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INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

STUDY OF THE USE OF NON-SIMULTANEOUS MEASUREMENTS IN TRIANGULATION

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LIST OF SYMBOLS

a - subscript indicating any station whose bearing reading is old

A - closest integer to $D_T \sin(\theta_2 - \phi)$

A_1, A_2 - regions of integration in the t_2, t_3 plane

A_U - smallest region within which the measured target position occurs with a specified probability

B - closest integer to $D_T \sin(\theta_2 - \phi)$

C - closest integer to $|\gamma_a|$

C_1, C_2, C_3 - constants

C_4, C_5, C_6 - constants

d_i - polar coordinate (distance) of Station i

d_{ij} - distance from Station i to the intersection of the bearing lines from Stations i and j

d_{ti} - polar coordinate (distance) of the target at the time the bearing measurement at Station i is performed

D - reference distance in the triangulation system (chosen to be the distance of each station from the center of the equilateral triangle formed by three stations)

D_N - distance that the target moves in the time, T, normalized with respect to D and $\sigma_{\epsilon B}$

D_T - distance that the target moves in the time, T, normalized with respect to D

E - position error

E_C - variable used to represent the square of E_N

E_D - position error normalized with respect to D

E_N - position error normalized with respect to D and $\sigma_{\epsilon B}$

E_o - an arbitrary value of E_N

E_R - computer address corresponding to E

E_s - transformation of E_c

E_x - expected value of E_D

E_{10} - position error which is exceeded only 10% of the time

K - a multiplier which is a function of C

K_1, K_2, K_3 - constants used to transform E_c into E_R

M - the number of targets under surveillance

n - the number of bearing measurement stations

N - total number of intersections of pairs of bearing lines

o - a subscript indicating a particular value of the variable to which it is affixed

p - a relative storage address

P - discrete probability distribution for the variable indicated as a subscript

$P(\ )$ - probability of a particular event, or the cumulative probability distribution for an event indicated within the parentheses

$P_c(\ )$ - discrete, conditional probability distribution for the variable and condition listed within the parentheses

$Q_{ij}, Q_{j\ell}, Q_{k\ell}$ - symbols used to represent trigonometric expressions

$Q'_{ij}, Q'_{j\ell}, Q'_{k\ell}$ - approximations of Q_{ij} , $Q_{j\ell}$ and $Q_{k\ell}$, respectively

r_{ij} - polar coordinate (distance) of the intersection of the bearing lines from Stations i and j

s - subscript indicating the smallest possible value of a variable

S - ratio of the standard deviation of U to the standard deviation of ϵ_B

t_i - age of the bearing reading from Station i at the time of its use

T - T_i when all T_i are the same

LIST OF SYMBOLS (CONT'D)

T_i - time between consecutive measurements of the bearing of a particular target at Station i

u - discrete variable representing ϵ_1

v - discrete variable representing ϵ_2

v' - discrete variable representing ϵ_{B2}

v'' - discrete variable representing ϵ_{t2}

V - speed of the target

w - discrete variable representing ϵ_3

w' - discrete variable representing ϵ_{B3}

w'' - discrete variable representing ϵ_{t3}

W_{ij} - weighting factor assigned to the intersection of a pair of bearing lines from Stations i and j

$\bar{x}$ - Cartesian coordinate of the centroid or weighted centroid of the N intersections

x_{ij} - Cartesian coordinate of the intersection of the bearing lines from Stations i and j

$\bar{y}$ - Cartesian coordinate of the centroid or weighted centroid of the N intersections

y_{ij} - Cartesian coordinate of the intersection of the bearing lines from Stations i and j

α_1, α_2 - angles between the bearing lines from Stations 1 and 2, respectively, and the line joining the stations

γ - a combination of other variables

Δ - a small change in the variable which follows it

ϵ - a value of ϵ_i

ϵ_{Bi} - error in the bearing measurement of Station i

ϵ_{Bim} - individual components of ϵ_{Bi}

ϵ_i - error in the bearing reading at Station i

LIST OF SYMBOLS (CONT'D)

ϵ_{ti} - component of the error in the bearing reading at Station i due to time delay and target motion

θ_i - polar coordinate (angle) of Station i

θ_{ij} - polar coordinate (angle) of the intersection of the bearing lines from Stations i and j

λ - an arbitrary constant

π - 3.14159. . .

ρ, ρ' - probability density-function for the variable indicated as a subscript

$\rho_c(\ )$ - conditional probability density-function for the variable and condition listed within the parentheses

σ - standard deviation of the distribution for the variable indicated as a subscript

$\sum_{ij}$ - a sum performed over all permutations of the values of i and j from 1 through n

$\sum_{ijkl}$ - a sum performed over all permutations of the values of i, j, k, and l from 1 through n

$\sum_{v_0''}$ - a sum over all possible values of v_0''

τ - arbitrary constant with the dimensions of time

τ_i - time required to perform an individual bearing measurement at Station i

ϕ - polar coordinate (angle) of a target traveling a straight path

ϕ_i - polar coordinate (angle) of a target at the time the bearing measurement at Station i is performed

CHAPTER I

DESCRIPTION OF THE PROBLEM

A need exists for electronic systems which can determine by triangulation, the position of many moving targets which exist simultaneously. Triangulation is the general method of calculating the position of a target from measurements of the angle of direction (bearing) to the target from a set of at least two known points. When all of the bearing measurements on a particular target are performed simultaneously, the error in the computed position of the target is unaffected by target motion. When the bearing measurements are not performed simultaneously, an error is introduced because of the motion of the target during the time between the measurements used in the calculation of target position.

When many targets are to be under surveillance, simultaneous bearing measurement, although desirable, is not always a practical design requirement. In order to obtain simultaneous bearing measurement at several bearing-measurement stations, the selection of a particular target must be made in advance at some central point within the system and then the selected target must be designated to each station. The selection may be made on the basis of an expected bearing angle at each station, a set of electromagnetic parameters of the expected signal, or both. When many targets are under surveillance, expected bearing angle will not uniquely define a target. For each signal received, the use of electromagnetic parameters to define a selected target requires that a measurement of the parameters be performed and the signal discarded if it is not the selected one. The rate at which data is obtained on each target by such a process is substantially less than the rate which would result if all the received information were used. Data rate is an important characteristic of a system which must handle many fast-moving targets. Consequently, triangulation systems which do not require simultaneous bearing measurements are of interest.

The use of non-simultaneous bearing measurements requires that the bearing-measurement data, as well as those signal characteristics which are used to distinguish among them, be stored for future use and that the triangulation system possess a capability for selecting the proper data for use in the calculation of target position.

A decision to use non-simultaneous bearing measurements in a triangulation system should be based on whether or not the additional error caused by their use can be tolerated. The study herein is aimed primarily toward providing a measure of this error in terms of normalized triangulation-system parameters. The measure is in the form of probability distributions for the magnitude of the error in the calculated position of the target, hereafter called position error. The results of the investigation provide a means for deciding whether or not to use non-simultaneous measurements and, in addition, provide a way of selecting some of the system design parameters. The results provide a way of evaluating existing and proposed systems as well as modifications to systems.

Because the study is concerned with many, simultaneous, moving targets, a situation chiefly associated with radio direction-finding networks, the details and examples which are considered herein are limited to the direction-finding case. This does not preclude the use of the methods employed nor the application of the results to optical, infrared, or any other type of system which uses triangulation.

The study described herein is restricted to the two-dimensional case, i.e., altitude has not been considered. The position error has been investigated in general for a triangulation system made up of an arbitrary number of bearing-measurement stations. Consideration is given to weighting the information from which the target position is calculated according to the expected accuracy of the information. Numerical analysis has been performed only for the case of three bearing-measurement stations, in a special case which is generally useful. The techniques used in the numerical analysis are such that they may be extended easily to other situations.

CHAPTER II

BACKGROUND INFORMATION

Triangulation is the general method of estimating the location of a target (any desired point) by measuring the angle of direction of the target from a set of known points. Estimation of the location of a target from the measured data may be performed in a variety of ways. The simplest procedure is the use of a plotting board on which the intersection of bearing lines from two known points (bearing-measurement stations) is selected as an estimate of the target location. It is only an estimate because the measurement cannot be performed without error. Both manual and automatic plotting boards have been used, or, of course, direct, trigonometric calculation can be substituted for them.

When more than two bearing-measurement stations are used, the bearing lines seldom intersect at a point. Using a plotting board in this case, a human observer may make an estimate of the location of the target on the basis of his experience or with the assistance of a mechanical aid, called a "spider", with which the set of intersections formed by the set of bearing readings is adjusted into a smaller region by adding a correction angle of equal magnitude to each bearing reading. Barfield(2) describes an interesting electromechanical device with which the most probable target location can be obtained. If exactly three bearing-measurement stations are used, an observer may compare the shape and location of the triangle formed by the bearing lines with the triangles prepared by Stansfield(19), which show the location of the "most probable point". Estimation of the location of the target may also be accomplished by use of digital computing apparatus. A computer may be designed to perform the estimate by making use of any one of a variety of criteria, and can also be used to compensate the measured data for known systematic error.

The general problems of radio direction-finding are extensively discussed by Bond(3), Keen(14), and Ross(17). Ross conveniently divides the problems into three groups, (1) those dealing with the instrument itself, (2) those dealing with phenomena occurring in the course of propagation of the waves, and (3) those concerned with the interpretation of the bearings once the readings have been obtained. The study described herein is concerned with the interpretation of bearing readings, and, therefore, some attention must be given to the character of the error in the readings.

Error in bearing measurement and means for reducing it has received considerable attention.[1] However, reliable measurements of the error due to individual sources of error are difficult to obtain because of the large number of such individual sources and because of their dependence upon many parameters. Ross(18) classifies errors into four groups: instrumental errors, site errors, propagation errors, and observational errors, and describes a method of estimating "a priori" the probable error of a given bearing.

Compensation for systematic error which can be measured in bearing-measurement equipment is accomplished by calibration. For the random error which remains, there is general agreement that, in practice, it is described by a probability distribution which differs very little

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[1] See, for example, Bowen(4), Horner(11), and a collection of papers devoted to direction-finding in The Journal of the Institution of Electrical Engineers, Vol. 94, Part III A, London; 1947.

from the normal or Gaussian distribution. Ross[18] cites the results of experimental trials which agree with the results predicted on the basis of a normal distribution. In studies of position error, the normal distribution has been used[8,10,19].

Position error is studied for two reasons: (1) to evaluate the confidence which can be placed in the estimated target locations obtained from particular triangulation systems as a function of the location of the target with respect to the bearing-measurement stations and as a function of other parameters and (2) to study the procedures by which the probability of error in each bearing reading can be used to obtain the best estimate of a target location. The evaluation will depend, of course, on the estimation procedure which is used. Stansfield[19] points out that the problem of the determination of the most probable point given by a set of position lines of unequal weight was considered by d'Ocagne[16] in 1893. Stansfield[19] developed an expression for the conditions which the coordinates of the estimated target position must satisfy in order to be a best estimate of the true target position by the principal of maximum likelihood. He considers the case of two dimensions and an arbitrary number of bearing-measurement stations, assuming a normal distribution for the error in each of the bearing measurements with the absence of systematic error. He also presents a geometric interpretation of his conditions for the case of three bearing-measurement stations. Even in this case, the application of the conditions to the problem of selecting the coordinates of the estimated target position is not simple. Stansfield also examines the probability distribution which describes the position error when his criterion is used. He presents graphs of the fifty percent-probability contours (ellipses)[1] as a function of the target location for the special cases of two, three, and four direction-finding stations, when the standard deviation of each bearing reading is two degrees.

Harkin(10) has considered the error in the three-dimensional triangulation problem with a normal, circular, bivariate distribution for each of the bearing lines. He does not consider a best estimate of target position but weights each of the bearing lines equally.

The studies of the position error which are referred to above consider that either the bearing measurements are performed simultaneously or that any motion of the target is negligible. As pointed out in Chapter I, the use of simultaneous measurement, although desirable, is not always practical. The author(8) has investigated the position error when non-simultaneous measurements are used for the case of two bearing-measurement stations, with the rather loose assumption that the error in a bearing reading because of its age and target motion can be approximated by a normal distribution with a mean of zero. A more realistic study requires the use of non-normal probability distributions with non-zero mean values.

The use of probability distributions which are not normal, or the combination of independent, random variables by a process other than addition often requires the use of numerical methods. Such is the case in the study described herein. A variety of numerical methods, including methods of sampling which are discussed below, are available for

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[1] If circular probability contours are desired, they may be obtained by use of a table of "Q-Functions" such as those prepared by Marcum(15).

performing convolution, the integration process by which the probability distribution for the sum of independent random variables is obtained.[1] When random variables are combined by a process other than addition, more complicated procedures are usually necessary to obtain the resultant distribution. The resultant distribution can be expressed in the form of an integral depending on a parameter, but as Kaplan[2] points out, it can easily happen that the integral cannot be expressed in terms of elementary functions, even when the distributions which describe the random variables are expressed in simple equation form. Teichroew(21) points out that even with the use of high speed computers, numerical integration is not always practical. "It is just as impractical to use a high-speed computer for a year to do an integration as it is to do it by hand in 10^5 years."

"Distribution Sampling" is another numerical method of obtaining the resultant distribution in which "the basic problem is expressed in probability terms and sampling has been used to solve it; the integral formulation is not necessary for the sampling procedure."[3] The value of each of the independent, random variables is sampled at random according to the probability distribution which describes it and the corresponding value of the dependent variable is calculated. The sampling process is repeated many times and the set of values obtained for the dependent variable is ranked in order of magnitude. The probability that the

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1 See, for example, Tustin(23) and Truxal(22).

2 Kaplan(13), p. 218.

3 Teichroew(21), p. 3.

dependent variable is less than some value is approximated by the ratio of the number of times a smaller value is obtained to the number of times the sampling process is employed. Teichroew(21) points out that this method was introduced by "Student"(20) in 1908. More recently this method has been used to evaluate definite integrals, to solve differential equations, and to invert matrices by analogy, i.e., by approximating the solution of a probability problem which can be formulated in the same way as the non-probability problem at hand. The use of distribution sampling in this application has been given the name "Monte Carlo Methods"[1], a name which has carried over to any use of distribution sampling.

Although the convergence of the approximate distribution obtained by Monte Carlo methods to the true distribution is relatively rapid in the vicinity of the mean of the distribution, convergence is quite slow at the "tails" of the distribution because only a small fraction of the total number of sets of samples yields values of the dependent variable in this region. Kahn(12) discusses six techniques that can be used with Monte Carlo methods to improve the accuracy of approximation for a given number of samples. Two of these techniques, systematic sampling and stratified sampling, are applicable to the approximate solution for a complete distribution, a major part of the study described herein.

In systematic sampling, the values of one of the independent variables in a multi-variable sampling problem are not determined by chance, but instead each of the values of this one variable is distributed among the total number of samples according to the probability associated

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[1] A collection of papers on Monte Carlo methods has been published in book form in Symposium on Monte Carlo Methods, edited by Meyer, H. A., John Wiley and Sons, New York; 1956.

with each value. This technique does not lead to substantial improvements in accuracy, but as Kahn points out, "it ordinarily does not cost anything to apply this technique, so that there is no point in not using it."[1]

In stratified sampling, the sample space is divided into non-overlapping sub-sets, for each of which the conditional probability is calculated. A representative sample of the same size is then taken from each sub-set, and the results from each sub-set are combined according to the conditional probability for each.[2]

The method used in the study herein to calculate an approximate resultant probability distribution cannot properly be called distribution sampling or a Monte Carlo method. However, the method used herein must be compared to Monte Carlo methods because of the current acceptance and interest in them.

In the method used herein, which might be called "complete systematic sampling", the probability space is divided into a large number of non-overlapping sub-sets by dividing each of the dimensions of the space into a set of non-overlapping intervals. The possible values of the independent, random variables (dimensions) are grouped and approximated by the value at the center of the interval, i.e., the probability that the value of the variable is within an interval is assigned to the value at the center of the interval. Each of the large number of sub-sets is sampled once, i.e., all of the possible combinations of intervals for all of the independent,

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1 Kahn(12), p. 154.

2 See Albert(1), p. 44 and Teichroew(21), pp. 17-20.

random variables are considered in a systematic way. Although the number of sub-sets considered in this way may be unusually large,[1] certain efficiencies are available. The random sampling process is eliminated and, therefore, the generation, storage, and use of random numbers is unnecessary. When a systematic selection procedure is used, the time-consuming process of computing the value of the dependent variable for each set of randomly selected values of the independent variables may be reduced to a simple calculation based on the value obtained on the previous trial. Such is the case in the study described herein.

The procedure described above is an extension of both systematic sampling and stratified sampling to the point where they are identical. It is an extension, applied in grand scale to a digital computer, of a simple technique for combining discrete probability distributions by considering all of the possible outcomes. The computer provides a means for systematically considering all of the possible outcomes as well as for performing the necessary calculations.

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[1] In the three-dimensional probability space considered herein, the number of sub-sets used was in excess of 125,000.

CHAPTER III

DESCRIPTION OF THE ANALYSIS AND THE ASSUMPTIONS

The position error in a triangulation system consisting of n bearing-measurement stations was investigated in general for any arrangement of the stations with respect to the target. An expression for the magnitude of the position error in a single measurement was obtained as a function of the geometry of the situation and the error in each bearing reading. In developing an expression for position error, it was assumed that the calculated position of the target is the centroid[1] of the set of all intersections formed by all the bearing readings taken two at a time. The centroid is used because of its suitability for high-speed, automatic computation, a necessity for triangulation systems that are used in determining the positions of many, high-speed targets. The use of a weighted centroid to reduce the expected value of the position error is also considered.

The probability distribution for the magnitude of the position error is obtained from the probability distributions for the error in the bearing reading at each of the bearing-measurement stations. Bearing-reading error is separated into two components: (1) error in the bearing-measurement itself, and (2) error due to motion of the target during the time between the measurements used in the calculation of target position. Error in the bearing measurement is assumed to be normally distributed for the reasons described in Chapter IV. The age of the measurement is

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[1] "Centroid" means the center of mass. In this case each intersection has the same mass. A "weighted centroid" means that unequal weights (or masses) may be assigned to each intersection.

assumed to be uniformly distributed over a finite interval because the triangulation system is assumed to operate in the following way.

Each bearing-measurement station is operated independently and targets are selected for measurement on the basis of their availability. When many targets are available simultaneously, measurements are performed by sequencing through them in any orderly fashion. At each central point in the system at which target position is calculated, bearing reports and target-identification data from each bearing-measurement station are stored according to the time of arrival of such reports. Each time a new report is received from one station, the most recent bearing information on the same target is selected from the storage associated with each of the other bearing-measurement stations and is used to calculate the position of the target. Therefore, in each calculation, one bearing reading is new; the age of each of the other bearing readings depends upon the rate at which bearing readings are performed at each of the stations. If T_i denotes the time between consecutive measurements of the bearing of a particular target at Station i, then the age of the bearing measurement at the time it is used is described by a probability distribution over the interval from zero to T_i. Because the bearing-measurement stations are operated independently, the distribution for each station is a uniform one. In addition, the age of each of the bearing readings used in the calculation of target position is independent of the others.

As explained below, the general method selected for numerical evaluation of probability distributions for position error is not based on the use of such simple probability distributions as the normal distribution for error in bearing measurement and the uniform distribution for age of the measurement. These distributions were selected because of their general utility and because they are appropriate for the triangulation system in question, and not for reasons of convenience in numerical analysis. The numerical method used is applicable to any theoretical or empirical probability distribution for the components of error, provided that the components are independent.

For use in the numerical analysis, the general expressions for the position error in terms of the error in bearing readings were approximated for the case of small bearing-reading error. The approximate expressions are simpler in form and, consequently, more convenient to use in numerical analysis. The method used, however, does not require that the approximation be made.

The special case of three bearing-measurement stations was selected for numerical analysis because it is an arrangement that is used frequently and because this arrangement is well suited for use in a system designed to cover a large area and made up of three-station units, such as those illustrated in Figure 3-1. The study of the three-station arrangement has been restricted to a study of a symmetrical arrangement with the stations located at the vertexes of an equilateral triangle because this arrangement or those differing only slightly from it are generally used. For this symmetrical arrangement, the variation in the position error is demonstrated to be small for a target located anywhere throughout a region surrounding the center which contains at least half of the area of the triangle. For this reason the numerical analysis has been restricted to this region. A number of probability

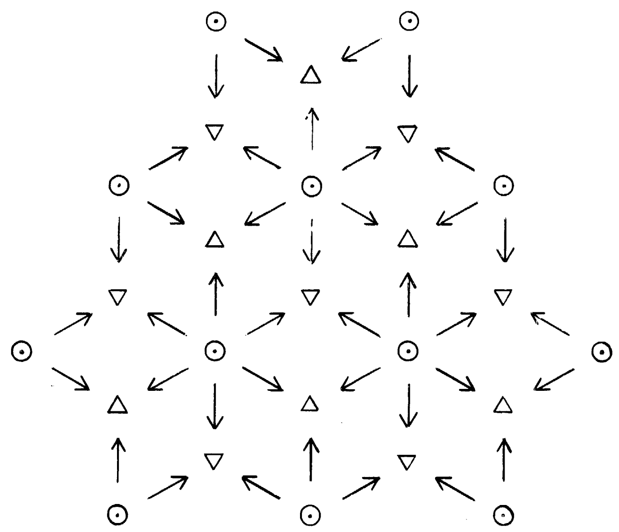

Figure 3-1 Triangulation System Made Up of Three-Station Units

LEGEND:

⊙ bearing measurement station

△ central station for calculation of target position

→ information flow

distributions for the normalized position error have been obtained in this case for several values of normalized triangulation-system parameters. The results are valid for any choice of the variance of the bearing-measurement error, provided that it is small, and for any choice of spacing between the bearing-measurement stations. Separate probability distributions were obtained for several values of the error due to target motion and age of bearing readings. This error is normalized with respect to the bearing-measurement error. Separate probability distributions for the normalized position error were obtained for each of 48[1] directions of target motion with respect to the location of the station whose bearing reading is new, so that the variation in position error with target direction is determined. The separate distributions were combined, assuming a uniform distribution for target direction, to provide one probability distribution which describes the position error for an arbitrary target direction.

A modification of the triangulation system consisting of three bearing-measurement stations was considered also. In this modification only two bearing readings, the new reading and the more recent of the other two, are used to calculate the position of the target. The reason for considering this modification is the recognition that in a system in which no attempt is made to compensate for the error due to target motion and age of the measurements, the use of the bearing reading with the greater age may increase rather than decrease the position error. This modified system was analysed in a manner similar to the analysis of the conventional system. The conditions under which this modification

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[1] Because of symmetry, only 12 different probability distribution curves are necessary for the 48 different target directions.

provides a reduction in the position error were determined by comparing the probability distributions which were obtained for both cases.

CHAPTER IV

ANALYSIS

Position Error in Terms of Bearing-Reading Error

A target is assumed to be located at the origin of a two-dimensional coordinate system. The locations of n bearing-measurement stations are specified by the set of polar coordinates, d_i and θ_i, in which the subscript i, which denotes the corresponding bearing-measurement station, takes on all integer values from one to n. Each pair of bearing lines intersects at a point denoted by r_{ij}, θ_{ij}, in which the double subscript indicates the pair of bearing-measurement stations involved and, of course, $i \neq j$. The total number of intersections, N, is given by

$$N = \frac{n(n-1)}{2} \quad . \tag{4-1}$$

The location of the target, as calculated from the bearing readings, is the centroid of the set of N intersections. Because the target is located at the origin of the coordinate system, the coordinates of the centroid of the intersections specify the error in the calculated location of the target. The magnitude of this error in terms of the coordinates of the intersections is obtained as follows.

In order to determine the centroid easily, the coordinates of the set of intersections are expressed temporarily in a Cartesian coordinate system in which $x = r \cos \theta$ and $y = r \sin \theta$. In this Cartesian coordinate system, the coordinates of the intersections, (x_{ij}, y_{ij}), are

$$x_{ij} = r_{ij} \cos\theta_{ij} \quad \text{and} \quad y_{ij} = r_{ij} \sin\theta_{ij} \, . \tag{4-2}$$

The coordinates of the centroid are

$$\bar{x} = \frac{1}{2N} \sum_{\substack{ij \\ j \neq i}} x_{ij} \tag{4-3}$$

and

$$\bar{y} = \frac{1}{2N} \sum_{\substack{ij \\ j \neq i}} y_{ij} \quad , \tag{4-4}$$

in which $\sum_{\substack{ij \\ j \neq i}}$ denotes a double sum performed over all permutations of the possible values of i and j. The use of permutations of values of i and j requires the factor of 1/2 in the expression for the coordinates of the centroid. The position error, E , i.e., the magnitude of the error in the calculated position of the target, can be expressed in terms of its square by

$$E^2 = \bar{x}^2 + \bar{y}^2 \quad . \tag{4-5}$$

The combination of Equations (4-2), (4-3), (4-4), and (4-5) yields:

$$4N^2E^2 = 4N^2(\bar{x}^2+\bar{y}^2) = (2N\bar{x})^2 + (2N\bar{y})^2$$

$$= \Big(\sum_{\substack{ij\\ j\neq i}} x_{ij}\Big)^2 + \Big(\sum_{\substack{ij\\ j\neq i}} y_{ij}\Big)^2$$

$$= \Big(\sum_{\substack{ij\\ j\neq i}} r_{ij}\cos\theta_{ij}\Big)^2 + \Big(\sum_{\substack{ij\\ j\neq i}} r_{ij}\sin\theta_{ij}\Big)^2$$

$$= \sum_{\substack{ijkl\\ j\neq i,\, l\neq k}} r_{ij} r_{kl}\cos\theta_{ij}\cos\theta_{kl} + \sum_{\substack{ijkl\\ j\neq i,\, l\neq k}} r_{ij} r_{kl}\sin\theta_{ij}\sin\theta_{kl}$$

$$= \sum_{\substack{ijkl\\ j\neq i,\, l\neq k}} r_{ij} r_{kl}\cos(\theta_{ij}-\theta_{kl}) \qquad (4\text{-}6)$$

in which $\sum\limits_{\substack{ijkl\\ j\neq i,\, l\neq k}}$ denotes a quadruple summation performed over all permutations of the subscripts, with the exceptions $j \neq i$ and $l \neq k$.

An expression for the coordinates of the intersections in terms of the coordinates of the bearing-measurement stations and the error in the bearing readings is obtained by use of Figure 4-1 which illustrates the geometry of an arbitrary pair of bearing-measurement stations. The error in the bearing reading taken at Stations i and j is denoted by ϵ_i and ϵ_j, respectively. An expression of the "law of

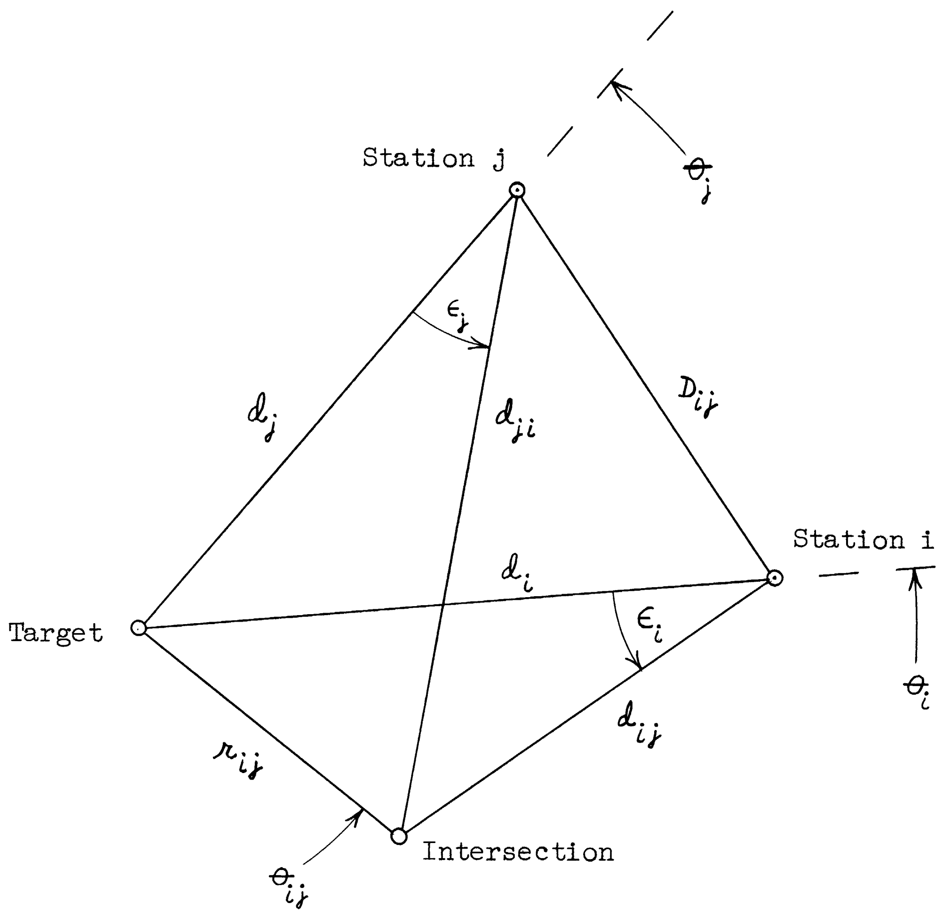

Figure 4-1 The Geometry of a Pair of Bearing-Measurement Stations

sines" for the two larger triangles is

$$\frac{r_{ij}}{\sin \epsilon_i} = \frac{d_i}{\sin(\pi + \theta_{ij} - \theta_i - \epsilon_i)} \tag{4-7}$$

and

$$\frac{r_{ij}}{\sin \epsilon_j} = \frac{d_j}{\sin(\pi + \theta_{ij} - \theta_j - \epsilon_j)} , \tag{4-8}$$

or

$$d_j \sin \epsilon_j \left[\sin(\theta_i + \epsilon_i) \cos \theta_{ij} - \cos(\theta_i + \epsilon_i) \sin \theta_{ij} \right]$$
$$= d_i \sin \epsilon_i \left[\sin(\theta_j + \epsilon_j) \cos \theta_{ij} - \cos(\theta_j + \epsilon_j) \sin \theta_{ij} \right] . \tag{4-9}$$

Equation (4-9) can be rearranged to provide

$$\tan \theta_{ij} = \frac{\sin \theta_{ij}}{\cos \theta_{ij}} = \frac{d_j \sin \epsilon_j \sin(\theta_i + \epsilon_i) - d_i \sin \epsilon_i \sin(\theta_j + \epsilon_j)}{d_j \sin \epsilon_j \cos(\theta_i + \epsilon_i) - d_i \sin \epsilon_i \cos(\theta_j + \epsilon_j)} . \tag{4-10}$$

Sin θ_{ij} and cos θ_{ij} can be obtained by using Equation (4-10) and the trigonometric identity

$$\cos^2 \theta_{ij} = \frac{1}{1 + \tan^2 \theta_{ij}} , \tag{4-11}$$

from which

$$\cos^2 \theta_{ij} = \frac{\left[d_j \sin \epsilon_j \cos(\theta_i + \epsilon_i) - d_i \sin \epsilon_i \cos(\theta_j + \epsilon_j) \right]^2}{d_j^2 \sin^2 \epsilon_j - 2 d_j d_i \sin \epsilon_j \sin \epsilon_i \cos(\theta_j - \theta_i + \epsilon_j - \epsilon_i) + d_i^2 \sin^2 \epsilon_i} . \tag{4-12}$$

The expressions for cos θ_{ij} and sin θ_{ij} for use in Equation (4-6) are obtained from Equations (4-10) and (4-12). The results are:

$$\cos\theta_{ij} = \frac{d_j \sin\epsilon_j \cos(\theta_i + \epsilon_i) - d_i \sin\epsilon_i \cos(\theta_j + \epsilon_j)}{\left[d_j^2 \sin^2\epsilon_j - 2 d_j d_i \sin\epsilon_j \sin\epsilon_i \cos(\theta_j - \theta_i + \epsilon_j - \epsilon_i) + d_i^2 \sin^2\epsilon_i\right]^{1/2}} \quad (4\text{-}13)$$

and

$$\sin\theta_{ij} = \frac{d_j \sin\epsilon_j \sin(\theta_i + \epsilon_i) - d_i \sin\epsilon_i \sin(\theta_j + \epsilon_j)}{\left[d_j^2 \sin^2\epsilon_j - 2 d_j d_i \sin\epsilon_j \sin\epsilon_i \cos(\theta_j - \theta_i + \epsilon_j - \epsilon_i) + d_i^2 \sin^2\epsilon_i\right]^{1/2}} . \quad (4\text{-}14)$$

An expression for r_{ij} for use in Equation (4-6) is obtained from Equations (4-8), (4-13), and (4-14) as follows:

$$r_{ij} = \frac{d_j \sin\epsilon_j}{\sin(\theta_j + \epsilon_j)\cos\theta_{ij} - \cos(\theta_j + \epsilon_j)\sin\theta_{ij}}$$

$$= \frac{d_j \sin\epsilon_j \left[d_j^2 \sin^2\epsilon_j - 2 d_j d_i \sin\epsilon_j \sin\epsilon_i \cos(\theta_j - \theta_i + \epsilon_j - \epsilon_i) + d_i^2 \sin^2\epsilon_i\right]^{1/2}}{\left[\begin{array}{l} d_j \sin\epsilon_j \cos(\theta_i + \epsilon_i)\sin(\theta_j + \epsilon_j) - d_i \sin\epsilon_i \cos(\theta_j + \epsilon_j)\sin(\theta_j + \epsilon_j) \\ - d_j \sin\epsilon_j \sin(\theta_i + \epsilon_i)\cos(\theta_j + \epsilon_j) + d_i \sin\epsilon_i \sin(\theta_j + \epsilon_j)\cos(\theta_j + \epsilon_j) \end{array}\right]}$$

$$= \frac{\left[d_j^2 \sin^2\epsilon_j - 2 d_j d_i \sin\epsilon_j \sin\epsilon_i \cos(\theta_j - \theta_i + \epsilon_j - \epsilon_i) + d_i^2 \sin^2\epsilon_i\right]^{1/2}}{\sin(\theta_j - \theta_i + \epsilon_j - \epsilon_i)} . \quad (4\text{-}15)$$

The proper selections of algebraic signs for cos θ_{ij} , sin θ_{ij} , and r_{ij} are determined by consideration of the special cases $\epsilon_i = 0$ and $\epsilon_j = 0$ and the geometry illustrated in Figure (4-1).

By use of Equations (4-13), (4-14), and (4-15), the summand of Equation (4-6) can be put in the form:

$$r_{ij} r_{kl} \cos(\theta_{ij} - \theta_{kl})$$

$$= r_{ij} r_{kl} \left(\cos\theta_{ij} \cos\theta_{kl} + \sin\theta_{ij} \sin\theta_{kl} \right)$$

$$= \frac{\left(\begin{array}{l} \begin{bmatrix} d_j \sin\epsilon_j \cos(\theta_i + \epsilon_i) \\ - d_i \sin\epsilon_i \cos(\theta_j + \epsilon_j) \end{bmatrix} \begin{bmatrix} d_l \sin\epsilon_l \cos(\theta_k + \epsilon_k) \\ - d_k \sin\epsilon_k \cos(\theta_l + \epsilon_l) \end{bmatrix} \\ + \begin{bmatrix} d_j \sin\epsilon_j \sin(\theta_i + \epsilon_i) \\ - d_i \sin\epsilon_i \sin(\theta_j + \epsilon_j) \end{bmatrix} \begin{bmatrix} d_l \sin\epsilon_l \sin(\theta_k + \epsilon_k) \\ - d_k \sin\epsilon_k \sin(\theta_l + \epsilon_l) \end{bmatrix} \end{array} \right)}{\sin(\theta_j - \theta_i + \epsilon_j - \epsilon_i) \sin(\theta_l - \theta_k + \epsilon_l - \epsilon_k)} \qquad (4\text{-}16)$$

Substituting Equation (4-16) in Equation (4-6) and rearranging the summation into a more useful form yields:

$$4N^2E^2 = 2\sum_{\substack{ijkl \\ j\neq i,\, l\neq k}} d_i \sin\epsilon_i \frac{\left(\begin{array}{l} \cos(\theta_j+\epsilon_j)\begin{bmatrix} d_k \sin\epsilon_k \cos(\theta_l+\epsilon_l) \\ -d_l \sin\epsilon_l \cos(\theta_k+\epsilon_k)\end{bmatrix} \\ +\sin(\theta_j+\epsilon_j)\begin{bmatrix} d_k \sin\epsilon_k \sin(\theta_l+\epsilon_l) \\ -d_l \sin\epsilon_l \sin(\theta_k+\epsilon_k)\end{bmatrix}\end{array}\right)}{\sin(\theta_j-\theta_i+\epsilon_j-\epsilon_i)\sin(\theta_l-\theta_k+\epsilon_l-\epsilon_k)}$$

$$= 2\sum_{\substack{ijkl \\ j\neq i,\, l\neq k}} d_i \sin\epsilon_i \frac{\begin{bmatrix} d_k \sin\epsilon_k \cos(\theta_j-\theta_l+\epsilon_j-\epsilon_l) \\ -d_l \sin\epsilon_l \cos(\theta_j-\theta_k+\epsilon_j-\epsilon_k)\end{bmatrix}}{\sin(\theta_j-\theta_i+\epsilon_j-\epsilon_i)\sin(\theta_l-\theta_k+\epsilon_l-\epsilon_k)}$$

$$= 4\sum_{\substack{ijkl \\ j\neq i,\, l\neq k}} \frac{d_i \sin\epsilon_i\, d_k \sin\epsilon_k \cos(\theta_j-\theta_l+\epsilon_j-\epsilon_l)}{\sin(\theta_j-\theta_i+\epsilon_j-\epsilon_i)\sin(\theta_l-\theta_k+\epsilon_l-\epsilon_k)} \qquad (4\text{-}17)$$

Therefore, the position error can be expressed in terms of the error in the bearing reading and the coordinates of the bearing-measurement stations with respect to the target as

$$N^2E^2 = \sum_{\substack{ijkl \\ j\neq i,\, l\neq k}} \frac{d_i\, d_k \sin\epsilon_i \sin\epsilon_k \cos(\theta_j - \theta_l + \epsilon_j - \epsilon_l)}{\sin(\theta_l - \theta_k + \epsilon_l - \epsilon_k)\sin(\theta_j - \theta_i + \epsilon_j - \epsilon_i)} \quad . \tag{4-18}$$

As would be expected from the geometry of a problem in which the angle coordinate of each bearing-measurement station is measured from an arbitrary reference, only the difference in angle coordinates enters the expression for error.

The Use of Weighting Factors

In order to obtain the best estimate of target position, each source of information should be weighted according to the possible error produced by it. In this way, the total error tends to be reduced by placing a greater emphasis on that information which is most likely to have small error and less emphasis on that information which is most likely to have large error.[1]

Stansfield(19) properly treats the problem of weighting in triangulation systems in terms of perpendicular distances from the estimated target position to each of the bearing lines. When the error in each of the bearing lines is described by a normal probability

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[1] Cramer(7), p. 234.

distribution with a mean of zero, the best estimate of the position of the target, on a maximum likelihood basis, is that position at which the sum of the weighted squares of the perpendicular distances is a minimum. The proper weight for each perpendicular distance is proportional to the reciprocal of the variance of the probability distribution which describes the possible values of each perpendicular distance. For small errors in bearing reading, this variance is approximately equal to the product of the variance of the bearing-reading error with the square of the distance along the corresponding bearing line from the station to the intersection with the perpendicular distance. This weighted, least-square estimate is generally used for convenience even when the probability distributions involved are not normal. However, the use of this criterion in the selection of the estimated target location is complicated.

It was pointed out in Chapter III that the centroid of the set of all possible intersections of pairs of bearing lines is convenient to use in the calculation of target position. In the use of a weighted centroid, a weight is assigned to each of the intersections. For a best estimate of the target position, the weight should not be assigned according to the possible error in each of the intersections because the error in each intersection is not independent of the error in the others. Even if the weights were to be assigned in this way, the determination of the variance for the distribution which describes the error in an intersection is not simple, as can be seen from Equation (4-15).

The weighting factor which is used in this study is obtained in the following intuitive way. Each intersection is formed by two bearing lines, to each of which is assigned the weighting recommended by Stansfield[19]. The possible error in each intersection is also proportional to the cosecant of the angle of intersection of the two bearing lines. The weight assigned to each intersection is the product of the weights assigned to each bearing line divided by the square of the cosecant of the angle of intersection, i.e.,

$$W_{ij} = \frac{D^4 \sin^2(\theta_j - \theta_i + \epsilon_j - \epsilon_i)}{\sigma_{\epsilon i}^2 \, \sigma_{\epsilon j}^2 \, d_{ij}^2 \, d_{ji}^2} , \tag{4-19}$$

in which

W_{ij} is the weighting factor assigned to the intersection of the pair of bearing lines from Stations i and j,

D is some reference distance, a constant,

$\sigma_{\epsilon i}^2$ is the variance of the error in the bearing reading at Station i, and

d_{ij} is the distance from Station i to the intersection formed by the bearing lines from Stations i and j,

as shown in Figure 4-1. The use of a weighted centroid using this weighting factor yields the same estimate of the target position as the estimate obtained by Stansfield[19] which is more difficult to apply.

When the target position is calculated by use of a "weighted centroid" the coordinates of the centroid are given by

$$\bar{x} = \sum_{\substack{ij \\ j \neq i}} W_{ij} x_{ij} \Big/ \sum_{\substack{ij \\ j \neq i}} W_{ij} \tag{4-20}$$

and

$$\bar{y} = \sum_{\substack{ij \\ j \neq i}} W_{ij}\, y_{ij} \Bigg/ \sum_{\substack{ij \\ j \neq i}} W_{ij} \,, \tag{4-21}$$

which are analagous to Equations (4-3) and (4-4) when an unweighted centroid is used. The magnitude of the position error is given by

$$\frac{E^2}{4}\left[\sum_{\substack{ij \\ j \neq i}} W_{ij}\right]^2 = \sum_{\substack{ijkl \\ j \neq i,\, l \neq k}} \frac{W_{ij} W_{kl}\, d_i d_k \sin\epsilon_i \sin\epsilon_k \cos(\theta_j - \theta_l + \epsilon_j - \epsilon_l)}{\sin(\theta_l - \theta_k + \epsilon_l - \epsilon_k)\sin(\theta_j - \theta_i + \epsilon_j - \epsilon_i)} \tag{4-22}$$

Probability Distribution for Bearing-Reading Error

In a triangulation system in which bearing measurements are not performed simultaneously, the error, ϵ_i , in the bearing reading at the time it is used in the calculation of the target location is made up of an error, ϵ_{Bi} , in the bearing measurement itself and an error, ϵ_{ti} , due to the motion of the target from the time the bearing measurement is performed to the time that the bearing measurement is used in the calculation of the target location.[1] These errors add directly:

$$\epsilon_i = \epsilon_{Bi} + \epsilon_{ti} \qquad (4\text{-}23)$$

ϵ_{Bi}, the error in the bearing measurement itself, is caused by many factors, such as errors in siting, time-varying propagation effects due to refraction of the atmosphere and the addition of reflected signals from local terrain features, and calibration errors as well as random errors present in the measurement apparatus. Some of these factors are randomly distributed, i.e., their values are subject to chance. Other factors, called systematic errors, are invariant during a set of repetitive measurements and, therefore, their effect cannot be reduced by averaging the result of repetitive measurements. If the value of a systematic error is known, the result of the measurement can be compensated and the effect of the "error" eliminated. If the value of a systematic error is known and the result is not compensated, the error in the result of the measurement due to the lack of compensation can be determined directly from the equation of the measurement, such as Equation (4-18) for the case of a triangulation system. If the value of a systematic error is unknown but the possible values are described by a probability distribution which expresses the lack of knowledge of the value, this probability distribution can be used to combine the

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[1] The position error considered in this study is the difference between the calculated position of the target and the true position of the target at the instant at which the most recent measurement used in the calculation was performed.

effects of this error with other systematic and random sources of error to determine the probability distribution which describes the total error in the result of a measurement. In such a combination, it is necessary to insure that invariant nature of the systematic error to repetitive measurement is properly treated. In a triangulation system in which <u>one</u> measurement from each bearing-measurement station is used to calculate the location of a moving target, the probability distributions which describe the lack of knowledge of the values of systematic error and the probability distributions which describe the random errors can be combined directly.

The error, ϵ_{Bi} , in the measurement of bearing at bearing-measurement Station i consists of a set of independent, random errors and systematic errors, as described above, denoted by ϵ_{Bim}. The total error, equal to the sum of these errors, is

$$\epsilon_{Bi} = \sum_{m} \epsilon_{Bim} \quad . \tag{4-24}$$

According to the central limit theorem[1], if the independent sources of error are described by probability distributions which have standard deviations which are finite, the probability distribution which describes the result of the summation approaches the normal distribution as the number of such sources of error becomes large. If no one source of error predominates, i.e., if the standard deviations for each of the

[1] Statements of the central limit theorem appear in slightly different form in Goode and Machol[(9)], p. 112, Cramér[(7)], pp. 114-116, and Woodward[(24)], p. 16.

major sources of error is of the same order of magnitude, the convergence is relatively rapid. Cramér points out that it often seems reasonable to assume that experimental errors combine in this way.[1] Goode and Machol point out that such an assumption may lead to pitfalls.[2] However, in a general study in which measuring equipment as yet unspecified is involved, the central limit theorem suggests that the error in measurement is best approximated by a normal distribution than by any other. Ross[(18)] cites experimental data in which approximation by a normal distribution was justified.

In the quantitative studies of a conventional and a modified three-station triangulation system described in Chapters V and VI, the total error in the measurement of bearing is assumed to be normally distributed. This assumption is reasonable for the reasons cited above. The adequacy of the application of the results of the quantitative general study to particular triangulation systems will depend upon how closely the probability distribution of the bearing-measurement error approximates the normal distribution. It is believed that of the total of all possible bearing-measurement apparatus that would be used in triangulation systems, the distributions for bearing-measurement error can be better approximated by a normal distribution than by any other distribution, especially if measures are taken to suppress the major sources of error in the apparatus. However, the method used to study quantitatively the error in three-station and two-station triangulation systems is not restricted to the use of the normal distribution; any theoretical

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[1] Cramér[(7)], pp. 120 and 230.

[2] Goode and Machol[(9)], p. 112.

or empirical distribution can be used, provided that the error is independent of the other parameters which are considered.

The error, ϵ_{ti} , in the bearing reading due to the motion of the target from the time the bearing measurement is performed to the time that the bearing measurement is used in the calculation of the target location, is a function of the velocity of the target, the location of the target with respect to the bearing-measurement stations, and t_i , the age of the bearing reading at the time of its use. Figure 4-2 describes the geometry of the situation. The "law of sines" for the triangle shown in Figure 4-2 is

$$\frac{d_{ti}}{\sin \epsilon_{ti}} = \frac{d_i}{\sin(\pi - \theta_i - \epsilon_{ti} + \phi_i)} , \qquad (4\text{-}25)$$

in which

d_i , θ_i are the polar coordinates of bearing-measurement Station i with respect to the true target position at the time the position is calculated,

d_{ti} , ϕ_i are the polar coordinates of the target location at the time the bearing measurement at Station i is performed, with respect to the same reference, and

ϵ_{ti} is the error in the bearing reading due to target motion and age of the bearing measurement at the time the target position is calculated.

Equation (4-25), when solved for ϵ_{ti} , is

$$\epsilon_{ti} = \tan^{-1} \left[\frac{d_{ti} \sin(\theta_i - \phi_i)}{d_i - d_{ti} \cos(\theta_i - \phi_i)} \right] . \qquad (4\text{-}26)$$

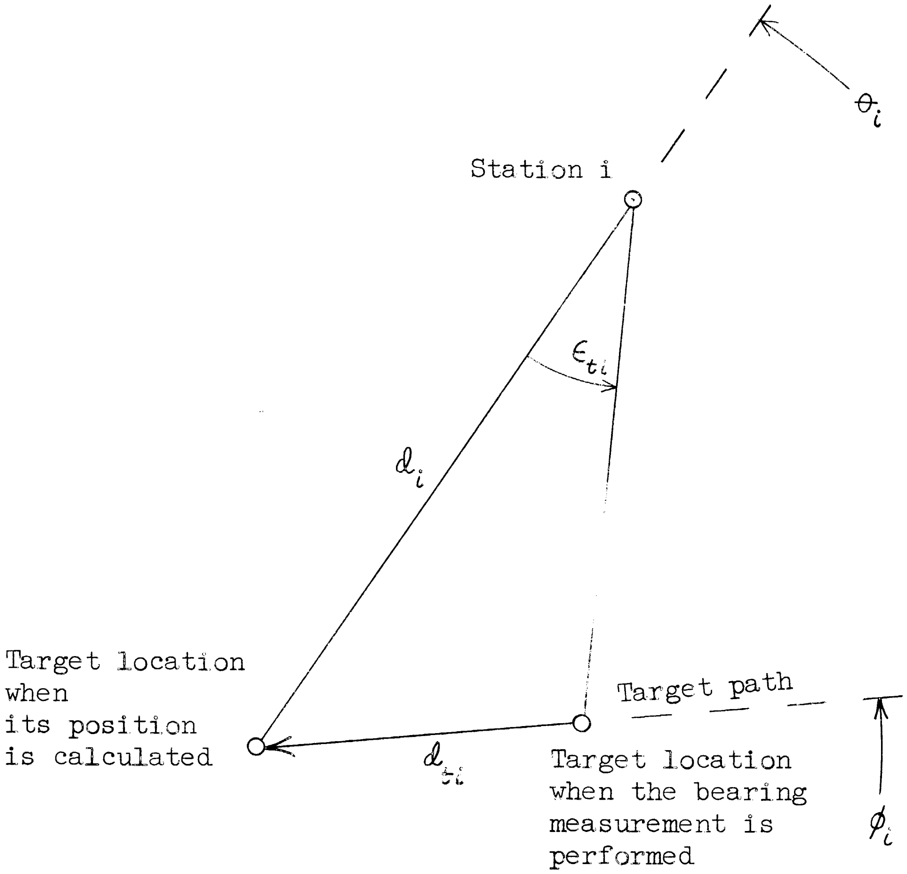

Figure 4-2 Error in Bearing Reading due to Target Motion and Non-Simultaneous Measurement

For a target which is traveling a straight line path, the value of ϕ_i is the same for each of the stations and may be denoted by ϕ. If in addition, the speed of the target, V, is constant, then

$$d_{ti} = Vt_i \quad , \tag{4-27}$$

and

$$\epsilon_{ti} = \tan^{-1}\left[\frac{Vt_i \sin(\theta_i - \phi)}{d_i - Vt_i \cos(\theta_i - \phi)}\right] \tag{4-28}$$

The triangulation system which is considered in this study is one in which each time a bearing measurement is performed at one bearing-measurement station, this measurement and those measurements already taken at each of the other stations are used to calculate the location of the target. The bearing measurements at each station are performed by cycling through the set of targets in some orderly fashion, independent of the measurements taken at the other stations. The time between consecutive measurements of the bearing of a particular target is denoted by T_i. T_i depends on τ_i, the time required to perform an individual measurement. If only one measurement is taken at a time, T_i depends directly upon M, the number of targets under surveillance. In this case,

$$T_i = M\tau_i \quad . \tag{4-29}$$

Because of the independence of measurement among the stations, age, t_i, of the measurement at Station i, when the measurement is used in a calculation initiated by another station, is described by a probability distribution which is uniformly distributed over the interval from 0 to T_i . T_i is a characteristic of the triangulation system and the capacity at which it is operating. Except in unusual systems, T_i will have the same value for each bearing measurement station and can be denoted by T .

The probability distribution for ϵ_{ti} can be obtained from the probability distribution for t_i by means of Equation (4-28). Although the set of variables denoted by ϵ_{ti} are independent, the parameters which describe the probability distribution of these variables are not independent; each distribution depends on the parameters V , ϕ , and T . If distributions rather than individual values of target parameters V and ϕ are to be considered, the probability distributions for the error in the calculated location of the target for individual values of the target parameters can be combined according to the probability distributions of those parameters.

In the quantitative study of a conventional and a modified three-station triangulation system described in Chapters V and VI, a uniform distribution for target direction is obtained by combining the results obtained for individual target directions. A uniform distribution for the age of bearing measurements is used although the method used in this study is not restricted to the use of uniform distributions. The method is restricted only to independence of the age of measurements at different stations.

Approximations for Small Bearing-Reading Error

Equation (4-18) for the magnitude of the position error, E , can be expanded into the form

$$N^2 E^2 = \sum_{\substack{ijkl \\ j\neq i,\, l\neq k}} \frac{d_i d_k \sin\epsilon_i \sin\epsilon_k Q_{jl}}{Q_{kl} Q_{ij}} \tag{4-30}$$

in which

$$Q_{jl} = \cos(\theta_j - \theta_l)\cos(\epsilon_j - \epsilon_l) - \sin(\theta_j - \theta_l)\sin(\epsilon_j - \epsilon_l) , \tag{4-31}$$

$$Q_{kl} = \sin(\theta_l - \theta_k)\cos(\epsilon_l - \epsilon_k) + \cos(\theta_l - \theta_k)\sin(\epsilon_l - \epsilon_k) , \tag{4-32}$$

and

$$Q_{ij} = \sin(\theta_j - \theta_i)\cos(\epsilon_j - \epsilon_i) + \cos(\theta_j - \theta_i)\sin(\epsilon_j - \epsilon_i) . \tag{4-33}$$

Equation (4-30) can be simplified by using approximate expressions for Q_{jl}, Q_{kl} and Q_{ij} which are valid when the bearing-reading error is small.

When the bearing-reading error is small, Q_{jl} can be approximated by Q'_{jl} , given by

$$Q'_{jl} = \cos(\theta_j - \theta_l)\cos(\epsilon_j - \epsilon_l) \approx Q_{jl} . \tag{4-34}$$

For the terms of the summation described by Equation (4-30) in which $\ell = j$, $\sin(\theta_j - \theta_\ell)$ is zero and the approximation given by Equation (4-34) is exact. For terms of the summation in which the value of $\theta_j - \theta_\ell$ is in the neighborhood of 0 or π, the error in the approximation is small because

$$\left|\sin(\theta_j - \theta_\ell)\sin(\epsilon_j - \epsilon_\ell)\right| \ll \left|\cos(\theta_j - \theta_\ell)\sin(\epsilon_j - \epsilon_\ell)\right| \tag{4-35}$$

For terms of the summation in which the value of $\theta_j - \theta_\ell$ is in the neighborhood of $\pm \pi/2$, the error in the approximation of each such term is large on a percentage basis, but the contribution to the summation on subscript ℓ of such terms is small because the corresponding value of $Q_{j\ell}$ is comparatively small. For example, when $\theta_j - \theta_\ell = \pm \pi/2$,

$$\left|Q_{j\ell}\right| = \left|\sin(\epsilon_j - \epsilon_\ell)\right| \approx \left|Q'_{j\ell}\right| = 0 , \tag{4-36}$$

whereas

$$\left|Q_{j\ell}\right| = \left|\cos(\epsilon_j - \epsilon_\ell)\right| \approx 1 \tag{4-37}$$

for those terms in which $\ell = j$ or in which $\theta_j - \theta_\ell = 0$ or π. In general, the percentage error of approximation is large in the terms with small values and small in the terms with large values.

$Q_{k\ell}$ and Q_{ij}, which appear in the denominator of the summand of Equation (4-30), can also be approximated when the bearing-reading error is small. These quantities are the same except for the subscript notation, so that a discussion of the approximation of one of them applies to the other. Q_{ij} is approximated by Q'_{ij}, which is given by

$$Q'_{ij} = \sin(\theta_j - \theta_i)\cos(\epsilon_j - \epsilon_i) \quad . \qquad (4\text{-}38)$$

For terms of the summation described by Equation (4-30) in which the value of $\theta_j - \theta_i$ is in the neighborhood of $\pm \pi/2$, the error in the approximation is small because

$$\left|\cos(\theta_j - \theta_i)\sin(\epsilon_j - \epsilon_i)\right| \ll \left|\sin(\theta_j - \theta_i)\cos(\epsilon_j - \epsilon_i)\right| \quad . \qquad (4\text{-}39)$$

For terms of the summation in which the value of $\theta_j - \theta_i$ is in the neighborhood of 0 or π, the error in the approximation of each such term is large on a percentage basis. Furthermore, the contribution to the summation on subscript ℓ of such terms is large because the corresponding value of Q_{ij} is small. These are terms which represent major contributions to the position error. The proper design of a triangulation system will eliminate them by eliminating from the computation those pairs of bearing readings which intersect at angles in the neighborhood of zero or π or by using the weighting factor given in Equation (4-19).

The result of these approximations for $Q_{j\ell}$, $Q_{k\ell}$, and Q_{ij} when the bearing-reading error is small is

$$N^2E^2 = \sum_{\substack{ijk\ell \\ j\neq i,\, \ell\neq k}} \frac{d_i\, d_k \sin\epsilon_i \sin\epsilon_k \cos(\theta_j - \theta_\ell)\cos(\epsilon_j - \epsilon_\ell)}{\sin(\theta_\ell - \theta_k)\cos(\epsilon_\ell - \epsilon_k)\sin(\theta_j - \theta_i)\cos(\epsilon_j - \epsilon_i)} \qquad (4\text{-}40)$$

Additional approximations can be made for small bearing-reading error. For the ϵ_i as large as five degrees:

$$\sin \epsilon_i \approx \epsilon_i \quad , \text{ and}$$
$$\sin \epsilon_k \approx \epsilon_k \tag{4-41}$$

with an error in each which does not exceed 0.13 percent; and

$$\cos(\epsilon_j - \epsilon_l) \approx 1 \quad ,$$
$$\cos(\epsilon_l - \epsilon_k) \approx 1 \quad , \text{ and}$$
$$\cos(\epsilon_j - \epsilon_i) \approx 1 \quad , \tag{4-42}$$

with an error in each which does not exceed 0.6 percent. Using these approximations, Equation (4-40) becomes

$$N^2 E^2 = \sum_{\substack{ijkl \\ j \neq i,\, l \neq k}} \frac{d_i d_k \epsilon_i \epsilon_k \cos(\theta_j - \theta_l)}{\sin(\theta_l - \theta_k) \sin(\theta_j - \theta_i)} \quad . \tag{4-43}$$

The entire set of approximations which have been made for small bearing-reading error may be interpreted geometrically as the approximation of the bearing lines from each Station i by lines which are located parallel to the true bearing lines with a displacement of $d_i \epsilon_i$ in the proper direction.[1]

When the intersection of a single pair of bearing lines is used to estimate the position of a target, the error of approximation due to the use of Equation (4-43) in place of Equation (4-30) can be examined easily by direct comparison. If the bearing-reading error is at most five degrees, the maximum error of approximation occurs when the geometric interpretation of the approximation indicates that the geometry is distorted most, i.e., when $\epsilon_1 = 5^\circ$ and $\epsilon_2 = -5^\circ$.

The error of approximation depends upon the angle at which the bearing lines intersect, $\theta_2 - \theta_1$, and the ratio of the distances from the target to the bearing-measurement stations. Figure 4-3 shows this error obtained by direct comparison as a function of $\theta_2 - \theta_1$ for the two limiting cases, $d_1 = d_2$ and d_1/d_2 or $d_2/d_1 = 0$. When bearing-reading error is at most five degrees, Figure 4-3 shows that the error of approximation does not exceed twenty percent if $50^\circ \leq |\theta_2 - \theta_1| \leq 140^\circ$.

When more than two bearing-measurement stations are used, a centroid or weighted centroid of all of the intersections of pairs of bearing readings is used to estimate target position. The position error in this case is no greater than the maximum of the errors in the individual intersections. Similarly, the error of approximation in calculating the position error by using Equation (4-43) instead of (4-30) is no greater than the maximum of the errors of approximation in the

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1 This geometric approximation has been used as a starting point in other studies such as Frese(8), Stansfield(19), and Harkin(10).

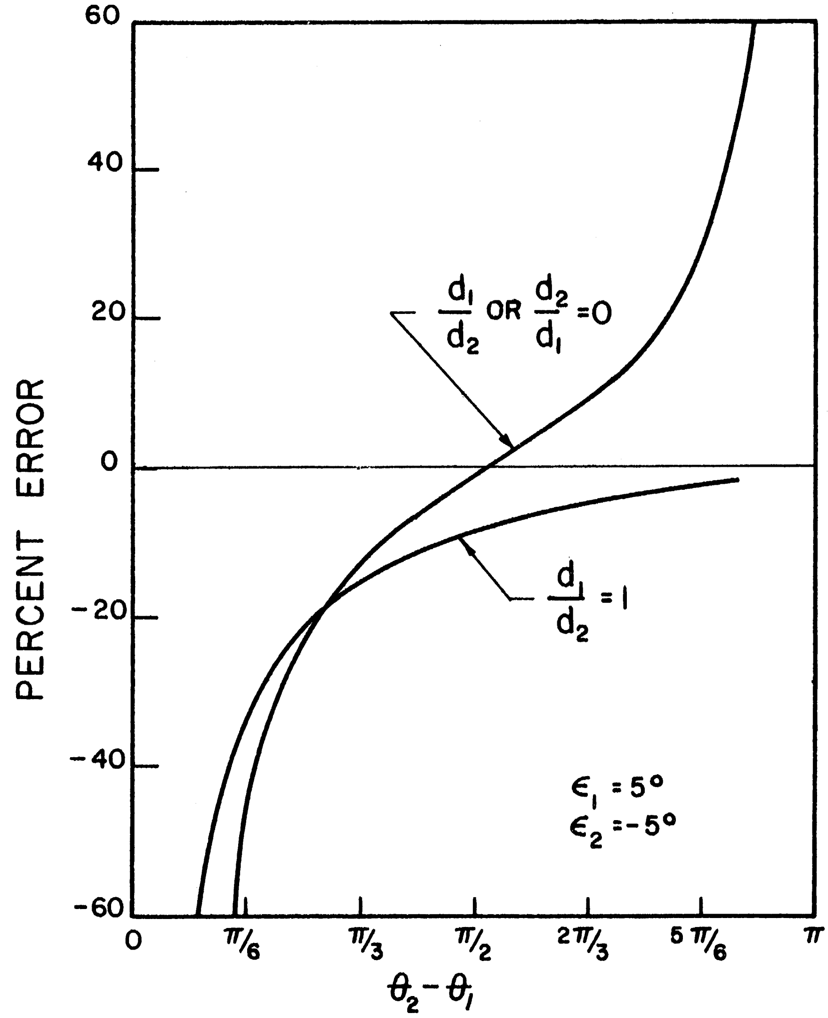

Figure 4-3 Error in the Approximate Expression for the Position Error

error in each of the individual intersections. If a target is so located with respect to more than two bearing-measurement stations that all of the $\theta_j - \theta_i$ are within the interval $50° \leq |\theta_j - \theta_i| \leq 140°$, the error of approximation in the position error does not exceed twenty percent for bearing-reading errors which are at most five degrees. For the special case of a target located at the center of an equilateral triangle formed by exactly three bearing-measurement stations $(\theta_2 - \theta_1 = 2\pi/3)$, Figure 4-3 shows that the error of approximation does not exceed six percent for bearing-reading errors which are at most five degrees.

When a weighting factor is used, approximations for small errors in bearing readings can also be made. The weighting factor is first expressed in terms of known distances, approximating the measured distances from each station to each intersection by the distance from the station to the target, i.e.,

$$d_i \approx d_{ij} \quad . \tag{4-44}$$

Equation (4-19) then becomes

$$W_{ij} = \frac{D^4 \sin^2(\theta_j - \theta_i + \epsilon_j - \epsilon_i)}{\sigma_{\epsilon i}^2 \sigma_{\epsilon j}^2 d_i^2 d_j^2} \quad . \tag{4-45}$$

This approximation is valid whenever the errors in the intersections are small compared with the distance of the target from the bearing-measurement stations. It is necessary if the equation for position error is to be a simple one.

Using Equation (4-45), the position error as given in Equation (4-22) becomes

$$\frac{E^2}{4}\left[\sum_{\substack{ij\\ j\neq i}} W_{ij}\right]^2 = \sum_{\substack{ijkl\\ j\neq i,\, l\neq k}} \frac{D^8 \sin\epsilon_i \sin\epsilon_k\, Q_{ij}\, Q_{kl}\, Q_{jl}}{\sigma_{\epsilon i}^2\, \sigma_{\epsilon j}^2\, \sigma_{\epsilon k}^2\, \sigma_{\epsilon l}^2\, d_i\, d_j^2\, d_k\, d_l^2} \tag{4-46}$$

in which Q_{ij} , Q_{kl} , Q_{jl} are the quantities defined in Equations (4-31), (4-32), and (4-33). When the bearing-reading error is small, these quantities can be approximated in the same way as they are approximated when no weighting factors are used. The approximations are given in Equations (4-34) and (4-38). The approximation of Q_{jl} is valid for the same reasons which were presented for the case of no weighting. The approximations of Q_{ij} and Q_{kl} are better when a weighting factor is used than when it is not, because when Q_{ij} and Q_{kl} are in the numerator of the summand, the percentage error is large in the terms with small values and small in the terms with large values.

By use of these approximations, Equation (4-46) becomes

$$\frac{E^2}{4}\left[\sum_{\substack{ij\\ j\neq i}} W_{ij}\right]^2$$

$$= \sum_{\substack{ijkl\\ j\neq i,\, l\neq k}} \frac{D^8 \sin\epsilon_i \sin\epsilon_k \sin(\theta_j-\theta_i)\cos(\epsilon_j-\epsilon_i)\sin(\theta_l-\theta_k)\cos(\epsilon_l-\epsilon_k)\cos(\theta_j-\theta_l)\cos(\epsilon_j-\epsilon_l)}{\sigma_{\epsilon i}^2\, \sigma_{\epsilon j}^2\, \sigma_{\epsilon k}^2\, \sigma_{\epsilon l}^2\, d_i\, d_j^2\, d_k\, d_l^2}\ . \tag{4-47}$$

By use of the approximation for Q_{ij}, the weighting factor as given in Equation (4-45) becomes

$$W_{ij} = \frac{D^4 \sin^2(\theta_j - \theta_i)}{\sigma_{\epsilon i}^2 \, \sigma_{\epsilon j}^2 \, d_i^2 \, d_j^2} \quad . \tag{4-48}$$

As in the case when no weighting factor is used, the trigonometric functions of the bearing-reading errors in Equation (4-47) can be approximated by the expressions given in Equations (4-41) and (4-42). Using these approximations, Equation (4-47) may be written as

$$E^2 = \frac{4 \displaystyle\sum_{\substack{ijkl \\ j \neq i,\, l \neq k}} \frac{D^8 \epsilon_i \epsilon_k \sin(\theta_j - \theta_i) \sin(\theta_l - \theta_k) \cos(\theta_j - \theta_l)}{\sigma_{\epsilon i}^2 \, \sigma_{\epsilon j}^2 \, \sigma_{\epsilon k}^2 \, \sigma_{\epsilon l}^2 \, d_i^2 \, d_j^2 \, d_k^2 \, d_l^2}}{\left[\displaystyle\sum_{\substack{ij \\ j \neq i}} \frac{D^4 \sin^2(\theta_j - \theta_i)}{\sigma_{\epsilon i}^2 \, \sigma_{\epsilon j}^2 \, d_i^2 \, d_j^2} \right]^2} \tag{4-49}$$

Two approximate expressions for the position error have been developed: Equation (4-43) for the case of no weighting factor and Equation (4-49) for the case of the particular weighting factor given in Equation (4-19). Chapters V and VI describe the use of these equations to determine the position error for the case of three bearing-measurement stations.

An approximation for small bearing-reading error can also be made for Equation (4-28) which describes the error in a bearing reading due to motion of the target. For the error, ϵ_{ti}, to be small for all values of $\theta_i - \phi$, Equation (4-28) shows that Vt_i must be much less than d_i. With this condition and by replacing $\tan \epsilon_{ti}$ by ϵ_{ti}, Equation (4-28) can be approximated by

$$\epsilon_{ti} = \frac{Vt_i \sin(\theta_i - \phi)}{d_i} \qquad (4\text{-}50)$$

For Vt_i/d_i as large as 0.05, the maximum error in this approximation is five percent and it occurs when $\theta_i - \phi$ is close to zero or π and ϵ_{ti} is very small. As $\theta_i - \phi$ approaches $\pm\pi/2$, the error of approximation is zero. Because t_i is uniformly distributed in the interval $0 \le t_i \le T$, as discussed on page 36, Equation (4-50) shows that ϵ_{ti} is uniformly distributed in the interval between zero and $VT\sin(\theta_i - \phi)/d_i$, as shown in Figure (4-4).

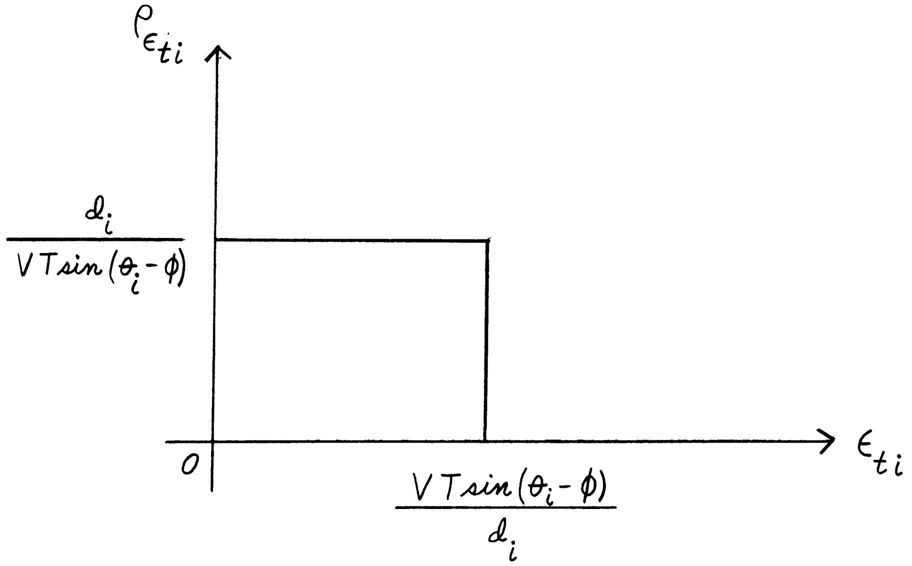

Figure 4-4 Probability Density-Function for ϵ_{ti}

Probability Distribution for Position Error

Expressions for the magnitude of the error in the calculated position of a target have been developed in general, Equation (4-18), for the case of a particular weighting function, Equation (4-22), and for both of these cases with an approximation for small bearing-reading errors, Equations (4-43) and (4-49). These expressions can be denoted by the general expression

$$E = E(d_i, \epsilon_i, \theta_i) \quad , \tag{4-51}$$

which indicates that E is a function of all of the d_i's , ϵ_i's , and θ_i's. Only the ϵ_i's are described by a probability distribution. Each ϵ_i is the sum of an ϵ_{Bi} and an ϵ_{ti} as given in Equation (4-23). Both ϵ_{Bi} and ϵ_{ti} have independent probability distributions. The probability distribution of each ϵ_{Bi} is specified directly. Each ϵ_{ti} is a function [Equation (4-28) or (4-50)] of the variables d_i, θ_i, V, ϕ, and t_i, of which only t_i is described by a probability distribution. This functional relationship can be denoted by the set of general expressions

$$\epsilon_{ti} = \epsilon_{ti}(d_i, \theta_i, V, \phi, t_i) \quad . \tag{4-52}$$

The probability distribution for each ϵ_{ti} can be obtained directly from Equation (4-52) by using the probability distribution for t_i and the values of the other parameters. The probability distribution for each ϵ_i can then be obtained by a convolution of the probability distributions of ϵ_{Bi} and ϵ_{ti} using the equation

$$\epsilon_i = \epsilon_{Bi} + \epsilon_{ti} \quad . \tag{4-53}$$

The probability distribution for the position error, E, can then be obtained by a combination of the probability distributions for each of the ϵ_i by using Equation (4-51) with values supplied for the parameters other than ϵ_i. The complexity of the combination process requires the use of automatic computing devices.

To investigate the error in a particular existing or proposed triangulation system which fits the general model described herein, the values of the parameters used in the analysis can be selected and the necessary combinations of the probability distributions can be performed. To investigate triangulation systems in a more general way, configurations of bearing-measurement stations possessing geometric symmetry can be selected and investigated using system parameters which are conveniently normalized. The probability distributions which describe ϵ_{Bi}, ϵ_{ti}, and finally the position error, E, can be determined in terms of these normalized parameters. This general type of investigation for a three-station triangulation system is described in the next chapter.

CHAPTER V

CONVENTIONAL THREE-STATION SYSTEM

A triangulation system which consists of three bearing-measurement stations is of particular interest because such an arrangement provides an efficient way of providing area coverage when the range of the bearing-measurement apparatus is limited. An example of such an arrangement is shown in Figure 3-1. The position error in a triangulation system consisting of three bearing-measurement stations is investigated in the following way.

An expression for the magnitude of the error in the computed location of a target in the special case of a triangulation system consisting of three bearing-measurement stations is obtained from the expressions developed in the general analysis (Chapter IV). This special case is specialized further by considering the three bearing-measurement stations to be located at the vertexes of an equilateral triangle.

A discrete and a normal probability distribution for the bearing-reading error are used to investigate the expected value of the position error when the target is located at the center of the equilateral triangle and at several other points. The results of this investigation demonstrate that the variation in the expected value of position error is small as the location of the target is moved within a region around the center of the equilateral triangle at least as large as half the area of the triangle. Because of this small variation, the case of the target located at the center of the equilateral triangle is selected for detailed investigation. The position error in this case is a good estimate of the position error in a large region around this point.

The probability distribution of the position error for a target which is located exactly at the center of the equilateral triangle formed by three bearing-measurement stations is obtained from the probability distributions for the error in each of the bearing readings. The complexity of both the individual probability distributions for the bearing-reading error and the expression for the square of the position error in terms of these distributions requires that the combination process which yields the resultant distribution be performed with the use of an automatic computing device.

The bearing error at each bearing-measurement station is grouped into a set of many small intervals and, thereby, the continuous probability distributions are approximated by discrete distributions. The probability distribution for the square of the position error is constructed by examining all of the possible ways that one interval can be selected from each of the three sets. For the particular digital computer used and for this type of problem, the use of all possible combinations of intervals has many advantages over the "Monte Carlo" method which was first considered for use in the solution of this problem.

The result of the digital computer study is a set of cumulative probability distributions for the position error, normalized with respect to both the distance of the bearing-measurement stations from the center of the equilateral triangle formed by them and with respect to the standard deviation of the error in the bearing measurement. The set of distributions consists of separate distributions for the two parameters: (1) target direction and (2) a normalized combination of target velocity and time delay. Probability distributions are also presented for the case of a uniform distribution for target direction.

Equations for the Position Error

For the special case of only three bearing-measurement stations, Equation (4-43), which expresses the magnitude of the position error when no weighting factor is used and when the error in bearing reading is small, can be expanded into the form:

$$E^2 = \frac{1}{9}\left[\begin{aligned}
&\epsilon_1^2 d_1^2\left[\frac{1}{\sin^2(\theta_2-\theta_1)}+\frac{1}{\sin^2(\theta_1-\theta_3)}-\frac{2\cos(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)\sin(\theta_1-\theta_3)}\right]\\
&+\epsilon_2^2 d_2^2\left[\frac{1}{\sin^2(\theta_2-\theta_1)}+\frac{1}{\sin^2(\theta_3-\theta_2)}-\frac{2\cos(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)}\right]\\
&+\epsilon_3^2 d_3^2\left[\frac{1}{\sin^2(\theta_3-\theta_2)}+\frac{1}{\sin^2(\theta_1-\theta_3)}-\frac{2\cos(\theta_2-\theta_1)}{\sin(\theta_3-\theta_2)\sin(\theta_1-\theta_3)}\right]\\
&+2\epsilon_2\epsilon_1 d_2 d_1\left[\frac{\cot(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)}+\frac{\cot(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)}-\frac{\cot(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)}-\frac{1}{\sin(\theta_1-\theta_3)\sin(\theta_3-\theta_2)}\right]\\
&+2\epsilon_3\epsilon_2 d_3 d_2\left[\frac{\cot(\theta_2-\theta_1)}{\sin(\theta_3-\theta_2)}+\frac{\cot(\theta_1-\theta_3)}{\sin(\theta_3-\theta_2)}-\frac{\cot(\theta_3-\theta_2)}{\sin(\theta_3-\theta_2)}-\frac{1}{\sin(\theta_2-\theta_1)\sin(\theta_1-\theta_3)}\right]\\
&+2\epsilon_1\epsilon_3 d_1 d_3\left[\frac{\cot(\theta_2-\theta_1)}{\sin(\theta_1-\theta_3)}+\frac{\cot(\theta_3-\theta_2)}{\sin(\theta_1-\theta_3)}-\frac{\cot(\theta_1-\theta_3)}{\sin(\theta_1-\theta_3)}-\frac{1}{\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)}\right]
\end{aligned}\right] \quad (5\text{-}1)$$

For the same case, but with <u>any</u> weighting factor used[1], Equation (4-22) can be expanded into the form:

$$
\frac{E^2}{4}\left[\sum_{\substack{ij\\ j\neq i}} W_{ij}\right]^2 = \left[
\begin{aligned}
& \frac{W_{21}^2}{\sin^2(\theta_2-\theta_1)}\left[\epsilon_1^2 d_1^2 + \epsilon_2^2 d_2^2 - 2\epsilon_1\epsilon_2 d_1 d_2 \cos(\theta_2-\theta_1)\right] \\
& + \frac{W_{32}^2}{\sin^2(\theta_3-\theta_2)}\left[\epsilon_2^2 d_2^2 + \epsilon_3^2 d_3^2 - 2\epsilon_3\epsilon_2 d_3 d_2 \cos(\theta_3-\theta_2)\right] \\
& + \frac{W_{13}^2}{\sin^2(\theta_1-\theta_3)}\left[\epsilon_1^2 d_1^2 + \epsilon_3^2 d_3^2 - 2\epsilon_1\epsilon_3 d_1 d_3 \cos(\theta_1-\theta_3)\right] \\
& + \frac{2W_{21}W_{32}}{\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)}\left[\begin{array}{l} -\epsilon_2^2 d_2^2 \cos(\theta_1-\theta_3) + \epsilon_2\epsilon_1 d_2 d_1 \cos(\theta_3-\theta_2) \\ +\epsilon_3\epsilon_2 d_3 d_2 \cos(\theta_2-\theta_1) - \epsilon_1\epsilon_3 d_1 d_3 \end{array}\right] \\
& + \frac{2W_{32}W_{13}}{\sin(\theta_3-\theta_2)\sin(\theta_1-\theta_3)}\left[\begin{array}{l} -\epsilon_3^2 d_3^2 \cos(\theta_2-\theta_1) + \epsilon_3\epsilon_2 d_3 d_2 \cos(\theta_1-\theta_3) \\ +\epsilon_1\epsilon_3 d_1 d_3 \cos(\theta_3-\theta_2) - \epsilon_2\epsilon_1 d_2 d_1 \end{array}\right] \\
& + \frac{2W_{13}W_{21}}{\sin(\theta_1-\theta_3)\sin(\theta_2-\theta_1)}\left[\begin{array}{l} -\epsilon_1^2 d_1^2 \cos(\theta_3-\theta_2) + \epsilon_1\epsilon_3 d_1 d_3 \cos(\theta_2-\theta_1) \\ +\epsilon_2\epsilon_1 d_2 d_1 \cos(\theta_1-\theta_3) - \epsilon_3\epsilon_2 d_3 d_2 \end{array}\right]
\end{aligned}
\right] \qquad (5\text{-}2)
$$

which is equivalent to Equation (5-1) when the weighting factors are one.

---

[1] In the numerical analyses to be described, only the weighting factor givèn in Equation (4-19) is considered.

For coverage of an area (Figure 3-1), triangulation systems composed of units of three bearing-measurement stations use arrangements of stations which differ very little from a symmetric arrangement with each station located at the vertex of an equilateral triangle, because it has been generally recognized that such an arrangement of three stations leads to minimization of the position error. In the following analysis of a three-station system, this symmetric arrangement is used because of its more general utility. The procedures which are used in this analysis are not restricted to this symmetrical arrangement and can be applied equally well to any other particular arrangement of bearing-measurement stations.

Because of the use of a symmetric arrangement of bearing-measurement stations, it is convenient to normalize all distances, including the position error, with respect to the distance, D , of each bearing-measurement station from the center of the equilateral triangle. This normalized position error is denoted by E_D . This distance, D , is used in the weighting factor also.

Variation in the Position Error

The variation of the normalized position error with variation in the target location is studied in order to justify the choice of a particular target location for use in a detailed study of the position error in a three-station triangulation system. In this study, both a simple, discrete probability distribution and a normal distribution are used to describe the error in bearing readings.

The simple, discrete probability distribution presented in Figure 5-1 shows that bearing-reading error is described by two equally likely values, $+\epsilon$ and $-\epsilon$. This discrete probability distribution for bearing-reading error is used to examine the position error at the four specific target locations, designated in Figure 5-2 as the points O, A, B and C. At each of these points, numerical values of the angles and distances in Equations (5-1) and (5-2) were calculated and substituted in these equations. All distances are normalized with respect to D, the distance of each bearing measurement station from the center of the equilateral triangle. At the point O, Equation (5-1) becomes the simple equation

$$\frac{E^2}{D^2} = E_D^2 = \frac{4}{9}\left[\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 - \epsilon_2\epsilon_1 - \epsilon_3\epsilon_2 - \epsilon_1\epsilon_3\right] \quad . \qquad (5\text{-}3)$$

Equation (5-3) is valid whether or not the weighting factor is used, because at the point O equal weights must be assigned to each intersection. Equal weights are assigned because the variances of the bearing readings are identical and because the distances of the stations to the point O are equal. For the other points of interest, the equations which describe the position error are more complicated than Equation (5-3).

For each of the points, A, B, C, and O, and for each of the possible combinations of values of ϵ_i given by the probability distributions for ϵ_i, numerical values of E_D were calculated. The expected values of E_D, E_X, based on these probability distributions, are listed in Table 5-1.[1] For the use of the weighting factor,

[1] The process for determining E_X is described in Chapter VI for a simpler case.

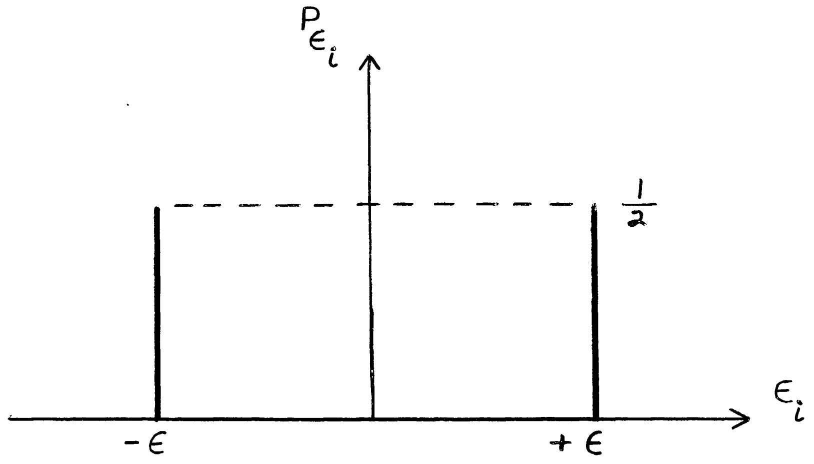

Figure 5-1 A Simple, Discrete Probability Distribution for Bearing-Reading Error

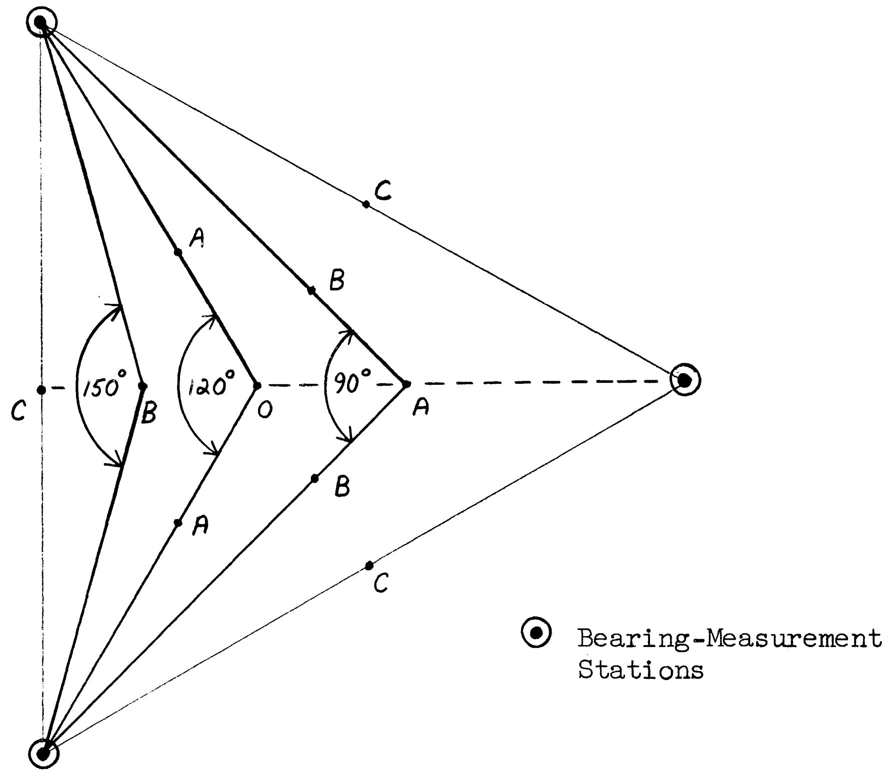

Figure 5-2 Target Locations Selected for the Examination of the Position Error

the numerical results presented in Table 5-1 demonstrate that the expected value of the position error for a target located as far from the center of the triangle as points A and B is little different from the expected value for a target located at the center. A maximum difference of eighteen percent occurs when the target is located at point B.

TABLE 5-1

EXPECTED VALUE OF THE NORMALIZED POSITION ERROR FOR A SIMPLE PROBABILITY DISTRIBUTION FOR BEARING-READING ERROR

	E_X (Expected value of the normalized position error, E_D)	
	Without Weighting	With Weighting
Point O	1.00ϵ	1.00ϵ
Point A	1.18ϵ	1.14ϵ
Point B	1.20ϵ	1.18ϵ
Point C	1.62ϵ [1]	1.62ϵ

[1] The value without weighting at point C was obtained assuming that the target is on the line joining the two stations when the measured bearing lines to the target do not intersect.

Although the expected values of the normalized position error listed in Table 5-1 are not realistic because the probability distribution for bearing-reading error that was used is not realistic, the demonstration that the variation of position error is small in a large region around the center of the triangle can be extended to any more realistic distribution, provided it is symmetric. Consider an arbitrary, symmetric distribution for bearing-reading error such as that shown in Figure 5-3. If this distribution is divided into intervals, the contribution to the position error corresponding to each pair of symmetric intervals can be examined separately in exactly the same way the simple, discrete probability distribution was used. If the variation in each contribution is small, the variation in the total position error is small.

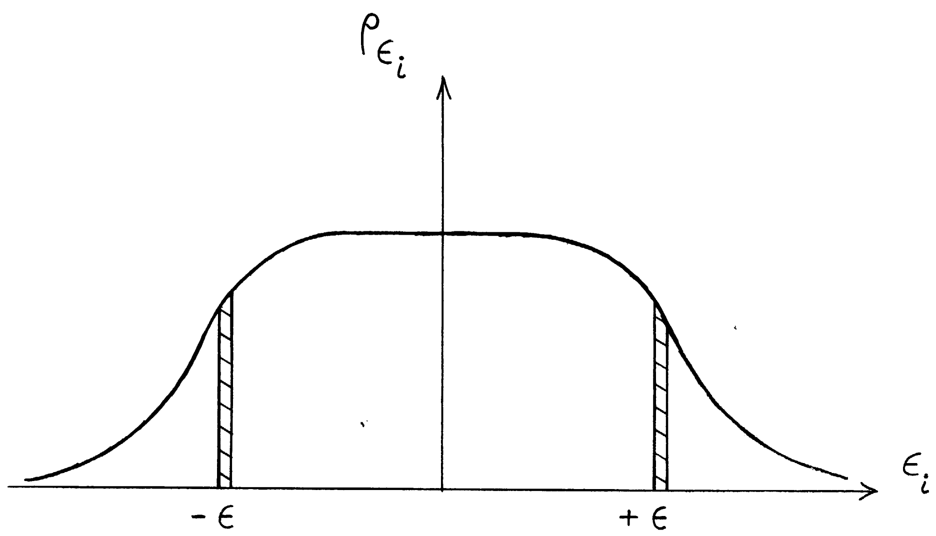

Figure 5-3 An Arbitrary, Symmetric Distribution for Bearing-Reading Error

The variation of the position error in the region around the center of the triangle can be examined using a normal distribution for bearing-reading error. In a previous report, the author has investigated the error in determining the location of a target when two bearing-measurement stations are used and when the error in bearing reading is described by a normal distribution.[1] The results of the investigation are presented in the form of an area of uncertainty, A_U , defined as the area of the smallest region surrounding the true target position[2] within which the measured target position will fall with a specified probability. The results show that

$$A_U \propto \frac{d_1 d_2}{|\sin(\theta_2 - \theta_1)|} \tag{5-4}$$

for any specific probability level and for any finite values of the variance of the normal distributions which describe ϵ_1 and ϵ_2 .[3] This proportionality can be used to demonstrate that the variation in position error is small.

The arc of a circle joining two bearing-measurement stations, as shown in Figure 5-4, is the locus of the points at which the bearing-measurement stations subtend a fixed angle, i.e., at which $\theta_2 - \theta_1$ is constant.

---

[1] Frese(8), pp. 2-22.

[2] The smallest region is an ellipse.

[3] Equation (5-4) indicates that A_U increases without bound as $\theta_2 - \theta_1$ approaches π . If the fact that bearing lines are semi-infinite instead of infinite had been taken into account, A_U would be bounded. For semi-infinite bearing lines, A_U is certainly no larger than the value obtained from Equation (5-4).

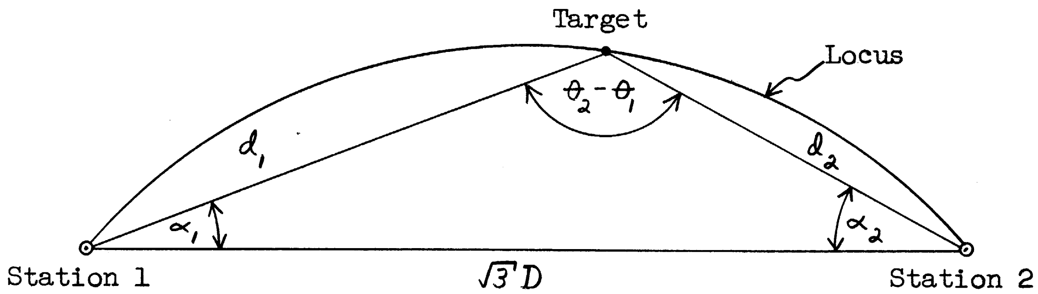

Figure 5-4 Locus of Points at Which $\theta_2 - \theta_1$ is Constant

From the "law of sines" for the triangle shown in Figure 5-4, the product of the distances d_1 and d_2 can be expressed as

$$d_1 d_2 = \frac{\sqrt{3}D \sin \alpha_2}{\sin(\theta_2 - \theta_1)} \cdot \frac{\sqrt{3}D \sin \alpha_1}{\sin(\theta_2 - \theta_1)} , \qquad (5\text{-}5)$$

in which α_1 and α_2 are the angles between the line joining the two bearing-measurement stations and the bearing lines from Stations 1 and 2, respectively, and $\sqrt{3}D$ is the spacing between the stations. Equation (5-5) can be simplified by use of trigonometric identities to the form

$$d_1 d_2 = \frac{3D^2 \left[\cos(\alpha_1 - \alpha_2) - \sin(\theta_2 - \theta_1)\right]}{\sin^2(\theta_2 - \theta_1)} . \qquad (5\text{-}6)$$

On the locus defined by a fixed value of $\theta_2 - \theta_1$, Equation (5-6) shows that $d_1 d_2$ is a maximum when $\alpha_1 = \alpha_2$, i.e., when d_1 equals d_2. Therefore, at any point on the locus defined by a fixed value of $\theta_2 - \theta_1$, A_u is no larger than its value at $d_1 = d_2$. Figure 5-5 is a graph of the maximum values of A_u as a function of the value of $\theta_2 - \theta_1$ which defines the locus. The maximum value of A_u for the locus, $\theta_2 - \theta_1 = 120^\circ$ is normalized to unity.

When three bearing-measurement stations are used, the area of uncertainty for the three intersections of pairs of bearing lines can be examined. Figure 5-6 shows three stations located at the vertexes of an equilateral triangle. The center of the triangle is the point at which the value of A_u was normalized. For each intersection of a pair of bearing lines, Figure 5-5 shows that A_u does not exceed twice its value at the center of triangle anywhere in the region between the loci defined by bearing lines which intersect at 71° and at 160°. The region (Figure 5-6) in which the value of A_u for each of the three intersections is no larger than twice its value at the center of the triangle is at least fifty percent of the area of the equilateral triangle. In terms of a distance rather than an area of uncertainty, the position error is no larger than the $\sqrt{2}$ times its value at the center throughout a region around the center which contains at least fifty percent of the area of the triangle. For a value of A_u no larger than three times its value at the center, the region contains at least seventy-five percent of the area of the triangle.

Thus, in a detailed investigation of the position error, the determination of the position error for a target located at the center

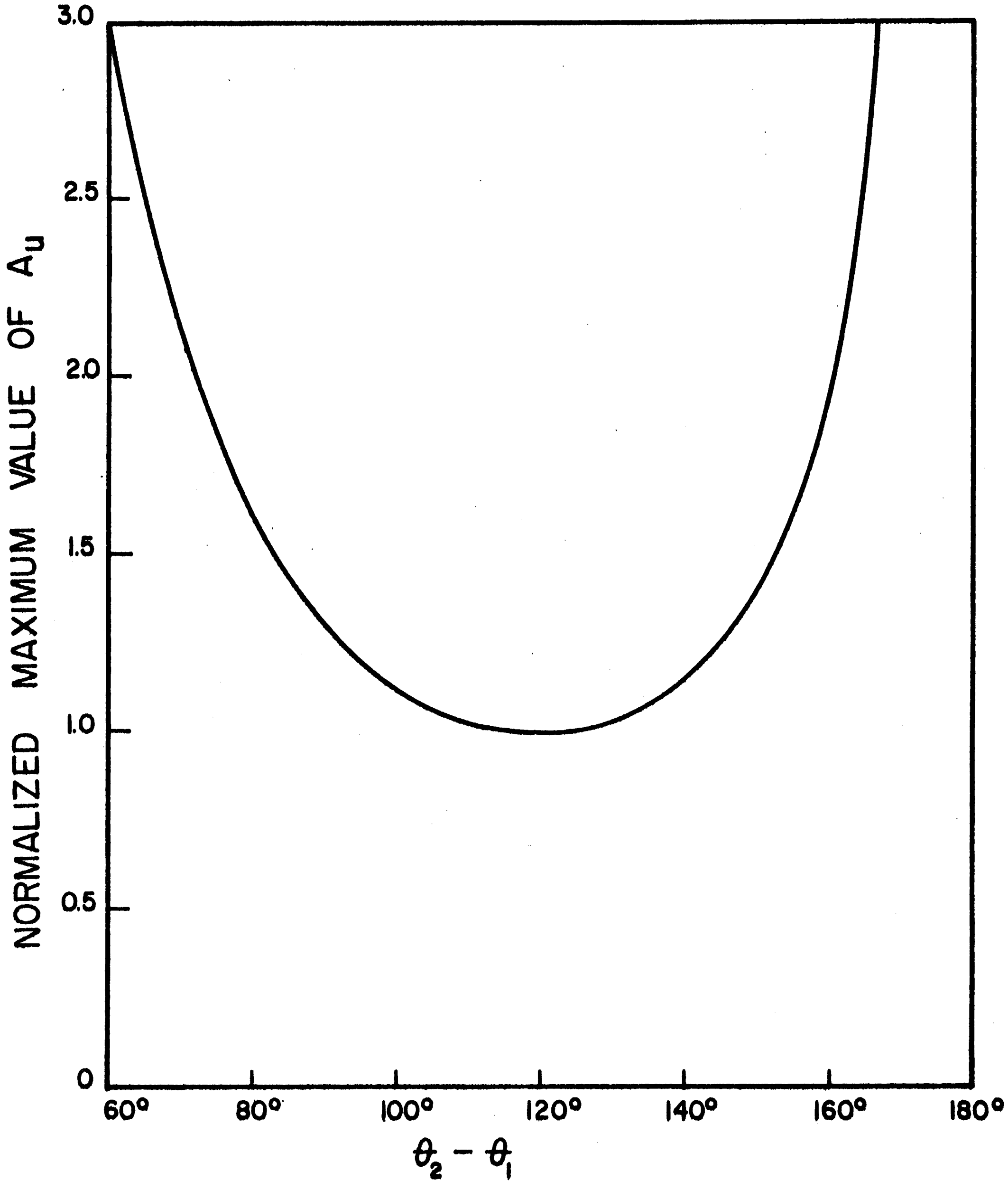

Figure 5-5 Maximum Values of A_U on the Loci of Constant Values of $\theta_2 - \theta_1$

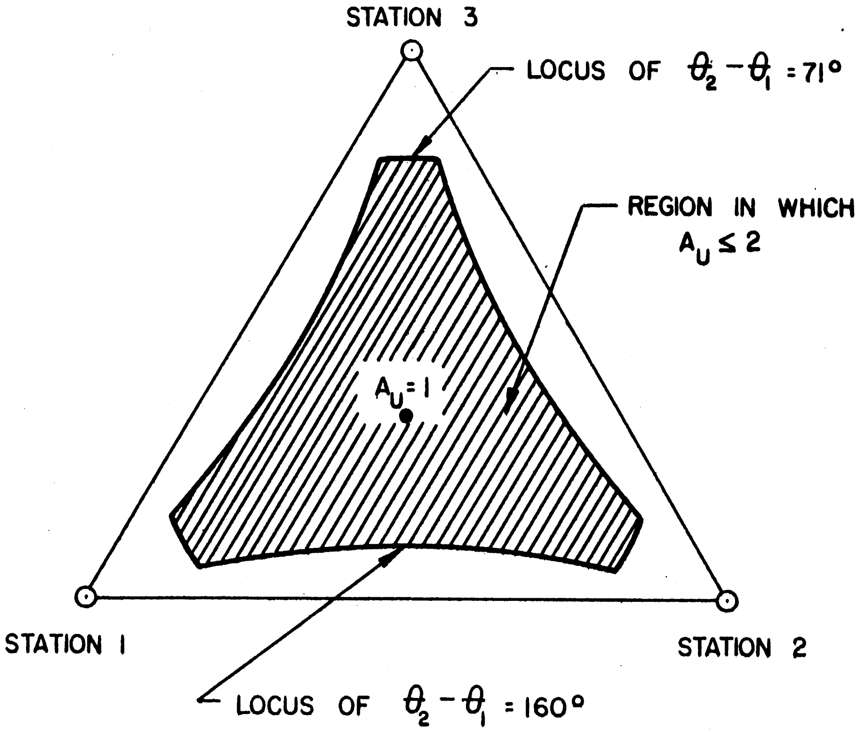

Figure 5-6 Region in which A_U Does Not Exceed Twice its Value at the Center

of the equilateral triangle is an adequate estimate of the position error for a target located in a region around the center at least as large as half the area of the triangle.

Probability Distribution for Bearing-Reading Error

At the time the position of the target is calculated in a triangulation system which contains three bearing-measurement stations, the bearing reading from one of the stations is new, and the age of each of the other two bearing readings is described by the uniform probability distribution discussed in Chapter IV. For the purpose of calculation, the new bearing reading is assumed to have been taken at bearing-measurement Station 1. The assumption is arbitrary because the error in bearing reading at each bearing-measurement station enters into the equation for the normalized position error in the same way.

Figure 5-7 is a sketch of the probability distributions for the error and the components of the error in the bearing readings at each of the bearing-measurement stations for a particular target path. The probability distribution for the error in bearing reading at Station 1, ϵ_1 , is the same as the probability distribution for the error in the bearing measurement itself, ϵ_{B1} , because the age of the measurement is zero. The quantity, ϵ_{B1} , and therefore ϵ_1 , are assumed to be normally distributed for the reasons discussed in Chapter IV. The standard deviations of these distributions are denoted by $\sigma_{\epsilon B}$.

The error in the bearing readings at Stations 2 and 3 is given in Equation (4-23) as

$$\epsilon_2 = \epsilon_{B2} + \epsilon_{t2} \tag{5-7}$$

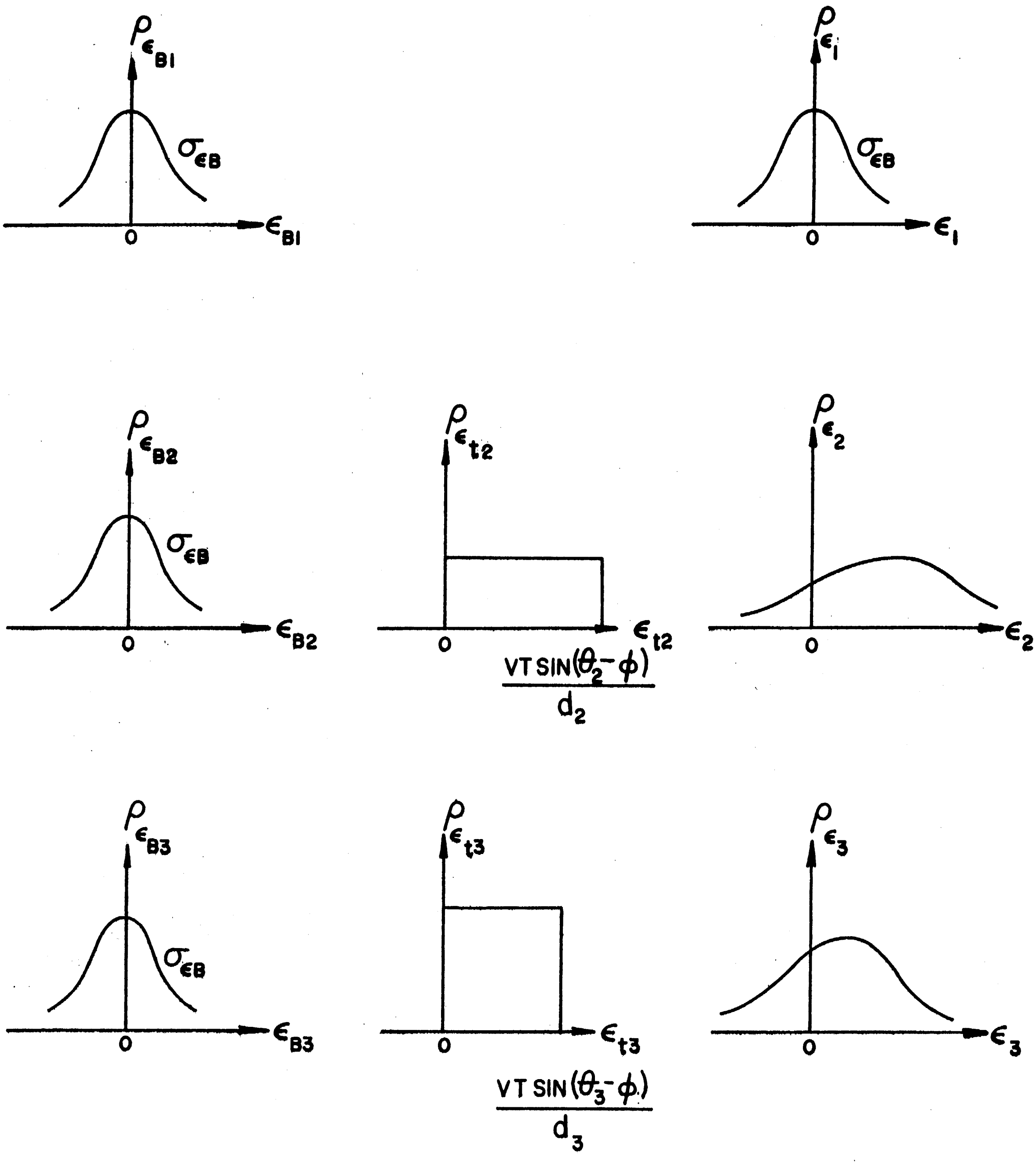

Figure 5-7 Probability Density-Functions for Bearing-Reading Error and the Components of Bearing-Reading Error

and

$$\epsilon_3 = \epsilon_{B3} + \epsilon_{t3} \quad \text{, in which} \tag{5-8}$$

the subscripts 2 and 3 refer to the bearing-measurement Stations 2 and 3, respectively, and in which

ϵ_i is the error in the bearing reading,

ϵ_{Bi} is the error in the bearing measurement, and

ϵ_{ti} is the error due to motion of the target from the time the bearing measurement is performed to the time that the bearing measurement is used in the calculation of the position of the target.

The error in bearing measurement at Stations 2 and 3 is assumed to be normally distributed for the reasons discussed in Chapter IV. If it is assumed also that identical bearing-measurement apparatus is used at each station, and if there are no specific reasons for the characteristics of the error in bearing measurement to differ among the three stations[1], the standard deviations of the probability distributions for ϵ_{B2} and ϵ_{B3} will also be equal to $\sigma_{\epsilon B}$.

The probability distributions for ϵ_{t2} and ϵ_{t3}, as shown in Figures 4-4 and 5-7, are uniformly distributed between the limits of zero and $VT \sin(\theta_i - \phi)/d_i$, in which i takes on the values 2 and 3, respectively.

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[1] Differences in the conductivity of the ground and other features of the local environment in addition to differences in the propagation paths of the signals received can cause differences in the characteristics of the error even though the measurement apparatus is identical.

For the special case of a target located at the center of the equilateral triangle,

$$d_2 = d_3 = D \quad . \tag{5-9}$$

If D_T is defined as the value of VT, normalized with respect to D, i.e.,

$$D_T = \frac{VT}{D} \quad , \tag{5-10}$$

then the limits of the probability distribution for ϵ_{t2} and ϵ_{t3} may be expressed as zero and $D_T \sin(\theta_i - \phi)$.

The construction of the probability distributions for ϵ_2 and ϵ_3 from the normal and uniform distributions of their components is the first part of a digital computer program for determining the probability distribution for the position error.

Probability Distribution for Position Error

The probability distribution for the position error for a target located at the center of the equilateral triangle formed by three bearing-measurement stations was obtained by a combination of the probability distributions for bearing-reading error at each station, making use of Equation (5-3). This combination is the second part of a digital computer program for numerically determining the probability distribution for the position error. The use of a digital computer requires that each of the probability distributions be approximated by discrete distributions.

The entire probability distribution for position is desired in order that the results be of general use to those who are interested in the value of position error which will not be exceeded except with a very small probability, as well as to those who are interested only in the mean value of position error. If that part of the distribution in the neighborhood of the arithmetic mean of the distribution for position error were of interest only, then the "Monte Carlo" method, which provides reasonably rapid convergence in this neighborhood could be used. Because the probability of values of position error remote from the arithmetic mean of the distribution are of interest, each possible combination, rather than a random sample, of the discrete values of the components must be considered. In addition, systematic consideration of the possible combinations reduces by manyfold the computing time required per combination over that required by the "Monte Carlo" method.

The succeeding sections of this chapter contain a brief description and a discussion of some of the details of the computer program used to obtain probability distributions for the position error.

Description of the Computer Program

The digital computer program which was used to determine the probability distribution for the position error (using the MIDAC computer[1]) is divided into two parts. In the first part, the probability distributions for bearing-reading error are constructed. In the second part.the probability distribution for the position error is determined.

In the first part of the program, the probability distributions which describe the bearing-reading error at each bearing measurement station were approximated by discrete probability distributions

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[1] MIDAC is the MIchigan Digital Automatic Computer, and is located at the University of Michigan.

of many ordinates each. The non-normal distributions which describe the bearing-reading error at Stations 2 and 3 were obtained by processing the normal distribution for the bearing-reading error at Station 1, a distribution which was contained in the computer program. The different distributions which describe the bearing-reading error at Stations 2 and 3 were obtained automatically for each selection of target direction.

In the second part of the program, for each possible combination of the ordinates of the three probability distributions, the joint incremental probability, which is the product of the three ordinates, and the corresponding value of the position error were calculated. The value of the position error for each combination of ordinates was used to obtain a storage address in the computer at which the values of the joint incremental probabilities were accumulated. The computer program was designed to cycle through all of the possible combinations of ordinates of the three probability distributions for bearing-reading error in an efficient manner so that the probability distribution for position error was obtained with a minimum of computing time.

The result of the computation is a set of numbers which is the desired probability distribution. Depending on the particular normalized parameters used in the computation, the location of a number in the set corresponds to a particular value of normalized position error.

Figure 5-8 summarizes, in block diagram form, the brief description of the computer program given above. For convenience in the digital computation, the set of possible values of ϵ_1 , ϵ_2 , and ϵ_3 were represented by integers and integers plus one half. The variables υ, ν , and ω are used to denote this representation. Thus, in

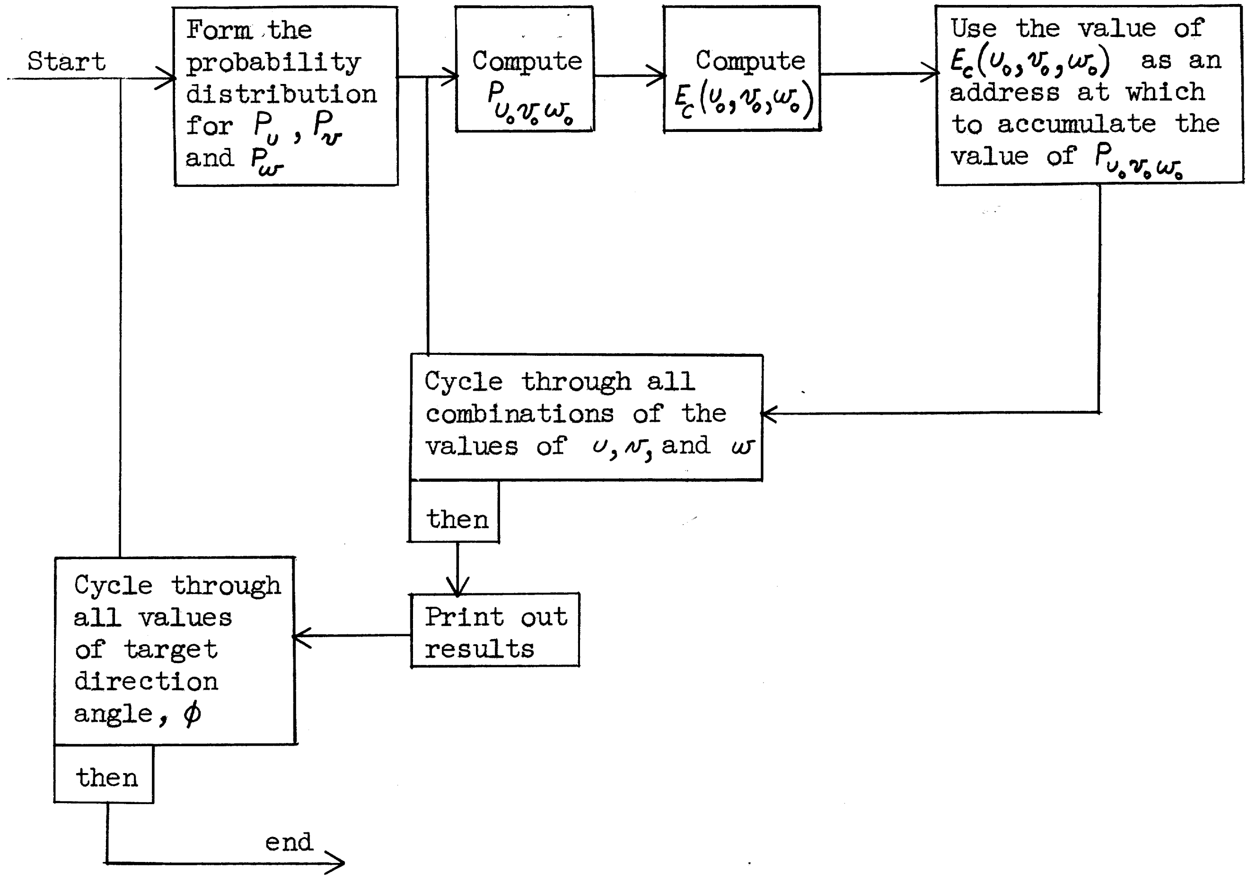

Figure 5-8 Simplified Block Diagram for the Computer Program for the Position Error in the Conventional System

Figure 5-8, P_{u} , P_{v} , and P_{w} are the three discrete probability distributions for the variables u , v , and w, respectively. The joint probability of a particular combination of values of u , v , and w is given by

$$P_{u_o v_o w_o} = P_{u_o} P_{v_o} P_{w_o} \quad . \tag{5-11}$$

The subscript, o , indicates particular values of the variables. E_c is used to represent the square of the position error.

Details of the Computer Program

Each of the probability distributions used in the digital calculation to describe the components of bearing-reading error was approximated by a discrete distribution of many ordinates. For convenience, the elements of the set of possible discrete values of these variables are represented by integers plus one half. The variables u , v', v'', w' , and w'' are used in the description of the computer program to denote these discrete values corresponding to the variables ϵ_1 , ϵ_{B2}, ϵ_{t2}, ϵ_{B3}, and ϵ_{t3}, respectively. The computer variables are related to the components of bearing-reading error by a scale factor, denoted by s and defined by

$$s = \frac{\sigma_u}{\sigma_{\epsilon B}} \tag{5-12}$$

in which

σ_u is the standard deviation of the discrete distribution used in the computer program to describe the variable u , and

$\sigma_{\epsilon B}$ is the standard deviation of the probability distribution for error in bearing measurement.

The variable U is related to ϵ_1, the error in the bearing reading from Station 1, by

$$U = s\epsilon_1 \tag{5-13}$$

If U is considered to be a continuous variable, it is normally distributed with a mean value of zero and a standard deviation, σ_U, which is related to $\sigma_{\epsilon B}$ by Equation (5-12). For use in the digital computation, the normal distribution for U is modified by truncation, i.e., by neglecting both "tails" of the distribution, as illustrated in Figure 5-9. The remainder of the distribution is divided into 50 equal intervals with integer values of U as boundaries and is approximated by discrete ordinates located at the center of the intervals. The discrete ordinates are equal to the area under the normal probability-density curve for the corresponding intervals. The values of the ordinates listed in the computer program were obtained from a table of areas.[1]

In the course of the digital computer study, two standard deviations for the probability distribution for the variable $U = s\epsilon_1$ were used. These are $\sigma_U = 10$ and $\sigma_U = 50/7$. In both cases, fifty ordinates were used for the distribution, corresponding to the ranges $-2.5\sigma_U \leq U \leq 2.5\sigma_U$ and $-3.5\sigma_U \leq U \leq 3.5\sigma_U$. The fifty ordinates were listed as constants in the computer program.

As previously discussed, the probability distributions for ϵ_{B1}, ϵ_{B2}, ϵ_{B3}, and ϵ_1 are assumed to be identical. Therefore, the probability distribution listed in the computer program for the discrete variable U can be used also as an approximation of the

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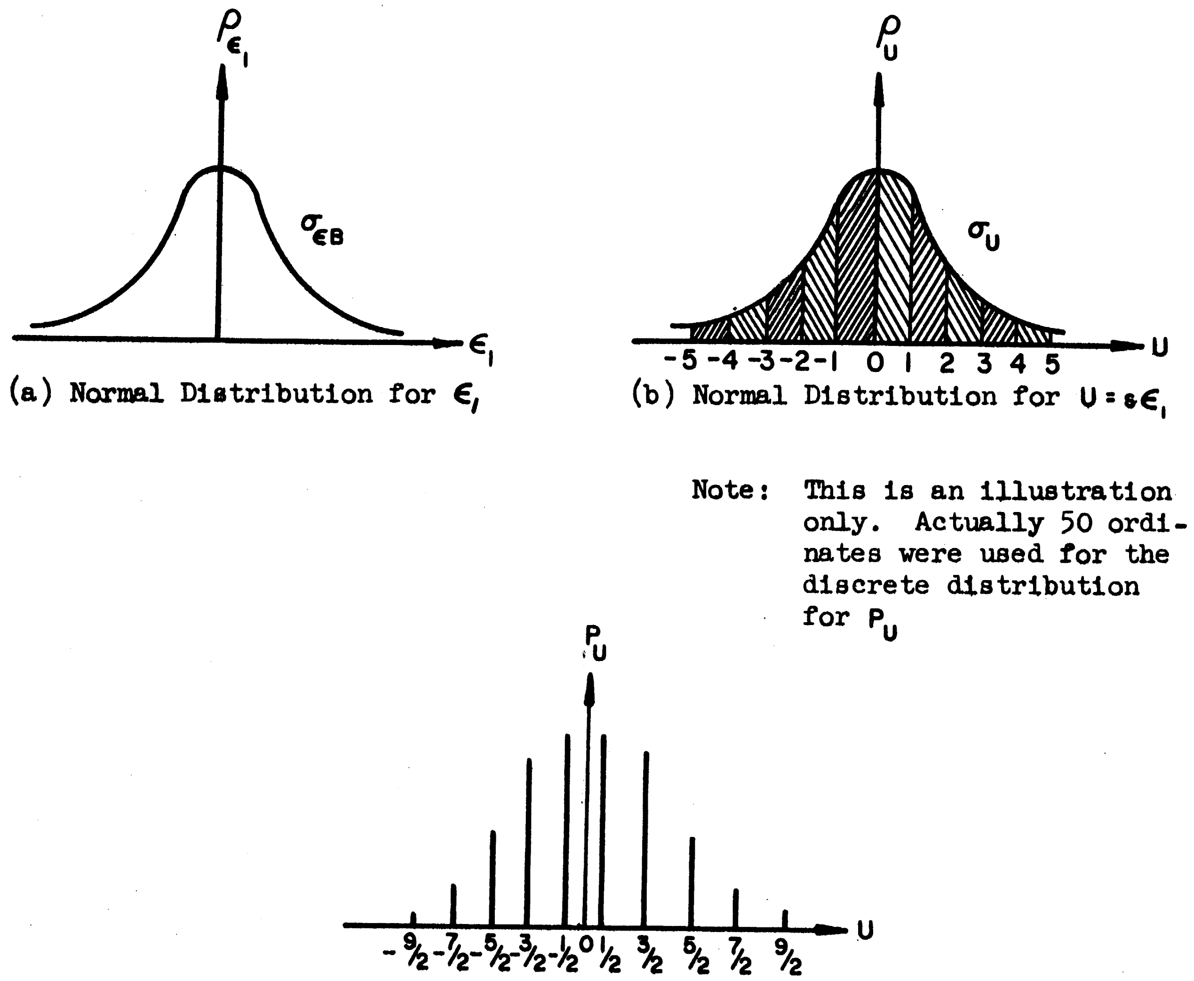
[1] Carver(6).



(a) Normal Distribution for ϵ_1

(b) Normal Distribution for $U = s\epsilon_1$

Note: This is an illustration only. Actually 50 ordinates were used for the discrete distribution for P_U

(c) Approximate Discrete Distribution for U

Figure 5-9 Approximation of the Probability Distribution for ϵ_1

probability distributions for the variables v' and w', which correspond to ϵ_{B2} and ϵ_{B3}. Although the distributions are the same, the variables are independent. Equations (5-14) and (5-15) define the variables v' and w':

$$v' = s\,\epsilon_{B2} \tag{5-14}$$

and

$$w' = s\,\epsilon_{B3} \tag{5-15}$$

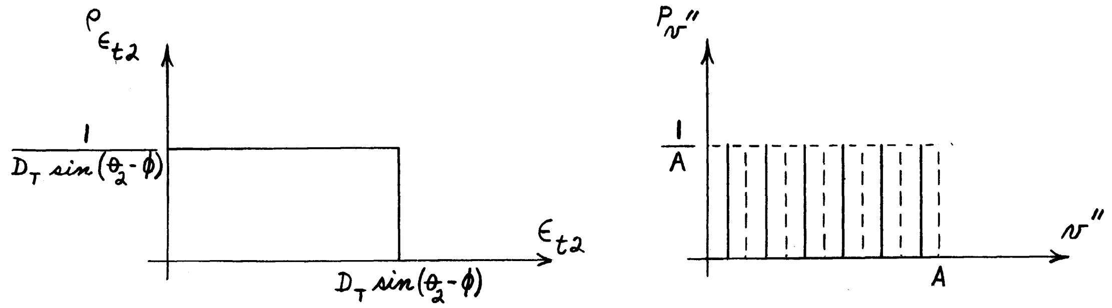
The probability distributions which describe ϵ_{t2} and ϵ_{t3}, the components of bearing-reading error due to target motion and age of the bearing measurement, are uniformly distributed between the limits of zero and $D_T \sin(\theta_i - \phi)$, as previously discussed. The variables v'' and w'' are related to ϵ_{t2} and ϵ_{t3} by

$$v'' = s\,\epsilon_{t2} \tag{5-16}$$

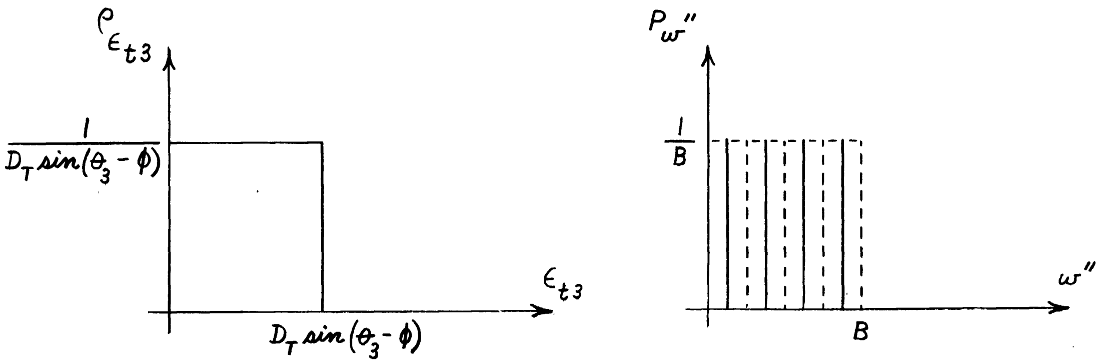
and

$$w'' = s\,\epsilon_{t3} \tag{5-17}$$

and are uniformly distributed between the limits of zero and $sD_T \sin(\theta_2 - \phi)$ and $sD_T \sin(\theta_3 - \phi)$, respectively. In order to approximate the continuous uniform distributions by usable discrete distributions, the non-zero limits are approximated by the closest integers, A and B, respectively. The intervals between zero and A and the interval between zero and B are divided into $|A|$ and $|B|$ sub-intervals, respectively. The continuous distributions are then approximated by discrete distributions with ordinates of amplitude $1/|A|$ and $1/|B|$, respectively, centered in the sub-intervals. This approximation is illustrated in Figure 5-10.



A = closest integer to $s\,D_T \sin(\theta_2 - \phi)$, in which $s = \sigma_U / \sigma_{\epsilon B}$



B = closest integer to $s\,D_T \sin(\theta_3 - \phi)$, in which $s = \sigma_U / \sigma_{\epsilon B}$

Figure 5-10 Approximation of the Probability Distributions for ϵ_{t2} and ϵ_{t3}

If v, v', and v'' were considered to be continuous variables with probability density-functions $\rho_v(v)$, $\rho_{v'}(v')$, and $\rho_{v''}(v'')$, respectively, the probability distribution for $v = v' + v''$ can be expressed in terms of the convolution integral

$$\rho_v(v) = \int_{-\infty}^{\infty} \rho_{v''}(v'')\, \rho_{v'}(v - v'')\, dv'' \quad . \qquad (5\text{-}18)$$

To describe the method used in the computer program to perform the convolution of the discrete probability distributions of the variables v' and v'', it is convenient to interpret the probability distribution of the variable v as the sum of the discrete conditional probability distributions of v for each value of v''. Let $P_c\,(v \mid v'' = v_o'')$ be the discrete, conditional probability distribution for the variable v for $v'' = v_o''$, a particular value of v''. Then

$$P_c\,(v \mid v'' = v_o'') = P_{v'}(v - v_o'') \quad . \qquad (5\text{-}19)$$

The probability distribution for P_v is the summation of the products of each of the conditional probabilities and the corresponding probability of the condition, i.e., the probability of the corresponding value of v_o''. Therefore,

$$P_v = \sum_{v_o''} P_{v''}(v_o'')\, P_c\,(v \mid v'' = v_o'')$$

$$= \sum_{v_o''} P_{v''}(v_o'')\, P_{v'}(v - v_o'') \quad , \qquad (5\text{-}20)$$

which is the equivalent of Equation (5-18) for discrete variables. Figure 5-11 illustrates the convolution by summation for the case of $A=3$. In the program for the computer study, sequential computer storage locations were assigned to sequential values of v over the entire range of values of v considered. For each value of A selected, the computer program was designed to have the following operations performed automatically:

(1) The probability distribution which was listed in the computer program and used for the computer variables U, v', and v'' was multiplied by $1/|A|$ to obtain the set of ordinates for $P_{v''}(v_0'')P_{v'}(v - v_0'')$.

(2) The storage location corresponding to v_s, the smallest possible value of v ($v_s = -24$ for $A > 0$ and $v_s = -24 + A$ for $A < 0$), was determined.

(3) The set of 50 ordinates for $P_{v''}(v_0)P_{v}(v - v_0'')$ was added to the contents (original contents were zero) of 50 sequential storage locations starting with the one corresponding to v_s. The addition was iterated a total of $|A|$ times with the initial storage location increased by one each time.

In this way, the probability distribution for the variable v was constructed. The probability distribution for the variable w was constructed in the same way with the simplification, $B > 0$, which results from the symmetry described in the following paragraphs.

For each value of sD_T selected for use in the digital calculation, a set of twelve values of A and B, corresponding to the set of twelve target directions considered, was listed in the computer program. Values of target direction, ϕ, over a range of only $\pi/2$ need be considered because of symmetry. To illustrate this symmetry, an expression for the position error in terms of the computer variables is utilized.

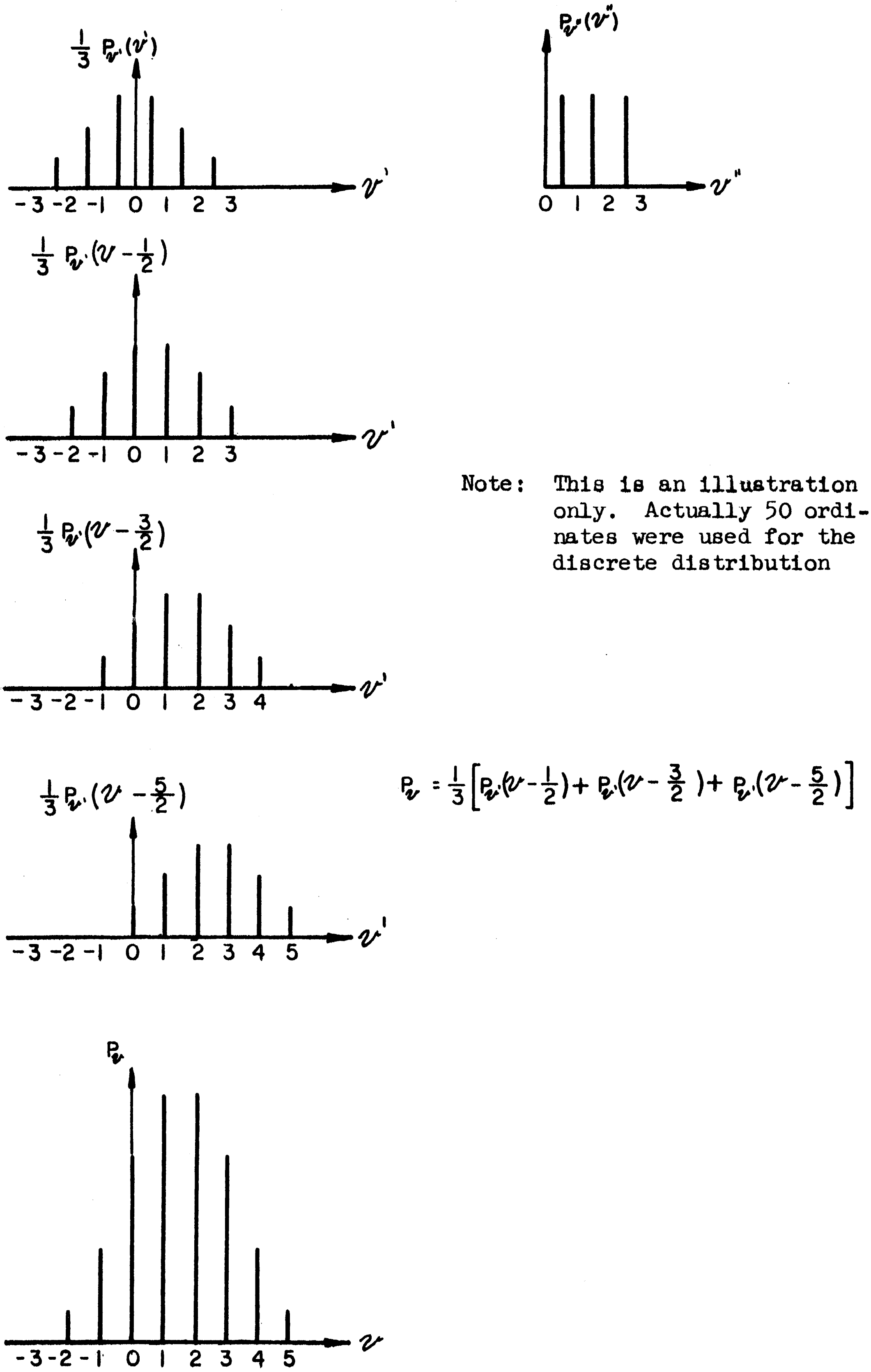


Figure 5-11 Convolution to Obtain the Probability Distribution for the Variable v

Equation (5-3) for the position error can be written in terms of the computer variables as follows:

$$E_c = \frac{9E^2S^2}{4D^2} = u^2 + v^2 + w^2 - uv - vw - wu \qquad (5\text{-}21)$$

in which:

$$v = v' + v'' , \qquad (5\text{-}22)$$

$$w = w' + w'' , \qquad (5\text{-}23)$$

and

E_c = the variable used in the digital computer program to represent the square of the position error.

E_c is a function of the variables u , v' , v'' , w' , w'' , and also ϕ which determines one of the limits of the range of possible values of v'' and w''. Therefore, E_c is denoted by $E_c(u, v', v'', w', w'', \phi)$.

The symmetry of E_c with respect to target direction can be examined with the aid of Figure 5-12. For the particular reference angle arbitrarily selected in Figure 5-12 (a), sin $(\theta_2 - \phi)$ and sin $(\theta_3 - \phi)$, to which A and B are proportional, have been sketched in Figure 5-12 (b). Because

$$\sin(\theta_2 - \phi - \pi) = -\sin(\theta_2 - \phi) \qquad (5\text{-}24)$$

and

$$\sin(\theta_3 - \phi - \pi) = -\sin(\theta_3 - \phi) , \qquad (5\text{-}25)$$

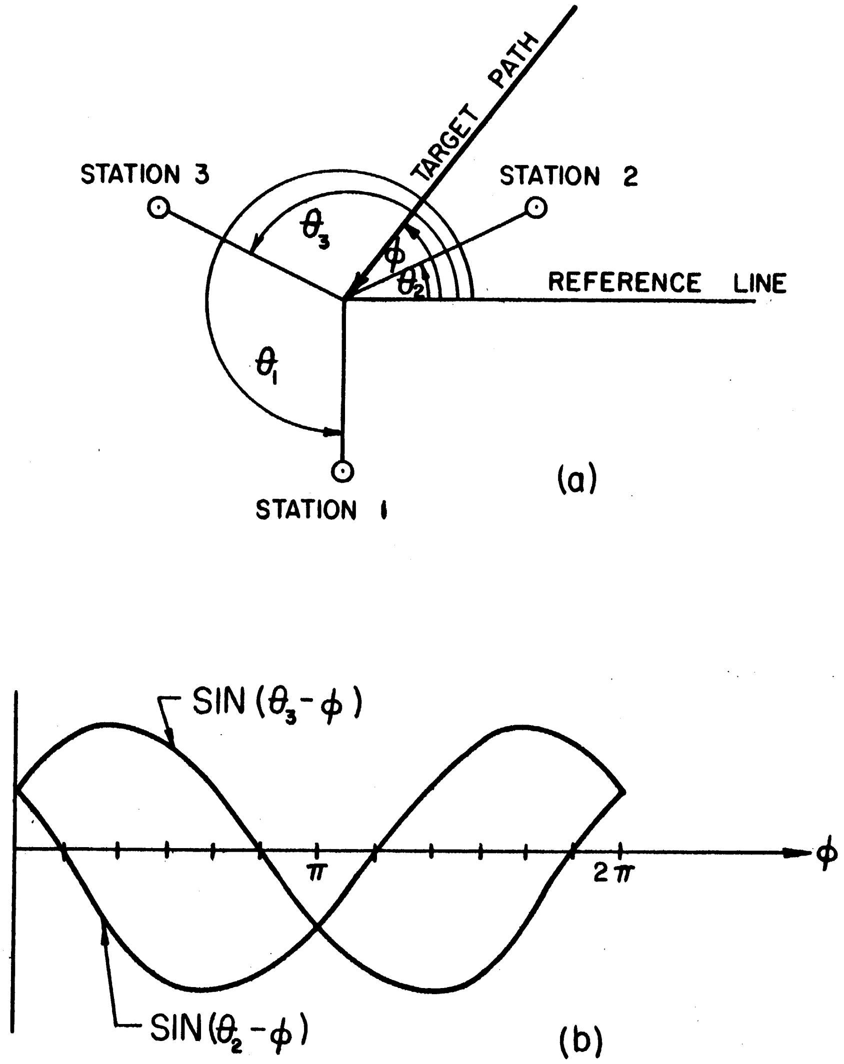


Figure 5-12 Symmetry with Respect to ϕ

the following is true:

$$E_c(v, v', v'', w', w'', \phi + \pi) = E_c(v, v', -v'', w', -w'', \phi) . \quad (5\text{-}26)$$

The probability distributions for v, v', and w' are even functions of these variables so that

$$E_c(v, v', -v'', w', -w'', \phi) = E_c(-v, -v', -v'', -w', -w'', \phi) . \quad (5\text{-}27)$$

Equation (5-21) shows that

$$E_c(-v, -v', -v'', -w', -w'', \phi) = E_c(v, v', v'', w', w'', \phi) , \quad (5\text{-}28)$$

so that

$$E_c(v, v', v'', w', w'', \phi + \pi) = E_c(v, v', v'', w', w'', \phi) . \quad (5\text{-}29)$$

Additional symmetry exists because v'' and w'' can be interchanged in Equation (5-21) with no effect on E_c . The fact that the probability distributions for v' and w' are identical makes this possible. In addition, because

$$\sin(\theta_3 - \phi) = -\sin(\theta_2 - \pi + \phi) \quad (5\text{-}30)$$

and

$$\sin(\theta_2 - \phi) = -\sin(\theta_3 - \pi + \phi) , \quad (5\text{-}31)$$

the following is true:

$$E_c(v, v', v'', w', w'', \pi - \phi) = E_c(v, v', -w'', w', -v'', \phi)$$

$$= E_c(-v, -v', -v'', -w', -w'', \phi)$$

$$= E_c(v, v', v'', w', w'', \phi) \quad . \qquad (5\text{-}32)$$

Therefore, only the values of ϕ in the range

$$0 \leq \phi \leq \pi/2 \qquad (5\text{-}33)$$

need be considered.

The process used in the computer program to calculate P_v and P_w for each value of ϕ considered is summarized in the block diagram for Part One of the computer program which is shown in Figure 5-13.

In the digital computer study, target direction is assumed to be uniformly distributed over the interval $0 \leq \phi \leq 2\pi$. The previous discussion points out that for purposes of calculation, only the range $0 \leq \phi \leq \pi/2$ need be considered. The uniform distribution over the latter interval was divided into 12 equal intervals, and approximated by 12 ordinates, centered in these intervals as shown in Figure 5-14. A probability distribution for the position error was calculated and printed out for this distribution of target direction. In the process, a separate probability distribution for each value of target position was also calculated and printed out.

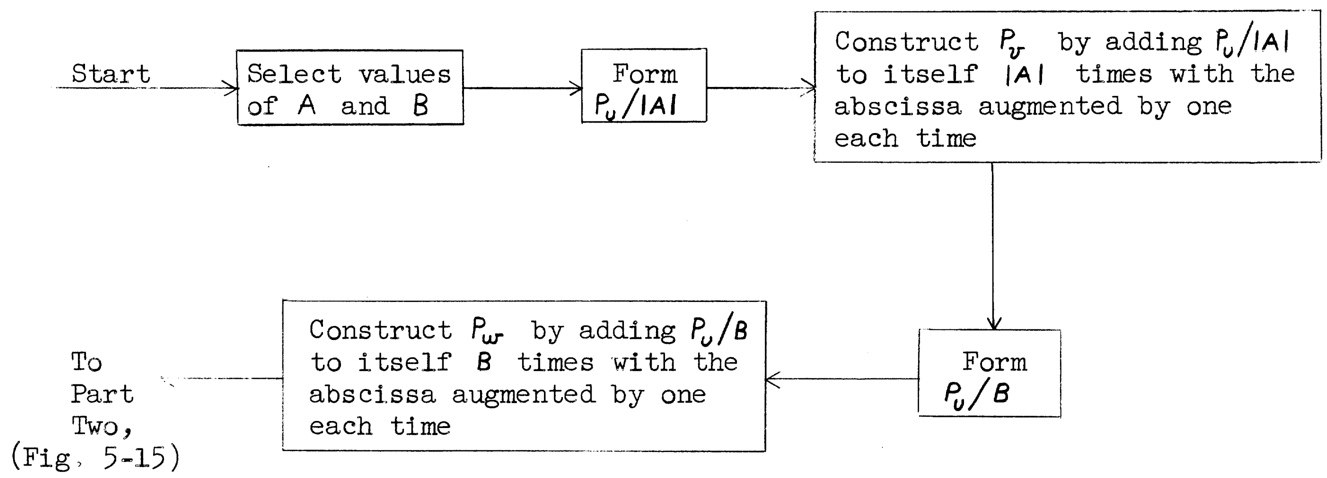


Figure 5-13 Part One of the Computer Program for the Position Error in the Conventional System

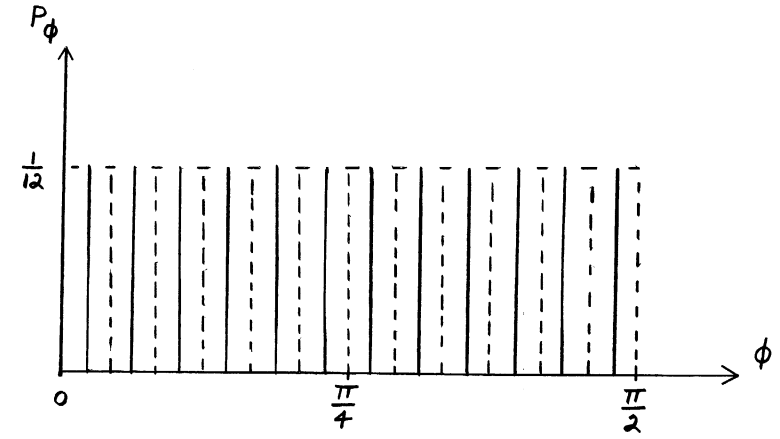


Figure 5-14 Discrete Distribution for Target Direction

For each set of normalized triangulation-system parameters considered, the probability distribution for the position error was constructed by examining every possible combination of values of the variables u, v, and w. For each combination, the value of $E_c(u, v, w)$ was calculated. Then, the value of $P_{uvw} = P_u P_v P_w$ was calculated and added to the contents of one of sixty storage locations determined by the value of $E_c(u, v, w)$. Thus, the probability distribution for position error was obtained as a set of numbers from the set of sixty storage locations. The position of a number in the set corresponded to a particular range of values of E.

For the computation, it is convenient to calculate the normalized position error in terms of its square, as given in Equation (5-21). If this form had been used directly to determine the storage address for P_{uvw}, the smaller values of position error would have been compressed into a relatively few storage locations and the larger values of position error would have been spread uneconomically among the majority of the storage locations. Rather than utilize a time-consuming routine to calculate the square root of E_c, the transformation

$$E_R = K_3 + \frac{K_1}{E_c + K_2} \quad , \tag{5-34}$$

in which K_1, K_2, and K_3 are constants, was used. K_1 and K_2 were chosen so that

$$0 \leq K_1/(E_c + K_2) < 60 \tag{5-35}$$

over the entire range of values of υ, v, and ω considered, and so that only the value of $K_1/(E_c + K_2)$ which corresponds to smallest discrete value of E_c would be within the interval

$$59 \leq K_1/(E_c + K_2) \leq 60 \quad . \tag{5-36}$$

The constant K_3 was selected to be the computer "address" of the first of the 60 storage locations used for the accumulation of the probability distribution of position error.

Inequality (5-35) can be rearranged to show the range of values of E corresponding to each storage location. For the p-th storage location, where $1 \leq p \leq 60$,

$$p-1 \leq K_1/(E_c + K_2) < p \quad . \tag{5-37}$$

K_1 , K_2 and E_c are all greater than zero so that Inequality (5-37) can be rearranged into the form

$$\frac{K_1}{p} - K_2 < E_c \leq \frac{K_1}{p-1} - K_2 \quad . \tag{5-38}$$

Using Equation (5-21), Inequality (5-38) can be further modified to the form

$$\frac{4D^2}{9s^2}\left[\frac{K_1}{p} - K_2\right] < E^2 \leq \frac{4D^2}{9s^2}\left[\frac{K_1}{p-1} - K_2\right] \tag{5-39}$$

in which $s = \sigma_U / \sigma_{\epsilon B}$. Inequality (5-39) expresses the position error corresponding to each storage location in terms of constants and the two triangulation-system parameters, D and $\sigma_{\epsilon B}$. For each value of the other parameters, D_T and ϕ, separate probability distributions were obtained.

Part Two of the computer program was designed to obtain the probability distribution for position error by examining every possible combination of U , v , and w in a systematic and efficient manner. A block diagram of Part Two of the program is presented in Figure 5-15, in which the subscript, s , is used to denote the smallest possible value of each variable. In the calculations the possible values were considered according to ascending numerical order.

Part Two of the computer program consists of three loops corresponding to three variables whose values must be cycled. In Step 1, v_s , the initial value of v is selected. In Steps 2 and 3, all of the

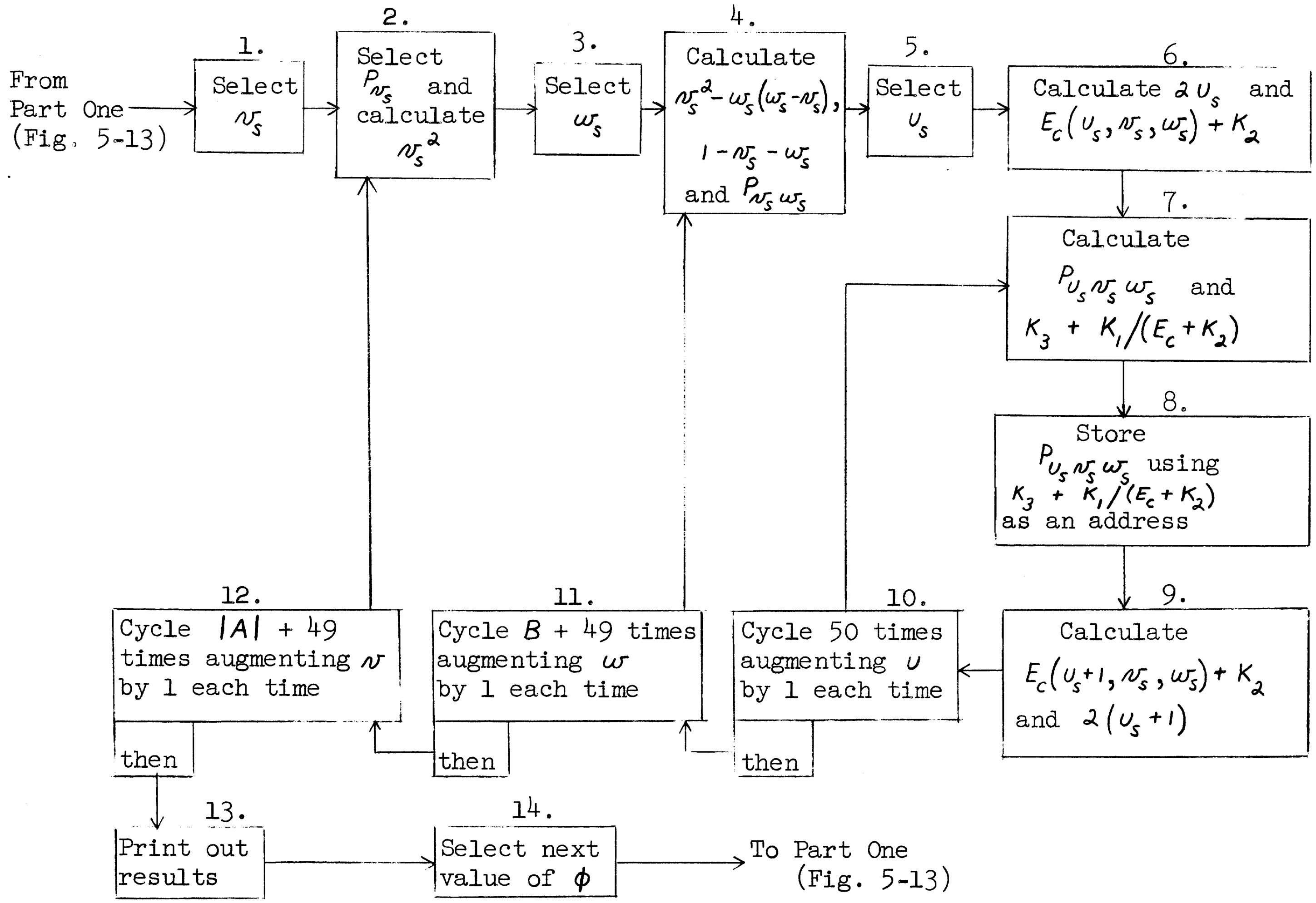


Figure 5-15 Part Two of the Computer Program for the Position Error in the Conventional System

computation that can be performed after the value of v has been selected is performed and, w_s , the initial value of w is selected. In Steps 4, 5, and 6, all of the additional computation that can be performed after the value of w has been selected is performed. Then, u_s , the initial value of u is selected, and $2u_s$, and $E_c(u_s, v_s, w_s)$ are calculated. In the inner loop, which consists of Steps 7, 8, 9, and 10, $P_{u_s v_s w_s}$ is calculated and properly stored, making use of $E_c(u_s, v_s, w_s)$. Then, $E_c(u_s+1, v_s, w_s)$ is calculated in a simple fashion, making use of $E_c(u_s, v_s, w_s)$ and other quantities calculated in steps outside of the inner loop, as indicated in the following equations in which the subscript, o , indicates any particular value of the variable. From Equation (5-21),

$$E_c(u_o, v_o, w_o) = u_o^2 + v_o^2 + w_o^2 - u_o v_o - v_o w_o - w_o u_o \qquad (5\text{-}40)$$

and

$$E_c(u_o+1, v_o, w_o) = (u_o+1)^2 + v_o^2 + w_o^2 - (u_o+1)v_o - v_o w_o - w_o(u_o+1) \ . \qquad (5\text{-}41)$$

Equation (5-41) can be rearranged into the form

$$E_c(u_o+1, v_o, w_o) = u_o^2 + v_o^2 + w_o^2 - u_o v_o - v_o w_o - w_o u_o + 2u_o + 1 - v_o - w_o \ , \qquad (5\text{-}42)$$

or

$$E_c(\upsilon_0+1, \nu_0, \omega_0) = E_c(\upsilon_0, \nu_0, \omega_0) + 2\upsilon_0 + 1 - \nu_0 - \omega_0 \quad . \tag{5-43}$$

Equation (5-43) was used to calculate E_c in the inner loop of the computer program because this method requires a minimum of computation time on MIDAC, the digital computer which was used.

In Steps 10, 11, and 12, the number of cycles through each of the loops is controlled: 50 times for the variable υ, $(B+49)$ times for the variable ω and $(|A|+49)$ times for the variable ν. A total of $50(B+49)(|A|+49)$ combinations of values of υ, ν and ω were considered to obtain each probability distribution for position error. After cycling through these combinations, the computer printed out the probability distribution.

In Step 13, the probability distribution was read out in the form of a cumulative probability function rather than a probability density function in order that the results be useful directly and to provide a check[1] on the results. In Step 14, the number of values of target direction which were considered was controlled.

The computer program is listed in Appendix A. The changes necessary in the program to change P_υ and D_T are listed in Appendixes D and E, respectively.

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[1] $\sum_{E_c} P_{E_c} = \left(\sum_{\upsilon} P_\upsilon\right)\left(\sum_{\nu} P_\nu\right)\left(\sum_{\omega} P_\omega\right)$, a known value.

CHAPTER VI

MODIFIED THREE-STATION SYSTEM

A modification of the three-station triangulation system was investigated also. In this modification, only two bearing readings are used to calculate the position of the target. When a bearing report from one of the bearing-measurement stations is received at the central station, the more recent of the reports on the same target from the other two bearing-measurement stations is selected from storage. The "new" reading and the "more recent" reading are then used to calculate the position of the target.

The numerical investigation of this modification proceeds in the same general fashion as the investigation of the unmodified case which was described in Chapter V. Details, of course, are different. The result of the digital computer study is a set of separate probability distributions for the magnitude of the position error for the parameters: (1) target direction and (2) a combination of target velocity and time delay. Probability distributions are also presented for the case of a uniform distribution for target direction. The results of the investigation of the modified case and the unmodified case can be compared directly.

Equation for the Position Error

In this modified case, for the purpose of calculation, the new bearing reading is assumed to have been taken at bearing-measurement Station 1. Thus, Equation (5-33), which expresses the magnitude of the position error when the error in bearing reading is small, can be expanded into the form

$$E^2 = \frac{d_1^2\epsilon_1^2 + d_a^2\epsilon_a^2 - 2d_1 d_a \epsilon_1 \epsilon_a \cos(\theta_a - \theta_1)}{\sin^2(\theta_a - \theta_1)} , \quad (6\text{-}1)$$

in which the subscript "a" assumes the values 2 or 3, depending upon which bearing-measurement station provides the more recent bearing reading. Of course, no weighting factor can be used because only one intersection is computed.

As in the unmodified case, the bearing-measurement stations are assumed to be located at the vertexes of an equilateral triangle. It is convenient to normalize all distances, including the position error, with respect to the distance (D) of each bearing-measurement station from the center of the equilateral triangle. The normalized position error, denoted by E_D, is obtained by modifying Equation (6-1) to the form

$$E_D^2 = \frac{E^2}{D^2} = \frac{\left(\frac{d_1}{D}\right)^2\epsilon_1^2 + \left(\frac{d_a}{D}\right)^2\epsilon_2^2 - 2\left(\frac{d_1}{D}\right)\left(\frac{d_a}{D}\right)\epsilon_1\epsilon_a \cos(\theta_a - \theta_1)}{\sin^2(\theta_a - \theta_1)} . \quad (6\text{-}2)$$

Variation in the Position Error

The variation in the normalized position error as the location of the target is varied throughout the triangle formed by the bearing-measurement stations is investigated by using the simple, discrete, probability distribution for bearing-reading error (Figure 5-1) that was used in the study of the unmodified case. For this investigation, only two bearing stations need be used. Stations 1 and 2 have been selected arbitrarily.

Using this simple, discrete distribution, four equally likely arrangements of bearing-reading error are possible:

Case I $\epsilon_1 = \epsilon \quad , \quad \epsilon_2 = \epsilon$

Case II $\epsilon_1 = \epsilon \quad , \quad \epsilon_2 = -\epsilon$

Case III $\epsilon_1 = -\epsilon \quad , \quad \epsilon_2 = \epsilon$

Case IV $\epsilon_1 = -\epsilon \quad , \quad \epsilon_2 = -\epsilon$

The expected value of E_D, denoted by E_X, is given by

$$E_X = \frac{1}{4}\left[E_I + E_{II} + E_{III} + E_{IV}\right] \tag{6-3}$$

in which the Roman numeral subscripts indicate the particular arrangements of bearing-reading error. From Equation (6-2) (Cases I and IV and Cases II and III yield identical values for E_D), E_X can be expressed as

$$E_X = \frac{\epsilon}{2\sin(\theta_2-\theta_1)}\left[\begin{array}{l}\left[\left(\frac{d_1}{D}\right)^2 + \left(\frac{d_2}{D}\right)^2 + \frac{2d_1d_2}{D^2}\cos(\theta_2-\theta_1)\right]^{1/2} \\ + \left[\left(\frac{d_1}{D}\right)^2 + \left(\frac{d_2}{D}\right)^2 - \frac{2d_1d_2}{D^2}\cos(\theta_2-\theta_1)\right]^{1/2}\end{array}\right] . \tag{6-4}$$

Reference to Figure 6-1, which describes the geometry of this situation, shows by the "cosine law" for a triangle that

$$\left(\frac{d_1}{D}\right)^2 + \left(\frac{d_2}{D}\right)^2 - \frac{2d_1d_2}{D^2}\cos(\theta_2 - \theta_1) = 3 \quad . \tag{6-5}$$

Equation (6-4) can be simplified, using Equation (6-5), to the form:

$$E_x = \frac{\epsilon}{2\sin(\theta_2 - \theta_1)}\left[\left[3 + \frac{4d_1d_2}{D^2}\cos(\theta_2 - \theta_1)\right]^{1/2} + \sqrt{3}\right] \quad . \tag{6-6}$$

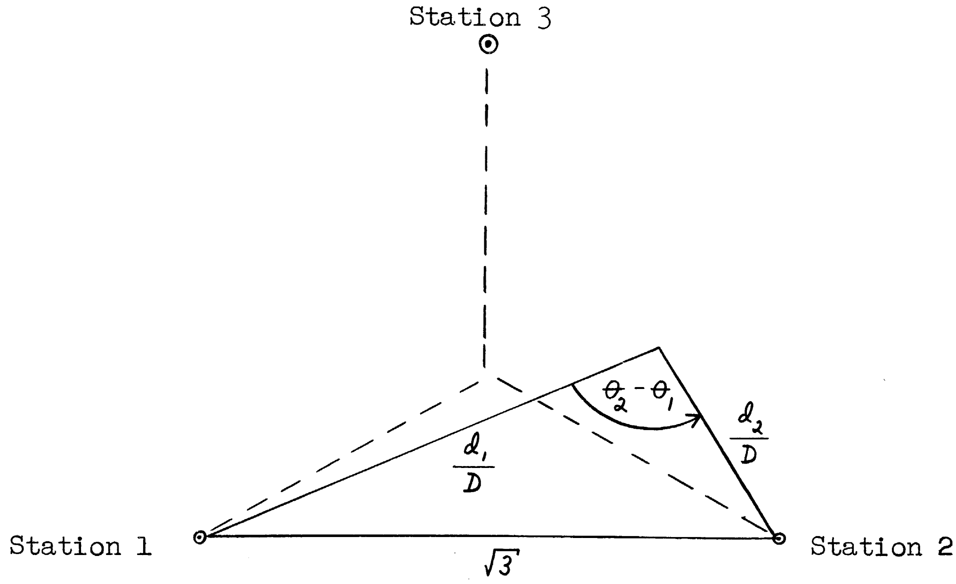


Figure 6-1 Normalized Geometry for the Two-Bearing-Reading Case

For numerical computation it is convenient to define the angles α_1 and α_2 as the angles between the bearing lines to the target and the line joining the bearing-measurement stations. The "law of sines" for the triangle is

$$\frac{d_1/D}{\sin\alpha_1} = \frac{d_2/D}{\sin\alpha_2} = \frac{\sqrt{3}}{\sin(\theta_2-\theta_1)} , \qquad (6\text{-}7)$$

in which $\alpha_1 + \alpha_2 + \theta_2 - \theta_1 = \pi$.

Therefore,

$$\frac{d_1 d_2}{D^2} = \frac{3\sin\alpha_1 \sin\alpha_2}{\sin^2(\theta_2-\theta_1)} . \qquad (6\text{-}8)$$

Substitution of Equation (6-8) into Equation (6-6) yields

$$E_x = \frac{\sqrt{3}\,\epsilon}{2\sin(\theta_2-\theta_1)}\left[\left[1+\frac{4\sin\alpha_1\sin\alpha_2\cos(\theta_2-\theta_1)}{\sin^2(\theta_2-\theta_1)}\right]^{1/2}+1\right] . \qquad (6\text{-}9)$$

Equation (6-9) was used to calculate values of E_x for a variety of target locations as shown in Figure 6-2. For a target located near the center of the equilateral triangle, the value of E_x is 1.59ϵ. As explained in Chapter V, the values of E_x calculated from Equation (6-9) are not realistic because the probability distributions for bearing reading that were used are not realistic. However, the calculations demonstrated that the variation of E_x is small in a region surrounding

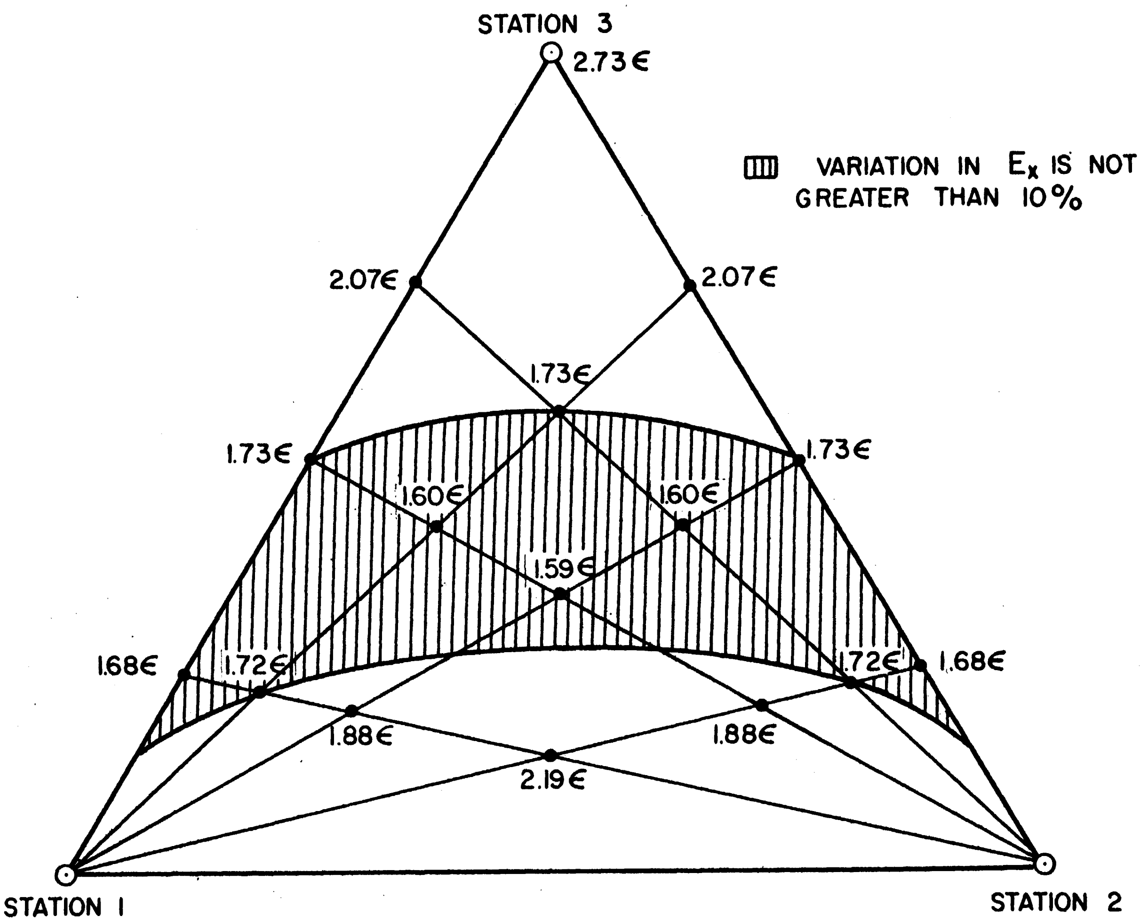


Figure 6-2 Variation of the Expected Value of the Position Error for a Simple, Discrete Distribution for Bearing-Reading Error

the center of the equilateral triangle. The region in which E_X does not exceed its value at the center of the triangle by more than ten percent is shaded in Figure 6-2. If the variation in the position error with variation in target location is investigated by using a normal distribution for bearing-reading error, the results obtained in Chapter V apply directly. The region in which the area of uncertainty is no larger than twice its value at the center of the triangle is at least 80 percent of the area of the equilateral triangle.

In the numerical investigation to determine the probability distribution for the position error using realistic probability distributions for the error in the bearing readings, only a target located at the center of the equilateral triangle has been considered. This provides an adequate estimate of the position error in a region surrounding the center.

Probability Distribution for Bearing-Reading Error

In the modification of the three-station triangulation system in which only two bearing readings are used to calculate the position of the target, the probability distribution which describes the component of the error in the bearing reading from each station due to error in the bearing measurement itself , ϵ_B , is the same as in the unmodified case. This component, ϵ_B , is described by a normal distribution with a mean of zero and a standard deviation $\sigma_{\epsilon B}$. The probability distribution which describes the component due to age of bearing measurement and target motion is obtained in the following way.

As already stated, the new bearing reading is assumed to have come from Station 1. The other bearing reading used in the calculation is from either Station 2 or 3, depending upon which of the stored readings from these stations is the more recent. The ages of the stored bearing readings from Stations 2 and 3 are denoted by t_2 and t_3. As discussed in Chapter IV, both t_2 and t_3 are described by a probability distribution which is uniform over the interval from zero to T, i.e.,

$$\rho_{t_2} = \begin{bmatrix} \frac{1}{T} \text{ for } 0 < t_2 < T \\ 0 \text{ elsewhere} \end{bmatrix} \tag{6-10}$$

and

$$\rho_{t_3} = \begin{bmatrix} \frac{1}{T} \text{ for } 0 < t_3 < T \\ 0 \text{ elsewhere} \end{bmatrix} \tag{6-11}$$

in which ρ_{t_2} and ρ_{t_3} are the probability density-functions for the variables t_2 and t_3, respectively. The joint probability density-function for t_2 and t_3 is denoted by $\rho_{t_2 t_3}$ and is given by

$$\rho_{t_2 t_3} = \rho_{t_2}\rho_{t_3} = \begin{bmatrix} \frac{1}{T^2} \text{ for } 0 < t_2 < T \text{ , } 0 < t_3 < T \\ 0 \text{ elsewhere} \end{bmatrix} . \tag{6-12}$$

If $a = 2$ and $a = 3$ denote the use of the stored bearing reading from Station 2 and Station 3, respectively, the probability of $a = 2$ is given by

$$P(a=2) = P(t_2 < t_3) = \int_{A_1} \rho_{t_2 t_3} \, dA_1 = \frac{A}{T^2} , \tag{6-13}$$

in which A_1 is the area within a region $0 < t_2 < T$ and $0 < t_3 < T$ such that $t_2 < t_3$. As shown in Figure 6-3, this area is

$$A_1 = \frac{T^2}{2} \quad , \tag{6-14}$$

and therefore,

$$P(a = 2) = \frac{1}{2} \quad . \tag{6-15}$$

By similar reasoning, or by use of the fact that

$$P(a = 3) = 1 - P(a = 2) \quad , \tag{6-16}$$

the following can be determined:

$$P(a = 3) = \frac{1}{2} \quad . \tag{6-17}$$

If τ is an arbitrary constant, the probability that the age of the bearing reading which is used, be it from Station 1 or Station 2, exceeds τ is given by

$$P(t_a > \tau) = P(t_2 > \tau, t_3 > \tau) = \int_{A_2} p_{t_2 t_3} \, dA_2 = \frac{A_2}{T^2} \quad , \tag{6-18}$$

in which A_2 is the area within the region $0 < t_2 < T$ and $0 < t_3 < T$ such that $t_2 > \tau$ and $t_3 > \tau$. As shown in Figure 6-3, this area is

$$A_2 = (T - \tau)^2 \quad , \tag{6-19}$$

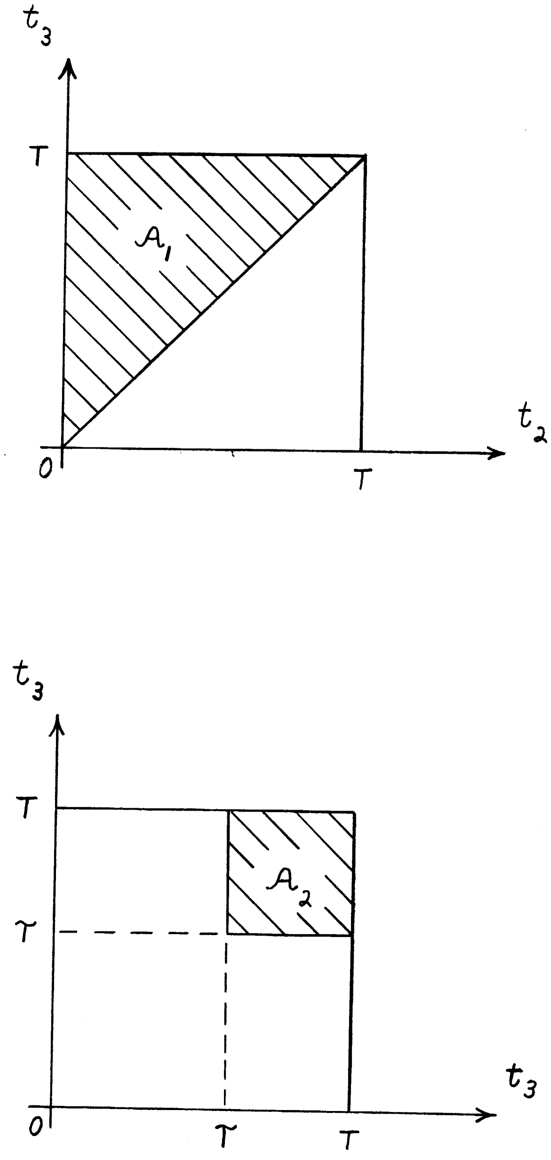


Figure 6-3 The Areas A_1 and A_2 in the t_2, t_3 Plane

and therefore,

$$P(t_a > \tau) = \frac{(T-\tau)^2}{T^2} \quad \text{for} \quad 0 < \tau < T \; . \tag{6-20}$$

The probability density-function for t_a is obtained by differentiation of Equation (6-20) with respect to τ.

$$\rho_{t_a} = \frac{\partial P(t_a < \tau)}{\partial \tau} = \frac{2(T-\tau)}{T^2} \qquad 0 < \tau < T \; . \tag{6-20a}$$

This probability density-function is illustrated in Figure 6-4. The relationship between the error in the bearing reading and the age of the bearing measurement when the error is small is provided by Equation (4-50) which is repeated here:

$$\epsilon_{ta} = \frac{V t_a \sin(\theta_a - \phi)}{d_a} \quad , \tag{6-21}$$

in which V, θ_a, ϕ and d_a are constants. The probability distribution for ϵ_{ta} is obtained by use of Equation (6-21) and is shown in Figure 6-5.

The error in the bearing reading which is old is given by

$$\epsilon_a = \epsilon_B + \epsilon_{ta} \; , \tag{6-22}$$

in which ϵ_B is the error in the bearing measurement itself. The probability distribution for the error in the bearing reading which is old is given by the convolution integral

$$\rho_{\epsilon_a} = \int_{-\infty}^{\infty} \rho_{\epsilon_B}(\epsilon_B)\, \rho_{\epsilon_{ta}}(\epsilon_a - \epsilon_B)\, d\epsilon_B \tag{6-23}$$

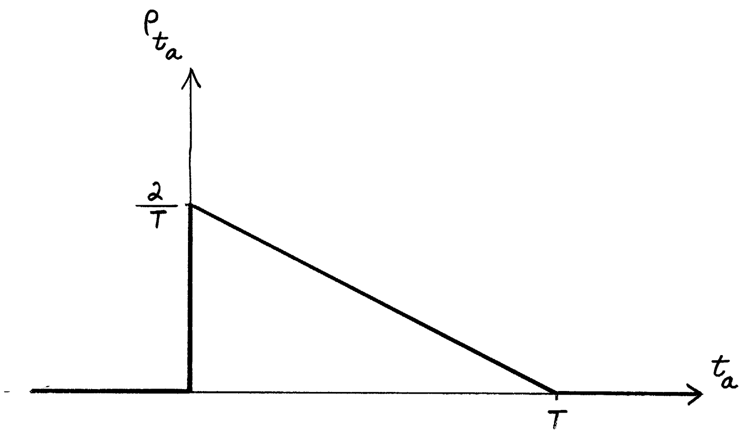


Figure 6-4 Probability Density-Function for t_a , the Age of the Bearing Reading which is Used

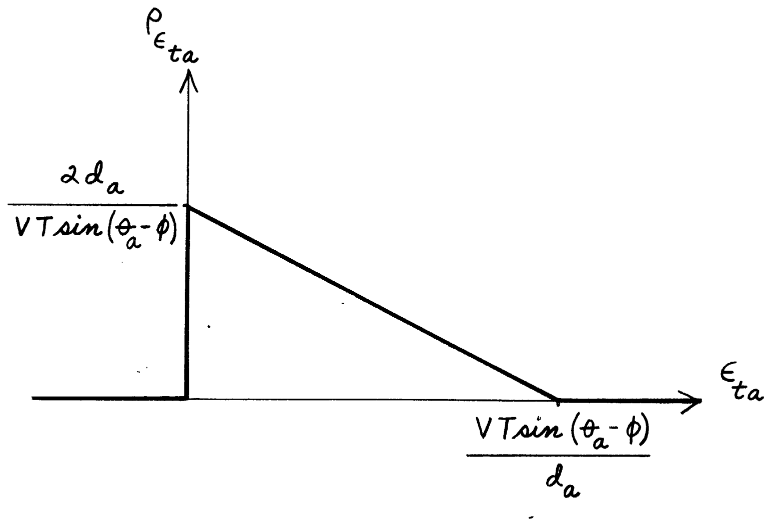


Figure 6-5 Probability Density-Function for ϵ_{ta}

The solution of this convolution integral is the first part of a digital computer program for determining the probability distribution for the position error.

<u>Probability Distribution for Position Error</u>

The probability distribution for the position error for a target located at the center of the equilateral triangle formed by the three bearing-measurement stations was obtained by a convolution of the probability distributions for bearing-reading error at each station. This convolution is the second and third parts of a digital computer program for numerically determining the probability distribution for the position error.

For a target located at the center of the equilateral triangle (as shown in Figure 5-12),

$$\frac{d_1}{D} = \frac{d_2}{D} = 1 \quad , \qquad (6\text{-}24)$$

$$\sin^2(\theta_2 - \theta_1) = \sin^2(\theta_3 - \theta_1) = \frac{3}{4} \quad , \qquad (6\text{-}25)$$

$$\cos(\theta_2 - \theta_1) = \cos(\theta_3 - \theta_1) = -\frac{1}{2} \quad , \qquad (6\text{-}26)$$

and, therefore, Equation (6-2) becomes

$$E_D^2 = \frac{4}{3}\left[\epsilon_1^2 + \epsilon_a^2 + \epsilon_1\epsilon_a\right] \quad , \qquad (6\text{-}27)$$

in which ϵ_a depends upon ϵ_B and ϵ_{ta} according to Equation (6-22).

The probability distribution for ϵ_{ta} (shown in Figure 6-5) can be described in terms of the quantity γ_a , which is defined as

$$\gamma_a = \frac{VT}{D} \sin(\theta_a - \phi) = D_T \sin(\theta_a - \phi) \quad . \qquad (6\text{-}28)$$

For a particular value of the constant, a , the conditional probability distribution for E_D , denoted by $\rho_c(E_D \mid a)$, is a function of γ_a The probability distribution for E_D may then be expressed in terms of the conditional distributions as

$$\rho_{E_D} = P(a=2)\rho_c(E_D \mid a=2) + P(a=3)\rho_c(E_D \mid a=3) \quad , \qquad (6\text{-}29)$$

which, by Equations (6-15) and (6-17), can be written as

$$\rho_{E_D} = \frac{1}{2}\sum_{a=2}^{3} \rho_c(E_D \mid a) \quad . \qquad (6\text{-}30)$$

Equation (6-30) suggests that for numerical computation it may be convenient to obtain ρ_{E_D} for each value of target direction, ϕ , from a pre-calculated set of conditional probability distributions corresponding to a set of values of γ_a. Because ρ_{ϵ_B} is an even function of ϵ_B , Equation (6-27) shows that only the absolute value of γ_a need be considered. Therefore, the second part of the digital computer study of the modified three-station arrangement consisted of tabulating a set of conditional probability distributions for a set of values of $|\gamma_a|$.

In the third part of the digital computer study, selected groups of conditional probability distributions were combined to obtain distri probability distribution for position error assuming a uniform distribution for target direction, ϕ . This latter distribution, denoted by ρ'_{E_D} , is given by

$$\rho'_{E_D} = \frac{1}{2\pi}\int_0^{2\pi} \rho_{E_D}\, d\phi \quad , \tag{6-31}$$

which can be expanded, by using Equation (6-30), into the form

$$\rho'_{E_D} = \frac{1}{4\pi}\int_0^{2\pi} \left[\rho_c(E_D \mid a = 2) + \rho_c(E_D \mid a = 3)\right] d\phi \quad . \tag{6-32}$$

The terms of the integrand depend on γ_a , as defined in Equation (6-28). Because the integration is performed over a complete cycle of $\sin(\theta_a - \phi)$, Equation (6-32) can be written as

$$\rho'_{E_D} = \frac{1}{2\pi}\int_0^{2\pi} \rho_c(E_D \mid a = 2)\, d\phi \quad . \tag{6-33}$$

Because only the absolute value of γ_a need be considered and because of the symmetry of the integrand, Equation (6-33) may be rewritten in the form

$$\rho'_{E_D} = \frac{2}{\pi}\int_0^{\pi/2} \rho_c(E_D \mid a = 2)\, d(\theta_2 - \phi) \quad . \tag{6-34}$$

In the numerical computation of ρ'_{E_D}, the interval $0 \leq \theta_2 - \phi \leq \frac{\pi}{2}$ was divided into a set of sub-intervals and the integral was evaluated by approximating the value of $\theta_2 - \phi$ within each sub-interval by its value at the center of the sub-interval.

Description of the Computer Program

The digital computer program which was used to determine the probability distribution for the position error is divided into three parts. In the first part, the probability distributions for bearing-reading error are constructed. In the second part, a set of conditional probability distributions for the position error are constructed and stored in the low-speed computer storage. These conditional probability distributions are applicable to both cases, $a = 2$ and $a = 3$ In the third part of the computer program, groups of conditional probability distributions are combined to obtain a probability distribution for position error which assumes a uniform distribution for the target direction angle.

Except for the third part, the computer program differs only slightly in detail from that used in the analysis on the unmodified three-station case. Because only two stations are involved, the required computation time is comparatively small. Figure 6-6 summarizes the computer program in block diagram form. For convenience in the digital computation, the set of possible values of ϵ_1 and ϵ_2 was represented by integers and integers plus one half. The variables u and v are used to denote this representation. Thus, in Figure 6-6, P_u and P_v are the probability distributions for the variables u and v, respectively. The joint probability of a particular combination of

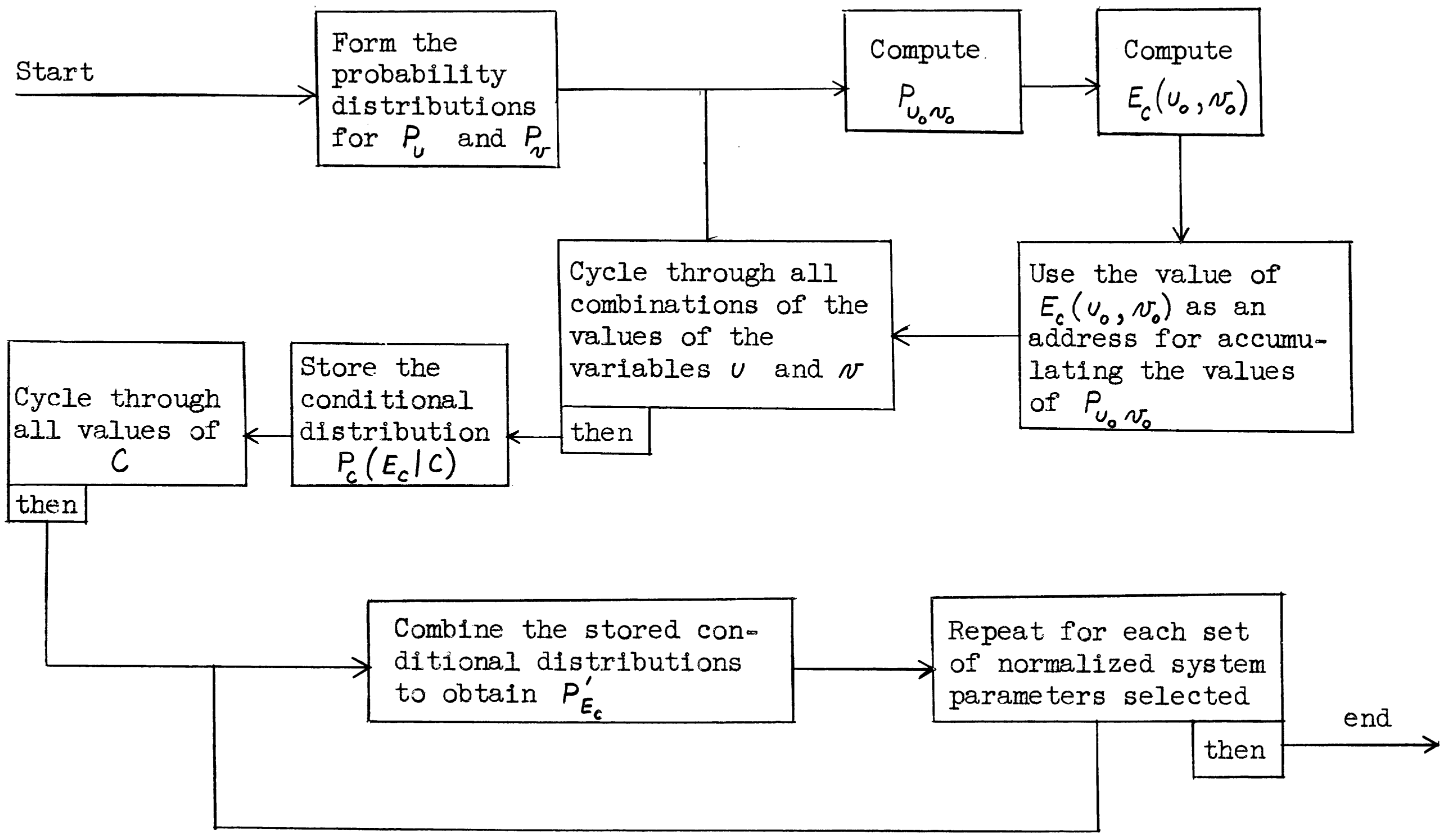


Figure 6-6 Simplified Block Diagram of the Computer Program for the Position Error in the Modified System

values of u and v is given by

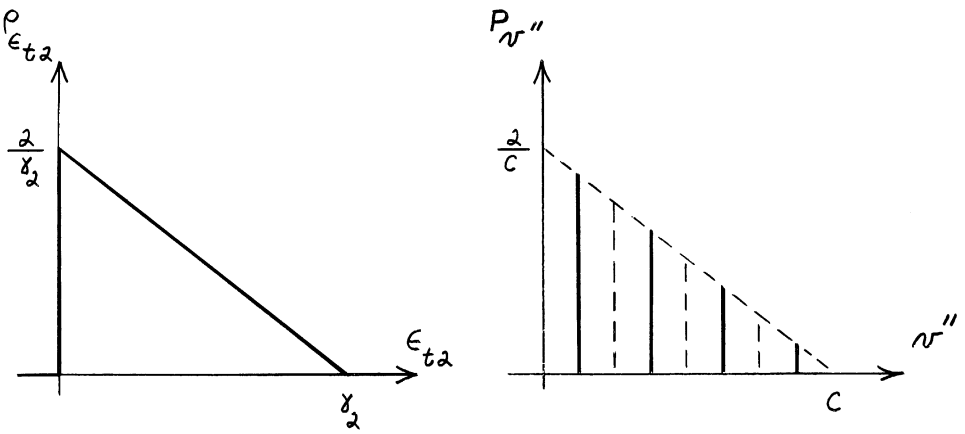
$$P_{u_o v_o} = P_{u_o} P_{v_o} \quad . \tag{6-35}$$

The subscript, o , indicates particular values of the variables. E_C is used to represent the square of the position error and C is used to represent $|\gamma_a|$.

Details of the Digital Computer Program

As in the study of the unmodified three-station triangulation system, each of the probability distributions which are used to describe the components of bearing-reading error for the modified system was approximated by a discrete distribution of many ordinates. For convenience, the set of possible discrete values of these variables is represented by integers plus one half. The variables u , v' and v'' are used in the computer program to denote these discrete values and are defined by Equations (5-13), (5-14), and (5-16), in which the scale factor s is defined by Equation (5-12). The same discrete probability distribution for the variables u and v' that was used in the unmodified case was used in this case also. This distribution is illustrated in Figure 5-9.

The discrete probability distribution which describes the variable v'' was obtained by approximation of the probability distribution for ϵ_{t2} , shown in Figure 6-5. The approximation is illustrated in Figure 6-7 in which γ_2 is used to describe the distribution for ϵ_{t2} . If v'' is considered to be a continuous distribution over the interval $0 \leq v'' \leq C$ in which C is a positive integer, the continuous



$$\nu'' = s\,\epsilon_{t2}$$

in which $s = \dfrac{\sigma_{\nu}}{\sigma_{\epsilon B}}$

Figure 6-7 Approximation of the Probability Distribution for ϵ_{t2}

distribution is divided into C equal intervals, which are replaced by C ordinates, each with a value equal to the area under the continuous distribution in the corresponding interval. The discrete distribution can be expressed in equation form as

$$P_{\nu''} = 2\left[\frac{1}{C} - \frac{\nu''}{C^2}\right] \tag{6-36}$$

for values of ν'' which are integers plus one-half in the interval $\frac{1}{2} \leq \nu'' \leq C - \frac{1}{2}$.

The discrete probability distribution for the variable $\nu = \nu' + \nu''$ was obtained by the same method used in the unmodified case described in Chapter V. For each value of C selected, the computer program was designed to have the following operations performed automatically:

(1) The probability distribution which was listed in the computer program and used for the computer variables U and v' was multiplied by the factor $P_{v''}(v'' = \frac{1}{2})$ to obtain the set of fifty ordinates for $P_{v''}(v'' = \frac{1}{2})\, P_{v'}(v - \frac{1}{2})$.

(2) The fifty ordinates were added to the contents (original contents were zero) of fifty sequential storage locations starting with the one corresponding to v_s, the smallest possible value of v, which is $v_s = -24$.

(3) Operations (1) and (2) were performed a total of C times, with the value of v'' and the address of the first storage location augmented by one each time.

In this way, the probability distribution for the variable v was constructed for each value of C selected. This process is summarized in the block diagram shown in Figure 6-8.

The probability distributions for the variables U and v were combined by the process described in Chapter V, with, of course, one less variable described by a probability distribution. The variable, E_C, used to represent the square of the position error in this case is defined as

$$E_C = \frac{3}{4} s^2 E_D^{\,2} = U^2 + v^2 + Uv \quad . \tag{6-37}$$

The probability distribution for position error was obtained as a set of numbers from a set of sixty computer storage locations. The position of a number in the set corresponded to the same particular range of

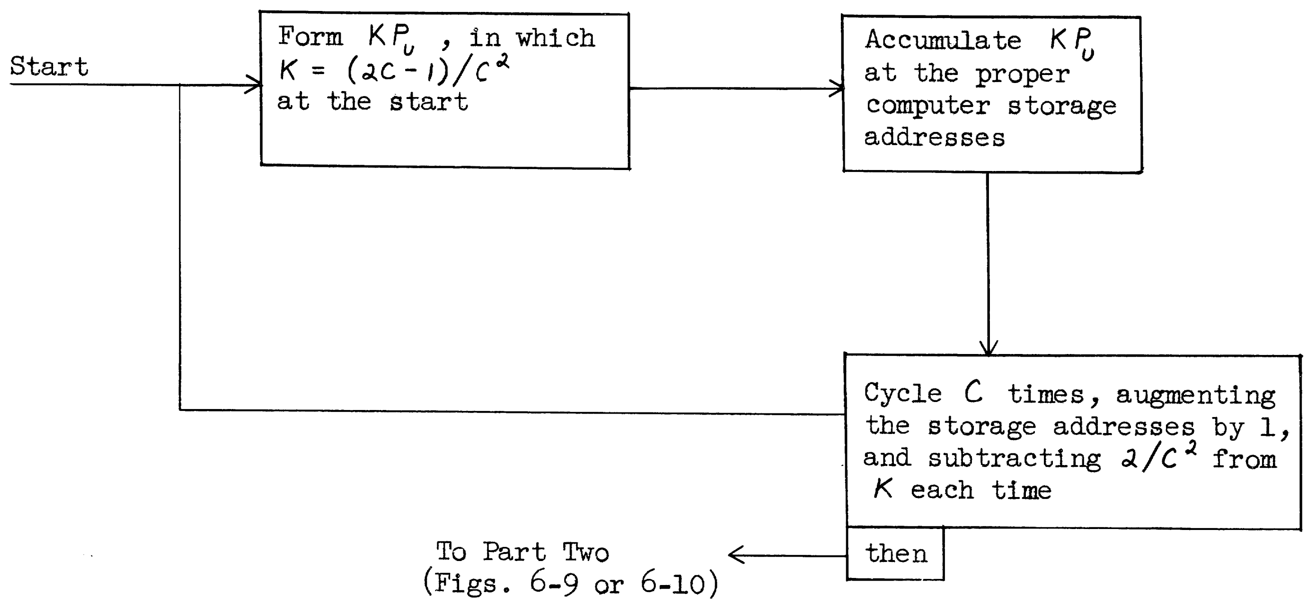


Figure 6-8 Part One of the Computer Program for the Position Error in the Modified System

values of E_c as described in Equation (5-38). Thus, the corresponding range of values of E is given by

$$\frac{4D^2}{3s^2}\left[\frac{K_1}{p} - K_2\right] < E^2 \leq \frac{4D^2}{3s^2}\left[\frac{K_1}{p-1} - K_2\right] , \qquad (6\text{-}38)$$

which is analogous to Equation (5-39) for the convention system.

Separate probability distributions were obtained and printed out for each value of C from one through fifty by use of Program I which is listed in Appendix B. The second part of this program is summarized in the block diagram shown in Figure 6-9.[1] The change necessary in this program to change P_U is listed in Appendix D.

In order to obtain a probability distribution for position error which assumes a uniform distribution for the target direction angle, Program II, which is listed in Appendix C, was used to change Program I. With this change, the separate distributions for each value of C were stored in the low-speed computer storage. For each set of 12 values of C corresponding to a particular value of D_T and twelve values of target direction angle, ϕ, the corresponding set of twelve probability distributions was selected from the computer storage and combined.

The second part of the program, when changed by Program II, is summarized in the block diagram shown in Figure 6-10. The sets of values of C which were used are listed in Appendix F.

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[1] The first part of this program is summarized in the block diagram shown in Figure 6-8.

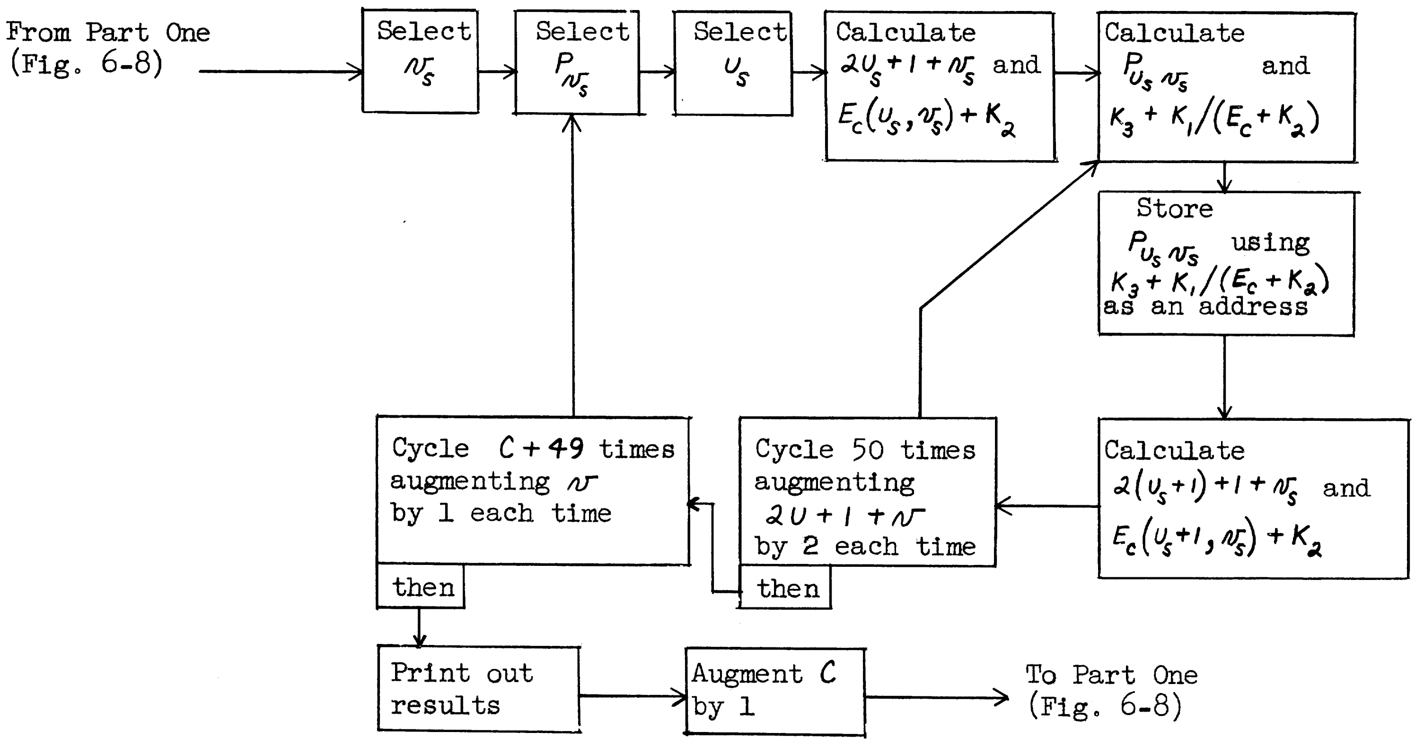

Figure 6-9 Part Two of the Computer Program for the Position Error in the Modified System

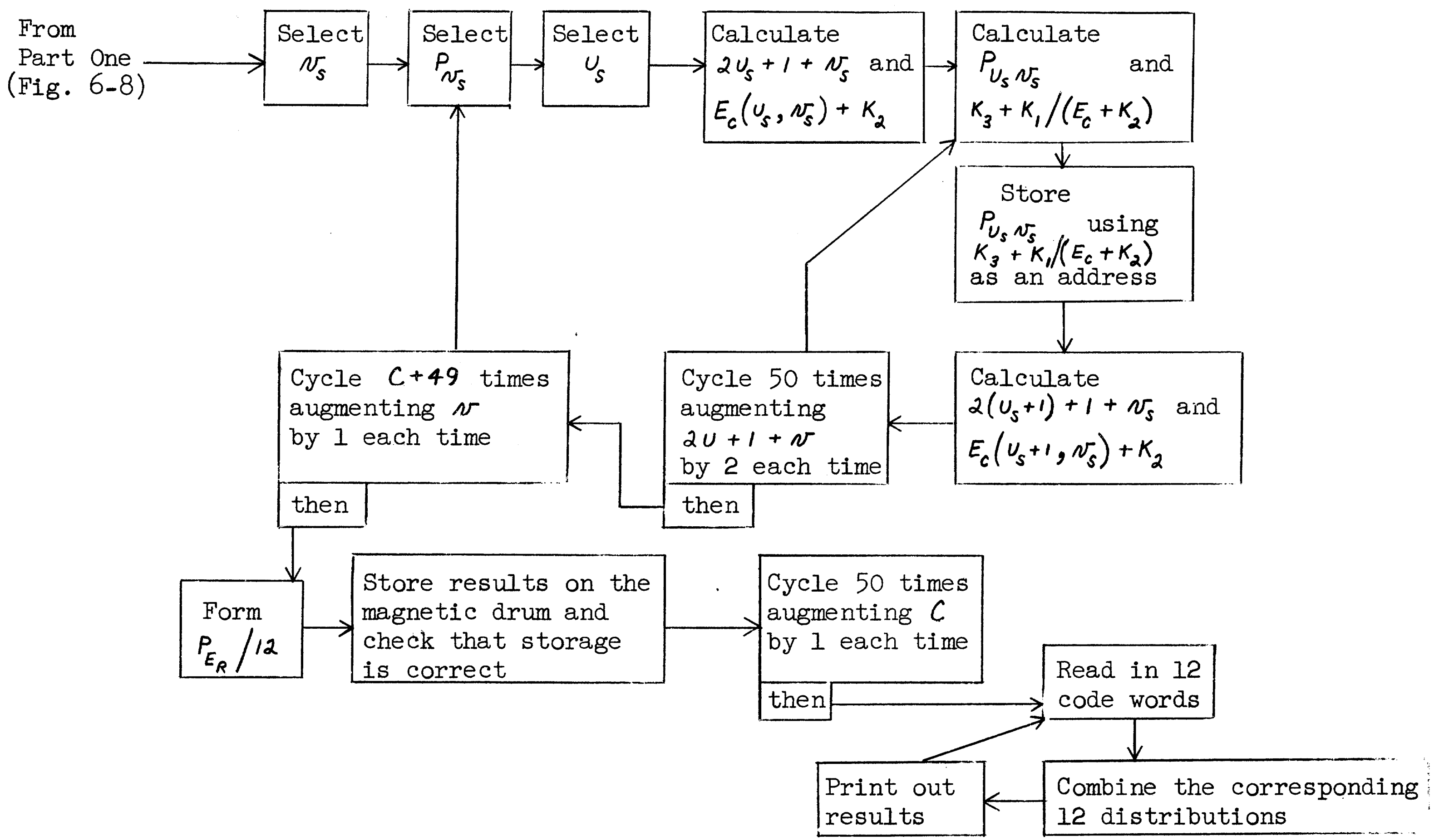

Figure 6-10 Modified Part Two of the Computer Program for the Position Error in the Modified System

Closed Form Solution

For the special case of $\sigma_{\epsilon B}$ very small in comparison with $\sigma_{\epsilon t 2}$, an expression for the position error can be obtained in closed form, even for the case of uniformly distributed target direction.

For a target located at the center of an equilateral triangle formed by three bearing-measurement stations, Equation (6-27) for the modified system is

$$E_D^2 = \frac{4}{3}\left[\epsilon_1^2 + \epsilon_a^2 + \epsilon_1\epsilon_a\right] . \qquad (6\text{-}39)$$

When

$$\sigma_{\epsilon B} \ll \sigma_{\epsilon t a} , \qquad (6\text{-}40)$$

Equation (6-39) can be approximated by

$$E_D^2 = \frac{4}{3}\epsilon_{ta}^2 \qquad (6\text{-}41)$$

and, therefore,

$$E_D = \frac{2}{\sqrt{3}}\epsilon_{ta} \quad \text{for} \quad \epsilon_{ta} \geq 0 . \qquad (6\text{-}42)$$

The probability distribution for E_D is obtained from the probability distribution for ϵ_{ta} and Equation (6-42). As explained previously, [Equations (6-32) and (6-33)] only the case $a = 2$ need be considered. Therefore, the probability distribution for E_D as obtained from Figure 6-7 and Equation (6-42) is as shown in Figure 6-11. The quantity, γ_2, is defined in Equation (6-28) as

$$\gamma_2 = D_T \sin(\theta_2 - \phi) . \qquad (6\text{-}43)$$

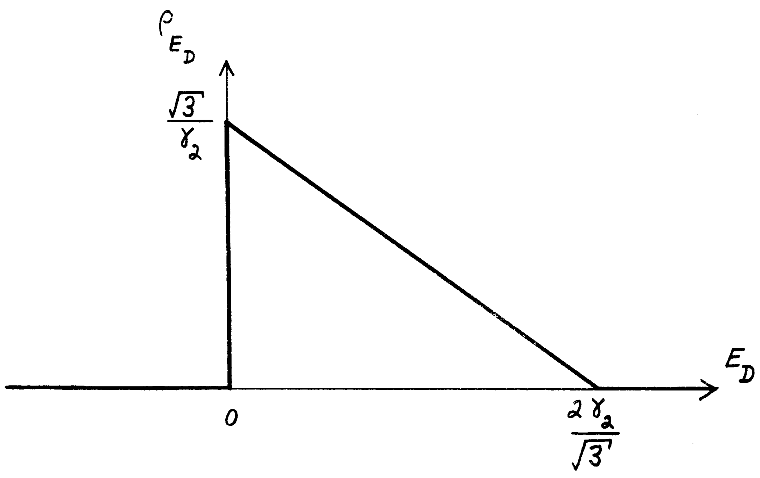

Figure 6-11 Probability Density-Function for E_D When $\sigma_{\epsilon B} \ll \sigma_{\epsilon ta}$

As also explained previously, only values of $\theta_2 - \phi$ within the interval $0 \leq \theta_2 - \phi \leq \frac{\pi}{2}$ need be considered. For uniformly distributed target direction, the joint probability distribution for E_D and $\theta_2 - \phi$ is as shown in Figure 6-12. This joint distribution has a triangular cross-section normal to the $\theta_2 - \phi$ axis and, of course, it satisfies the condition that

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{\sqrt{3}} D_T \sin(\theta_2 - \phi)} \rho_{E_D, \theta_2 - \phi} \, dE_D \, d(\theta_2 - \phi) = 1 \quad . \tag{6-44}$$

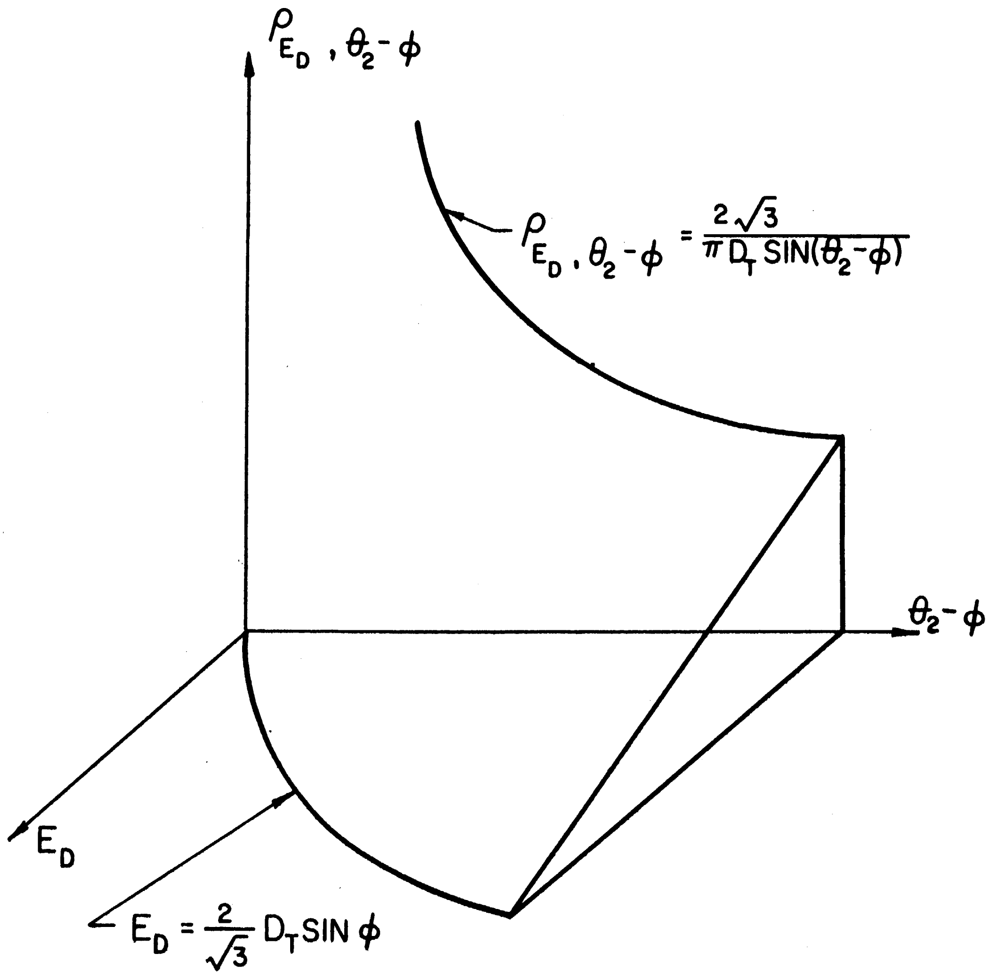

Figure 6-12 Joint Probability Density-Function for E_D and $\theta_2 - \phi$ when $\sigma_{\epsilon B} \ll \sigma_{\epsilon ta}$

Within the region in which it is non-zero, $\rho_{E_D, \theta_2 - \phi}$ can be expressed as

$$\rho_{E_D, \theta_2-\phi} = \frac{1}{\pi}\left[\frac{2\sqrt{3}}{D_T \sin(\theta_2-\phi)} - \frac{3E_D}{D_T^2 \sin^2(\theta_2-\phi)}\right] . \tag{6-45}$$

The probability that E_D exceeds some arbitrary value, λ , is

$$P(E_D > \lambda) = \frac{1}{\pi}\int_{\sin^{-1}\left(\frac{\sqrt{3}\lambda}{2D_T}\right)}^{\frac{\pi}{2}} \int_{\lambda}^{\frac{2}{\sqrt{3}}D_T \sin(\theta_2-\phi)} \left[\frac{2\sqrt{3}}{D_T \sin(\theta_2-\phi)} - \frac{3E_D}{D_T^2 \sin^2(\theta_2-\phi)}\right] d_{E_D}\, d(\theta_2-\phi) . \tag{6-46}$$

Performing the first integral yields:

$$P(E_D > \lambda) = \frac{1}{\pi}\int_{\sin^{-1}\left(\frac{\sqrt{3}\lambda}{2D_T}\right)}^{\frac{\pi}{2}} \left[2 - \frac{2\sqrt{3}\lambda}{D_T \sin(\theta_2-\phi)} + \frac{3\lambda^2}{2D_T^2 \sin^2(\theta_2-\phi)}\right] d(\theta_2-\phi) . \tag{6-47}$$

Performing this integral yields:

$$P(E_D > \lambda) = 1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\sqrt{3}\,\lambda}{2D_T}\right) + \frac{3\lambda^2}{2\pi D_T^2}\sqrt{\frac{4D_T^2}{3\lambda^2} - 1}$$

$$+ \frac{2\sqrt{3}\,\lambda}{\pi D_T}\log\left[\frac{\frac{\sqrt{3}\,\lambda}{2D_T}}{1 + \sqrt{1 - \frac{3\lambda^2}{4D_T^2}}}\right] . \qquad (6\text{-}48)$$

Values of $P(E_D > \lambda)$ were calculated and are plotted in Figure 7-5 using notation developed in Chapter VII. This graph is a limiting case of a set of graphs obtained by using the digital computer.

CHAPTER VII

NUMERICAL RESULTS

The position error in a triangulation system consisting of three bearing-measurement stations located at the vertexes of an equilateral triangle was calculated and is presented in the form of cumulative probability distributions.

For the presentation of the results, normalized system parameters are used which are independent of the choice of σ_U in the computer program. These parameters are:

$$E_N = \frac{E}{D\sigma_{\epsilon B}} \tag{7-1}$$

and

$$D_N = \frac{VT}{D\sigma_{\epsilon B}} = \frac{D_T}{\sigma_{\epsilon B}} , \tag{7-2}$$

in which

- E_N is the normalized magnitude of the position error,
- E is the magnitude of the position error,
- D is the distance from the center of the equilateral triangle to each of the bearing-measurement stations,
- $\sigma_{\epsilon B}$ is the standard deviation of the error in the bearing measurements at each station,
- D_N is the normalized distance that the target moves in the time between consecutive bearing measurements of a target at each bearing-measurement station,
- V is the speed of a target traveling a straight-line path,

T is the time between consecutive bearing measurements of a target at each bearing-measurement station, and

D_T is the distance the target moves in the time T .

Figure 7-1 shows the cumulative probability distribution for the normalized position error for the three-station triangulation system in which all three bearing measurements are used to calculate the position of the target. Target direction is uniformly distributed. Figure 7-2 is an expanded view of the same graph for the higher values of position error. In both figures, the ordinate is $P(E_N > E_o)$, the probability that E_N will exceed the corresponding value of the abscissa. Four graphs are presented for four values of D_N . The value $D_N = 0$ corresponds to the use of simultaneous measurements. Figures 7-1 and 7-2 describe the magnitude of the position error if the system parameters are known. These figures clearly show the additional position error which results when non-simultaneous measurements are used. These figures also provide a way of specifying system parameters to meet particular restrictions on the position error. Examples of these uses are presented in Chapter VIII.

Figure 7-3 shows the cumulative probability distribution for the normalized position error for a modified three-station triangulation system in which only two bearing readings, the new reading and the more recent of the other two, are used to calculate the position of the target. Figure 7-4 is an expanded view of the same graphs for the higher values of position error. Eight graphs are presented for eight values of D_N . These graphs can be used in the same way as the graphs of Figures 7-1 and 7-2.

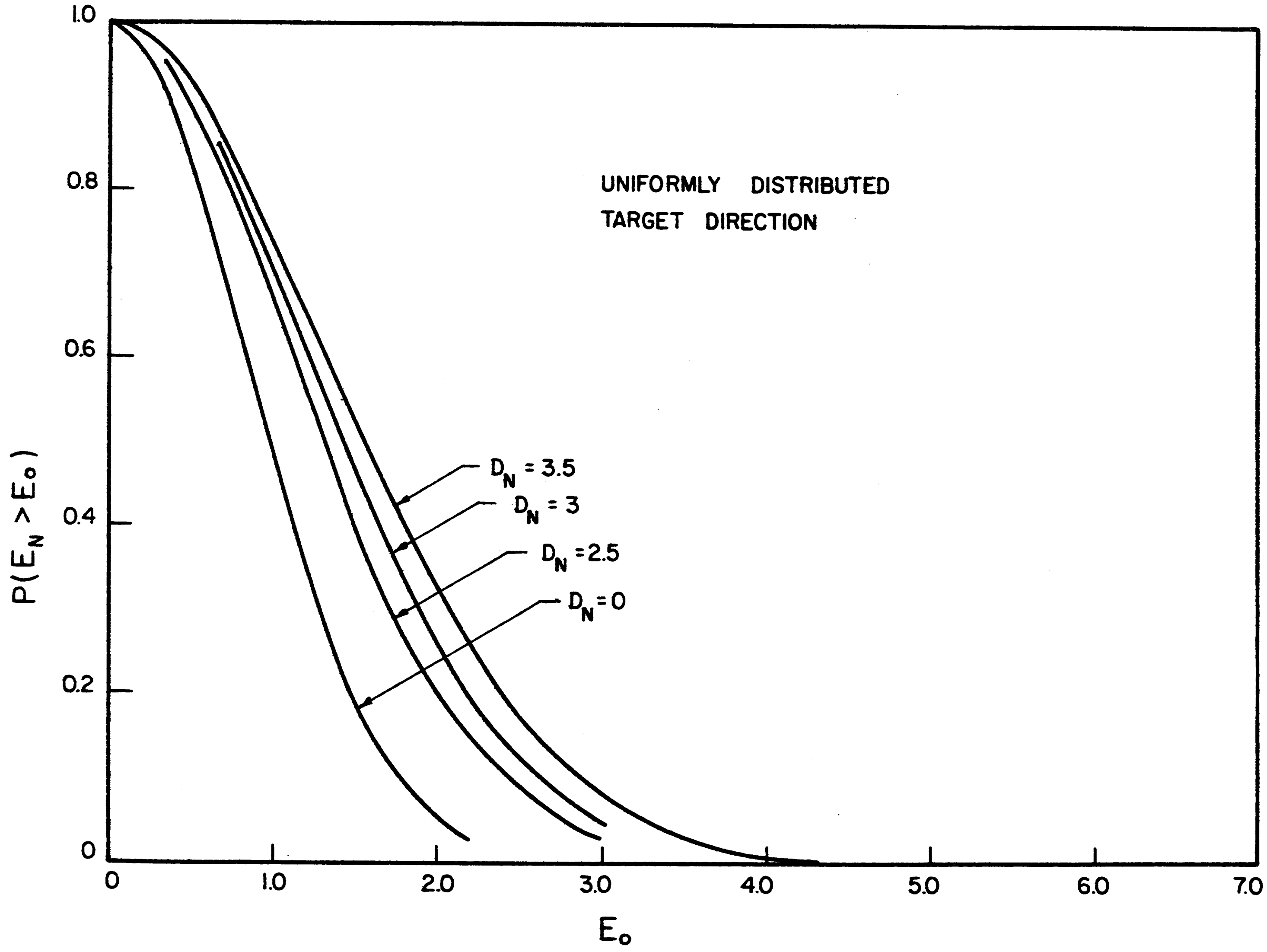

Figure 7-1 Normalized Position Error for a Three-Station Triangulation System

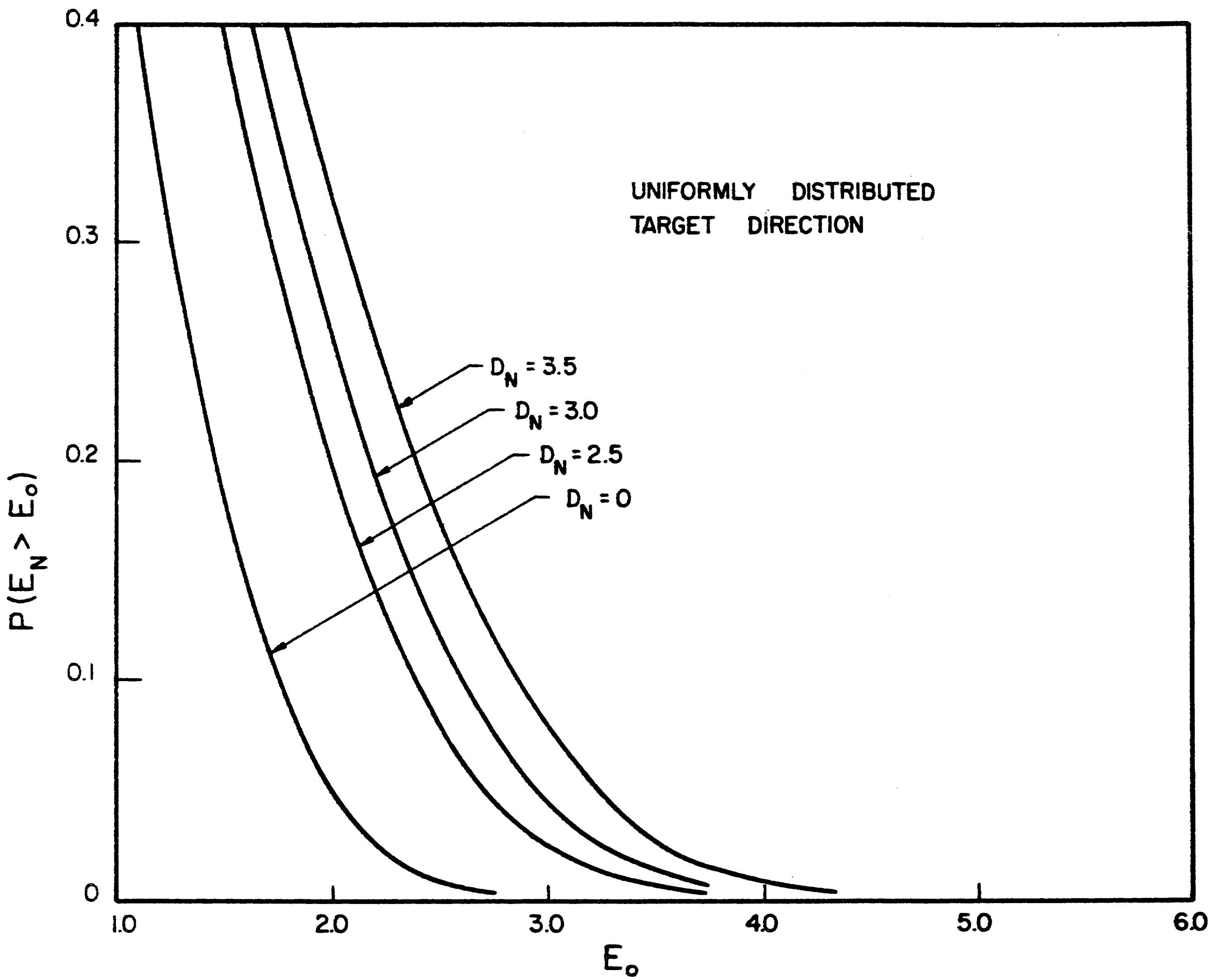

Figure 7-2 Normalized Position Error for a Three-Station Triangulation System

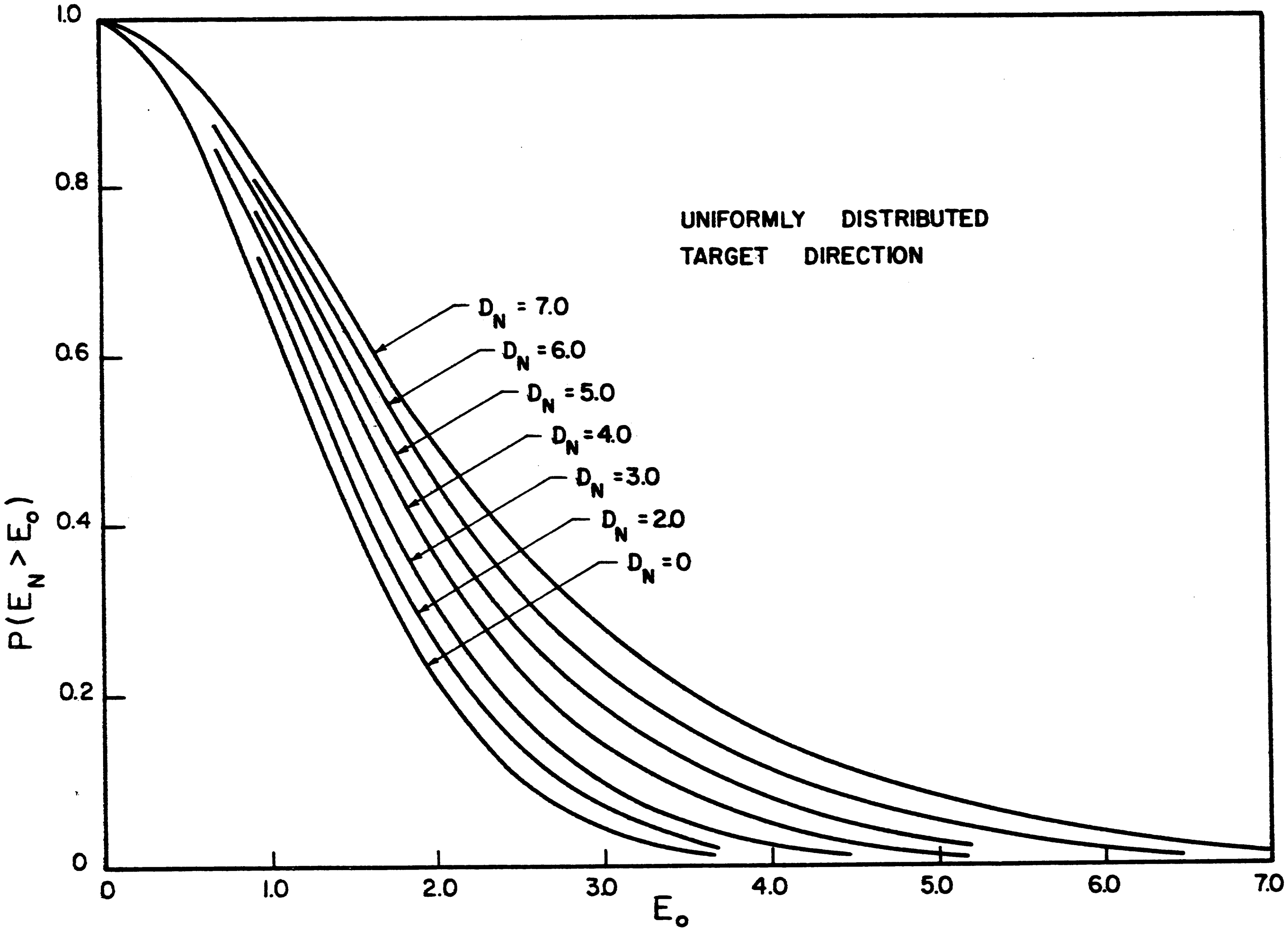

Figure 7-3 Normalized Position Error for a Modified Three-Station Triangulation System

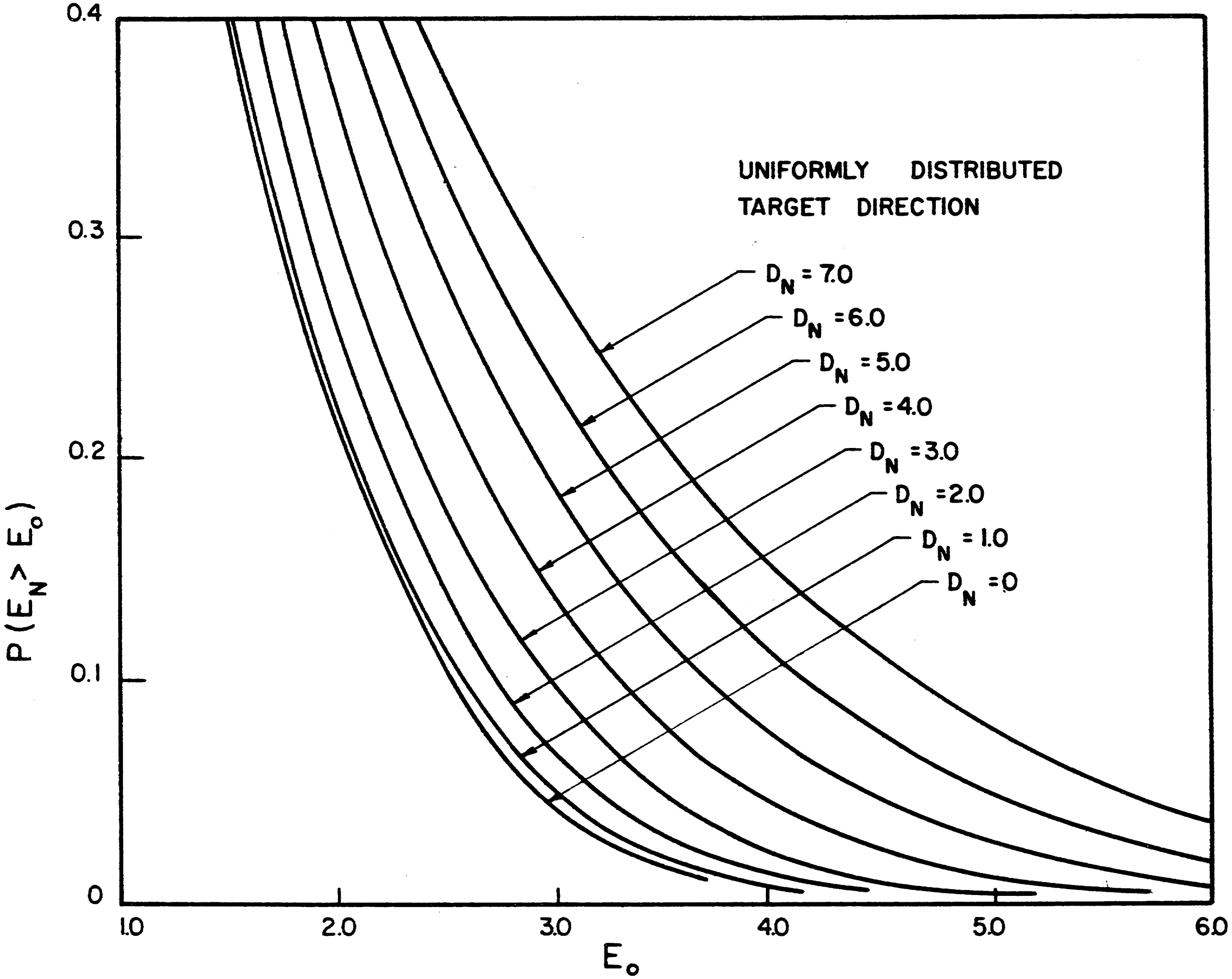

Figure 7-4 Normalized Position Error for a Modified Three-Station Triangulation System

In Figure 7-5, the abscissas of the curves presented in Figures 7-3 and 7-4 have been divided by D_N. The probability distributions which describe the position in the modified three-station system are given in terms of $P([E_N/D_N] > [E_o/D_N])$ or $P([E/VT] > [E_o/D_N])$, which are identical according to Equations (7-1) and (7-2). By presenting the distributions in this form, they can be compared with a graph of Equation (6-48), which is a solution in closed form for the probability distribution when the value of D_N approaches infinity.[1] The graph for this limiting case has been added to Figure 7-5. When compared with the set of calculated probability distributions, this graph of the limiting distribution demonstrates that little would be gained in using a digital computer to calculate many distributions for values of D_N greater than seven.

A comparison of the graphs for $D_N = 0$ (e.g., no target motion) shows, as would be expected, that for any selected value of E_o, $P(E_N > E_o)$ is greater for the modified three-station system than for the conventional system. As the value of D_N is increased in both cases, $P(E_N > E_o)$ increases less rapidly for the modified system because the information which is most likely to be in large error is omitted in the calculation of target position in the modified system. In the range of values of D_N from 3.5 to 4.0, $P(E_N > E_o)$ is approximately the same for both systems. For values of D_N greater than 4.0, $P(E_N > E_o)$ is less for the modified system than for the conventional system. Thus,

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[1] The approach of D_N to infinity corresponds to the case of position error due to error in bearing measurement being negligible in comparison with the error due to time delay and target motion.

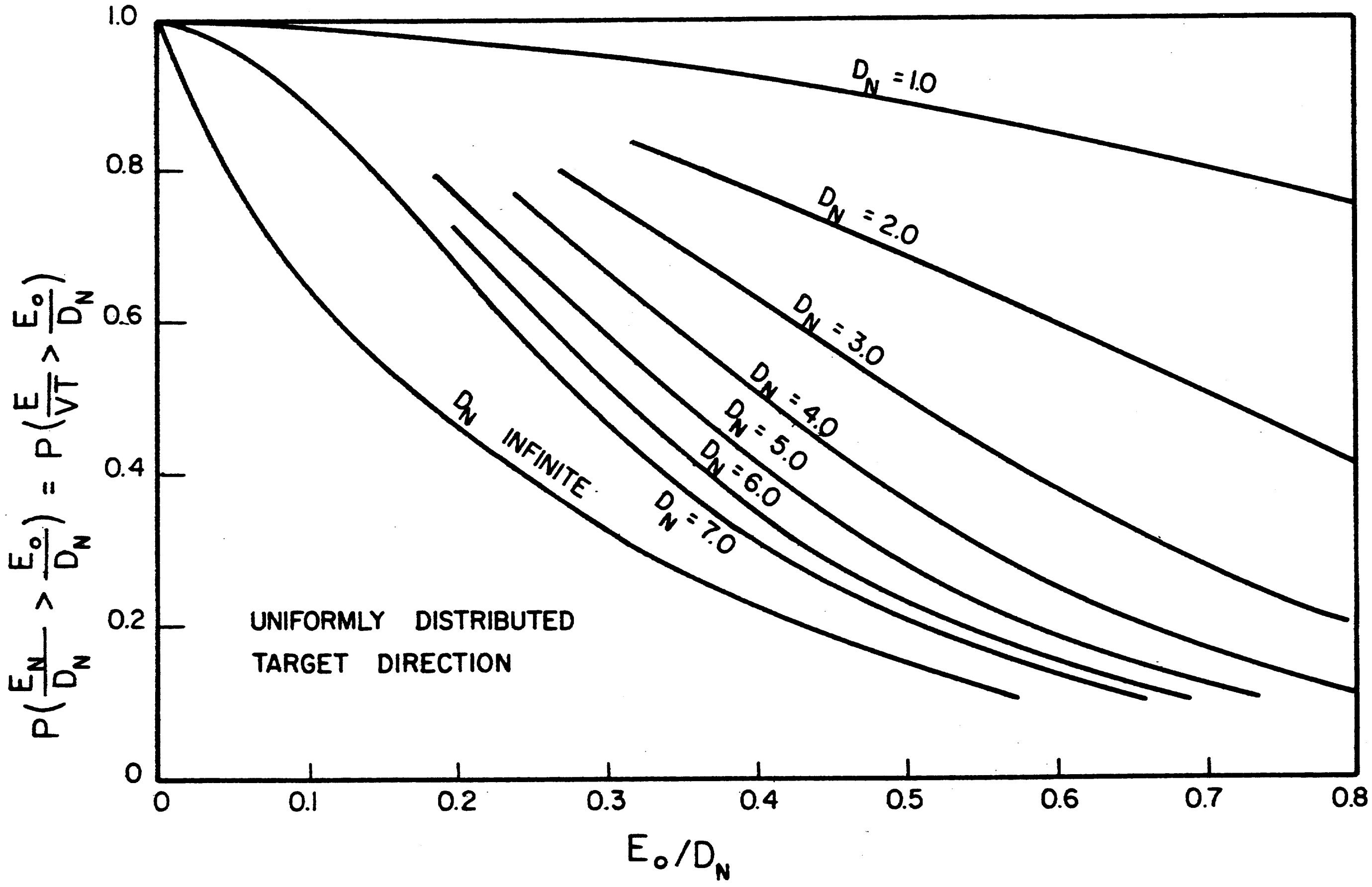

Figure 7-5 Normalized Position Error for a Modified Three-Station Triangulation System

from the standpoint of minimizing the position error, these graphs indicate that the modified type of system should be employed if a three-station triangulation system is to operate in an environment in which D_N is greater than 4.0.

The comparison of the conventional and modified systems is illustrated further in Figures 7-6 and 7-7. The same data used in the previous figures have been used to construct a family of curves for both systems for several values of constant $P(E_N > E_o)$. As D_N increases for both systems, a greater range of values of position error is included at each probability level. For a selected probability, the value of D_N at which the range of values of position error is the same for both systems is quite evident. Minor extrapolation is required because the range of values of D_N investigated for the conventional system was somewhat restricted. For this reason, the calculated points on these graphs have been indicated.

The data previously presented are for the case of target direction which is uniformly distributed. Of interest also, are the probability distributions which describe the position error for particular target directions. For the conventional three-station system, a set of such distributions has been plotted in Figure 7-8 for the case of $D_N = 3$ and for several values of $|\theta_1 - \phi|$, the absolute value of the angle between the target path and the line joining the center of the equilateral triangle with the station whose bearing reading is new. Table 7-1 lists the values of $|\theta_1 - \phi|$ for each of the cases listed in Figure 7-8. The curves in Figure 7-8 indicate the variation in position error that can be expected in a sequence of measurements as each of the stations becomes

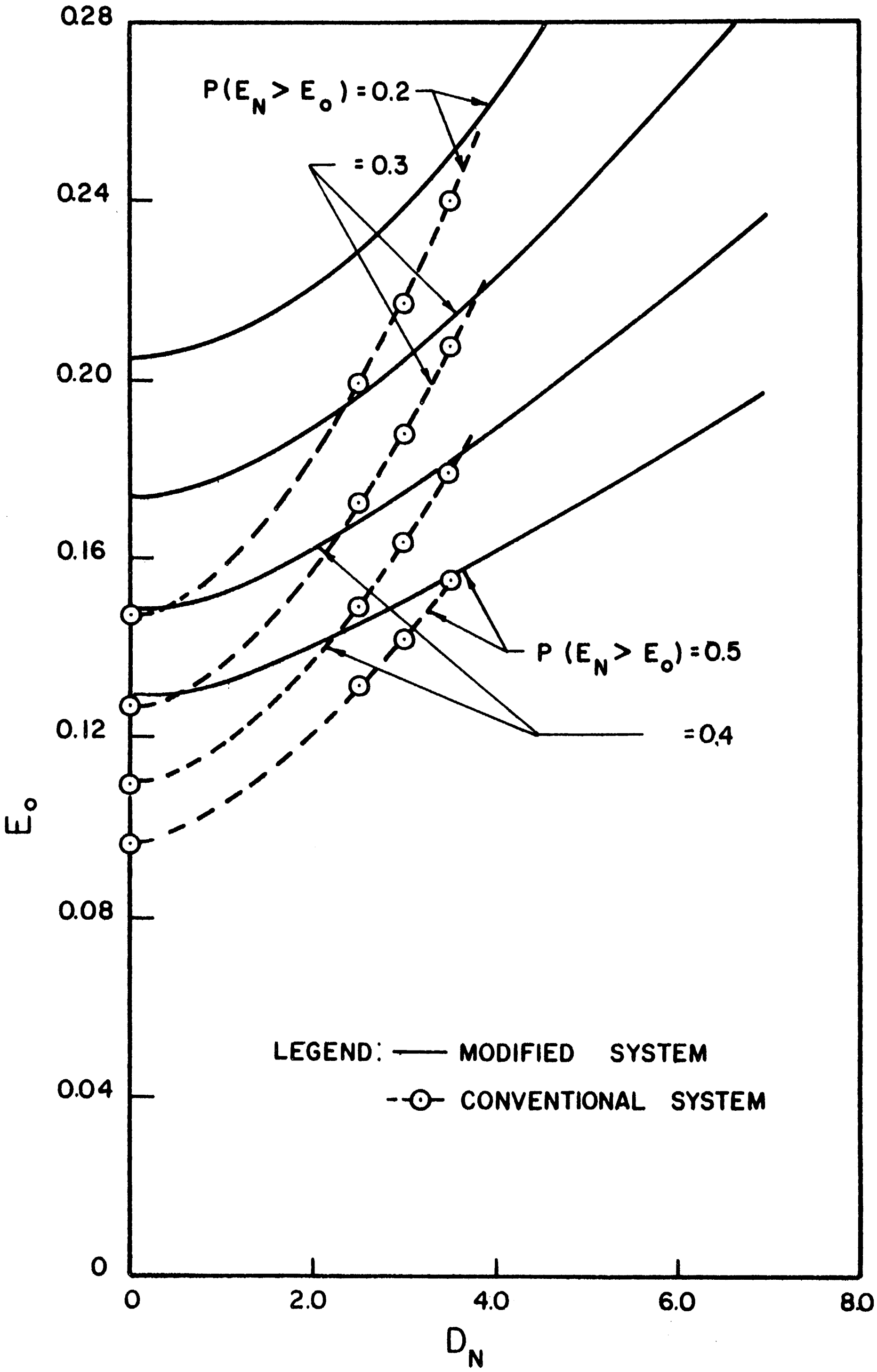

Figure 7-6 Loci of Constant $P(E_N > E_o)$ for Uniformly Distributed Target Direction

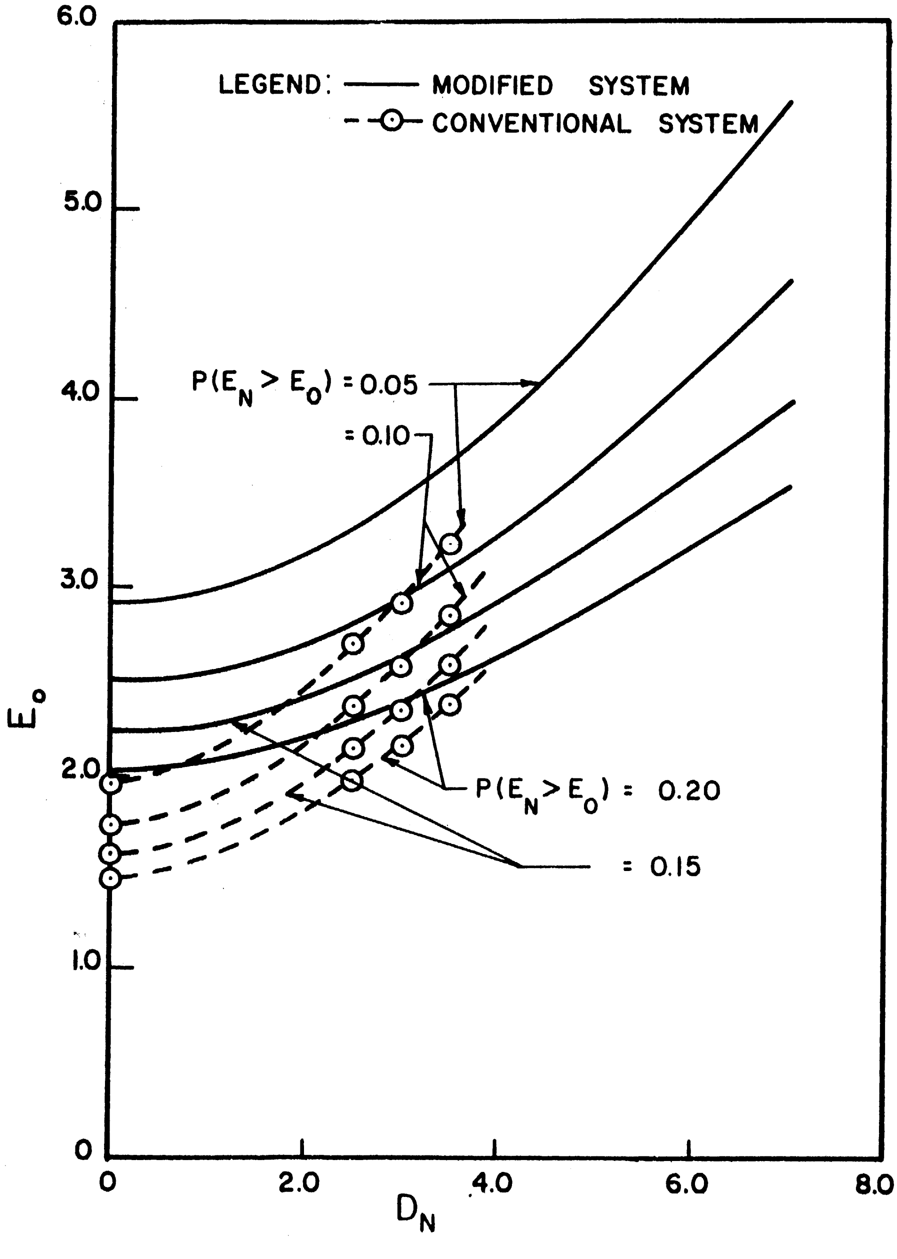

Figure 7-7 Loci of Constant $P(E_N > E_o)$ for Uniformly Distributed Target Direction

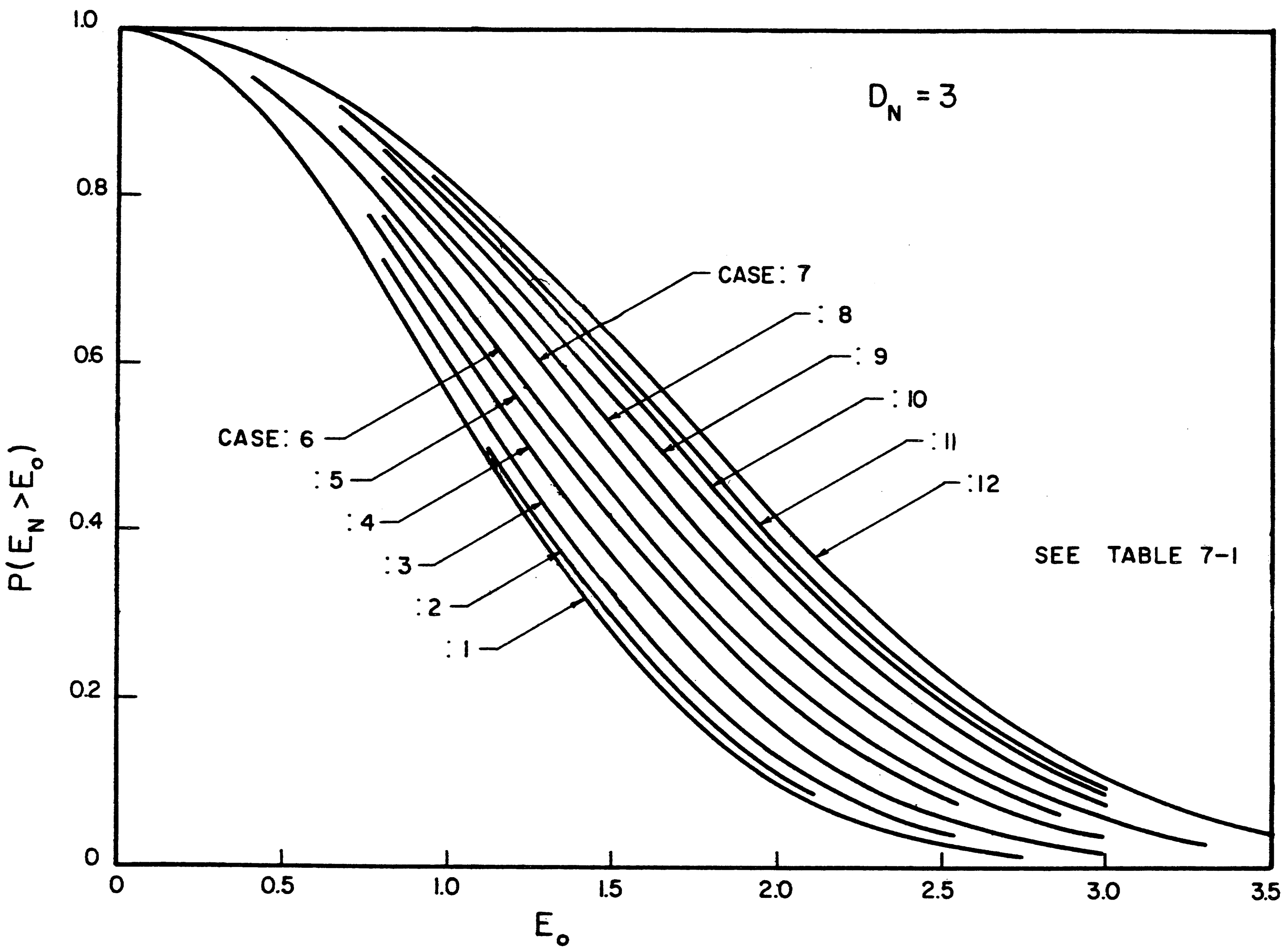

Figure 7-8 Effect of Target Direction on the Normalized Position Error for a Three-Station Triangulation System

TABLE 7-1

GEOMETRY REPRESENTED BY THE TWELVE CASES IN FIGURE 7-8

Case	$\lvert \theta_1 - \phi \rvert$ [1]			
1	86.25°	93.75°	266.25°	273.75°
2	78.75°	101.25°	258.75°	281.25°
3	71.25°	108.75°	251.25°	288.75°
4	63.75°	116.25°	243.75°	296.25°
5	56.25°	123.75°	236.25°	303.75°
6	48.75°	131.25°	228.75°	311.25°
7	41.25°	138.75°	221.25°	318.75°
8	33.75°	146.25°	213.75°	326.25°
9	26.25°	153.75°	206.25°	333.75°
10	18.75°	161.25°	198.75°	341.25°
11	11.25°	168.75°	191.25°	348.75°
12	3.75°	176.25°	183.75°	356.25°

[1] When Station 1 provides the more recent bearing reading

the one to provide the new reading. The observed, extreme variation with target direction corresponds in magnitude to the variation observed in the probability distributions for uniformly distributed target direction when D_N is varied between zero and 3.5. However, if target direction with respect to the system rather than with respect to the station providing the new bearing readings is considered, each station is equally likely to be the one providing the new reading. In this case, the position error is described by a combination of the three conditional probability distributions for the absolute values of the angles between the target path and the lines joining the center of the equilateral triangle with each of the three bearing-measurement stations. The variation with target direction among the resultant distributions is negligible. The limiting curves of the set of probability distributions which describe the position error differ by at most 0.02 in the $P(E_N > E_0)$ coordinate and they can be approximated by the corresponding curve for uniformly distributed target direction.

For the modified three-station triangulation system, a set of probability distributions which describes the variation in position error with target direction has been plotted in Figure 7-9 for the case of $D_N = 3$. For the modified system, target direction affects the position error by way of $|\theta_a - \phi|$, the absolute value of the angle formed by the intersection of the target path and the line joining the center of the equilateral triangle with the station which provides the more recent of the two bearing readings which are old. The entire range of variation of the probability distributions is covered by the four curves plotted for four values of this angle. Comparison of Figures 7-8 and 7-9

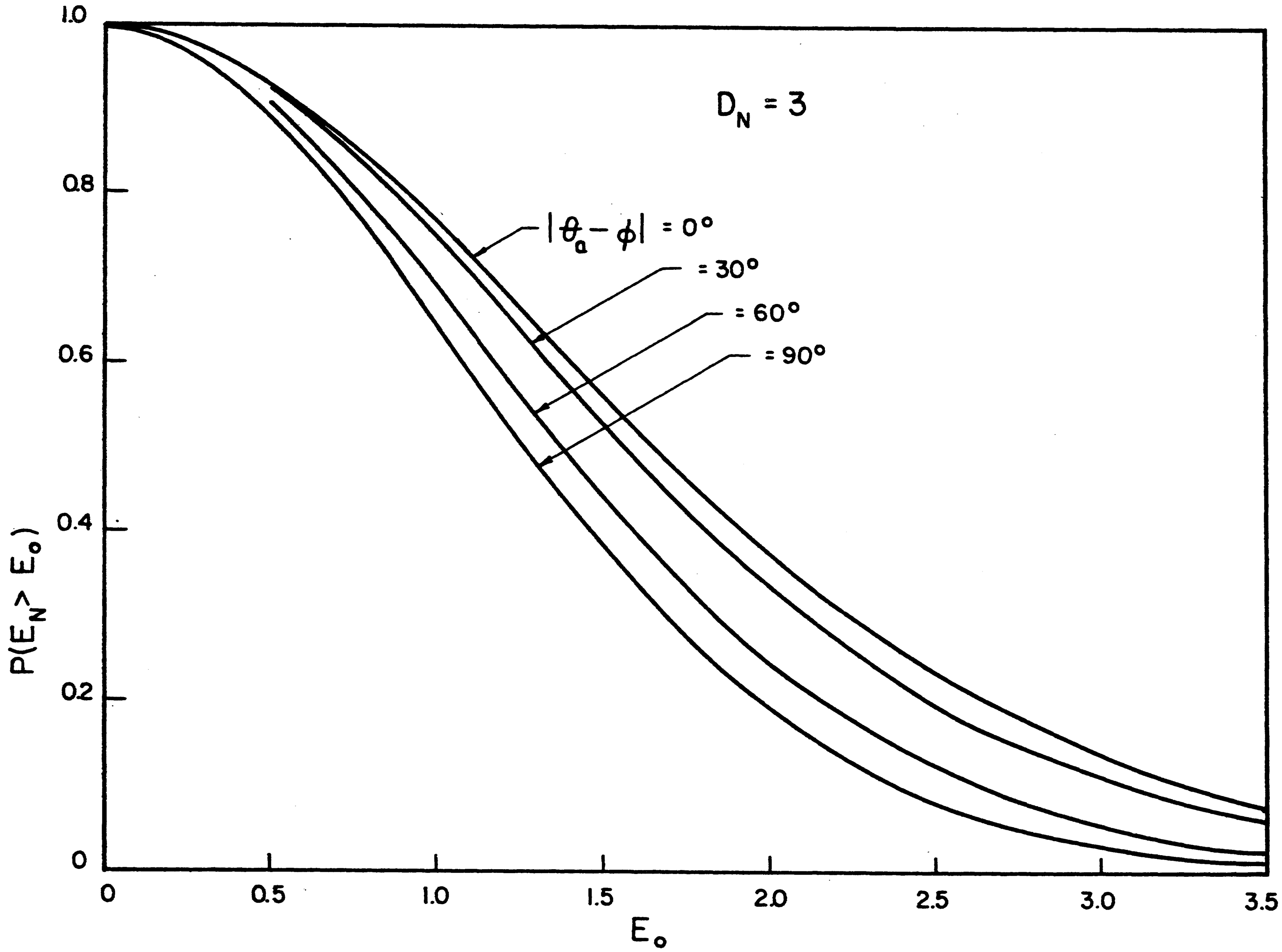

Figure 7-9 Effect of Target Direction on the Normalized Position Error for a Modified Three-Station Triangulation System

shows that when $D_N = 3$, the variation due to target direction is approximately twice as great for the conventional system. Thus, the modified system has some advantage if the probability of large error is to be minimized, even at values of D_N as low as 3.

If target direction with respect to the system rather than with respect to the station providing the old bearing reading is investigated, considering that each station is equally likely to provide the old reading which is used, the variation in the probability distributions which describe the position error is negligible, as in the case of the unmodified system. The limiting curves of the set of probability distributions which describe the position error in this case differ by at most 0.04 in the $P(E_N > E_0)$ coordinate, and they can be approximated by the corresponding curve for uniformly distributed target direction.

The results of this study are based on a normal distribution for the error in bearing measurements. The numerical results of this study are in error because the normal distribution was truncated and then approximated by a discrete probability distribution.[1] In order to estimate this error, the probability distribution for the position error for D_N = 2.5 in the conventional three-station system was calculated using two different discrete probability distributions for the error in bearing measurements. The distributions differ in the points of truncation. One of the distributions results from truncation at $u = \pm 3.5\sigma_u$, the other at $u = \pm 2.5\sigma_u$.[2] Both of the discrete distributions consist

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[1] The points of truncation determine the range of values of bearing-measurement error considered.

[2] The variable u is used in the digital computer program to represent the error in the bearing measurements. The quantity σ_u is the standard deviation of the continuous distribution from which the approximate discrete distribution for the variable u is obtained.

of a set of probability values for the same fifty values of υ . Because of the difference in the points of truncation, the sets of probability values and the relationships of the variable υ to the error in bearing measurement are different for the two distributions.

The discrete distribution which results from truncation at $\upsilon = \pm 3.5\sigma_{\upsilon}$ was used to obtain the results presented in the previous figures. The particular distribution for position error for $D_N = 2.5$ for the conventional three-station system (Figure 7-2) is repeated in Figure 7-10, using an expanded abscissa. For the same case, the probability distribution obtained by use of the discrete distribution which results from truncation at $\upsilon = \pm 2.5\sigma_{\upsilon}$ is presented for comparison in the same figure. The difference in the two probability distributions for position error is small. As expected, the discrete distribution which was truncated at $\upsilon = \pm 3.5\sigma_{\upsilon}$ produces a higher probability that large position error will occur. It is believed that most of the difference between the two probability distributions for position error is due to the difference in the points of truncation rather than to the use of discrete distributions. The percent difference in the values of E_o for the cumulative distributions shown in Figure 7-11 was calculated and plotted as a function of the probability level. In the range of probability values in which these calculations can be performed accurately (0.05 to 0.9), the difference does not exceed three percent.

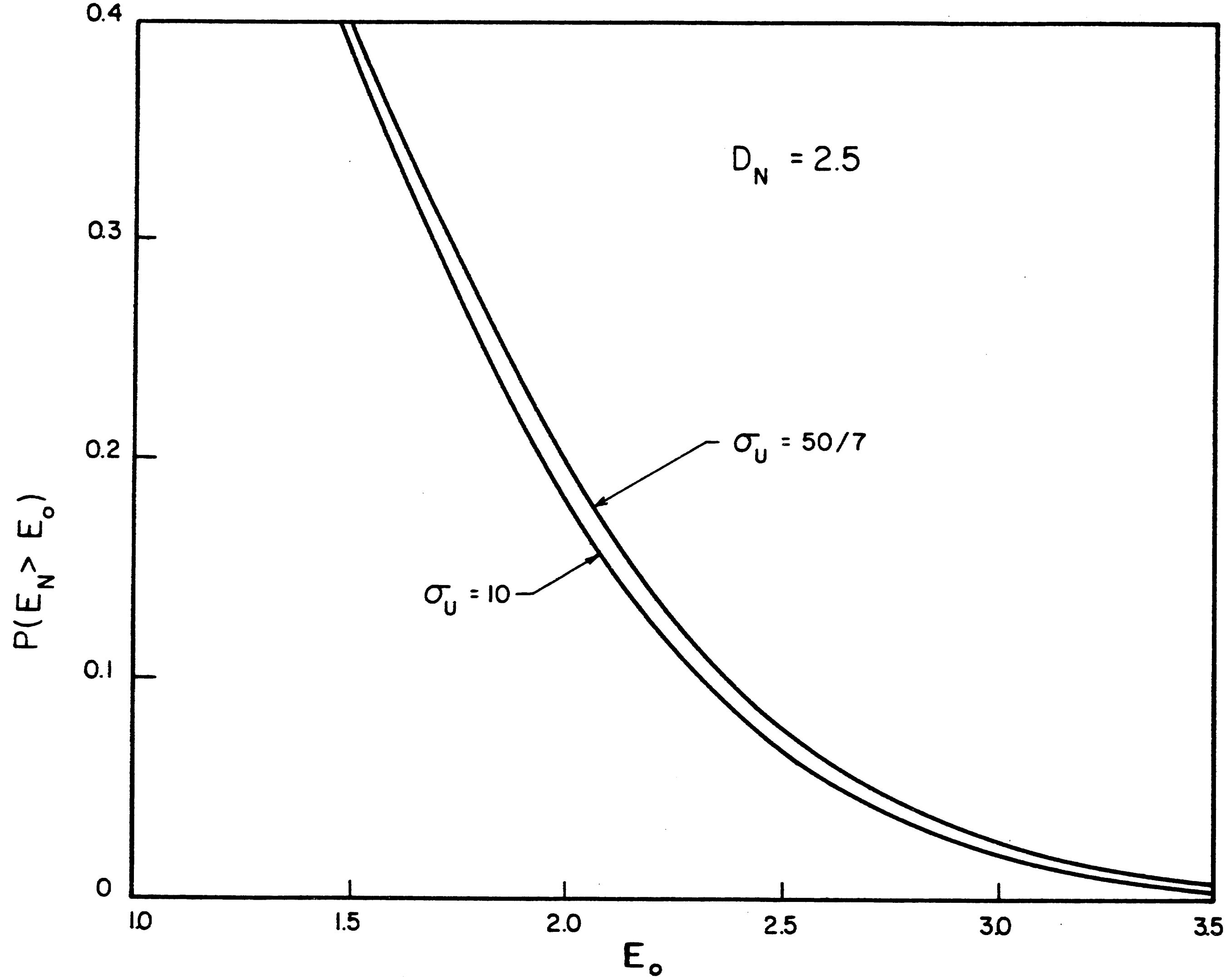

Figure 7-10 Comparison of Results Obtained Using Different Probability Distributions

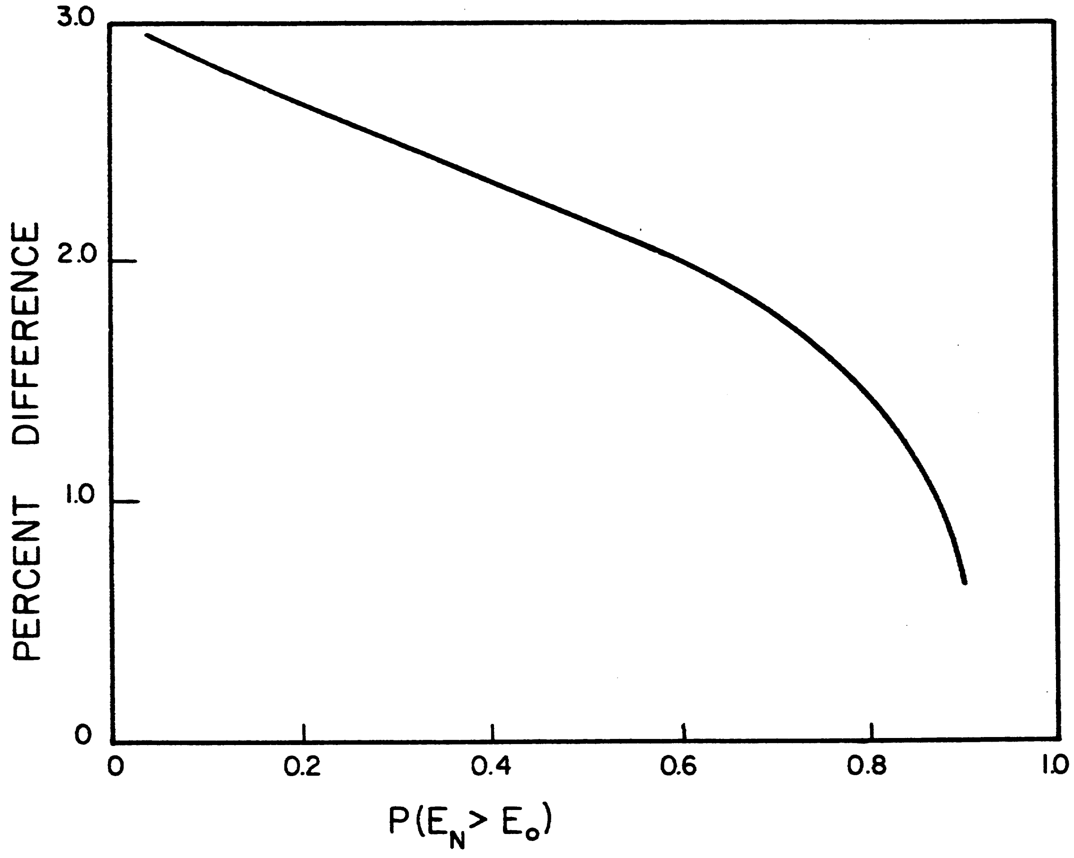

Figure 7-11 Percent Difference Between the Values of E_o Obtained by Using Two Different Discrete Distributions for Bearing-Measurement Error

CHAPTER VIII

APPLICATIONS

The numerical results presented herein can be used (1) to estimate the position error in the evaluation of existing or proposed three-station triangulation systems, (2) to study the effect on position error of proposed modifications to the system, and (3) to specify system design parameters when requirements on position error have been established. The following fictitious example illustrates these uses.

A triangulation system consists of three bearing-measurement stations located at the vertexes of an equilateral triangle. The distance, D , from the center of the triangle to each of the bearing-measurement stations is fifty miles. The standard deviation, $\sigma_{\epsilon B}$, of the error in the bearing measurement at each station is 0.057 radians. The time required at each station to perform one measurement is 5.0 seconds. Targets with speeds of 0.2 miles per second are expected. The characteristics of this fictitious triangulation system are summarized in Table 8-1. It is assumed that this system may operate in either the conventional or the modified mode.

For a target located near the center of the triangle, it is desired to determine the magnitude of the position error that will be exceeded only ten percent of the time when ten targets are under surveillance. From Equation (4-29), the time between consecutive bearing measurements of a target at each bearing-measurement station is $T = M\tau = 50$ seconds. From Equation (7-2), the normalized distance that the target moves in the time T is $VT/D\sigma_{\epsilon B} = 3.5$. From the graph for $P(E_N > E_0) = 0.1$ in Figure 7-7, the value of E_0 which corresponds to $D_N = 3.5$ is 2.86.

TABLE 8-1

CHARACTERISTICS OF A FICTITIOUS THREE-STATION TRIANGULATION SYSTEM

D	=	50 miles
$\sigma_{\epsilon B}$	=	0.057 radians
τ	=	5.0 seconds
M	=	10
T	=	50 seconds
V	=	0.2 miles per second
D_N	=	$VT/D\sigma_{\epsilon B}$ = 3.5
E_o	=	2.86 (from Figure 4-8)
E	=	$E_o D \sigma_{\epsilon B}$ = 8.16 miles

From Equation (7-1), the magnitude of the position for this value of E_o is given by $E = E_o D \sigma_{\epsilon B}$ = 8.16 miles. Thus, the magnitude of the position error will exceed 8.16 miles with a probability of 0.1. The results of this analysis have been included in Table 8-1.

Study of the effect of minor modifications to the system is most easily accomplished by use of an equation for the position error and perhaps a Taylor series expansion of the equation about the values of the parameters which represent the present system. Again, for the purpose of illustration, it is assumed that the position error which will be exceeded only ten percent of the time is of interest. Therefore, the graphs in Figure 7-7 of the locus of $P(E_N > E_o)$ = 0.1 for both the conventional and modified three-station system are used. The locus for the modified three-station system is observed to have zero slope at $D_N = 0$ and has, for large values of D_N, an asymptote which is a straight line with a slope of 0.58. This latter observation is made from the graph of $P([E_N/D_N] > [E_o/D_N])$ for $D_N = \infty$ in Figure 7-5. This asymptote, expressed in equation form is

$$E_o \approx C_1 + C_2 D_N \quad , \tag{8-1}$$

in which C_2 equals 0.58 and C_1, estimated from Figure 7-7, is 1.0.

In spite of some similarity, the locus is not a hyperbola. The entire locus could be approximated by a quotient of higher degree polynomials; a quotient of low degree polynomials is not a satisfactory approximation. However, for the purpose to be served herein, a simple second degree polynomial is sufficient. For both the conventional and modified three-station systems, that part of each locus represented by the graphs plotted in Figure 7-7 can be approximated reasonably well[1] by the parabola

$$E_o \approx C_3 + C_4 D_N + C_5 D_N^2 \quad , \tag{8-2}$$

in which all of the constants are positive. $C_3 = 1.75$, $C_4 = 0.0965$, and $C_5 = 0.0624$ for the conventional three-station system, and $C_3 = 2.52$, $C_4 = 0.0400$, and $C_5 = 0.0376$ for the modified three-station system. From Equations (7-1) and (7-2), the position error which is exceeded

[1] The deviation of the approximate equation from the points plotted in Figure 7-7 does not exceed 0.03 (1.2 percent) in the E_o coordinate.

only ten percent of the time can be expressed in the form:

$$E_{10} \approx C_3 D \sigma_{\epsilon B} + C_4 V T + \frac{C_5 V^2 T^2}{D \sigma_{\epsilon B}} \quad . \qquad (8\text{-}3)$$

The partial derivatives of E_{10} with respect to D, $\sigma_{\epsilon B}$, and T, as obtained from the approximate Equation (8-3), assuming these variables to be independent, are:

$$\frac{\partial E_{10}}{\partial D} \approx C_3 \sigma_{\epsilon B} - \frac{C_5 V^2 T^2}{D^2 \sigma_{\epsilon B}} = \left[C_3 - C_5 D_N^2\right] \sigma_{\epsilon B} , \qquad (8\text{-}4)$$

$$\frac{\partial E_{10}}{\partial \sigma_{\epsilon B}} \approx C_3 D - \frac{C_5 V^2 T^2}{D \sigma_{\epsilon B}^2} = \left[C_3 - C_5 D_N^2\right] D , \text{ and} \qquad (8\text{-}5)$$

$$\frac{\partial E_{10}}{\partial T} \approx C_4 V + \frac{2 C_5 V^2 T}{D \sigma_{\epsilon B}} = \left[C_4 + 2 C_5 D_N\right] V . \qquad (8\text{-}6)$$

For values of D_N less than $\left(C_3/C_5\right)^{1/2}$, each of the partial derivatives is greater than zero, which indicates that a small increase in D, $\sigma_{\epsilon B}$, or T causes an increase in E_{10}. For the parameters D and $\sigma_{\epsilon B}$, the effect on E_{10} is a maximum at $D_N = 0$ and is smaller for larger values of D_N. For the parameter T, the effect on E_{10} is a minimum at $D_N = 0$ and is larger at larger values of D_N.

Equations (8-4) and (8-5) indicate that the partial derivatives of E_{10} with respect to D and $\sigma_{\epsilon B}$ are zero at $D_N = \left(C_3/C_5\right)^{1/2}$

and negative for D_N in excess of this value. Obviously the approximation is not valid in this region; $D_N = (C_1/C_3)^{1/2}$ corresponds to $D_N = 5.4$ for the conventional, and $D_N = 8.2$ for the modified three-station system, values which are beyond the range of points plotted in Figure 7-7 from which the approximation was made.

For the modified three-station system, Equation (8-1) can be used to examine the partial derivatives of E_{10} at large values of D_N. Assuming the variables D, $\sigma_{\epsilon B}$, and T to be independent, these partial derivatives are:

$$\frac{\partial E_{10}}{\partial D} = C_1 \sigma_{\epsilon B} , \tag{8-7}$$

$$\frac{\partial E_{10}}{\partial \sigma_{\epsilon B}} = C_1 D , \text{ and} \tag{8-8}$$

$$\frac{\partial E_{10}}{\partial T} = C_2 V , \tag{8-9}$$

all of which are positive.

If $\Delta D/D$, $\Delta\sigma_{\epsilon B}/\sigma_{\epsilon B}$, and $\Delta T/T$ denote small fractional changes in the parameters D, $\sigma_{\epsilon B}$, and T, respectively, the corresponding small change in E_{10} is approximately

$$\Delta E_{10} \approx \left[C_3 - C_5 D_N^2\right] \sigma_{\epsilon B} D \left[\frac{\Delta D}{D} + \frac{\Delta\sigma_{\epsilon B}}{\sigma_{\epsilon B}}\right] + \left[C_4 + 2C_5 D_N\right] D_N \sigma_{\epsilon B} D \frac{\Delta T}{T} \tag{8-10}$$

for values of D_N which are small, and

$$\Delta E_{10} \approx C_1 \sigma_{\epsilon B} D \left[\frac{\Delta D}{D} + \frac{\Delta \sigma_{\epsilon B}}{\sigma_{\epsilon B}} \right] + C_2 D_N \sigma_{\epsilon B} D \frac{\Delta T}{T} \qquad (8\text{-}11)$$

for the modified system when the value of D_N is large. For the modified system, the coefficients C_1 and $C_2 D_N$ in Equation (8-11) are the limiting values for large values of D_N of the coefficients $C_3 - C_5 D_N^2$ and $(C_4 + 2C_5 D_N) D_N$, respectively, in Equation (8-10). The relative effect on E_{10} of the same fractional change in D, $\sigma_{\epsilon B}$, and T can be compared by comparing the coefficients in Equation (8-10). The relative effects are the same when

$$C_3 - C_5 D_N^2 \approx C_4 + 2C_5 D_N \qquad (8\text{-}12)$$

or when

$$D_N \approx \sqrt{\frac{C_3 - C_4}{C_5} + 1} - 1 \; . \qquad (8\text{-}13)$$

For the conventional three-station system, Equation (8-13) yields $D_N \approx 4.2$, which indicates that throughout the range of values of D_N in which the conventional system should be used in preference to the modified system, greater reduction in position error will be accomplished by a reduction in D or $\sigma_{\epsilon B}$ than by the same fractional reduction in T.[1]

---

[1] When D, $\sigma_{\epsilon B}$, and T are independent variables, the equality of the relative effects of D and $\sigma_{\epsilon B}$ on E_{10} is a consequence of the definition of E_N and D_N.

For the modified three-station system, Equation (8-13) yields $D_N \approx 7.2$, which indicates that in the range $0 \le D_N \le 7$, E_{10} is more sensitive to a fractional change in D or $\sigma_{\epsilon B}$ than to the same fractional change in T.

Depending on the nature of the triangulation system, the parameters D, $\sigma_{\epsilon B}$, and T may or may not be independent. Variation in the parameter D corresponds to variation in the area of surveillance. If T is directly proportional to the number of targets under surveillance, and if the expected number of targets under surveillance is directly proportional to the area of surveillance, then

$$T = C_6 D^2 , \qquad (8\text{-}14)$$

in which C_6 is a proportionality constant. T must be considered dependent upon any variation in D but must also be considered to be independent when D is constant because T may be changed in other ways. For this case, the partial derivative of E_{10} with respect to D, as obtained from the approximate Equation (8-3), is:

$$\frac{\partial E_{10}}{\partial D} \approx C_3 \sigma_{\epsilon B} + C_4 V C_6 2 D + \frac{C_5 V^2 C_6^2 3 D^2}{\sigma_{\epsilon B}}$$

$$= \left(C_3 + 2 C_4 D_N + 3 C_5 D_N^2 \right) \sigma_{\epsilon B} \quad . \qquad (8\text{-}15)$$

For this case, the partial derivative of E_{10} with respect to D as obtained from Equation (8-1), which is the limit for large values of D_N, is

$$\frac{\partial E_{10}}{\partial D} \approx C_1 \sigma_{\epsilon B} + C_2 V C_6 2 D$$

$$= \left(C_1 + 2 C_2 D_N\right) \sigma_{\epsilon B} \quad . \qquad (8\text{-}16)$$

The partial derivatives of E_{10} with respect to $\sigma_{\epsilon B}$ and T are the same as those obtained for the case of all independent parameters. In terms of small fractional changes in the parameters, the change in E_{10} is

$$\Delta E_{10} \approx \left(C_3 + 2C_4 D_N + 3C_5 D_N^2\right) \sigma_{\epsilon B} D \frac{\Delta D}{D}$$

$$+ \left(C_3 - C_5 D_N^2\right) \sigma_{\epsilon B} D \frac{\Delta \sigma_{\epsilon B}}{\sigma_{\epsilon B}}$$

$$+ \left(C_4 + 2C_5 D_N\right) \sigma_{\epsilon B} D \frac{\Delta T}{T} \qquad (8\text{-}17)$$

for values of D_N which are small, and

$$\Delta E_{10} \approx \left(C_1 + 2C_2 D_N\right) \sigma_{\epsilon B} D \frac{\Delta D}{D}$$

$$+ C_1 \sigma_{\epsilon B} D \frac{\Delta \sigma_{\epsilon B}}{\sigma_{\epsilon B}}$$

$$+ C_2 D_N \sigma_{\epsilon B} D \frac{\Delta T}{T} \qquad (8\text{-}18)$$

for only the modified system when the values of D_N are large. The coefficients of Equation (8-17) and (8-18) demonstrate that ΔE_{10} is most sensitive to fractional changes in D at all values of D_N.

Dependence among parameters may exist in other ways, depending upon the exact nature of the triangulation system. For example, the standard deviation of the bearing-measurement error may be related to the time T, the distance D, or both. If functional relationships which describe the dependence are obtained, the effect of small changes in the parameters can be examined in the same way as those considered herein.

Study of the effect of modifications to the system in which large changes in parameter values are involved requires direct use of the graphs presented in Chapter VII. To illustrate the effect of large changes, the following modifications to the system described in Table 8-1 have been considered:

Modification A	D = 25 miles,
Modification B	$\sigma_{\epsilon B}$ = 0.285 radians, and
Modification C	T = 25 seconds.

In each case, the parameter value has been decreased by half. Each modification and combination of the parameters was analysed in the same way as the original system was analysed; the results are tabulated in Table 8-2. Dependence of T upon D, as described in Equation (8-14), has been considered also in a modification designated by the symbol A'. In all of the cases, it has been assumed that the system operates as a conventional three-station system for $D_N < 3.5$ and as a modified three-station system for $D_N > 3.5$.

TABLE 8-2

EFFECT OF LARGE CHANGES IN PARAMETER VALUES ON THE MAGNITUDE OF THE POSITION ERROR (WHICH IS EXCEEDED WITH A PROBABILITY OF 0.1)

		D_N	E_0	E (miles)
Unmodified System		3.5	2.86	8.16
System with the listed modifications and combinations of modifications:	A	7.0	4.64	6.61
	B	7.0	4.64	6.61
	C	1.75	2.0	5.71
	AB	14.0	8.1*	5.76*
	AC	3.5	2.86	4.08
	BC	3.5	2.86	4.08
	ABC	7.0	4.64	3.30
	A'	1.75	2.0	2.86
	A'B	3.5	2.86	2.04
	A'C	0.875	1.86	2.65
	A'BC	1.75	2.0	1.43

* estimated

Knowledge of the position error as a function of the system parameters and knowledge of the importance of changes in each parameter provide a foundation upon which values of these parameters can be specified in design when requirements on position error have been established. The state of the art determines the minimum value of $\sigma_{\epsilon B}$ that can be used and partially limits the minimum value of T for a fixed number of targets under surveillance. A value of D can then be selected to satisfy the position error requirements. If the value of D so selected is impractical, i.e., does not provide a sufficient area of surveillance, then advances in the state of the art are required. The relative importance of the parameters suggests which of the parameters should receive the most attention in advancing the state of the art in order to satisfy the system requirements.

CHAPTER IX

CONCLUSIONS AND POSSIBLE EXTENSIONS

This study provides a measure of the additional error in locating a target by triangulation when non-simultaneous, rather than simultaneous, bearing measurements are used. It extends the previous work on the analysis of the error in triangulation systems and investigates the effect of a geometric approximation used in previous studies. The results suggest areas in which future study would be profitable.

This study extends the work of the author[1], who first considered the error in triangulation systems which use non-simultaneous bearing measurements by loosely approximating the effect of the non-simultaneous measurements. The approximation consisted of increasing the variance of a normal distribution used to describe the error in the bearing measurements. This present study uses a separate, realistic probability distribution for the components of bearing-reading error due to non-simultaneous measurements.

In previous studies[(8,10,19)], the geometry in the vicinity of the target has been approximated in order to simplify the mathematical expressions used in the studies. This study determines the error in this approximation by deriving the equations for position error without approximating the geometry.

This study demonstrates the advisability of using a weighted centroid of the intersections of all possible pairs of bearing lines, whereas previous studies[(19)] have considered a weighting procedure which is more complicated.

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[1] Frese[(8)].

For a triangulation system consisting of three bearing-measurement stations, this study determines the conditions under which greater accuracy is obtained when only two, rather than three, bearing readings are used to calculate the location of the target. These conditions had not been determined previously.

The methods used in this study are applicable to future studies of the error in triangulation systems. These future studies include numerically investigating the position error when more than three stations are used and extending the range and scope of numerical investigations of the three-station system.

For triangulation systems which employ four or more bearing-measurement stations, the general expression for the position error which was developed in Chapter IV, and the numerical procedures presented herein can be used to calculate probability distributions for the position error. Each additional station considered adds an additional dimension to the probability space. In such a study, if the number of intervals used in the approximations of the probability distributions for the components of bearing-reading error is the same as the number of intervals used in the study described herein, the computing time required would be increased by a factor of approximately sixty for each additional station considered.[1] Reduction of the number of intervals used will decrease the accuracy of the calculations. However, the

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[1] The computer (MIDAC), which was used to obtain the numerical results presented herein, performs multiplication in approximately three milliseconds and, when most efficiently programmed, performs addition and logical operations in 0.43 milliseconds. The computing time required to calculate each of the probability distributions presented in Figure 7-1 (with the exception of $D_N = 0$) was from 8-1/2 to 9 hours.

results of this study as presented in Figures 7-10 and 7-11 indicate that the number of intervals may be reduced a small amount with little effect on the overall accuracy because the error due to truncation appears to predominate.

For the conventional three-station system, probability distributions for the magnitude of the position error can be obtained for values of D_N greater than those considered herein in order to obtain a quantitative measure of how much reduction in position error is produced by the use of the modified three-station system. The calculations of each such probability distribution would require computing time only slightly in excess of the time required to calculate the distributions presented herein. However, the calculation of such distributions does require the use of a digital computer with a high-speed storage capacity of greater than 512 words in order to accommodate component distributions over a larger range of arguments if the size of the approximation intervals is not to be increased.[1]

For any number of bearing-measurement stations, the equations and numerical methods presented herein can be used to describe the magnitude of the position error in any combination of the following situations:

1. a target located at a point other than at the center of the system,

2. a non-symmetrical arrangement of the bearing-measurement stations,

---

[1] The computer program, listed in Appendix A, utilizes 511 of the 512 high-speed storage cells of the MIDAC computer. Only a small saving in the use of these cells could be accomplished by changes in the program.

3. a non-normal probability distribution for bearing-measurement error,

4. a non-uniform distribution for the age of the bearing measurements, and

5. large rather than small bearing-reading error.

Four times as much computing time would be required to obtain each probability distribution for uniformly distributed target direction in the first and second situations than in the special case numerically treated herein because of differences in symmetry. The third situation requires only a change of the numerical distribution which is stored in the program and does not require additional computation time. The fourth situation may not require additional computing time nor additional computer storage space if the distribution to be used for age can be described by a simple equation. Changes in the computer program are necessary, of course. If an empirical distribution is to be used in the fourth situation, additional storage space in the computer is required. The fifth situation would substantially increase the required computing time because of the additional arithmetic operations to be performed.

APPENDIX A

PROGRAM FOR THE CONVENTIONAL SYSTEM

The program listed below was used with the MIDAC computer to calculate a set of cumulative probability distributions for the normalized position error for the case of uniformly distributed target direction. Intermediate results for each of twelve target directions were obtained also. This particular program applies to the cases $\sigma_U = 10$ and $sD_T = 25$. A modification to this program for $\sigma_U = 50/7$ is presented in Appendix D. Modifications for $sD_T = 125/7$ and $150/7$ are presented in Appendix E.

MIDAC, the digital computer which was used to calculate the probability distributions for position error, is a three-address, serial, general purpose computer.[1] The programs presented herein make use of the Magic I system of MIDAC, an automatic programming system which translates a program which is written in floating address form into the correct computer language for computation.[2]

A MIDAC instruction word is made up of four parts; α , β , γ , and an operation symbol, which are written in the order listed. Table A-1 contains a simplified description of the logical and arithmetic operations performed by MIDAC in terms of these four parts of each

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[1] MIDAC is described in detail by Carr and Scott(5), Appendix IX.1.

[2] The Magic I system, which includes other features such as automatic error diagnoses and a library of frequently used sub-routines, is described by Brown, J. H., "Programming for the Magic I System," Section II.3 of a book edited by Carr and Scott(5).

TABLE A-1

SIMPLIFIED DESCRIPTION OF THE MIDAC OPERATIONS

Operation Symbol	Description
ba	Augment C_B by $-\alpha$. If $\beta > C_B, \rightarrow \gamma$. If $\beta \leq C_B$, set $C_B = 0$ and continue.
ex	$[\gamma] = [\alpha] \oplus [\beta]$.
ad	$[\gamma] = [\alpha] + [\beta]$.
ri	Read in α words from input station or drum address β to α cells starting with address γ.
su	$[\gamma] = [\alpha] - [\beta]$.
cn	If $[\beta] > [\alpha], \rightarrow \gamma$. If $[\beta] \leq [\alpha]$, continue.
sn	$[\gamma] = [\alpha] \times 2 \exp\left([\beta] \times 2^{-44}\right)$.
cm	If $\|[\beta]\| > \|[\alpha]\|, \rightarrow \gamma$. If $\|[\beta]\| \leq \|[\alpha]\|$, continue.
fi	Store $C_i + 1$ in γ of α and $\rightarrow \gamma$.
dv	$[\gamma] = [\beta]/[\alpha]$.
mr	$[\gamma] = [\beta][\alpha]$, (rounded).
mℓ	$[\gamma] = [\beta][\alpha] \times 2^{-44}$.
ro	Read out α words from α cells starting with address β to output station or drum address γ.
bd	$[\gamma]$ is a binary coded decimal number equal to $[\alpha]$.

LEGEND:

C_B is the base counter, a counter used in cycling.

C_i is the instruction counter, which is set to the address of the instruction being performed.

$\rightarrow$ means "set C_i to".

[ ] means "the contents of".

$\oplus$ means a logical "and" rather than arithmetic addition.

instruction word. The symbols α , β , and γ are used to designate either numbers or the addresses of storage cells, depending on the nature of the operation. By affixing a minus sign to an address, the address is automatically augmented by either of two counters, the base counter if the operation symbol is prefixed by a minus sign, or the instruction counter if not. A minus sign affixed to a number in a "read in" or "read out" instruction specifies that the contents of the cells are coded typewriter symbols and not numerical quantities. The address which precedes the instruction words is the floating address which is not assigned to a computer address until translation takes place.

PROGRAM

cd					
faa04	000	000	000	ba	clear the base counter
faa05	1e01	e02	e00	ex	set T_ϕ , the ϕ tally, to its initial value (usually zero)
faa06	e00	b00	a07	ad	put unmodified instruction in a07
faa07	000	000	000	ri	put selected value of $A \times 2^{-36}$ and $B \times 2^{-36}$
	-001	002	a07	ba	in c00 and 1c00, respectively
	c00	e02	2c00	ex	$\|A\| \times 2^{-36}$ in 2c00
faa08	-d01	-d01	-d01	-su	clear 155 cells from address d01
	-001	155	a08	ba	through address 73d02
	1b00	e02	a12	ex	put unmodified instructions
	2b00	e02	b05	ex	in a12 and b05
	2e01	3e01	c01	su	$v_s = -24 \times 2^{-36}$ in c01
	f00	c00	a09	cn	if $A > 0 \rightarrow$ a09
	c00	1e02	3c00	sn	$A \times 2^{-24}$ in 3c00
	c00	a12	a12	ad	add A to the addresses in
	3c00	a12	a12	ad	γ and β of instruction a12
	c00	c01	c01	ad	$(v_s + A) \times 2^{-36}$ in c01
faa09	2e01	2c00	a10	cm	if $\|A\| > 1 \rightarrow$ a10
	e02	e02	4c00	ex	$1 - 2^{-44} \approx 1/\|A\|$ in 4c00
	512	001	a11	fi	$\rightarrow$ a11
faa10	2c00	2e01	4c00	dv	$1/\|A\|$ in 4c00
faa11	4c00	-d00	-d03	-mr	$P_c(v \| v'' = v_0'') = P_{v''}(v_0'') P_u(v - v_0'')$
	-001	050	a11	ba	stored in 50 cells starting with address d03
	1e00	1e00	1e00	su	clear T_A , the "A" tally
	a12	e26	b05	ex	use instruction a12 (already modified) to modify instruction b05 so that the use of P_v starts at $P_v(v_s)$
faa12	000	000	000	ri	accumulate P_v at the proper
	-001	050	a12	ba	address determined by instruction a12
	2e01	1e00	1e00	ad	augment T_A by 1
	4e01	a12	a12	ad	augment β and γ of instruction a12 by 1, i.e., augment v_0'' by 1
	1e00	2c00	a12	cm	if $\|A\| > T_A \rightarrow$ a12
faa13	3b00	e02	a16	ex	put unmodified instruction in a16
faa14	1c00	2e01	5c00	dv	1/B in 5c00
faa15	-d00	5c00	-d03	-mr	$P_c(\omega \| \omega'' = \omega_0'') = P_{\omega''}(\omega_0'') P_u(\omega - \omega_0'')$
	-001	050	a15	ba	in 50 cells starting with address d03
	2e00	2e00	2e00	su	clear T_B , the "B" tally
faa16	000	000	000	ri	accumulate P_ω at the proper address
	-001	050	a16	ba	determined by instruction a16
	2e01	2e00	2e00	ad	augment T_B by 1
	4e01	a16	a16	ad	augment β and γ of instruction a16 by 1, i.e., augment ω_0'' by 1
	2e00	1c00	a16	cm	if $B > T_B \rightarrow$ a16
faa17	e04	-d00	-d03	-mr	P_u /12 in 50 cells starting
	-001	050	a17	ba	with address d03
faa18	2c00	5e01	c02	ad	$(49 + \|A\|) \times 2^{-36}$ in c02
	1c00	5e01	1c02	ad	$(49 + B) \times 2^{-36}$ in 1c02
	3e00	3e00	3e00	su	clear T_v , the "v" tally
fab05	000	000	000	ri	$P_v(v_0)$ in 3c01
	c01	c01	2c01	mℓ	$v_0^2 \times 2^{-28}$ in 2c01
	4b00	e02	b06	ex	put unmodified instruction in b06
	2e01	3e01	1c01	su	$\omega_s = -24 \times 2^{-36}$ in 1c01
	4e00	4e00	4e00	su	clear T_ω , the "ω" tally
fab06	000	000	000	ri	$P_v(v_0) P_\omega(\omega_0)$ in 2e27
	1c01	c01	4c01	su	$(\omega_0 - v_0) \times 2^{-36}$ in 4c01
	4c01	1c01	5c01	mℓ	$\omega_0(\omega_0 - v_0) \times 2^{-28}$ in 5c01
	5c01	2c01	6c01	ad	$[v_0^2 + \omega_0(\omega_0 - v_0)] \times 2^{-28}$ in 6c01
	6e01	3e01	c04	su	$u_s = -24.5 \times 2^{-36}$ in c04
	1c01	c01	1c04	ad	$(\omega_0 + v_0) \times 2^{-36}$ in 1c04
faa19	c04	1c04	2c04	su	$(u_s - \omega_0 - v_0) \times 2^{-36}$ in 2c04
	2c04	c04	3c04	mℓ	$u_s(u_s - \omega_0 - v_0) \times 2^{-28}$ in 3c04
	3c04	6c01	4c04	ad	$E_c(u_s, v_0, \omega_0) \times 2^{-28}$ in 4c04

faa20	4c04	e01	c05	sn	$E_c \times 2^{-13}$ in c05
	2e01	1c04	5c04	su	$(1 - v_0 - w_0) \times 2^{-36}$ in 5c04
	e05	c05	e25	ad	$(E_c + K_2) \times 2^{-13}$ in e25
	5c04	3e02	5e25	sn	$(1 - v_0 - w_0) \times 2^{-13}$ in 5e25
	c04	6e02	4e25	sn	$u_0 \times 2^{-12}$ in 4e25
faa22	a27	001	a25	fi	→ a25, the start of the inner loop subroutine
	2e01	4e00	4e00	ad	augment T_w by 1
	2e01	1c01	1c01	ad	augment w_0 by 1
	7e01	b06	b06	ad	
	4e00	1c02	b06	cm	if 49 + B > T_w → b06
faa23	2e01	3e00	3e00	ad	augment T_v by 1
	2e01	c01	c01	ad	augment v_0 by 1
	7e01	b05	b05	ad	
	3e00	c02	b05	cm	if 49 + $\lvert A \rvert$ > T_v → b05
faa24	c06	c06	c06	su	clear cell c06, the cell at which the cumulative distribution is accumulated
	001	1e01	001	ro	print out code words which identify the results
	001	e00	001	ro	
faa28	-1c4	c06	c06	-ad	construct the cumulative distribution
	c06	001	1c06	bd	convert the result to a decimal number
	001	1c06	001	ro	print out the result
	-001	060	a28	ba	→ a28 until all 60 results are printed out
faa29	4e02	e00	e00	ad	augment T_ϕ by 2
	e00	5e02	a06	cm	if 24 > T_ϕ → a06
	000	000	000	ri	HALT
faa25	e25	e29	1e29	dv	$E_R(u_0, v_0, w_0) \times 2^{-24} = K_1 \times 2^{-37}/(E_c + K_2) \times 2^{-13}$ in 1e29
	-d03	2e27	1e27	-mr	$P_{uvw}(u_0, v_0, w_0)$ in 1e27
	1e29	e26	1e25	ex	β_R, i.e., β digits of $E_R \times 2^{-24}$ in 1e25
	1e25	1e26	3e25	ad	$(452 + \beta_R) \times 2^{-24}$ in 3e25
	3e25	2e26	e27	sn	$(452 + \beta_R) \times 2^{-36}$ in e27
	e27	4326	a26	ex	modify instruction a26 so that the results are accumulated properly
	3e25	3e26	a26	ex	
	5e25	4e25	2e25	ad	$(2u_0 + 1 - v_0 - w_0) \times 2^{-13}$ in 2e25
	2e25	e25	e25	ad	$[E_c(u_0 + 1) + K_2] \times 2^{-13} = [E_c(u_0) + K_2] \times 2^{-13} + (2u_0 + 1 - v_0 - w_0) \times 2^{-13}$ in e25
	4e25	e28	4e25	ad	$2(u_0 + 1) \times 2^{-13} = 2u_0 \times 2^{-13} + 2 \times 2^{-13}$ in 4e25
faa26	1e27	000	000	ad	accumulate P_{uvw} in the proper cell
	-001	050	a25	ba	→ a25, i.e., cycle through all values of u in order to accumulate $P_c(E_R \mid v = v_0, w = w_0)$ in 60 cells starting with cell number 452
faa27	512	001	000	fi	→ 1a22, i.e., return to the main program
fae25					assign six cells for temporary storage
ac6e25					
fae28					
da001	001				2×2^{-13}
fae26					
da005	000fff				constants entered as hexadecimal numbers
	0001c4				
	-0000000000c				
	000fff				
	000000fff				
fae27					assign three cells for temporary storage
ac3e27					
fae29					
da001	000000d6f2c				$K_1 \times 2^{-37}$
ac2e29					assign one cell for temporary storage
cd					
fab00	f00	-e03	-c00	-ad	fixed instructions which are modified in use elsewhere in the program
	-d03	-21d01	-21d01	-ad	
	f00	000	3c01	ad	
	-d03	-d02	-d02	-ad	
	3c01	d02	2e27	mr	

```
fac00          } assign six cells    }
ac6c00         }                     }
fac01          } assign seven cells  }
ac7c01         }                     }
fac02          } assign two cells    }  for temporary storage
ac2c02         }                     }
fac04          } assign six cells    }
ac6c04         }                     }
fac05            assign one cell     }
fac06          } assign two cells    }
ac2c06         }                     }
def
fad00   .1988    -2d   0b
        .2526    -2d   0b
        .3179    -2d   0b
        .3961    -2d   0b
        .4886    -2d   0b
        .5967    -2d   0b
        .7213    -2d   0b
        .8635    -2d   0b
        .10234   -1d   0b
        .12008   -1d   0b
        .13950   -1d   0b
        .16043   -1d   0b
        .18270   -1d   0b
        .20596   -1d   0b
        .22989   -1d   0b
        .25405   -1d   0b
        .27795   -1d   0b
        .30109   -1d   0b
        .32289   -1d   0b
        .34285   -1d   0b
        .36040   -1d   0b
        .37511   -1d   0b
        .38651   -1d   0b    the set of constants, P_v(v) , entered as decimal fractions
        .39432   -1d   0b
        .39828   -1d   0b
        .39828   -1d   0b
        .39432   -1d   0b
        .38651   -1d   0b
        .37511   -1d   0b
        .36040   -1d   0b
        .34285   -1d   0b
        .32289   -1d   0b
        .30109   -1d   0b
        .27795   -1d   0b
        .25405   -1d   0b
        .22989   -1d   0b
        .20596   -1d   0b
        .18270   -1d   0b
        .16043   -1d   0b
        .13950   -1d   0b
        .12008   -1d   0b
        .10234   -1d   0b
        .8635    -2d   0b
        .7213    -2d   0b
        .5967    -2d   0b
        .4886    -2d   0b
        .3961    -2d   0b
        .3179    -2d   0b
        .2526    -2d   0b
        .1988    -2d   0b
```

fad01		
ac81d01		assign eighty-one cells (for temporary storage)
fad02		
ac74d02		assign seventy-four cells (for temporary storage)
fad03		
ac50d03		assign fifty cells (for temporary storage)
fae00		
ac5e00		assign five cells (for temporary storage)
fae01		
da008	0000000000f	used to multiply by 2^{15} by shifting
	0	initial value of T_ϕ
	000000001	1×2^{-36}
	000000019	25×2^{-36}
	000001001	used to modify an instruction
	000000031	49×2^{-36}
	0000000008	0.5×2^{-36}
	000001	used to modify an instruction
fae02		
da007	ffffffffff	extractor
	0000000000c	used to multiply by 2^{12} by shifting
	00000000009	used to multiply by 2^{9} by shifting
	00000000017	used to multiply by 2^{23} by shifting
	000002	2×2^{-24}
	000018	24×2^{-24}
	00000000018	used to multiply by 2^{24} by shifting
fae03		
da024	-000000015	values of A and B
	000000016	
	-000000013	
	000000018	
	-000000010	
	000000019	
	-00000000e	
	000000019	
	-00000000b	
	000000019	
	-000000008	
	000000019	
	-000000005	
	000000018	
	-000000001	
	000000016	
	000000001	
	000000015	
	000000005	
	000000013	
	000000008	
	000000010	
	00000000b	
	00000000e	
fae04		
def	.83333333333 -1d 0b	1/12
fae05		
da001	03a08	$K_2 \times 2^{-13}$
bca04		computation begins with the instruction at address a04

APPENDIX B

MODIFIED SYSTEM, PROGRAM I

The program listed below, in floating address form, was used with the MIDAC computer[1] to calculate and print out a set of fifty cumulative probability distributions for the magnitude of the position error for the modified three-station system for values of C from one through fifty. In this program, $\sigma_u = 10$. Appendix D contains a modification to this program for $\sigma_u = 50/7$.

[1] See Appendix A for a brief description of MIDAC operations and programming.

PROGRAM

cd					
faa04	000	000	000	ba	clear the base counter
faa08	-d01	-d01	-d01	-su	clear 99 cells starting
	-001	099	a08	ba	with address d01
	b00	e02	a12	ex	put unmodified instruction
	1b00	e02	b05	ex	in cells a12 and b05
	e01	1e01	c01	su	$\nu_s = -24 \times 2^{-36}$ in c01
	e01	e00	a10	cm	if $C > 1 \rightarrow$ a10
faa09	-d00	e02	-d01	-ex	$P_\nu = P_\upsilon(\nu - \frac{1}{2})$ in 50 cells
	-001	050	a09	ba	starting with d01
	512	001	a12	fi	$\rightarrow$ a18
faa10	e00	1e02	c00	sn	$2C \times 2^{-28}$ in c00
	e00	e00	1c00	mℓ	$C^2 \times 2^{-28}$ in 1c00
	c00	2e01	2c00	su	$(2C - 1) \times 2^{-28}$ in 2c00
	1c00	2c00	3c00	dv	$(2C-1)/C^2 = P_{\nu''}(1/2)$ in 3c00
	1c00	3e01	4c00	dv	$2/C^2$ in 4c00
	1e00	1e00	1e00	su	clear $T_c \times 2^{-26}$, the "C" tally
faa11	-d00	3c00	d03	-mr	$P_c(\nu \mid \nu'' = \nu_o'') = P_{\nu''}(\nu_o'') P_\upsilon(\nu - \nu_o'')$
faa12	000	000	000	ri	accumulated at the proper address
	-001	050	a11	ba	determined by instruction a12
	3c00	4c00	3c00	su	$P_{\nu''}(\nu_o'' + 1) = P_{\nu''}(\nu_o'') - 2/C^2$ in 3c00
	e01	1e00	1e00	ad	augment T_c by 1
	4e01	a12	a12	ad	augment ν_o'' by 1
	1e00	e00	a11	cm	if $C > T_c \rightarrow$ a11
faa18	e00	5e01	c02	ad	$(49 + C) \times 2^{-36}$ in c02
	2e00	2e00	2e00	su	clear $T_\nu \times 2^{-36}$, the "ν" tally
fab05	000	000	000	ri	$P_\nu(\nu_o)$ in 2e27
	c01	c01	1c01	mℓ	$\nu_o^2 \times 2^{-28}$ in 1c01
faa19	6e01	1e01	c04	su	$\upsilon_s = -24.5 \times 2^{-36}$ in c04
	c01	c04	1c04	ad	$(\upsilon_s + \nu_o) \times 2^{-36}$ in 1c04
	1c04	c04	2c04	mℓ	$\upsilon_s(\upsilon_s + \nu_o) \times 2^{-28}$ in 2c04
	1c01	2c04	3c04	ad	$[\nu_o^2 + \upsilon_s(\upsilon_s + \nu_o)] \times 2^{-28} = E_c(\upsilon_s, \nu_o)$ in 3c04
	3c04	2e02	c05	sn	$E_c(\upsilon_s, \nu_o) \times 2^{-13}$ in c05
	e05	c05	e25	ad	$(E_c + K_2) \times 2^{-13}$ in e25
faa20	c04	1e02	1c05	sn	$2\upsilon_s \times 2^{-28}$ in 1c05
	e01	c01	2c05	ad	$(1 + \nu_o) \times 2^{-36}$ in 2c05
	2c05	3e02	3c05	sn	$(1 + \nu_o) \times 2^{-28}$ in 3c05
	3c05	1c05	4c05	ad	$(2\upsilon_s + 1 + \nu_o) \times 2^{-28}$ in 4c05
	4c05	2e02	2e25	sn	$(2\upsilon_s + 1 + \nu_o) \times 2^{-13}$ in 2e25
faa22	a27	001	a25	fi	$\rightarrow$ a25, the start of the inner loop subroutine
	e01	2e00	2e00	ad	augment T_ν by 1
	e01	c01	c01	ad	augment ν_o by 1
	7e01	b05	b05	ad	
	2e00	c02	b05	cm	if $49 + C > T_\nu \rightarrow$ b05
faa24	c06	c06	c06	su	clear cell c06, the cell at which the cumulative distribution is accumulated
	001	e00	001	ro	read out C } identifies results
	-001	e06	001	ro	carriage return } identifies results
faa28	-e10	c06	c06	-ad	construct the cumulative distribution
	c06	001	1c06	bd	convert the result to a decimal number
	001	1c06	001	ro	print out the result
	-001	060	a28	ba	$\rightarrow$ a28 until all 60 results are printed out
faa29	e01	e00	e00	ad	augment C by 1
faa30	-e10	-e10	-e10	-su	clear 60 cells starting with
	-001	060	a30	ba	address e10
	-004	e06	001	ro	read out four carriage return instructions
	e00	8e01	a04	cm	if $50 > C \rightarrow$ a04
	000	000	000	ri	HALT

faa25	e25	e29	1e29	dv	$E_R(\upsilon_0, \nu_0) \times 2^{-24} = K_1 \times 2^{-37} / (E_c + K_2) \times 2^{-13}$
	1e29	e26	1e25	ex	β_R, i.e., β digits of $E_R \times 2^{-24}$, in 1e25
	1e25	1e26	3e25	ad	$(452 + \beta_R) \times 2^{-24}$ in 3e25
	3e25	2e26	e27	sn	$(452 + \beta_R) \times 2^{-36}$ in e27
	e27	4e26	a26	ex	modify instruction a26 so that the
	3e25	3e26	a26	ex	results are accumulated properly
	2e25	e25	e25	ad	$[E_c(\upsilon_0 + 1) + K_2] \times 2^{-13} = [E_c(\upsilon_0) + K_2] \times 2^{-13} + (2\upsilon_0 + 1 + \nu_0) \times 2^{-13}$ in e25
	2e25	e28	2e25	ad	$(2[\upsilon_0 + 1] + 1 + \nu_0) \times 2^{-13} = (2\upsilon_0 + 1 + \nu_0) \times 2^{-13} + 2^{-12}$ in 2e25
	-d00	2e27	1e27	-mr	$P_\upsilon(\upsilon_0) P_\nu(\nu_0)$ in 1e27
faa26	1e27	000	000	ad	accumulate $P_\upsilon P_\nu$ in proper cell
	-001	050	a25	ba	→ a25 and cycle through all values of υ in order to accumulate $P_c(E_R \mid \nu = \nu_0)$ in 60 cells starting with cell number 452
faa27	512	001	000	fi	→ 1a22, i.e., return to main program
fae25					assign six cells for temporary storage
ac6e25					
fae28					a constant, 2^{-12}
da001	001				
fae26					constants entered as hexadecimal numbers
da005	000fff				
	0001c4				
	-0000000000c				
	000fff				
	000000fff				
fae27					assign three cells for temporary storage
ac3e27					
fae29					
da001	000000d6f2c				$K_1 \times 2^{-37}$
ac2e29					assign one cell for temporary storage
cd					
fab00	d03	-d01	-d01	-ad	fixed instructions used at
	f00	d01	2e27	ad	addresses a12 and b05
fac00					assign five cells (for temporary storage)
ac5c00					
fac01					assign two cells (for temporary storage)
ac2c01					
fac02					assign one cell (for temporary storage)
fac04					assign four cells (for temporary storage)
ac4c04					
fac05					assign five cells (for temporary storage)
ac5c05					
fac06					assign two cells (for temporary storage)
ac2c06					
def					
fad00	.1988	-2d	0b		the set of constants, $P_\upsilon(\upsilon)$, entered as decimal fractions
	.2526	-2d	0b		
	.3179	-2d	0b		
	.3961	-2d	0b		
	.4886	-2d	0b		
	.5967	-2d	0b		
	.7213	-2d	0b		
	.8635	-2d	0b		
	.10234	-1d	0b		
	.12008	-1d	0b		
	.13950	-1d	0b		
	.16043	-1d	0b		
	.18270	-1d	0b		
	.20596	-1d	0b		
	.22989	-1d	0b		
	.25405	-1d	0b		
	.27795	-1d	0b		

(Continued on the following page)

	.30109	-1d	0b	
	.32289	-1d	0b	
	.34285	-1d	0b	
	.36040	-1d	0b	
	.37511	-1d	0b	
	.38651	-1d	0b	
	.39432	-1d	0b	the set of constants, $P_{\nu}(\nu)$, entered as decimal fractions
	.39828	-1d	0b	
	.39828	-1d	0b	
	.39432	-1d	0b	
	.38651	-1d	0b	
	.37511	-1d	0b	
	.36040	-1d	0b	
	.34285	-1d	0b	
	.32289	-1d	0b	
	.30109	-1d	0b	
	.27795	-1d	0b	
	.25405	-1d	0b	
	.22989	-1d	0b	
	.20596	-1d	0b	
	.18270	-1d	0b	
	.16043	-1d	0b	
	.13950	-1d	0b	
	.12008	-1d	0b	
	.10234	-1d	0b	
	.8635	-2d	0b	
	.7213	-2d	0b	
	.5967	-2d	0b	
	.4886	-2d	0b	
	.3961	-2d	0b	
	.3179	-2d	0b	
	.2526	-2d	0b	
	.1988	-2d	0b	
fad01				assign ninety-nine cells
ac99d01				for temporary storage
fad03				assign one cell for temporary storage
fae00				
da001	000000001			$C \times 2^{-36}$, initially set at $C = 1$
ac3e00				assign two cells for temporary storage
fae01				
da009	000000001			2^{-36}
	000000019			25×2^{-36}
	0000001			2^{-28}
	0000002			2×2^{-28}
	000001001			used to modify an instruction
	000000031			49×2^{-36}
	0000000008			0.5×2^{-36}
	000001			used to modify an instruction
	000000032			50×2^{-36}
fae02				
da004	ffffffffff			extractor
	00000000009			used to multiply by 2^{9} by shifting
	0000000000f			used to multiply by 2^{15} by shifting
	00000000008			used to multiply by 2^{8} by shifting
fae05				
da001	03a08			$K_2 \times 2^{-13}$
fae06				
da004	ec			
	ec			code for carriage return, used for the format of the results
	ec			
	ec			
ac452				
fae10				assign one cell for temporary storage
bca04				computation begins with the instruction at address a04

APPENDIX C

MODIFIED SYSTEM, PROGRAM II

The program listed below provides a change in Program I by which the set of fifty cumulative probability distributions for the position error is stored on the magnetic drum of the computer instead of being printed. Each of these cumulative distributions may be interpreted as a conditional distribution for the corresponding value of C . The program then provides that any sub-set of twelve conditional distributions may be selected (with replacement) and combined to obtain a resultant distribution assuming that each of the selected values of C is equally likely. This combinatorial procedure is used to obtain cumulative probability distributions for the position error for an arbitrary target direction. The selection of the twelve distributions is accomplished by listing the twelve corresponding values of C on an auxiliary tape which is processed by the computer. These values are listed in Appendix D for the several values of sD_T which were used.

PROGRAM

change					
aca24					
cd	512	001	a34	fi	→ to drum storage subroutine
ac4e06					
cd					
faa34	e11	e11	e11	su	clear cell used as temporary storage for distribution sum
	e00	1e13	1e12	sn	$8C \times 2^{-36}$ in 1e12
	a37	1e12	2e12	ad	$(1064m + 8C) \times 2^{-36}$, drum address for distribution storage, in 2e12
	2e12	2e13	3e12	sn	$(1064m + 8C) \times 2^{-24}$ in 3e12
	2e12	4e26	a32	ex	add proper drum address to instruction a32
	3e12	e26	1a32	ex	add proper drum address to instruction 1a32
faa31	-e10	c10	-1e11	-mr	multiply distribution by 1/12
	e11	-1e11	e11	-su	obtain the distribution sum
	-001	060	a31	ba	and place in cell e11
	a36	001	a35	fi	→ distribution sum check subroutine
	f00	c11	a34	cm	→ a34 if distribution sum check is not correct
faa32	061	e11	000	ro	store the distribution plus the distribution sum on the drum starting with drum address (1064m + 8C)
	061	000	451	ri	read the distribution plus the distribution sum from the drum starting with address 451
	b36	001	b35	fi	→ distribution sum check subroutine
	f00	c11	a32	cm	→ a32 if distribution sum check is not correct
	e11	001	e12	bd	print out distribution sum
	001	e12	001	ro	to indicate progress of computation
faa33	-e10	-e10	-e10	-su	clear cells used for storage
	-001	060	a33	ba	of the distribution
	e01	e00	e00	ad	augment C by 1
	e00	3e13	a04	cm	if $51 > C$ → a04
fab20	013	001	e14	ri	read in a code word followed by twelve values of $C \times 2^{-12}$ from an auxiliary tape
	f00	e28	e12	ad	set T_D, the distribution tally, to 1×2^{-12}
fab21	-1e11	-1e11	-1e11	-su	clear 60 cells starting
	-001	060	b21	ba	with address 1e11
fab24	a38	e12	b25	ad	add T_D to α of instruction a38 and put the result at address b25
fab25	000	000	000	ri	select the $(T_D - 1)$ 'th value of $C \times 2^{-12}$ which was read in from the auxiliary tape and put it at address 1e12 with a scale change to $C \times 2^{-21}$
	a37	1e12	2e12	ad	add $8C$ to β of the instruction at address a37 and put the result at address 2e12
	2e12	e26	b22	ex	put the drum address $(1064m + 8C)$ in β of the instruction at address b22
fab22	061	000	451	ri	read the distribution plus the distribution sum corresponding to the $(T_D - 1)$ 'th value of C from the drum to 61 cells starting with address 451
	b36	001	b35	fi	→ distribution sum check subroutine
	f00	c11	b22	cm	→ b22 if distribution sum check is not correct
fab23	-452	-1e11	-1e11	-ad	accumulate the sum of the twelve distributions
	-001	060	b23	ba	in 60 cells starting with address 1e11
	e28	e12	e12	ad	augment T_D by 1
	e12	4e13	b24	cm	if $13 > T_D$ → b24
fab26	001	e14	001	ro	print out a code word which identifies the results
	-001	e06	001	ro	read out a carriage return instruction
	e11	e11	e11	su	clear cell e11

```
fab27   -lell    ell     ell     -ad    construct the cumulative distribution
         ell     001     cll     bd     convert result to a decimal number
         001     cll     001     ro     print out the decimal result
        -001     060     b27     ba     → b27 until all 60 results are printed out
        -004     e06     001     ro     read out four carriage return instructions
         512     001     b20     fi     → b20 and repeat the distribution combination process for the
                                        next set of twelve values of C . (end of main program)
faa35    cll     cll     cll     su     clear cell used for temporary storage
        -ell     cll     cll     -ad)  (obtain the distribution sum
        -001     061     la35    ba )  (and place it in cell cll
faa36    512     001     000     fi     return to the main program
fab35    cll     cll     cll     su     clear cell used for temporary storage
        -451     cll     cll     -ad)  (obtain the distribution sum and
        -001     061     lb35    ba )  (place it in cell cll
fab36    512     001     000     fi     return to the main program
faa37    000     1064m   1064m   ri     dummy instruction used only for the address it contains
faa38    el4     el3     lel2    sn     fixed instruction used at address b25
fael2                               )  (assign four cells for
ac4el2                              )  (temporary storage
fael3
da005   -00000000009                    used to multiply by 2^-9 by shifting
         00000000003                    used to multiply by 2^3 by shifting
         0000000000c                    used to multiply by 2^12 by shifting
         000000033                      51 x 2^-36
         00d                            13 x 2^-12
facl0
def     .83333333333     -1      0b     1/12
facll                                   assign one cell for temporary storage
fael4                               )
acl3el4                             )   assign thirteen cells for temporary storage
faell                                   assign one cell for temporary storage
bca04                                   computation begins with the instruction at address a04
```

APPENDIX D

OTHER VALUE OF σ_v

The program listed below was used to change the computer programs listed in Appendixes A and B for σ_v = 50/7.

PROGRAM

```
change
acd00
def      .157      -3d   0b
         .251      -3d   0b
         .394      -3d   0b
         .606      -3d   0b
         .914      -3d   0b
         .1352     -2d   0b
         .1961     -2d   0b
         .2788     -2d   0b
         .3889     -2d   0b
         .5319     -2d   0b
         .7134     -2d   0b
         .9381     -2d   0b
         .121      -1d   0b
         .15301    -1d   0b
         .18977    -1d   0b
         .23078    -1d   0b
         .27522    -1d   0b
         .32186    -1d   0b
         .36911    -1d   0b
         .4151     -1d   0b
         .45776    -1d   0b
         .49503    -1d   0b
         .52496    -1d   0b
         .54591    -1d   0b
         .5567     -1d   0b
         .5567     -1d   0b
         .54591    -1d   0b
         .52496    -1d   0b
         .49503    -1d   0b
         .45776    -1d   0b
         .4151     -1d   0b
         .36911    -1d   0b
         .32186    -1d   0b
         .27522    -1d   0b
         .23078    -1d   0b
         .18977    -1d   0b
         .15301    -1d   0b
         .121      -1d   0b
         .9381     -2d   0b
         .7134     -2d   0b
         .5319     -2d   0b
         .3889     -2d   0b
         .2788     -2d   0b
         .1961     -2d   0b
         .1352     -2d   0b
         .914      -3d   0b
         .606      -3d   0b
         .394      -3d   0b
         .251      -3d   0b
         .157      -3d   0b
bca04
```

APPENDIX E

OTHER VALUES OF A AND B

The following sets of values of the constants A and B were used in the program listed in Appendix A at address fae03 when the distribution for the variable v is described by $\sigma_v = 50/7$. The value of sD_T for each set of values of A and B is listed at the head of each column. The set of values listed in Appendix A are used for $sD_T = 3.5$.

$sD_T = 2.5$	$sD_T = 3.0$
000000008	000000009
00000000a	00000000c
000000006	000000007
00000000c	00000000e
000000003	000000004
00000000d	000000010
000000001	000000001
00000000f	000000012
-000000001	-000000001
000000010	000000014
-000000003	-000000004
000000011	000000014
-000000006	-000000007
000000012	000000015
-000000008	-000000009
000000012	000000015
-00000000a	-00000000c
000000012	000000015
-00000000c	-00000000e
000000012	000000015
-00000000d	-000000010
000000011	000000014
-00000000f	-000000012
000000010	000000014

APPENDIX F

CODE WORDS FOR PROGRAM II

Each group of twelve code words (values of C) listed below was used in Program II to select and combine twelve probability distributions for position error in order to obtain a distribution for the case of uniformly distributed target direction. The code words apply to the case of $\sigma_U = 50/7$. The values of sD_T are listed at the head of each column.

0.5	1.0	1.5	2.0	2.5	3.0	3.5
001	001	001	001	001	001	002
001	001	002	003	003	004	005
001	002	003	005	006	007	008
002	003	005	006	008	009	00b
002	004	006	008	00a	00c	00e
002	005	007	009	00c	00e	011
003	005	008	00b	00d	010	013
003	006	009	00c	00f	012	015
003	006	00a	00d	010	013	016
003	007	00a	00e	011	014	018
004	007	00a	00e	012	015	019
004	007	00b	00e	012	015	019

4.0	4.5	5.0	5.5	6.0	6.5	7.0
002	002	002	003	003	003	003
006	007	007	008	008	009	00a
009	00a	00b	00d	00e	00f	010
00d	00e	010	011	013	015	016
010	012	014	016	018	01a	01c
013	015	018	01a	01c	01f	021
015	018	01b	01e	020	023	026
018	01b	01e	021	024	027	02a
01a	01d	020	023	026	02a	02d
01b	01e	022	025	029	02c	02f
01c	020	023	027	02a	02e	031
01c	020	024	027	02b	02e	032

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