ constant null 3-surfaces $\mathcal{N}_2$ and $\mathcal{N}_1$ with $\mathrm{d}S_\mu = l_\mu\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$, and $r =$ constant timelike 3-surfaces Σ_2 and Σ_1 with $\mathrm{d}S_\mu = (n_\mu - \frac{1}{2}Al_\mu)\,\mathrm{d}u\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$. The 4-volume R_4 can be pictured as a tube surrounding a timelike cylinder containing the central source. An orthogonal cross section of R_4 would have sides $\mathcal{N}_1, \mathcal{N}_2, \Sigma_1, \Sigma_2$, forming a rhomboid with Σ_1 bounding R_4 away from the central source. Thus

$$\int_{\mathcal{N}_2-\mathcal{N}_1} J^\mu l_\mu\sqrt{-g}\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi + \int_{\Sigma_2-\Sigma_1} J^\mu\left(n_\mu - \tfrac{1}{2}Al_\mu\right)\sqrt{-g}\,\mathrm{d}u\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = 0$$

with specific form

$$\int_{\mathcal{N}_2-\mathcal{N}_1} \rho\sqrt{-g}\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi - \int_{\Sigma_2-\Sigma_1} \mathcal{D}(\partial\rho/\partial r)\sqrt{-g}\,\mathrm{d}u\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = 0. \tag{44}$$

To first understand the string mass, we examine a static string fluid. Metric (1) includes static string fluids when $m(u, r)$ is restricted to $m(r)$. We can use the density solution equation (34) with $\mathcal{D} = 0$ so that there is no diffusion. For $\rho = \rho_0 + k_1/r$, we have

$$4\pi\int_{\mathcal{N}_2-\mathcal{N}_1} (\rho_0 r^2 + k_1 r)\,\mathrm{d}r = 0. \tag{45}$$

Thus the string mass is static with $m_{string}(r) = \int_{\mathcal{N}_1} = \int_{\mathcal{N}_2}$:

$$m_{string}(r) = m_0 + 4\pi\left(\tfrac{1}{3}\rho_0 r^3 + \tfrac{1}{2}k_1 r^2\right) \tag{46}$$

on null surface $\mathcal{N}_1$ and at a later time on $\mathcal{N}_2$. This is the mass in equation (38) with $\mathcal{D} = 0$.

Now consider the case with time dependence where the mass is $m(u, r)$ and there is a Vaidya fluid of short photons. Again using $\rho = \rho_0 + k_1/r$, we find

$$4\pi\int_{\mathcal{N}_2-\mathcal{N}_1} (\rho_0 r^2 + k_1 r)\,\mathrm{d}r + 4\pi\int_{\Sigma_2-\Sigma_1} k_1\mathcal{D}\,\mathrm{d}u = 0. \tag{47}$$

This yields the total mass of strings and null fluid in equation (38) where

$$m(u, r) = \int_{\mathcal{N}_2} - \int_{\Sigma_1} = \int_{\mathcal{N}_1} - \int_{\Sigma_2}. \tag{48}$$

We use our knowledge of the static case to identify the string mass as that part of the total mass integrated over the null 3-surface $\mathcal{N}$, and identify the flux through the timelike surface Σ as resulting from both the energy carried by the Vaidya photons and the diffusing strings. Photons enter R_4 from the central source through Σ_1 while string bits diffuse through Σ_1 toward the source, with the opposite happening at Σ_2. There the photons leave R_4 while string bits enter. Thus $m(u, r) = m_{string} + m_{flux}$ where

$$m_{string} = 4\pi\left(\tfrac{1}{3}\rho_0 r^3 + \tfrac{1}{2}k_1 r^2\right)$$
$$m_{flux} = -4\pi k_1\mathcal{D}u.$$

Of course m_{string} need not be static. The density examples with $\rho(u, r)$ will have $m_{string}(u, r)$.

7. Horizons

The topological 2-spheres (ϑ, φ) nested in an $r =$ constant surface at time u have outgoing null geodesic normal l^μ and incoming null geodesic normal n^μ. The spheres become trapped surfaces when both $l^\mu r_{,\mu}$ and $n^\mu r_{,\mu}$ are positive (negative for +2 signature). The marginally trapped surface is the outer boundary of all trapped surfaces at time u, and the apparent horizon is the time history of the marginally trapped surface.

Here $l^\mu r_{,\mu} = 1$. The fluid atmosphere has $n^\mu r_{,\mu} = -A/2$, negative until $m(u, r) \geqslant r/2$. At that time the mass has become compact enough to trap light.

We will analyse mass expression (38) for trapped surfaces. With $L_0 := 2m_0$, $L_1^{-3} := (4\pi\rho_0)/(3m_0)$, $L_2^{-2} := 2\pi k_1/m_0$ and $1/T_0 := 4\pi k_1 \mathcal{D}/m_0$, we search for zeros of the polynomial

$$\frac{r^3}{L_1^3} + \frac{r^2}{L_2^2} - \frac{r}{L_0} + 1 - \frac{u}{T_0} = 0 \tag{49}$$

at time $u =$ constant. The four parameters (T_0, L_0, L_1, L_2) are positive. Suppose there is a root at $r = \alpha L_0$, $\alpha > 0$. Then, at $u = 0$, equation (49) becomes

$$\alpha^3 (L_0/L_1)^3 + \alpha^2 (L_0/L_2)^2 + 1 - \alpha = 0$$

or

$$\alpha^2 (2m_0)^2 [\alpha(8\pi\rho_0/3) + 2\pi k_1/m_0] + 1 - \alpha = 0. \tag{50}$$

If k_1 and ρ_0 are zero then $\alpha = 1$ and we find the Schwarzschild horizon at $r = 2m_0$. For non-zero parameters equation (50) has no real roots, which is consistent with the outgoing short-wavelength photons.

At time $u = 0$ and for some short time after $1 - u/T_0$ is positive. After more time Δu, $1 - u/T_0$ becomes negative and then there is only one sign change in equation (49). Descartes' rule of signs tells us that, for $u > \Delta u$, there is at most one real root of equation (49) and the fluid will have a trapped surface at $r = 2m(u, r)$.

8. Conclusion

That $\hat{v}^\mu$ is hypersurface orthogonal implies $\hat{v}_\mu\, \mathrm{d}x^\mu = f(u, r)\, \mathrm{d}t$. If metric (1) were the Schwarzschild metric then $A = 1 - 2m_0/r$, $u = t - r - 2m_0 \ln(r - 2m_0)$ and $\hat{v}_\mu\, \mathrm{d}x^\mu = A^{-1/2}\, \mathrm{d}t$. Next in simplicity is the Vaidya metric, where A of metric (1) would be $A = 1 - 2m(u)/r$. The Vaidya metric is not static and one cannot coordinate transform back to Schwarzschild. With the Vaidya metric $r - 2m(u)$ is a spacelike hypersurface lying outside the local null cone [25], unlike the Schwarzschild hypersurface $r - 2m_0$ which is null.

The traditional view of the Vaidya metric places it outside a spherical star which is losing mass via Vaidya's short-wavelength photons. Vacuum Schwarzschild geometry is joined smoothly to Vaidya at its radiative boundary.

The system described here with $m(u, r)$ continues to have $\hat{v}^\mu$ hypersurface orthogonal, so the t in $\hat{v}_\mu\, \mathrm{d}x^\mu = f(u, r)\, \mathrm{d}t$ labels spacelike hypersurfaces with energy–momentum given by equation (15) above. Because of the implicit equation of state, $\rho + p_r = 0$, we interpreted the fluid as an 'atmosphere' of open strings which form a string fluid. This view has also been discussed by 't Hooft [6] and Maldacena [5]. This fluid was taken as a source of energy–momentum. As a modelling choice, we took the strings to interact diffusively with energy carried away by short-wavelength photons. It was necessary to use the approximation of short-wavelength photons since $(2\dot{m}/r^2) l_\mu l_\nu$ is not an exact solution of Maxwell's equations.

Quantum field theory requires the quantum vacuum to be Lorentz invariant [26], which constrains the energy–momentum tensor on a Minkowski background to have the form $T_{\mu\nu} = \rho\eta_{\mu\nu}$, and the energy density to transform as

$$\rho' = \frac{\rho + p(v^2/c^2)}{1 - (v^2/c^2)}$$

under change of inertial frame. The string equation of state $p_r = -\rho$ satisfies the required transformation property and so the vacuum outside a relativistic star could indeed include a string atmosphere.

This work necessarily forms an incomplete picture of the evolution of the astrophysical system we model here. Strings may exist at the Planck length scale, with a large number of them possibly providing a macroscopic classical string, and forming a visible atmosphere until they lie within a trapped surface.

Acknowledgments

ENG was partially supported by an NSERC of Canada grant. Computations were verified using MapleV.4 (Waterloo Maple Software, Waterloo, Ontario) and GRTensorII rel 1.59 (P Musgrave, D Pollney and K Lake, Queens University, Kingston, Ontario).

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