A DECENTRALIZED APPROACH TO SYSTEM OPTIMAL ROUTINGS IN DYNAMIC TRAFFIC NETWORKS

ALFRED GARCIA
Department of Industrial and Operations engineering
University of Michigan, Ann Arbor, MI 48109, USA

DANIEL REAUME
Department of Industrial and Operations engineering
University of Michigan, Ann Arbor, MI 48109, USA

ROBERT L. SMITH
Department of Industrial and Operations engineering
University of Michigan, Ann Arbor, MI 48109, USA

Technical Report 97-15
November 24, 1997

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Alfredo Garcia, Daniel Reaume and Robert L. Smith
Department of Industrial and Operations Engineering
University of Michigan, Ann Arbor, MI 48109

November 23, 1997

Abstract

We introduce a novel procedure to compute system optimal routings in a dynamic traffic network. Fictitious play is utilized within a game of identical interests wherein vehicles are treated as players with the common payoff of "average trip time experienced" in the network. Encouraging results from a large scale computational test on a real network are presented.

1 Introduction

We consider the problem of finding a *system-optimal* routing vector in a dynamic traffic network. In other words, we seek a set of routes that minimizes the average trip time experienced by the vehicles in the network A closely related problem is that of finding a *user-optimal* routing vector. that is. one such that no single driver may reduce her/his expected travel time. by unilaterally deviating from their assigned route.

This latter problem can also be reformulated to be the determination of a Nash equilibrium solution for the dynamic traffic network game. In such a game, players are identified with vehicles and payoffs are computed through an *assignment* mapping that, given routing decisions for all players. calculates the resulting dynamic travel times. A *routing* mapping assigns to each vehicle a time-dependent shortest path from origin to destination given the route choices of all other vehicles. In other words. the routing mapping is a *best reply* to the routing decisions of the other vehicles.

Fictitious play is an iterative procedure in which at each step players compute best replies assuming that opponents decisions are distributed according to the historical frequency of their previous decisions (Brown [1951]). In the dynamic traffic network context, this procedure can be interpreted as an iterative routing-assignment algorithm, in which at each

*This work was supported in part by the Intelligent Transportation Systems Research Center of Excellence at University of Michigan.

step, for each player, the routing mapping computes time-dependent shortest paths given that other players decisions are distributed according to the historical frequency of routing decisions. This algorithm has been implemented and widely tested (see Kaufman, Smith and Wunderlich[1992]). Experimental results for this procedure have been favorable but difficult to justify since a proof of convergence has yet to be developed.

Recent theoretical advances in game theory (see Monderer and Shapley [1996]), yielded conditions that may be applied to the dynamic vehicle routing problem. Monderer and Shapley demonstrate that when players (in this case, vehicles) share a common objective function, such as minimizing the average trip times of all vehicles in the network, then fictitious play converges in to an equilibrium in *beliefs*, i.e historical frequencies of routing decisions. Although, this convergence result requires interpretation in the dynamic traffic network context, it motivates a potentially attractive algorithm. Since an equilibrium solution to an artificial dynamic traffic game in which players share the above mentioned common objective is probably a *good* system optimal routing.

2 Preliminaries.

2.1 Notation.

We introduce the dynamic traffic network game where :

- $N = \{1, 2, ..., n\}$ is the index set of vehicles.
- Every vehicle has a finite number of routing choices, that is to say: for every $i \in N$ there is a set $Y_i = \{r_{i1}, r_{i2}, ..., r_{im_i}\}$ of possible routes to take. Let us denote by $Y = \prod_{i \in N} Y_i$
- $A : Y \to R^n$ is the *assignment* mapping. For any $y \in Y$, i.e a routing policy vector, $A_i(y)$ is the sum over the path determined by y_i of the resulting dynamic travel times per link. In other words, $A_i(y)$ is the expected total travel time for vehicle i , given that the other vehicles take routes y_j, $j \neq i$. We will denote by Δ^i the set of mixed routing decisions, i.e :

$$\Delta^i = \{f^i : Y_i \to [0, 1] \text{ such that } \sum_{y_i \in Y_i} f^i(y_i) = 1\}$$

  The extreme points of Δ^i are just the elements of Y_i .
  Let $\Delta = \prod_{i \in N} \Delta^i$. We extend the domain of the assignment mapping so that for $f \in \Delta$, we have:

$$A_i(f) = \sum_{y \in Y} A_i(y) \cdot f^1(y_1) \cdot f^2(y_2) \cdots f^n(y_n)$$

- Let $\Delta^{-i} = \prod_{j \neq i} \Delta^j$, the cartesian product of the sets of mixed routing decisions for all vehicles other than i.

$R_i : \Delta \rightarrow \Delta^i$ is the routing mapping or "best response" for vehicle i to a mixed routing decision followed by the other vehicles. Given $f \in \Delta$, $R_i(f)$ is the set of routes for vehicle i that yield the least expected travel time assuming the other vehicles $j \neq i$ follow the routing decisions as prescribed by f. Formally :

$$R_i(f) = \arg \min_{f^i \in \Delta^i} [\sum_{y \in Y} A_i(y) \cdot f^1(y_1) \cdot \cdot f^i(y_i) \cdots \cdot f^n(y_n)]$$

3 Nash Equilibrium.

We say that a mixed routing vector f^* is user-optimal if for every vehicle $i \in N$ the probabilities assigned to routes for vehicle i by f^* yield the minimum expected total travel time, provided that f^*_{-i}, the mixed routing choice of all other vehicles is held fixed, i.e :

$$f_i^* \in \arg \min_{f^i \in \Delta^i} A_i(f_i, f^*_{-i})$$

Now, recalling our definition of $R_i(.)$, the routing mapping, we note the previous equation can also be expressed as :

$$f_i^* \in R_i(f^*)$$

Finally, in vector notation the last equation can be rewritten as :

$$f^* = R(f^*)$$

which is the very familiar definition of a fixed point. It is worth pointing out that we only differ from Rosenthal's [1973] formulation in that to take into account time varying link travel times, we do not make any analytical assumptions on the *assignment* mapping.

Moreover, as opposed in the static setting (see Haurie and Marcotte[1985]) in this dynamic setting, there is not a straightforward relationship between Wardrops' *user optimality* and the Nash equilibrium concept.

3.1 Existence of Nash Equilibrium in Mixed Decisions.

Recall that a mixed routing decision $f \in \Delta$ is a "lottery" combination of routing decisions in the set Y.

A straightforward conclusion from the classical Nash equilibrium existence theorem is :

Theorem 1 :A dynamic traffic network has a Nash equilibrium in mixed routing decisions.

Proof: It simply follows Nash[1950], given the finiteness of the set of routes for each player and the way payoffs are defined for mixed strategies. ∎

4 Fictitious Play Convergence.

We now briefly review Monderer and Shapley 's result :

4.1 Notation.

Let us denote by $K \subset \Delta$ the equilibrium set for the artificial traffic game above presented and $\|.\|$ any fixed euclidean norm on Δ .
For $\delta > 0$ let $\mathbf{B}_\delta(K)$ the open ball with radii δ , i.e :

$$\mathbf{B}_\delta(K) = \{g \in \Delta : \min_{f \in K} \|g - f\| < \delta\}$$

A *path* is a sequence $\mathbf{y} = \{y(t)\}_{t=1}^{\infty}$ of elements of Y .
A *belief path* is a sequence $\mathbf{f} = \{f(t)\}_{t=1}^{\infty}$ in Δ .

Definition 1 : *We say that a belief path converges to equilibrium (or converges in beliefs) if each limit point is an equilibrium point. Formally, for every $\delta > 0$ there exists T such that $f(t) \in \mathbf{B}_\delta(K)$ for all $t \geq T$.*

To every path $\mathbf{y}$ we can associate a belief path $f_{\mathbf{y}}$ by simply computing the historical frequency of the various routing decisions, for given $r \in Y_i$:

$$f^i_{\mathbf{y},t}(r) = \frac{\#\{1 \leq s \leq t : y_i(s) = r\}}{t}$$

Note that if we define $I^i_{\mathbf{y},t}(r)$ to be the indicator function of the route r in the path then :

$$f^i_{\mathbf{y},t+1}(r) = f^i_{\mathbf{y},t}(r) + \frac{(I^i_{\mathbf{y},t}(r) - f^i_{\mathbf{y},t}(r))}{t+1}$$

where :

$$I^i_{\mathbf{y},t}(r) = \begin{matrix} 1 & \text{if } y_i(t) = r \\ 0 & \text{otherwise} \end{matrix}$$

Definition 2 : *A path $\mathbf{y}$ is a fictitious play process if for every $i \in N$ and every t :*

$$y_i(t+1) \in \arg\min_{y_i \in Y_i} [A_i(y_i, f^{-i}_{\mathbf{y},t})]$$

In words, at each t the route prescribed for player i ,is the pure best response to the mixed strategy for all other vehicles consisting of the historical frequency of routes they have chosen.

Definition 3 : *A Game has the Fictitious Play Property if every fictitious play process converges in beliefs .*
We now state the important Monderer and Shapley's result.

Theorem 2 : If all players have the same payoff function, then the game has the Fictitious Play Property.

Proof : See Monderer and Shapley [1996]. ∎
We now introduce the artificial dynamic traffic game that will, by construction, posess the fictitious play property.

5 The Artificial Dynamic Traffic Game.

In order to apply Monderer and Shapley's convergence result, we redefine the traffic game by artificially imposing the same payoff function to every player in the game. Specifically, we will use in this artificial traffic game (which we shall refer to as *ATDG*) the "average trip time" as the commmon payoff function for all vehicles, $U : \Delta \rightarrow R$:

$$U(f) = \sum_{i \in N} \frac{A_i(f)}{n}$$

Let us now examine the meaning of an equilibrium with respect to this "artificial" game.

$$f_i^* \in \arg\min_{f^i \in \Delta^i} U(f_i, f_{-i}^*)$$

Intuitively, given that all other vehicles $j \neq i$ follow f_j^*, vehicle i can not reduce any further the "average trip time" experienced by the vehicles in the network by deviating from the prescribed routing f_i^*. In other words, for the optimization problem :

$$(P) \quad \min_{f \in \Delta} U(f)$$

the mixed routing f^* is some sort of a "local" optimal solution. However, there may be optimal solutions to problem (P) that are not neccesarily equilibrium routings for the *ATDG* game. In other words;

$$\text{Optimal Solution Set}(P) \subset \text{Equilibrium Set}(ADTG)$$

5.1 An Algorithm.

We formally present the algorithm motivated by Monderer and Shapley's result. It presents, however, a major difficulty: Theorem 2 only asserts that for a *converging sequence of mixed strategies* generated by fictitious play the limit is a Nash equilibrium of the original game. It is in a sense, a "limsup" set convergence result, and for computational purposes we need a "liminf" type of result. However, it is worth pointing out that whenever the equilibrium set of our artificial game is a singleton, the algorithm is guaranteed to converge. In any other case, the algorithm will compute routings that will be arbitrarily close to the equilibrium set, then continuity of the *assignment* mapping, a reasonable assumption, will ensure a good approximation.

Algorithm

1. Pick an initial "pure" routing vector f_0.
2. Compute best reply :

$$y_i(t+1) \in \arg\min_{y_i \in Y_i} [U_i(y_i, f_t^{-i})]$$

3. Update historical frequencies of route choices, f_t .
4. If $\|f_{t-1} - f_t\| \leq \varepsilon$ then Stop, otherwise go to 2.

5.2 Implementation.

We have implemented the above algorithm in a software package called *Alliance*. To ensure tractability, we made several simplifications to the original algorithm. First, it is extremely difficult to, for each vehicle i, analytically compute a best response to the historical frequencies of the routings of the other vehicles. Instead we simulate the passage of these other vehicles through the network where each vehicle chooses its route with probability in accordance with the historical frequencies of its routings. Using the time dependent link travel time profile produced by this simulation, vehicle i may then be assigned to a path minimizing the increase to total system travel time. To further simplify matters, since a single vehicle will have little effect on congestion, we also simulate the the travel of vehicle i in this simulation, thus avoiding the need to run a separate simulation for each vehicle. It is hoped that for a heavily congested network with many vehicles leaving at the same time with the same origin and destination this approach will allow to adequately approximate true best responses.

The second simplification we perform is to discretize time into a sequence of slices. Within each slice, vehicles are routed by the simulation according to the routing tables. These tables assign, for each slice s and node n, the probability distribution with which a vehicle arriving at node n with destination d at time s, will choose its next link. Note that this simplification allows vehicles to follow routes they may have historically never taken, contradicting the algorithm's scheme. Here again, we appeal to the large number of vehicles flowing through the network to justify this simplification since the effective congestion should remain relatively unchanged while greatly reducing storage and computation requirements.

5.3 Computational Tests.

To validate *Alliance* we applied the algorithm to the Troy, Michigan traffic network. Approximately 16500 vehicles were allowed to flow into the network in 24 minutes according to travel patterns approximating those actually observed in Troy. After 24 minutes the flow into the network was halted and the vehicles were allowed to to travel for further 36 minutes, thus allowing the network to clear.

To account for the impact of different market penetration levels of ITS technologies we defined three types of vehicles classes: *Class 1*, consisted of those vehicles following the free flow shortest path, *Class 2*, consisted of those vehicles that perform a periodic update of the free flow shortest paths, and finally, *Class 3* vehicles were guided by the *Alliance* algorithm. The initial routing given to all classes corresponded to shortest path under free flow.

In the first test, with high market penetration (i.e. Class 3 vehicles account for 25% of the total number of vehicles) we observe that *Alliance* computes routings which are as good as those computed with *SAVaNT* in terms of system average trip time, in considerable less number of iterations and c.p.u time.

Test 1

	C1(50%)	C2(25%)	C3(25%)	# Iterations
Alliance	8.85988	8.85677	8.71779	14
SAVaNT	8.87245	8.84866	8.68266	34

In the second text, with low market penetration (i.e. Class 3 vehicles account for 5% of the total number of vehicles) we can observe reductions in travel time for intelligent vehicles, here again at a substantially lower computational effort when compared to *SAVaNT*.

Test 2

	C1(95%)	C3(5%)	Average	# Iterations
Alliance	17.30430	15.59930	17.21905	20
SAVaNT	17.49250	15.49160	17.39240	68

6 Conclusions.

Through the application of recent results in the theory of learning in games and the extension of Rosenthal's[1973] framework to the formulation of the dynamic congestion game, we have implemented a decentralized iterative procedure to compute system optimal routings.

By focusing on discrete routing decisions and with the help of a dynamic travel time simulator we avoided the technicalities of a more thorough analytical development.

First large scale results are encouraging, mostly by the substantial reductions in computational requirements.

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