INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **23** (2006) 4433–4451 doi:10.1088/0264-9381/23/13/009

Non-existence of black-hole solutions for the electroweak Einstein–Dirac–Yang/Mills equations

Yann Bernard

Department of Mathematics, The University of Michigan, 2072 East-Hall, Ann Arbor, Michigan 48109-1003, USA

E-mail: yannb@umich.edu

Received 23 November 2005
Published 9 June 2006
Online at stacks.iop.org/CQG/23/4433

Abstract
We consider a static, spherically symmetric system of a Dirac particle interacting with classical gravity and an electroweak Yang–Mills field. It is shown that the only black-hole solutions of the corresponding coupled equations must be the extreme Reissner–Nordström solutions, locally near the event horizon. This work generalizes a series of papers published by F Finster, J Smoller and S-T Yau.

PACS numbers: 03.65.Pm, 04.20.Jb, 11.15.Kc

1. Introduction

The coupling of gravity to classical force fields has given rise to many unexpected results and to various solutions of Einstein's field equations, thereby providing some insight into the intricate nature of the nonlinear interactions. The discovery by Bartnik and McKinnon [4] of particle-like solutions of the Einstein–Yang/Mills equations for a non-Abelian gauge field has triggered much research and further interesting discoveries. For the solutions numerically derived by Bartnik and McKinnon, and whose existence was rigorously established in [27], the repulsive effect of the Yang–Mills field compensates the attractive force of gravitation. This balance is so fragile that the generated solutions happen to be unstable, as was shown in [28]. This led Finster *et al* to introduce quantum fields in the problem, hoping the repulsion due to the Heinsenberg uncertainty principle would further counteract gravitational forces, sufficiently at least to allow the formation of stable bound states. In [15], they obtain numerical evidence pointing in that direction. In [14], the authors consider the same coupling, but focus attention instead on black-hole solutions. They prove the only globally normalizable (in some sense to be soon explained) such solutions are the Bartnik–McKinnon black holes. In other words, the spinors representing the state of the Dirac particle must vanish identically. In [13], the coupling of a gravitational field to Dirac particles and an Abelian $U(1)$ gauge field is investigated. The authors numerically construct particle-like solutions for a static, spherically symmetric singlet system. They find these solutions cease to exist if the rest-mass of the fermions exceeds a certain threshold value. One may conjecture that the gravitational interaction becomes so strong that it can no longer be compensated by repulsive electromagnetic forces, and it is thus legitimate to expect the formation of a black hole. This suggests that there should exist black-hole solutions of the coupled system for large fermionic masses. The work presented in [12] indicates however that this intuitive picture is erroneous. As in [14], the authors find that there exist no globally normalizable black-hole solutions of the coupled Einstein–Dirac–Maxwell equations.

We consider in this paper the coupling of a gravitational field to Dirac particles, and to both electromagnetic and weak forces, our original motivation being the hope of observing, either rigorously or numerically, the existence of black-hole solutions. The background is a four-dimensional Lorentzian spacetime, static and spherically symmetric in the sense that it is an S^2-bundle over a static two-dimensional pseudo-Riemannian manifold, and that the action of the isometry group $SO(3)$ has the 2-spheres as its orbits. It is well-known (cf [9]) that in such setting one may introduce local coordinates (t, r, θ, φ) in which the metric takes the form

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu := T^{-2}(r)\,\mathrm{d}t^2 - A^{-1}(r)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\varphi^2 \qquad (1)$$

for two positive radial functions $T(r)$ and $A(r)$.

The volume-element relative to this metric is $\sqrt{|g|} = r^2A^{-1}T^{-2}|\sin\theta|$.

We model electroweak forces by the gauge group $U(1)_{\text{Coulomb}} \times SU(2)_{\text{Magnetic}}$, so the Yang–Mills potential takes the form

$$\mathcal{A} = \Phi(r)\,\mathrm{d}t \oplus W(r)\tau^1\,\mathrm{d}\theta + (\cos(\theta)\tau^3 + W(r)\sin(\theta)\tau^2)\,\mathrm{d}\varphi. \qquad (2)$$

Here (τ^1, τ^2, τ^3) is the standard 2-dimensional basis of $\mathfrak{su}(2)$. The functions $\Phi(r)$ and $W(r)$ are real-valued.

The 'ansatz' (2) deserves some explanation. When an $SU(N)$ Yang–Mills field is coupled to the gravitational field of a spherically symmetric metric, it is natural to expect the Yang–Mills connection $\mathcal{A}$ to satisfy certain compatibility relations to the 'natural' Levi-Civita connection on spacetime. These compatibility criteria have received much attention in the literature, and we now have at our disposal a complete arsenal of consistent and explicit ansätze describing the specific form of an $SU(N)$ Yang–Mills potential $\mathcal{A}$ on a (static) spherically symmetric spacetime (although next to nothing is known for larger spacetime symmetry classes). For more details on this matter, the reader is invited to consult [2, 3, 19, 22, 30]. When quantum fields are introduced, it is *a priori* unclear why these ansätze should remain valid. In particular, the derivation leading to the specific form of $\mathcal{A}$ as given in (2) relies on the fact that when quantum effects are ignored, the Yang–Mills current is identically equal to zero (cf [2]). This is no longer true when Dirac particles generating a Dirac current are present. Remarkably, however, it can be shown that the 'classical' ansatz (2) continues to be correct when quantum particles are considered. A publication soon to appear will be devoted in parts to that goal (cf [5]). We therefore admit that (2) is the correct ansatz which the Yang–Mills connection should satisfy.

The general Einstein–Dirac–Yang/Mills equations are obtained by varying over metrics of the form (1), Yang–Mills connections of the form (2), and Dirac wavefunctions ψ, the action

$$\mathcal{S} = \int \left(\frac{1}{16\pi}R + \overline{\psi}(G - m)\psi - \frac{1}{16\pi\epsilon^2}\,\text{Trace}\,(F_{\mu\nu}F^{\mu\nu})\right)\sqrt{|g|}\,\mathrm{d}^4x. \qquad (3)$$

Here R is the scalar curvature relative to the metric (1) $\overline{\psi}$ is the Dirac conjugate of ψ; G is the Dirac operator; and $\mathcal{F}$ is the Yang–Mills field;

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.$$

The constant ϵ is the Yang–Mills coupling constant, and m is the rest-mass of the fermion. Note that we have deliberately omitted a gravitational coupling constant, for it will be of no use to our purpose. The specific form of the Dirac operator G is quite intricate, and we shall not dwell on it any further in this paper. Details may be found in [5, 10–12]. A few words about the wavefunction ψ are however in order. If the Dirac particle considered has *a priori* no internal symmetries (such as isospin or colour), then its wave function ψ is a complex-valued 4-vector on spacetime. In our problem, the introduction of the Yang–Mills potential may be thought of as arising from internal symmetries in the particle. This is the standard 'postulate' of quantum gauge field theory (cf [5, 20, 31]). Typically, if the particle has N internal degrees of symmetries, the wavefunction ψ is a $4N$-vector, and the Yang–Mills gauge group has Lie algebra $\mathfrak{su}(N)$. Thus, in our situation, ψ is an 8-vector. The electromagnetic part of the gauge group does not increase the number of components of ψ; it merely introduces a phase factor, which has no significance in quantum mechanics.

Naturally, we need to introduce a way of measuring the probability of presence of a particle in spacetime. To this end, we let $\mathcal{H}$ be a space-like hypersurface, and ν be a future-directed vector field normal to $\mathcal{H}$. We define an inner-product on solutions of the Dirac equation $(G - m)\psi = 0$ by

$$(\phi, \psi) := \int_{\mathcal{H}} \overline{\phi}\Gamma^j \psi \nu_j \,\mathrm{d}\mu, \tag{4}$$

where μ is the invariant measure on $\mathcal{H}$ induced by the metric on spacetime. It can be shown (see [10]) that this inner-product is positive definite and independent of $\mathcal{H}$.

In direct analogy to the special relativistic setting (cf [20]), we interpret the integrand of (ψ, ψ) as the probability density of presence in spacetime of the particle whose wavefunction is ψ. As explained in [11, 13], we require that the spinors decay fast enough at infinity

$$0 < \int_{t=\mathrm{cst.}|r>r_0} \overline{\psi}\Gamma^j \psi \nu_j \,\mathrm{d}\mu < \infty \qquad \text{for all} \quad r_0 > \rho, \tag{5}$$

where $\rho > 0$ is the location of a metric singularity.

For the variational process described above to be carried out appropriately, the various unknown fields involved must satisfy some 'asymptotic' conditions, which are discussed at length in [5]. See also [9, 27] and the references therein. We content ourselves in this discussion with noting that the metric should be asymptotically Minkowskian with finite ADM mass

$$\lim_{r\to\infty} r(1 - A(r)) < \infty \qquad \text{and} \qquad \lim_{r\to\infty} T(r) = 1.$$

In addition, the Coulomb and Yang–Mills 'potentials' must satisfy

$$\lim_{r\to\infty} \Phi(r) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} (|W(r)|, W'(r)) = (1, 0).$$

2. The coupled electroweak equations

Once the setting described in the previous section has been established, deriving the system of equations is rather straightforward. The introduction of the Yang–Mills field with gauge group $SU(2)_{\mathrm{Magnetic}}$ is treated as the addition of angular momentum to the usual orbital and intrinsic (spin) angular momenta, while, as mentioned above, $U(1)_{\mathrm{Coulomb}}$ has merely the effect of modifying the wavefunction by a phase factor. As is done in the classical setting (cf, for example [20, 26]), one finds an explicit basis of eigenstates, which then allows one to separate the radial from the angular variables. Confining the study to the simplest case of wavefunctions belonging to the kernel of J^2 (where J is the total angular momentum operator), only two real-valued radial functions $\alpha(r)$ and $\beta(r)$ (called *spinors*) are needed to describe the state of the Dirac particle. The full set of equations thus reduces to a system of ODEs where the independent variable is the radial coordinate r.

The Einstein–Dirac–Maxwell and Einstein–Dirac–Yang/Mills systems corresponding respectively to the gauge groups $U(1)_{\text{Coulomb}}$ and $SU(2)_{\text{Magnetic}}$ are derived in [13, 15]. We shall thus not elaborate in this paper on the derivation of the electroweak Einstein–Dirac–Maxwell–Yang/Mills (EDYMM) equations relative to the group $U(1)_{\text{Coulomb}} \times SU(2)_{\text{Magnetic}}$, as they may readily be deduced by appropriately combining the other two. More details on this procedure may be found in [5, 15], where larger gauge groups are considered. Eventually, one arrives at the following system of equations

$$\sqrt{A}\alpha' = \frac{W}{r}\alpha - (m + \Psi T)\beta \tag{6}$$

$$\sqrt{A}\beta' = -\frac{W}{r}\beta - (m - \Psi T)\alpha \tag{7}$$

$$rA' = 1 - A - \left(\frac{1-W^2}{er}\right)^2 - \frac{2}{e^2}A(W')^2 - \frac{1}{\widehat{e}^2}(rT\sqrt{A}\Psi')^2 - 2\Psi T^2(\alpha^2+\beta^2) \tag{8}$$

$$2rA\frac{T'}{T} = A - 1 + \left(\frac{1-W^2}{er}\right)^2 - \frac{2}{e^2}A(W')^2 + \frac{1}{\widehat{e}^2}(rT\sqrt{A}\Psi')^2 - 2\Psi T^2(\alpha^2+\beta^2)$$
$$- 2mT(\alpha^2 - \beta^2) + \frac{4}{r}TW\alpha\beta \tag{9}$$

$$\left(\frac{\sqrt{A}}{T}W'\right)' = \frac{1}{r^2T\sqrt{A}}((W^2-1)W + e^2rT\alpha\beta) \tag{10}$$

$$(r^2T\sqrt{A}\Psi')' = \widehat{e}^2\frac{T}{\sqrt{A}}(\alpha^2+\beta^2). \tag{11}$$

Note that we have introduced a new constant $\widehat{e}$ to describe the 'strength' of the Coulombic coupling, and thereby distinguish it from the Yang–Mills coupling constant e.

For notational convenience, we have also replaced the Coulomb potential Φ appearing in (2) by

$$\Psi(r) := \omega - \widehat{e}\Phi(r). \tag{12}$$

The constant ω (totally irrelevant in this paper) appears in the original derivation presented by Finster *et al*; it can be thought of as the total non-relativistic energy of the particle; it arises as the eigenvalue of a suitable Hamiltonian operator.

Remark 2.0.1. Letting $W(r) \equiv 1$ and $e = 0$ in the system (6)–(11), one recovers the Einstein–Dirac–Maxwell equations investigated in [12, 13]. Similarly, setting $\Psi(r) \equiv \omega$ and $\widehat{e} = 0$, the $SU(2)_{\text{Magnetic}}$ Einstein–Dirac–Yang–Mills system considered in [14, 15] is recovered.

We are concerned with studying (6)–(11) over the interval $[\rho, \infty)$, where $\rho > 0$ is the 'location' of a metric singularity. We shall often call ρ the *event horizon*. In this paper, we demand that a black-hole solution satisfy

$$A(\rho) = 0 \qquad \text{and} \qquad A(r) > 0, \qquad T(r) > 0 \quad \text{for} \quad r > \rho. \tag{13}$$

In addition, we demand that a black-hole solution be *globally normalizable* in the sense of (5). It can be shown (cf [11]) that this condition translates into the requirement that

$$0 < \int_{r_0}^{\infty} \frac{T}{\sqrt{A}}(\alpha^2+\beta^2)\,\mathrm{d}r < \infty, \qquad \text{for every} \quad r_0 > \rho. \tag{14}$$

Finally, the system (6)–(11) must be supplemented by a suitable set of 'regularity' hypotheses, valid locally around the event horizon[1], namely (cf section 3.1 for notational conventions)

$$W,\ \Psi',\ A(W')^2 \qquad \text{belong to} \quad \mathcal{L}^\infty(EH) \tag{15}$$

$$AT^2 \quad \text{and} \quad (AT^2)^{-1} \qquad \text{belong to} \quad \mathcal{W}^{1,\infty}(EH). \tag{16}$$

As explained in [4, 5], the condition (15) expresses the fact that the Yang–Mills potential is assumed to be regular. The second condition (16) guarantees that the volume-element of the metric (1) be sufficiently regular on the horizon. We believe the Lipschitzean condition given in (16) could be somewhat improved, although we were unable to weaken it. Perhaps replacing $\mathcal{W}^{1,\infty}(EH)$ by $\mathcal{W}^{1,p}(EH)$, for some suitable finite p, might suffice.

We draw the reader's attention to the fact that (16) is considerably weaker than the regularity assumptions which have thus far been imposed in papers dealing with similar problems. In [12], the authors demand that the volume-element and its inverse be infinitely differentiable on the horizon. In addition, they impose a local power-law assumption on the metric coefficient $A(r)$ of the form

$$A(r) = A_0(r-\rho)^s + \mathcal{O}((r-\rho)^{s+1}),$$

for some positive constants A_0 and s.

In [14], the volume-element and its inverse are chosen to be continuously differentiable on the horizon, while $A(r)$ is required to be monotone increasing outside of and near the event horizon.

Remark 2.0.2. One could *empirically* defend that the volume-element (and its inverse) should belong to $\mathcal{C}^1(EH)$ as follows. An event horizon should be a pure coordinate singularity, so that after a suitable coordinate transformation, the metric should be smooth. If this is possible, an observer freely falling into the black hole does not 'feel anything' while crossing the horizon (although, of course, he will feel strong gravitational forces shortly afterwards, when being sucked in the singularity located at the centre of the black hole). In the case of the Schwarzschild metric, the change of coordinates in question is the well-known Kruskal extension (see, for example, [1, 29]). For the extreme Reissner–Nordström metric (cf section 4), Carter found a similar transformation (see [6]). In both cases, the volume-element is smooth on the horizon in polar coordinates, and this is essential for the Kruskal and Carter transformations to work. One could postulate this feature holds in more generic settings, although we are unaware of any rigorous proof of this fact. In order to make this picture precise, one could consider the time-like geodesic of an observer free-falling into the black hole, take a tubular neighbourhood around it, and consider the regularity of the metric in this coordinate system. We have however not investigated this technique in detail, and caution the reader that it may be flawed.

The fact that our hypotheses are considerably weaker than those found in [12, 14], especially considering that our system generalizes those studied in these papers, has benefits beyond the mere mathematical performance. Indeed, in [8], Dafermos proposes a substantially different argument from that followed by Finister *et al* to analyse similar coupled systems. His method relies on working with different coordinates than the standard spherical frame naturally suggested by the foliation of a spherically symmetric space-time (our choice). This choice allows him to analyse very general gravity-matter field couplings.

[1] In this paper, the phrases 'near the event horizon' and 'around the event horizon' mean 'on any interval $[\rho, \rho+\delta]$, for $\delta > 0$'.

Dafermos argues that his method is more generic than that proposed by Finster *et al*, and he expresses concerns regarding the seemingly artificial hypotheses imposed by them on the local behaviour of the metric coefficient $A(r)$ near the singularity. In actuality, the hypotheses appearing in the paper [8] are quite different from those chosen by Finster *et al* in their works, and one cannot view the results established in the latter as special cases of those proved in the former. Along these lines, the interested reader is invited to consult the latter's 'rebuttal response' [17] to the critique made in [8].

In this paper, we present a variant of the method of Finister *et al* involving no *de facto* assumptions on the local behaviour of $A(r)$ near the singularity. All such hypotheses having been removed, the concerns raised in [8] become obsolete.

3. Protocol

The goal of this section is to set forth the notational conventions that will be used throughout the rest of this work, as well as to establish a number of elementary facts that will repeatedly be called upon in our demonstrations.

3.1. Notational conventions

We shall find it convenient to have at our disposal a few shorthands that will help render our discussion more fluent.

Spaces. We shall denote by $\mathcal{X}(EH)$ the space of functions which belong to $\mathcal{X}$ on any interval of the form $[\rho, \rho + \epsilon]$, for $\epsilon > 0$. For example, $\mathcal{L}^p(EH)$ is the space of functions which are locally Lebesgue p-integrable near the event horizon.

Order Relations. Let $F(r)$ and $G(r)$ be two functions.

(i) By $F(r) \simeq G(r)$, it will be meant that the ratio $|F(r)/G(r)|$ remains bounded from above and below near the event horizon.
(ii) By $F(r) \lesssim G(r)$, it will be meant that the ratio $|F(r)/G(r)|$ remains bounded from above near the event horizon.

Brackets. We shall often have to deal with functions $F(r)$ of the form

$$F(r) = a(r)(\alpha^2 + \beta^2)(r) + b(r)(\alpha^2 - \beta^2)(r) + c(r)(2\alpha\beta)(r),$$

for some functions $a(r), b(r)$, and $c(r)$. For notational convenience, we shall use the abbreviation

$$F = [a, b, c].$$

3.2. Elementary facts

We enumerate here a few useful facts that may be immediately inferred from the EDYMM equations, and that will be of frequent use in what follows.

Fact 3.2.1. Hypothesis (16) guarantees that both AT^2 and its inverse have finite limits on horizon. Accordingly, these limits must be non-zero, and AT^2 is thus bounded from above and below on the event horizon. This simple fact will be used repeatedly in our work. First and foremost, observe that it immediately yields $T^{-1}(\rho) = 0$, owing to the fact that $A(\rho) = 0$.

Fact 3.2.2. Combining the Dirac equations (6) and (7) produces

$$(\alpha^2 + \beta^2)' = \frac{2}{\sqrt{A}}\left[0, \frac{W}{r}, -m\right] \tag{17}$$

$$(\alpha^2 - \beta^2)' = \frac{2}{\sqrt{A}}\left[\frac{W}{r}, 0, -\Psi T\right] \tag{18}$$

$$(2\alpha\beta)' = \frac{2}{\sqrt{A}}[-m, \Psi T, 0]. \tag{19}$$

Fact 3.2.3. Adding together Einstein's equations (8) and (9) yields

$$\frac{r}{2T^2}(AT^2)' = -\frac{2}{e^2}A(W')^2 + \left[-2\Psi T, m, \frac{W}{r}\right]T. \tag{20}$$

Fact 3.2.4. We now derive an identity which will be of central importance all along our discussion. We begin by defining the functions

$$\mathcal{R} := \frac{1}{e^2}A(W')^2 - \frac{1}{2}\left(\frac{1-W^2}{er}\right)^2 \qquad \text{and} \qquad \mathcal{K} := \left[-\Psi T, m, \frac{W}{r}\right]T. \tag{21}$$

Using (17)–(19), it is a simple matter to verify that for all functions $f(r)$,

$$(f\mathcal{K})' = \left[(-f\Psi T^2)', m(fT)', \left(fT\frac{W}{r}\right)'\right]. \tag{22}$$

Next, let us recast (9) in the form

$$\frac{1}{2\hat{e}^2}(r^2T\sqrt{A}\Psi')^2 = \frac{r^2}{2}\left(1 - A + 2rA\frac{T'}{T}\right) - r^2\mathcal{K} + r^2\mathcal{R}. \tag{23}$$

We differentiate both sides of (23) starting with its left-hand side

$$\left\{\frac{1}{2\hat{e}^2}(r^2T\sqrt{A}\Psi')^2\right\}' = [r^2T^2\Psi', 0, 0], \tag{24}$$

where we have used Maxwell's equation (11).

As explained above, we have

$$(r^2\mathcal{K})' = [(-r^2\Psi T^2)', m(r^2T)', (rTW)']. \tag{25}$$

With the help of the Yang–Mills equation (10), we find

$$e^2(r^2\mathcal{R})' = 2W'\left(e^2rT\alpha\beta + r^2AW'\frac{T'}{T}\right) + 2rA(W')^2. \tag{26}$$

Substituting (24)–(26) into (23) yields, after a few elementary manipulations

$$[0, mrT, 0] = \left\{\frac{r^2}{2}\left(1 - A + 2rA\frac{T'}{T}\right)\right\}' - \frac{r}{T}(rT)'\left\{\left[-2\Psi T, m, \frac{W}{r}\right]T - \frac{2}{e^2}A(W')^2\right\}. \tag{27}$$

Observe that the second curly-bracketed term on the right-hand side of (27) is equal to $\frac{r}{2T^2}(AT^2)'$, as stated in (20). Whence we deduce

$$[0, mrT, 0] = \left\{\frac{r^2}{2}\left(1 - A + 2rA\frac{T'}{T}\right)\right\}' - \frac{r^2}{2T^3}(rT)'(AT^2)'. \tag{28}$$

Alternately, the latter may be recast in the form

$$(T^{-2})'' = \frac{2}{r^2 Z} - \frac{2}{r^2 T^2} - \frac{2A'}{rZ} - \frac{(r^4 Z)'}{2r^4 Z}(T^{-2})' - \frac{2m}{r^2\sqrt{Z}}\frac{(\alpha^2 - \beta^2)}{\sqrt{A}}, \tag{29}$$

where, for notational convenience, we have set $Z := T\sqrt{A}$. It should also be noted that (29) is not a mere 'artefact' deriving from the system of equations. Indeed, this identity is the third (and fourth) field equation, as can be shown by carefully inspecting the way the system is derived. For more details on the derivation of the system, the reader is referred to [5, 11, 14].

4. Non-existence of black-hole solutions

The contribution of the German aeronautical engineer Hans Reissner to general relativity, although rather limited, proved to be very important, for he was, as early as 1916, the first scientist to couple gravity to another force field (charged point-mass), derive a consistent system, and solve it exactly (see [25]). At the same time, the Finnish theoretical physicist Gunnar Nordström was conducting research aimed at developing a theory which would simultaneously describe gravity and electromagnetism (his ideas were the first example of what is nowadays known as Kaluza–Klein theory). In 1918, he successfully solved Einstein's field equations for a spherically symmetric charged body, thereby generalizing Reissner's results for a point-charge (see [24]). The metric for a (non-rotating) charge distribution is nowadays known as the *Reissner–Nordström metric*,

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi). \tag{30}$$

In addition, there is an external electromagnetic potential (solution to Maxwell's equation) of the form $(-\phi, \vec{0})$, with Coulomb potential

$$\phi(r) = \frac{q}{r}.$$

When $q < m$, the Reissner–Nordström metric (30) has two singularities. This is the so-called *non-extreme case*. When $q = m$, both singularities coincide and the metric has only one event horizon located at $r = m$. This is the *extreme Reissner–Nordström metric*. Finally, when $q > m$, the metric does not describe a black hole.

In our setting, the metric takes (in standard spherical coordinates) the static spherically symmetric form given in (1):

$$\mathrm{d}s^2 = T^{-2}(r)\,\mathrm{d}t^2 - A^{-1}(r)\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi),$$

for two positive radial functions $T(r)$ and $A(r)$. Since $r = \rho$ is an event horizon for our metric, we have $A(\rho) = 0$. Inspired by (30), we say that our metric is of the *extreme Reissner–Nordström type* provided

$$\lim_{r\searrow\rho}(r-\rho)^{-2}A(r) \in (0, \infty). \tag{31}$$

Equivalently, as explained in fact 3.2.1 this condition reads

$$\lim_{r\searrow\rho}(r-\rho)T(r) \in (0, \infty).$$

Black holes of the extreme Reissner–Nordström type have appeared in many different settings, and they are frequently encountered in the specialized literature. Aside from the original instance (i.e. the coupling of gravity to an electromagnetic field) in which they were discovered, they were also found to form a large class of solutions for problems involving the coupling of gravity to generic $SU(N)$ Yang–Mills fields, as shown in [23]. In [8], the author studies a number of situations in which extreme Reissner–Nordström black holes naturally arise from problems involving the coupling of gravity to various matter fields. Another important instance was discovered by Hawking. Namely, Extreme Reissner–Nordström black holes have zero temperature (cf [21]), and they may thus be considered as the asymptotic state of black holes emitting Hawking radiation. The main purpose of this paper is to demonstrate that our system supplemented with its set of regularity hypotheses has no globally normalizable solutions (in the sense of (14)) unless the metric is of the extreme Reissner–Nordström type nearby the event horizon. More precisely, we shall establish

Theorem 4.0.1. *If the system (6)–(11) is to satisfy the set of hypotheses (14)–(16), then it must be true that*

$$\lim_{r\searrow\rho}(r-\rho)^{-2}A(r)=\frac{1}{\rho^2}. \tag{32}$$

Setting $Z = T\sqrt{A}$, the conclusion (32) of theorem 4.0.1 is equivalent to

$$\lim_{r\searrow\rho}(r-\rho)T(r)=\rho Z(\rho),$$

since the limit $Z(\rho)$ of $Z(r)$ as $r \searrow \rho$ exists by (16).

The remainder of this paper is devoted to the demonstration of theorem 4.0.1 We attract the reader's attention to the fact that theorem 4.0.1 does not guarantee that there exists no black-hole solution to the EDYMM equations. To obtain a complete non-existence result, one would also have to dismiss the extreme Reissner–Nordström case. We shall have more to say about this in what follows, in sections 4.3 and 5.

4.1. Preliminary results

In this section we develop the skeleton of the proof of theorem 4.0.1, and defer the body of the demonstration to section 4.2.

Suppose there exists $\bar{r} > \rho$ for which

$$(\alpha^2+\beta^2)(\bar{r})=0,$$

so that $\alpha(\bar{r})$ and $\beta(\bar{r})$ both vanish (recall the spinors α and β are real-valued). Since, however, the Dirac system (6)–(7) is linear, it follows that $\alpha(r) \equiv 0 \equiv \beta(r)$ for all r outside of the event horizon, which violates the global normalization condition (14). We may consequently take for granted that

$$(\alpha^2+\beta^2)(\bar{r})\neq 0 \qquad \text{for all} \quad \bar{r}>\rho. \tag{33}$$

This fact will eventually allow us to reach an absurd statement, unless the extreme Reissner–Nordström condition (32) holds.

We open our derivations with a simple yet quite useful fact. It can be seen from (13) and (33) that the right-hand side of the Maxwell equation (11) is strictly positive outside of the event horizon. We thus deduce that the function $r^2T\sqrt{A}\Psi'$ is locally of bounded variation near the horizon (its boundedness follows from (15) and (16)). Whence its derivative (the right-hand side of (11)) is locally integrable:

$$\frac{T}{\sqrt{A}}(\alpha^2+\beta^2)\in\mathcal{L}^1(EH). \tag{34}$$

We may invoke fact 3.2.1 to express (34) in the alternate forms

$$A^{-1}(\alpha^2+\beta^2)\in\mathcal{L}^1(EH) \qquad \text{and} \qquad T^2(\alpha^2+\beta^2)\in\mathcal{L}^1(EH). \tag{35}$$

In addition, still from the Maxwell equation, we infer that the function $r^2 T\sqrt{A}\Psi'$ has a finite limit on the horizon, which, owing to fact 3.2.1, implies that $\Psi(r)$ belongs to $\mathcal{C}^1(EH)$.

We continue our study by proving that the metric coefficients $A(r)$ and $T^{-2}(r)$ are continuously differentiable on the horizon.

Lemma 4.1.1. *The functions $A(r)$ and $T^{-2}(r)$ belong to $\mathcal{C}^1(EH)$.*

Proof. From (35), (15) and the fact that $T^{-1}(\rho) = 0$ (cf the end of fact 3.2.1), we conclude that all of the summands on the right-hand sides of (8) and of (9) are locally Lebesgue integrable near the event horizon (some of these terms are actually bounded). Whence we conclude:

$$A'(r) \in \mathcal{L}^1(EH) \qquad \text{and} \qquad (T^{-2})'(r) \in \mathcal{L}^1(EH), \tag{36}$$

where, for the second part, we applied fact 3.2.1 to obtain

$$A\frac{T'}{T} \simeq \frac{T'}{T^3} \simeq (T^{-2})'. \tag{37}$$

Let us next consider (28), namely

$$\left\{\frac{r^2}{2}\left(1 - A + 2rA\frac{T'}{T}\right)\right\}' = -mrT(\alpha^2 - \beta^2) - \frac{r^2}{2T^3}(rT)'(AT^2)'. \tag{38}$$

Using again fact 3.2.1 along with (16), it follows from (36) that the second term on the right-hand side of (38) is locally Lebesgue integrable near the horizon. Indeed, we have

$$\left|\frac{r^2}{2T^3}(rT)'(AT^2)'\right| \simeq |A + (T^{-2})'||(AT^2)'| \lesssim A + |(T^{-2})'|. \tag{39}$$

In addition, since $T^{-1}(\rho) = 0$, (35) guarantees that the first summand on the right-hand side of (38) is also locally Lebesgue integrable near the event horizon. We accordingly deduce that $A\frac{T'}{T}$ has a finite limit as $r \searrow \rho$. It then follows, owing to (37), that $(T^{-2})'$ is continuous on the horizon. Finally, to obtain the first part of the desired assertion, it suffices to note that (16) yields

$$A' = (AT^2)'T^{-2} + AT^2(T^{-2})'. \tag{40}$$

□

It turns out that not only, as stated in the previous lemma, $A(r)$ and $T^{-2}(r)$ are continuously differentiable on the horizon, but actually their derivatives both vanish at $r = \rho$. To establish this fact, an important intermediary result is required.

Lemma 4.1.2. *We have* $\lim_{r\searrow\rho}(\alpha^2 + \beta^2)(r) = 0$.

Proof. Identity (17) shows that

$$(\alpha^2 + \beta^2)' \lesssim \frac{\alpha^2 + \beta^2}{\sqrt{A}}, \tag{41}$$

where we have used (15) (namely the boundedness of W near the horizon).

Since $A(\rho) = 0$, (35) applied to (41) implies that $(\alpha^2 + \beta^2)$ has a finite limit as $r \searrow \rho$. Assume, for the sake of contradiction, that this (necessarily non-negative) limit is non-zero. Then we have

$$\frac{1}{A} \lesssim \frac{\alpha^2 + \beta^2}{A},$$

which, using (35), shows that $A^{-1} \in \mathcal{L}^1(EH)$. From the boundedness of A' on the horizon (cf lemma 4.1.1), we deduce that the function $\log(A)$ belongs to $\mathcal{W}^{1,1}(EH)$. This is however impossible, since $A(\rho) = 0$. Whence the desired contradiction. □

Lemma 4.1.2 has important corollaries for our future purposes. We examine below a couple of these interesting results.

Corollary 4.1.3. $A'(\rho) = 0 = (T^{-2})'(\rho)$.

Proof. Observe first that (33) allows (41) to be recast in the form

$$(\log(\alpha^2 + \beta^2))' \lesssim \frac{1}{\sqrt{A}}.$$

Since $(\alpha^2 + \beta^2)(\rho) = 0$, as was shown in lemma 4.1.2, this last identity implies thus that $A^{-1/2} \notin \mathcal{L}^1(EH)$.

The fact that $A'(r)$ has a finite limit on the horizon was established in lemma 4.1.1 Assume, for the sake of contradiction, that $\lim_{r\searrow\rho} A'(r)$ is non-zero. Then we reach the contradictory fact

$$\frac{1}{\sqrt{A}} \simeq \frac{1}{\sqrt{r-\rho}} \in \mathcal{L}^1(EH).$$

Accordingly, indeed, it must be true that $A'(\rho) = 0$.

The second part of the claimed assertion follows from (40) combined to (16) and fact 3.2.1. □

For notational convenience, it is helpful to set $F(r) := r^2 T\sqrt{A}\Psi'(r)$. Recall that this function is continuous on and outside of the event horizon (cf comment following (35)). It is actually monotone outside of the black hole, as shown by the Maxwell equation (11). We now prove a central ingredient necessary to obtain the proof of theorem 4.0.1.

Corollary 4.1.4. *For all $r > \rho$, the following relation holds:*

$$\frac{(\alpha^2 + \beta^2)(r)}{r-\rho} \lesssim F(r) - F(\rho).$$

Proof. Observe first that (41) may be recast in the form

$$(\sqrt{\alpha^2 + \beta^2})' \lesssim \sqrt{\frac{\alpha^2 + \beta^2}{A}}. \tag{42}$$

Consider next the Maxwell equation

$$F' = \widehat{e}^2 \frac{T}{\sqrt{A}}(\alpha^2 + \beta^2) \simeq \frac{\alpha^2 + \beta^2}{A}$$

(the second part of this last equation follows from fact 3.2.1).

Incorporating (42) yields

$$\{(\sqrt{\alpha^2 + \beta^2})'\}^2 \lesssim F'. \tag{43}$$

We may integrate (43) calling upon Jensen's inequality and lemma 4.1.2 to obtain

$$\begin{aligned}\frac{(\alpha^2 + \beta^2)(r)}{r-\rho} &= (r-\rho)\left(\frac{1}{r-\rho}\int_\rho^r (\sqrt{\alpha^2 + \beta^2})'\,\mathrm{d}s\right)^2 \\ &\leqslant \frac{1}{r-\rho}\int_\rho^r ((\sqrt{\alpha^2 + \beta^2})')^2\,\mathrm{d}s \\ &\lesssim \int_\rho^r F'(s)\,\mathrm{d}s = F(r) - F(\rho),\end{aligned}$$

as announced. □

We now turn to another quite important fact, namely $\Psi(\rho) = 0$. Paired up to corollary 4.1.4, this fact will ultimately allow us to reach theorem 4.0.1 The proof of $\Psi(\rho) = 0$ is done by means of contradiction. Because it involves a certain level of trickery, we have chosen to present it step by step.

Lemma 4.1.5. *If $\Psi(\rho) \neq 0$, then $\frac{1}{\Psi T}$ belongs to $\mathcal{W}^{1,1}(EH)$.*

Proof. All terms appearing in Einstein's first equation (8) are locally bounded near the event horizon (including A', as was shown in lemma 4.1.1), safe for $\Psi T^2(\alpha^2 + \beta^2)$, which we thus infer to also be locally bounded near the horizon. If $\Psi(\rho) \neq 0$, it follows from the continuity of $\Psi(r)$ that $T^2(\alpha^2 + \beta^2)$ belongs to $\mathcal{L}^\infty(EH)$. Using fact 3.2.1, we have equivalently

$$\frac{\alpha^2 + \beta^2}{A} \in \mathcal{L}^\infty(EH).$$

Since $A(\rho) = 0$, we deduce at once that

$$\lim_{r\searrow\rho} \frac{\alpha^2 + \beta^2}{\sqrt{A}} = 0. \tag{44}$$

Let us next consider identity (29). Incorporating (16), lemma 4.1.3 and (44), it follows that $(T^{-2})''$ is continuous on the horizon. More precisely

$$\lim_{r\searrow\rho}(T^{-2})''(r) = \frac{2}{\rho^2 Z(\rho)}. \tag{45}$$

With the help of lemma 4.1.3 and the fact that $T^{-1}(\rho) = 0$, we may integrate twice (45) to infer that

$$(T^{-2})'(r) \simeq \frac{2}{\rho^2 Z(\rho)}(r-\rho) \qquad \text{and} \qquad T^{-2}(r) \simeq \frac{1}{\rho^2 Z(\rho)}(r-\rho)^2 \tag{46}$$

hold nearby the horizon. It whence follows that $(T^{-1})'$ belongs to $\mathcal{C}(EH)$.

Returning now to the statement of the lemma, we note that if $\Psi(\rho) \neq 0$, then

$$\left(\frac{1}{\Psi T}\right)' = \frac{(T^{-1})'}{\Psi} - \frac{\Psi'}{\Psi^2 T}$$

is continuous on and around the event horizon (recall that $\Psi \in \mathcal{C}^1(EH)$), and it thus surely belongs to $\mathcal{L}^1(EH)$, as claimed. □

Corollary 4.1.6. *Suppose that $\Psi(\rho) \neq 0$. Then $\frac{W}{r\Psi T}$ belongs to $\mathcal{W}^{1,1}(EH)$.*

Proof. Observe that

$$\left(\frac{W}{r\Psi T}\right)' = \frac{\sqrt{A}W'}{r\Psi T\sqrt{A}} - \frac{W}{r^2\Psi T} + \frac{W}{r}\left(\frac{1}{\Psi T}\right)'. \tag{47}$$

Hypotheses (15) and (16) guarantee that the first two summands on the right-hand side of (47) remain bounded on and around the event horizon. In addition, the local boundedness of W paired to the result from lemma 4.1.5 shows the last summand is locally Lebesgue integrable nearby the event horizon. The desired statement follows accordingly. □

Lemma 4.1.5 and corollary 4.1.6 will now enable us to prove that $\Psi(\rho) = 0$. Consider the function $G := -\frac{\mathcal{K}}{\Psi T}$ (the function $\mathcal{K}$ was defined in (21)). Setting $a = \Psi T$ and $b = W/r$, we may write $G = \langle U, MU\rangle$, with

$$U = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix}1 - m/a & -b/a\\ -b/a & 1 + m/a\end{pmatrix}.$$

Let us assume, for the sake of contradiction, that $\Psi(\rho) \neq 0$. Owing to the fact that $T^{-1}(\rho) = 0$, we then note that $1/a$ tends to zero as $r \searrow \rho$. Whence, since b is bounded on the event horizon (cf (15)), it follows that matrix M behaves locally near the horizon like the identity. We may thus find two positive constants q and ϵ such that

$$\frac{1}{q}|U(r)|^2 \leqslant G(r) \leqslant q|U(r)|^2 \qquad \text{for} \quad \rho < r < \rho + \epsilon. \tag{48}$$

As seen in (22), we have $G' = \langle U, M'U \rangle$, so that

$$|G'(r)| \leqslant |U(r)|^2 H(r), \qquad \text{where} \quad H(r) := \left| \left(\frac{m}{\Psi T} \right)' \right| + \left| \left(\frac{W}{r\Psi T} \right)' \right|. \tag{49}$$

Note that $H(r)$ is locally integrable near the event horizon, as shown in lemma 4.1.5 and corollary 4.1.6

Bringing altogether (48) and (49) produces the inequalities

$$-qH(r) \leqslant \frac{G'(r)}{G(r)} \leqslant qH(r). \tag{50}$$

Integrating (50) over the interval $(\rho, \rho + \epsilon)$, exponentiating and substituting the yield in (48) gives

$$\widetilde{q} \leqslant |U(r)|^2 \leqslant (\widetilde{q})^{-1} \qquad \text{on} \quad \rho < r < \rho + \epsilon, \tag{51}$$

for some constant $0 < \widetilde{q} < 1$.

By definition, $|U(r)|^2 = (\alpha^2 + \beta^2)$, so that (51) shows $\underline{\lim}_{r \searrow \rho} (\alpha^2 + \beta^2)(r) > 0$. This, however, violates the result of lemma 4.1.2 We have therefore established the veracity of

Lemma 4.1.7. *The Coulomb potential $\Psi(r)$ satisfies*

$$\Psi(\rho) = 0.$$

4.2. Main theorem

Thanks to our findings from section 4.1, we are now sufficiently geared to finish the proof of theorem 4.0.1. We distinguish two cases which we shall handle separately. Before all, we find helpful to have at our disposal.

Proposition 4.2.1. *Suppose that the function $A^{-1/2}(\alpha^2 + \beta^2)$ tends to zero as $r \searrow \rho$. Then the conclusion of theorem 4.0.1 holds.*

Proof. As seen in the proof of lemma 4.1.5, if $\lim_{r \searrow \rho} A^{-1/2}(\alpha^2 + \beta^2) = 0$, then nearby the horizon

$$T^{-2}(r) \simeq \frac{1}{\rho^2 Z(\rho)}(r - \rho)^2. \tag{52}$$

Because $Z = T\sqrt{A}$, we deduce at once from (52) that near the horizon

$$A(r) \simeq \frac{1}{\rho^2}(r - \rho)^2,$$

as claimed. □

The proof of theorem 4.0.1 shall therefore be complete once it is established that the hypothesis of proposition 4.2.1 holds. As mentioned above, we break our proof into two distinct cases.

Proposition 4.2.2. *If* $\Psi'(\rho) \neq 0$, *then* $\lim_{r\searrow\rho} A^{-1/2}(\alpha^2+\beta^2) = 0$.

Proof. If the derivative of the function Ψ is non-zero on the horizon, we conclude from lemma 4.1.7 that

$$|\Psi(r)| \simeq r - \rho. \tag{53}$$

As explained in the beginning of the proof of lemma 4.1.5, it is known that $\Psi T^2(\alpha^2+\beta^2)$ is locally bounded near the horizon. Whence we deduce from (53) that

$$(r-\rho)T^2(\alpha^2+\beta^2) \in \mathcal{L}^\infty(EH).$$

Equivalently, from fact 3.2.1, we may recast the latter in the alternate form

$$(r-\rho)\frac{\alpha^2+\beta^2}{A} \in \mathcal{L}^\infty(EH). \tag{54}$$

Using the result from corollary 4.1.4, (54) shows that

$$\frac{\alpha^2+\beta^2}{\sqrt{A}} \equiv \left((r-\rho)\frac{\alpha^2+\beta^2}{A}\right)^{1/2}\left(\frac{\alpha^2+\beta^2}{r-\rho}\right)^{1/2} \lesssim \left(\frac{\alpha^2+\beta^2}{r-\rho}\right)^{1/2}$$

tends to zero as $r \searrow \rho$. □

Proposition 4.2.3. *If* $\Psi'(\rho) = 0$, *then* $\lim_{r\searrow\rho} A^{-1/2}(\alpha^2+\beta^2) = 0$.

Proof. Firstly, we infer from $\Psi'(\rho) = 0$ that $\Psi(r)$ is strictly positive for all values of $r > \rho$. Indeed, we know from Maxwell's equation (11) that $r^2T\sqrt{A}\Psi'$ is an increasing function. Since $T\sqrt{A}$ is positive (cf (13)), it follows that Ψ is also an increasing function. Finally, the positiveness of Ψ follows from lemma 4.1.7.

From corollary 4.1.4 and fact 3.2.1 we deduce next that

$$\frac{\alpha^2+\beta^2(r)}{\Psi(r)} \lesssim (r-\rho)\frac{\Psi'(r)}{\Psi(r)}.$$

Because Ψ is continuously differentiable on and around the event horizon, the latter implies that

$$\frac{\alpha^2+\beta^2}{\Psi} \in \mathcal{L}^\infty(EH). \tag{55}$$

As explained in the beginning of the proof of lemma 4.1.5, it is known that $\Psi T^2(\alpha^2+\beta^2)$ is locally bounded near the horizon. Whence we deduce from (55) that

$$T(\alpha^2+\beta^2) \equiv \left(\frac{\alpha^2+\beta^2}{\Psi}\right)^{1/2}(\Psi T^2(\alpha^2+\beta^2))^{1/2} \in \mathcal{L}^\infty(EH). \tag{56}$$

We infer from the boundedness of W (cf (15)) and of $T(\alpha^2+\beta^2)$ that the function $(\alpha^2+\beta^2)(r)$ is locally Lipschitzean near the event horizon (see (17)).

For convenience, let us again denote by Z the function $T\sqrt{A}$. The results from fact 3.2.1, (56) and (17) may be combined altogether to deduce that whenever $r \in (\rho, \rho+\epsilon)$, there exist constants a and b (depending on the arbitrary $\epsilon > 0$, and whose signs are *a priori* unknown) satisfying

$$b(\alpha^2+\beta^2)' > \frac{2}{r^2Z} - \frac{2m}{r^2\sqrt{Z}}\frac{(\alpha^2+\beta^2)}{\sqrt{A}} > a(\alpha^2+\beta^2)'. \tag{57}$$

Furthermore, using our findings from lemma 4.1.3, and the fact that $T^{-1}(\rho) = 0$, we obtain from (29) that

$$\lim_{r\searrow\rho}\left\{(T^{-2})'' - \frac{2}{r^2Z} + \frac{2m}{r^2\sqrt{Z}}\frac{(\alpha^2+\beta^2)}{\sqrt{A}}\right\} = 0. \tag{58}$$

Altogether, (57) and (58) show that on $(\rho, \rho+\epsilon)$:

$$b(\alpha^2+\beta^2)' > (T^{-2})'' > a(\alpha^2+\beta^2)'.$$

We may integrate the latter on $(\rho, r) \subset (\rho, \rho+\epsilon)$, and invoke lemmas 4.1.2 and 4.1.3 to obtain

$$bT(\alpha^2+\beta^2) > 2(T^{-1})' > aT(\alpha^2+\beta^2). \tag{59}$$

Applying (56) to (59) therefore shows that $(T^{-1})'$ remains bounded near the event horizon.

Observe next that (17) gives the inequality:

$$|(T(\alpha^2+\beta^2))'| = \left|-(T^{-1})'T^2(\alpha^2+\beta^2) + \frac{2}{Z}T^2\left[0, \frac{W}{r}, -m\right]\right| \lesssim T^2(\alpha^2+\beta^2), \tag{60}$$

where we have used fact 3.2.1 along with (15).

Finally, (35) combined to (60) yields at once that $T(\alpha^2+\beta^2)$ has a finite limit on the horizon. If that limit were non-zero, we would have (using fact 3.2.1)

$$T^2(\alpha^2+\beta^2) \gtrsim \frac{1}{\sqrt{A}}.$$

This is however impossible, since the left-hand side of this equation is locally Lebesgue integrable near the horizon (as seen in (35)), while the right-hand side is not (cf the proof of corollary 4.1.3). Whence

$$\lim_{r\searrow\rho} T(\alpha^2+\beta^2) = 0.$$

Equivalently, because $T\sqrt{A}$ is bounded from below on and around the event horizon, we obtain the desired

$$\lim_{r\searrow\rho} \frac{\alpha^2+\beta^2}{\sqrt{A}} = 0.$$

□

4.3. The extreme Reissner–Nordström horizon

As stated in theorem 4.0.1, the unknown metric coefficient functions $A(r)$ and $T(r)$ admit, around the event horizon $r = \rho$, the local expansions

$$A(r) = \frac{1}{\rho^2}(r-\rho)^2 + \mathrm{o}((r-\rho)^2) \tag{61}$$

$$T(r) = T_0(r-\rho)^{-1} + \mathrm{o}((r-\rho)^{-1}), \tag{62}$$

where T_0 is a positive constant.

The goal of this section consists in deriving similar local expansions for the remaining unknowns of our problem, namely $\alpha(r)$, $\beta(r)$, $W(r)$ and $\Psi(r)$.

Recall that the function $\Psi(r)$ belongs to $\mathcal{C}^1(EH)$. Whence, it follows from (62) that ΨT is locally bounded around the horizon. Using the results from propositions 4.2.2 and 4.2.3, we deduce

$$\lim_{r\searrow\rho} \Psi T^2(\alpha^2+\beta^2) = 0. \tag{63}$$

Accordingly, taking limits on both sides of (20) implies

$$\lim_{r\searrow\rho} A(W')^2 = 0. \tag{64}$$

In turn, letting $r\searrow\rho$ on both sides of (8) now shows that

$$\lim_{r\searrow\rho}\left\{\frac{1}{\widehat{e}^2}(rT\sqrt{A}\Psi')^2+\left(\frac{1-W^2}{er}\right)^2\right\}=1. \tag{65}$$

In particular, since $T\sqrt{A}$ and Ψ' are continuous on the horizon, (65) implies the Yang–Mills 'potential' function $W(r)$ has a finite limit as $r\searrow\rho$.

Lemma 4.3.1. $\lim_{r\searrow\rho} W(r)\in\{-1,0,+1\}$.

Proof. Observe that the Yang–Mills equation (10) may be recast in the form

$$(\sqrt{A}W')'=\frac{(W^2-1)W}{r^2\sqrt{A}}+e^2\frac{T}{r\sqrt{A}}\alpha\beta+\sqrt{A}\frac{T'}{T}W'. \tag{66}$$

Fact 3.2.1 shows that

$$\sqrt{A}\frac{T'}{T}\simeq\frac{T'}{T^2}\simeq(T^{-1})',$$

which is known to be bounded near the black hole (cf proof of proposition 4.1.3). Whence the last summand on the right-hand side of (66) is locally integrable (although not necessarily absolutely integrable) near the event horizon. The same is true for the second summand, as explained in (34). Finally, the left-hand side of (66) is also integrable locally around the horizon, as shown by (64). We therefore conclude that the remaining term

$$\frac{(W^2-1)W}{\sqrt{A}}$$

must be integrable around the event horizon. However, since $A^{-1/2}\notin\mathcal{L}^1(EH)$ (cf the beginning of the proof of lemma 4.1.3), and since W is continuous on and around the horizon, we conclude at once that

$$\lim_{r\searrow\rho}W(r)\in\{-1,0,+1\},$$

as claimed. □

Let us now return to (65). Suppose that $\Psi'(\rho)=0$. Incorporating the result from lemma 4.3.1, we see that $W(\rho)$ must be equal to zero, and in that case,

$$e=\frac{1}{\rho}. \tag{67}$$

A priori, the Yang–Mills coupling constant e and the location of the black hole ρ are arbitrary, while their numerical values may always be adjusted via a rescaling of units. It is thus legitimate to assume that condition (67) is too restrictive to be of any interest. We shall accordingly henceforth focus only on the case when $\Psi'(\rho)\neq 0$. Interestingly enough, one will note that $\Psi'(r)$ *cannot* be equal to zero in the electromagnetic case with gauge $U(1)$. This can be seen by setting $W(r)\equiv 1$ in our problem and considering (65). Altogether, we have shown that around the horizon the following expansion holds:

$$\Psi(r)=\Psi_0(r-\rho)+\mathrm{o}((r-\rho)),\qquad\text{where}\quad\Psi_0\neq 0. \tag{68}$$

We consider next the Dirac system (6)–(7). It may be recast in the form

$$\frac{\mathrm{d}}{\mathrm{d}q}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}-W/r & m+\Psi T\\ m-\Psi T & W/r\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \tag{69}$$

where

$$q(r) := \int_r^{2\rho} A^{-1/2}(s)\,\mathrm{d}s.$$

Owing to (61), we note that $\lim_{r\searrow\rho} q(r) = \infty$.

Combining (62) to (68) and invoking lemma 4.3.1 show that the matrix appearing on the right-hand side of (69) remains bounded as $r \searrow \rho$, i.e. as $q \nearrow \infty$. It also clearly depends continuously on the variable q. We may thus call upon Grönwall's lemma (cf [7]) to conclude that $(\alpha\beta)$ decays exponentially in q, as $q \nearrow \infty$. Equivalently, comparing the local expansion (61) to the definition of $q(r)$, we find that locally around the event horizon, the functions $\alpha(r)$ and $\beta(r)$ must satisfy

$$\alpha(r) \simeq \alpha_0(r-\rho)^k \qquad \text{and} \qquad \beta(r) \simeq \beta_0(r-\rho)^k, \tag{70}$$

for some positive constant k, and non-zero constants α_0 and β_0.

From propositions 4.2.2 and 4.2.3 along with (61), it follows immediately that $k > \frac{1}{2}$. To obtain more precise information about the value of k, we may substitute the local expansions (61), (62), (68) and (70) into the Dirac system (6)–(7). This easily yields the relationship

$$k^2 = W(\rho)^2 + \rho^2\big(m^2 - \Psi_0^2 T_0^2\big). \tag{71}$$

Using (65) and the result from lemma 4.3.1, we refine (71) to

$$k^2 = \begin{cases} 1 + \rho^2(m^2 - \widehat{e}^2), & \text{if } W(\rho) = \pm 1 \\ (\widehat{e}/e)^2 + \rho^2(m^2 - \widehat{e}^2), & \text{if } W(\rho) = 0. \end{cases} \tag{72}$$

The local expansions and parameter relations given above enabled us to produce numerical simulations in the extreme Reissner–Nordström case. The system is implemented starting at $r = \rho + \varepsilon$, and the routine is run for decreasing values of $\varepsilon > 0$. It is not possible, however, to initiate the computations at $r = \rho$, since the system is singular at that point. We verified that the numerical simulations obtained by this method are stable. Due to the large number of parameters involved in the problem, it takes a substantial amount of simulations in order to observe the emergence of 'patterns'. For that reason, we have chosen not to include in this paper any of our numerics. The simulations have shown that there is no globablly normalizable solutions of (6)–(11). We observed that either a singularity develops in the metric in finite radius (i.e. $A(r_0) = 0$ for some $\rho < r_0 < \infty$), or else the metric fails to be asymptotically Minkoswkian (cf section 1) and $A(r)$ grows out of bound. Thus our numerics confirm that EDYMM has no globally normalizable black-hole solutions.

5. Conclusion

For an extreme Reissner–Nordström background, it is rigorously established in [16] that the solutions of the coupled Einstein–Dirac equations all violate the normalization condition (14). It would thus be interesting to know whether the introduction of a Yang–Mills field could allow the formation of black-hole solutions by making the integral in (14) finite. Similarly to the electroweak case treated here, when only electromagnetic forces are taken into account, Finster *et al* have numerically shown that the Einstein–Dirac–Maxwell equations have no globally normalizable black-hole solutions (cf [12]). Unfortunately, there is to this day no rigorous argument supporting these numerical findings, even in the simpler electromagnetic case. This is a difficult analytic problem, because one must control the global behaviour of the solution from the sole knowledge of its local behaviour near the event horizon. We have extensively explored possible ways to solve this problem, but our investigations and attempts have so far been fruitless.

It is interesting to note that when only weak forces are considered (gauge $SU(2)_{\mathrm{Magnetic}}$) an argument similar to our proof shows that there is no globally normalizable black-hole solutions, and that the extreme Reissner–Nordström case is not distinguished, thereby contrasting with the electromagnetic and electroweak couplings. A proof of this result may be found in [14]. Careful readers will, however, note that in [14] Finster *et al* impose *de facto* on the metric coefficient $A(r)$ slightly more than the announced local monotonicity. More precisely, it is implicitly supposed at the end of the proof of lemma 3.5 that $\lim_{r\searrow\rho}(r-\rho)^{-1}A(r)$ exists. Nevertheless, the argument presented in [14] is founded on correct ideas.

The $SU(2)_{\mathrm{Magnetic}}$ system is obtained from the electroweak system (6)–(11) by setting $\widehat{e}=0$ and $\Psi(r)\equiv\omega$ (cf (12)). Finster *et al* demonstrate in [14] that the constant ω must be zero. This fundamental fact allows them to rule out all black-hole solutions with non-identically vanishing spinors, including the local extreme Reissner–Nordström case which we have encountered. In contrast, in the $U(1)_{\mathrm{Coulomb}}$ and in the $U(1)_{\mathrm{Coulomb}}\times\mathrm{SU}(2)_{\mathrm{Magnetic}}$ settings, the local extreme Reissner–Nordström metric emerges as a distinguished case, and it is not known whether the total non-relativistic energy ω vanishes. The presence of the Coulomb potential $\Psi(r)$ makes the analysis of these systems far more complex than their $SU(2)_{\mathrm{Magnetic}}$ counterpart.

Nonetheless, owing to the results from [12] and [14], it is legitimate to conjecture that our numerical evidence faithfully render account of a general fact, namely the non-existence of black-hole solutions for this class of coupling problems.

Let us finally point out that our findings place strong restrictions on the behaviour of possible black-hole solutions (should any exist) near the horizon. The condition $k>1/2$ shows the wavefunctions decay so fast in as $r\searrow\rho$ that they have no influence on the asymptotic form of the metric and of the electroweak fields. We have indeed seen in theorem 4.0.1, lemma 4.3.1 and (68) the latter behave near the black hole like a vacuum solution. The restriction to the extremal Reissner–Nordström case means that the electric repulsion due to the charge of the black hole is sufficiently large to balance gravity, thereby preventing the Dirac particle from 'sinking into' the singularity. As observed by Finster *et al*, this is certainly not the physical situation which one would expect in the gravitational collapse of a stellar body in the universe. Nevertheless, as explained in section 4, extreme Reissner–Nordström black holes may be considered as the asymptotic states of black holes emitting Hawking radiations. This renders interesting the problem of deciding whether our local expansions can lead to global solutions of EDYMM.

References

[1] Adler R, Bazin M and Schiffer M 1975 *Introduction to General Relativity* 2nd edn (New York: McGraw-Hill)

[2] Bartnik R 1992 The spherically symmetric Einstein–Yang/Mills equations *Relativity Today: Proc. 3rd Hungarian Relativity Workshop (1989)* ed Z Perjés (Tihany, Commack, NY: Nova Science) pp 221–40

[3] Bartnik R 1997 The structure of spherically symmetric $SU(N)$ Yyang–Mills fields *J. Math. Phys.* **38** 3623–38

[4] Bartnik R and McKinnon J 1988 Particle-like solutions of the Einstein–Yang/Mills equations *Phys. Rev. Lett.* **61** 141–4

[5] Bernard Y The coupling of gravitation to $SU(N)$ and $U(1)\times\mathrm{SU}(N)$ quantum gauge fields, in preparation (expected submission 12/2005)

[6] Carter B 1966 The complete analytic extension of the Reissner–Nordström metric in the special case $e^2=m^2$ *Phys. Lett.* **21** 423–4

[7] Coddington E and Levinson N 1955 *Theory of Ordinary Differential Equations* (New York: McGraw-Hill)

[8] Dafermos M 2003 On 'time-periodic' black-hole solutions to certain spherically symmetric Einstein-matter systems *Commun. Math. Phys.* **238** 411–27 (*Preprint* gr-qc/0210025)

[9] Dubrovin B A, Fomenko A T and Novikov S P 1985 *Modern Geometry—Methods and Applications. Part II: The Geometry and Topology of Manifolds* (Berlin: Springer)

[10] Finster F 1998 Local $U(2, 2)$ symmetry in relativistic quantum mechanics *J. Math. Phys.* **39** 6276–90

[11] Finster F, Smoller J and Yau S-T 1999 Particle-like solutions of the Einstein–Dirac equations *Phys. Rev. Lett.* D **59**

[12] Finster F, Smoller J and Yau S-T 1999 Non-existence of black-hole solutions for a spherically symmetric static Einstein–Dirac–Maxwell system *Commun. Math. Phys.* **205** 249–62

[13] Finster F, Smoller J and Yau S-T 1999 Particle-like solutions of the Einstein–Dirac–Maxwell equations *Phys. Lett.* A **259** 431–6

[14] Finster F, Smoller J and Yau S-T 2000 The interaction of Dirac particles with non-Abelian gauge fields and gravity – black holes *Michigan Math. J.* **47** 199–208

[15] Finster F, Smoller J and Yau S-T 2000 The interaction of Dirac particles with non-Abelian gauge fields and gravity—bound states *Nucl. Phys.* B **584** 387–414

[16] Finster F, Smoller J and Yau S-T 2000 Non-existence of time-periodic solutions of the dirac equation in a Reissner–Nordström black hole background *J. Math. Phys.* **41** 2173–94

[17] Finster F, Smoller J and Yau S-T 2002 Non-existence of black-hole solutions to static, spherically symmetric Einstein–Dirac systems—a critical discussion *Preprint* gr-qc/0211043

[18] Finster F, Smoller J and Yau S-T 2000 Absence of static spherically symmetric black-hole solutions for Einstein–Dirac–Yang/Mills equations with complete fermion shells *Adv. Theor. Math. Phys.* **4** 1231–57

[19] Forgács P and Manton N S 1980 Spacetime symmetries in gauge theories *Commun. Math. Phys.* **72** 15–35

[20] Gross F 1993 *Relativistic Quantum Mechanics and Field Theory* (New York: Wiley)

[21] Hawking S W 1975 Particle creation by black holes *Commun. Math. Phys.* **43** 199–220

[22] Künzle H P 1991 SU(N)-Einstein–Yang/Mills fields with spherical symmetry *Class. Quantum Grav.* **8** 2283–97

[23] Künzle H P 1994 Analysis of the static spherically symmetric SU(N) Einstein–Yang/Mills equations *Commun. Math. Phys.* **162** 371–97

[24] Nordström G 1918 On the energy of the gravitational field in Einstein's theory *Proc. K. Ned. Akad. Wet.* **20** 1238–45

[25] Reissner H J 1916 Über die eigengravitation des elektrischen feldes nach Einsteinschen theorie *Ann. Phys.* **59** 106–20

[26] Sakurai J J 1967 *Advanced Quantum Mechanics* (Reading, MA: Addsion-Wesley)

[27] Smoller J and Wasserman A G 1995 Regular solutions of the Einstein–Yang/Mills equations *J. Math. Phys.* **36** 4301–23

[28] Strautmann N and Zhou Z 1990 Instability of the Bartnik–McKinnon solutions *Phys. Rev. Lett.* B **237** 353–6

[29] Wald R 1984 *General Relativity* (Chicago, IL: The University of Chicago Press)

[30] Witten E 1977 Some exact multipseudoparticle solutions in classical Yang–Mills theory *Phys. Rev.* D **38** 121–4

[31] Yang C and Mills R 1954 Conservation of isotopic spin and isotopic gauge invariance *Phys. Rev.* **96** 191–5