MINIMAL FORECASE HORIZONS THROUGH MONOTONICITY OF OPTIMAL SOLUTIONS

ALFRED GARCIA
Department of Industrial and Operations engineering
University of Michigan, Ann Arbor, MI 48109, USA

ROBERT L. SMITH
Department of Industrial and Operations engineering
University of Michigan, Ann Arbor, MI 48109, USA

Technical Report 97-16
November 24, 1997

Minimal Forecast Horizons through Monotonicity of Optimal Solutions*

Alfredo Garcia and Robert L. Smith
Department of Industrial and Operations Engineering
University of Michigan, Ann Arbor MI 48109

November 24, 1997

Abstract

Monotonicity of optimal solutions to finite horizon dynamic optimization problems is used to prove the existence of a forecast horizon, i.e a long enough planning horizon that ensures that a first period optimal action for the infinite horizon and the finite horizon problem agree, regardless of problem parameter changes in the tail. The existence of extremal monotone optimal solutions, as in the context of production planning with convex costs, motivates a stopping rule to detect the minimal forecast horizon.

1 Introduction.

Infinite Horizon planning models are motivated by the difficulty in establishing a rationale for an a priori fixed study horizon. If an arbitrarily chosen finite horizon is used, end of horizon effects can alter the validity of the model in question. Hence, it has been argued that an infinite horizon is a better way to model instances in which, for dynamic optimization problems, the decision makers do not have a clear and prespecified ending date.

However, the gains in modeling accuracy afforded by an infinite horizon model are severely compromised by the technical difficulties that render intractable the analysis. This is particularly troublesome in instances in which the parameters are not known precisely and are assumed to possibly vary in time. In other words, the case with nonstationary parameters.

This last consideration motivates the problem of finding a finite horizon such that the first optimal decision for such a horizon coincides with the infinite horizon counterpart. If such a horizon exists (which is called a solution horizon), it not only provides a rationale to offer such a horizon as the decision makers planning horizon, but interestingly enough motivates a finite algorithm to solve an infinite problem via a rolling horizon procedure.

Nonetheless, the solution horizon concept is of little practical interest, for its computation may potentially require an infinite forecast of data. Thus, the concept of a *forecast* horizon

*This work was supported in part by NSF grants DDM-9214894 and DMI-9713723.

(see for example, Bès and Sethi (1987) and Haurie and Sethi (1988)), that is, a long enough planning horizon that entails the insensitivity of first period optimal actions with respect to parameter changes in the tail, is very attractive to practitioners. In brief, in order to compute the first period optimal action, the planner need only forecast a finite amount of data, regardless of tail variations.

In this paper we make use of monotonicity of optimal solutions to prove the existence of forecast horizons for a general class of dynamic optimization problems. Such monotonicity is a pervasive feature of many applications (see for example, Heyman and Sobel (1984)). We build on the work by Morton (1978) who exploited monotonicity in the context of the nonstationary periodic review inventory model with stationary linear costs to obtain upper and lower bounds for first period optimal decisions that are monotone in planning horizon. We extend Morton's work to cover any dynamic optimization problem with the property that optimal solutions that are monotone to parametrized variations in the state transition function exist. Such a property together with the principle of optimality allow to optimally embed the finite horizon problem in the infinite horizon setting as a parametric variation at the tail. In other words, finite horizon optimal solutions can be seen as infinite horizon optimal solutions to problem with a stationary trivial tail of parameters. This focus on early decision monotonicity has also been recently exploited by Smith and Zhang (1997) in the context of production planning with convex costs to develop a closed form formula for a forecast horizon under more restrictive monotonicity assumptions. We close our paper by presenting a stopping rule to yield optimal early production decisions for the infinite horizon production planning problem with convex costs that is guaranteed to detect the *minimal* forecast horizon. The existence of *extremal* monotone optimal solutions (see Topkis (1978)) is the key to this result.

2 Preliminaries.

2.1 Framework for parametric analysis of Dynamic Optimization Problems.

As in Bès and Sethi(1991), we define a parametrized family of discrete time dynamic optimization problems as follows :

- **(Canonical Four-tuple** (A_t, S_t, c_t, f_t)**)** At time period $t \in \mathcal{N}$, $A_t \subset R^+$ is the compact set of all possible actions where $\mathcal{N} = \{0, 1, 2, ...\}$ and R^+ stands for the nonnegative real line. $S_t \subset R^+$ is the set of attainable states, and finally $A_t(s_t) \subseteq A_t$ is the nonempty closed subset of feasible actions given current state s_t.
  If an action $a_t \in A_t(s_t) \subseteq A_t$ is taken given state $s_t \in S_t$, a cost $c_t(s_t, a_t) \in R^+$ is incurred and the next state to be attained is $s_{t+1} = f_t(s_t, a_t)$ where the state dynamics mapping $f_t : S_t \times A_t \rightarrow S_{t+1}$ and the cost function $c_t : S_t \times A_t \rightarrow R^+$ are assumed continuous.

- **(Forecasts $\mathcal{F}_t$)** We denote by $\mathcal{F}_t$, the set of all possible forecasts for time period t; that is for every element $p \in \mathcal{F}_t$, there exists a unique four-tuple of the form :

$$(A_t(\cdot;p) \subseteq A_t, S_t(p) \subseteq S_t, c_t(\cdot,\cdot;p) : S_t \times A_t \to R^+, f_t(\cdot,\cdot;p) : S_t \times A_t \to S_{t+1})$$

associated with forecast p, where $A_t(\cdot;p), S_t(p), c_t(\cdot,\cdot;p), f_t(\cdot,\cdot;p)$ stand for time period t feasible action correspondence, set of attainable states, cost and transition functions, respectively[1].

- **(Null Parameter θ)**We convene to define the *null* parameter, $\theta \in \mathcal{F}_t$ to which we associate for all $t \in \mathcal{N}$ the four-tuple :

$$(A_t(\cdot;\theta) = \{0\}, S_t, c_t(\cdot,\cdot;\theta) = 0, f_t(\cdot,\cdot;\theta) = 0)$$

- We assume that each $\mathcal{F}_t$ is a finite set endowed with a partial ordering " $\succeq_t$", according to which there is a "minimum" say $\underline{p}_t$, and a "maximum" forecast $\bar{p}_t$.

- We denote by $\mathcal{F}(T)$ and $\mathcal{F}$, the set of $T-$horizon forecast and the set of infinite horizon forecasts, respectively, i.e:

$$\mathcal{F}(T) = \prod_{t=0}^{T-1} \mathcal{F}_t \quad \mathcal{F} = \prod_{t=0}^{\infty} \mathcal{F}_t$$

- **(Forecast Metric Space $(\mathcal{F}, d)$)**We endow each $\mathcal{F}_t$ with the discrete topology by means of the metric $\rho_t : \mathcal{F}_t \times \mathcal{F}_t \mapsto \{0,1\}$ defined as follows, for $p_t, q_t \in \mathcal{F}_t$:

$$\rho_t(p_t, q_t) = \begin{matrix} 1 & \text{if } p_t = q_t \\ 0 & \text{otherwise} \end{matrix}$$

By means of this metric, we define a metric $d : \mathcal{F} \times \mathcal{F} \mapsto \mathcal{R}$, on $\mathcal{F}$ as follow, for $p = (p_0, p_1, ...,), q = (q_0, q_1, ...) \in \mathcal{F}$:

$$d(p, q) = \sum_{t=0}^{\infty} \frac{\rho_t(p_t, q_t)}{2^t}$$

We remark that this metric induces the product topology and that by Tychonoff's Theorem the space $(\mathcal{F}, d)$ is compact.

Given forecast $p \in \mathcal{F}$, the $T-$long horizon optimization problem induced by it, for a given initial state $s_0$is :

$$\begin{aligned} \min \quad & \sum_{t=0}^{T-1} c_t(s_t, a_t; p) \\ \text{s.t} \quad & s_{t+1} = f_t(s_t, a_t; p) \\ & a_t \in A_t(s_t; p) \subseteq A_t \quad t = 0, 1, 2..., T-1 \end{aligned}$$

[1]Note that the set of assumptions on the canonical four-tuple must hold for every indexed four-tuple.

Moreover, we shall denote by $C_T^*(p)$ and $A_T^*(p)$, the optimal value and optimal solution set for this problem when data follows forecast p for the first $T-1$ periods. Similarly, the Infinite Horizon Problem according to forecast $p \in \mathcal{F}$ is :

$$\begin{array}{lll} \min & \limsup_{T\to\infty} \sum_{t=0}^{T-1} c_t(s_t, a_t; p) & \\ \text{s.t} & s_{t+1} = f_t(s_t, a_t; p) & \\ & a_t \in A_t(s_t; p) \subset A_t & t = 0, 1, 2, ... \end{array}$$

As above, we shall denote by $C^*(p)$ and $A^*(p)$, the optimal value and optimal solution set for the infinite horizon problem as prescribed by forecast $p \in \mathcal{F}$.

2.2 Standing Assumption.

Throughout our analysis we will assume that the limit of a converging sequence of finite horizon optimal solutions is an optimal solution for the infinite horizon problem :

Assumption 1 For every $p \in \mathcal{F}$ and every indexed collection $\{a^T\}_{T \in \mathcal{N}}$ such that $a^T \in A_T^*(p)$ for each T and $\lim_{T\to\infty} a^T = a$ we have $a \in A^*(p)$.

For a thorough study of sufficient conditions that imply Assumption 1, the reader is refered to Flam and Fougères (1992) and Schochetman and Smith (1989) and (1992).

3 Examples.

In this section we give two classes of dynamic optimization problems that belong to the family introduced in the previous section, with suggestions on how to apply the abstract parametric framework above presented.

3.1 Production Planning.

The $T-$long horizon time varying production planning problem, with given initial inventory level I_0, is to find a production schedule that satisfies demand at minimum cost.

$$\begin{array}{lll} \min & \sum_{t=0}^{T-1} \alpha^t (C_t(x_t) + h_t(I_{t+1})) & \\ s.t & I_{t+1} = I_t + x_t - d_t & \\ & M_t \geq x_t \geq 0, I_t \geq 0 & \\ & x_t, I_t \text{ integer} & t = 0, 1, 2, ..., T-1 \end{array}$$

where x_t is the production level at time period t, I_t is the inventory level at hand at the start of time period t, $C_t(x_t)$ is the $t-$th-period production cost function and $h_t(I_{t+1})$ is the inventory holding cost. M_t is the maximal production capacity at time period t and $\alpha \in (0, 1)$ is the discount factor.

Now let us assume that for any time period, demands can take integer values in the range $\{\underline{d}, \underline{d}+1, ..., \bar{d}\}$ and that cost functions do not vary in time. As an illustration of parametric analysis, we can define our parameter set to be $\mathcal{F}_t = \{\underline{d}, \underline{d}+1, ..., \bar{d}\}$. With this convention, an element $p \in \mathcal{F}$ is simply an infinite sequence of demands which take values on the defined range.
In this context, the *null* parameter can be interpreted as *zero* demand. Notice, without loss of optimality that for a $T-$planning horizon, one must end with zero inventory. If we append to the $T-$long production plan an infinite tail of zero production, this would be an optimal solution to the Infinite Horizon Production Planning Problem with demand $(d_0, d_1, ..., d_{T-1}, 0, 0, ...)$.

3.2 Optimal Exploitation of Renewable Natural Resources.

We are given an initial stock s_0 of a natural resource. We have to choose a consumption level c_t at time period t, from which we experience a net reward $r_t(c_t)$. The remaining stock, if the available amount of resource is s_t, is then $s_t - c_t$ which shall renew at a pace $f_t(s_t - c_t)$ (with the convention that $f_t(0) = 0$). The T-Planning Horizon problem is to choose exploitation levels so to maximize total discounted reward :

$$\begin{array}{lll} \max & \sum_{t=0}^{T-1} r_t(c_t) \cdot \beta^t & \\ \text{s.t} & s_{t+1} = f_t(s_t - c_t) & \\ & c_t \in [0, s_t] & t = 0, 1, 2, ..., T-1 \end{array}$$

where $\beta \in (0, 1)$ is the discount factor.

As an illustration of the parametric analysis, let us assume that the stock renewal dynamics $f_t(.)$ can take any of the forms described in the finite set $\{f^0(\cdot) = 0, f^1(\cdot), ..., f^m(\cdot)\}$, where by f^0 we denote the *no-renewal* function. Intuitively, stock renewal dynamics can vary in time due to seasonality patterns, pollution, technological improvements etc...
With this convention our parameter set is $\mathcal{F}_t = \{0, 1, 2, .., m\}$ and to any element $p \in \mathcal{F}$ there is associated an infinite trend of stock dynamics. Here again, any finite horizon optimal consumption plan. can be trivially extended with an infinite tail of zero consumption so that it is an optimal solution of the infinite horizon problem with *no-renewal* dynamics at the tail.

4 Existence of Solution Horizon.

With a slight abuse of notation let us write $p \in \mathcal{F}(T)$ to mean that we are only concerned with the first T parameters in the infinite vector $p \in \mathcal{F}$. Let us assume now that for every $T-$period horizon optimization problem there exists an optimal solution such that the first period decision behaves monotonically with respect to parameters $p \in \mathcal{F}(T)$. A straightforward but very useful existence of solution horizon result follows :

Theorem 1: (Solution Horizon Existence) Under assumption 1 and assuming that there exist a doubly indexed collection :

$$\{a^{T,p} \in A^*_T(p)\}_{T \in \mathcal{N}, p \in \mathcal{F}}$$

such that the first time period actions are monotonically increasing in $p \in \mathcal{F}(T)$ i.e :

$$p \succeq q \Rightarrow a_0^{T,p} \geq a_0^{T,q}$$

then there exists an infinite horizon optimal solution $a^p \in A^*(p)$ such that for every $\varepsilon \geq 0$, there exist a planning horizon $\bar{T}_\varepsilon$ such that for $T \geq \bar{T}_\varepsilon$ we have :

$$\left| a_0^{T,p} - a_0^p \right| < \varepsilon$$

Such horizon is called a "solution" horizon.

Proof: Without loss of generality let us assume that $\underline{p}_t = 0$ for all $t \in \mathcal{N}$. If not, one can always add the *null* parameter to $\mathcal{F}_t$ and monotonicity still holds since associated to the *null* parameter one can only have *zero* action. Let us now append the *null* action to $a^{T,p}$ at period T, and denote it by $\underline{a}^{T+1}$i.e :

$$\underline{a}^{T+1} = (a_0^{T,p}, a_1^{T,p}, ..., a_{T-1}^{T,p}, 0)$$

By the Principle of Optimality, we have that $\underline{a}^{T+1}$must be an optimal solution to the $T+1$–planning horizon problem with parameters $(p_0, p_1, ..., p_{T-1}, 0)$, formally :

$$\underline{a}^{T+1} \in A^*_{T+1}(p_0, p_1, ..., p_{T-1}, \theta)$$

and by definition of the *null* parameter :

$$(p_0, p_1, ..., p_{T-1}, \theta) \preceq (p_0, p_1, ..., p_{T-1}, p_T)$$

Hence, by monotonicity it follows that :

$$a_0^{T,p} = \underline{a}_0^{T+1} \leq a_0^{T+1,p}$$

In words, the first period optimal action sequence for forecast $p \in \mathcal{F}$ is monotonically increasing.

Recall that $A \subset \prod_{t=0}^{\infty} A_t$, the subset of infinite feasible sequences of actions is assumed closed and by Tychonoff's theorem $\prod_{t=0}^{\infty} A_t$ is compact in the product topology, hence A is compact. Let us now embed every element in the collection $\{a^{T,p} \in A^*_T(p)\}_{T \in \mathcal{N}}$ into A as follows :

$$a^{T,p} \hookrightarrow (a_0^{T,p}, a_1^{T,p}, ..., a_{T-1}^{T,p}, 0, 0, ...,) =^{def} \tilde{a}^{T,p}$$

By Assumption 1, we know that using this embedding :

$$(a_0^{T,p}, a_1^{T,p}, ..., a_{T-1}^{T,p}, 0, 0, ...,) \in A^*(p_1, p_2, ..., p_{T-1}, \theta, \theta, ...)$$

By compactness the collection $\{\tilde{a}^{T,p}\}_{T\in\mathcal{N}}$ has a converging subsequence, let us denote by a^p the limit of such subsequence :

$$\lim_{k\to\infty} \tilde{a}^{T_k,p} = a^p$$

By assumption 1, $a^p \in A^*(p)$. By monotonicity of first period decision and compactness, the sequence $\{a_0^{T,p}\}$ converges, moreover, since convergence in the product topology is componentwise convergence :

$$\lim_{T\to\infty} a_0^{T,p} = a_0^p$$

Or equivalently; for any $\varepsilon \geq 0$, there exist a planning horizon $\bar{T}_\varepsilon$ such that for $T \geq \bar{T}_\varepsilon$ we have :

$$\left|a_0^{T,p} - a_0^p\right| < \varepsilon$$

∎

In the proof of the above theorem we have constructed a well defined function $a_0^p : \mathcal{F} \to R^+$, the next corollary establishes that when action spaces are finite, this function inherits the monotonicity property.

Corollary: Under the same assumptions of the previous theorem and assumming that action sets are finite, there exists an infinite horizon optimal solution $a^p \in A^*(p)$ such that the first period action is monotone and continuous in $\mathcal{F}$, i.e for $p, q \in \mathcal{F}$: $p \succeq q \Longrightarrow a_0^p \geq a_0^q$.

Proof: By the Solution Horizon Existence Theorem, for every $\varepsilon \geq 0$, there exist a planning horizon $\bar{T}_\varepsilon^p$ such that for $T \geq \bar{T}_\varepsilon^p$ we have :

$$\left|a_0^{T,p} - a_0^p\right| < \varepsilon$$

Similarly, there exist a planning horizon $\bar{T}_\varepsilon^q$ such that for $T \geq \bar{T}_\varepsilon^q$ we have :

$$\left|a_0^{T,q} - a_0^q\right| < \varepsilon$$

Let us pick, $\varepsilon < 1$ and $\bar{T} \geq \max\{\bar{T}_\varepsilon^p, \bar{T}_\varepsilon^q\}$. By this choice we have that for $T \geq \bar{T}$:

$$a_0^{T,p} = a_0^p \quad a_0^{T,q} = a_0^q$$

But by monotonicity hypothesis :

$$a_0^p \geq a_0^q$$

Moreover, since :

$$|a_0^p - a_0^q| \leq \left|a_0^p - a_0^{T,p}\right| + \left|a_0^{T,p} - a_0^{T,q}\right| + \left|a_0^{T,q} - a_0^q\right|$$

For any $\delta > 0$, by continuity[2] of $a_0^{T,p}$ in $p \in \mathcal{F}$ and the sequential construction of a_0^p there exists $\varepsilon > 0$ and T such that if $d(p, q) < \varepsilon$ then

$$|a_0^p - a_0^q| \leq \left|a_0^p - a_0^{T,p}\right| + \left|a_0^{T,p} - a_0^{T,q}\right| + \left|a_0^{T,q} - a_0^q\right| \leq \delta$$

Hence, the map $a_0^p : \mathcal{F} \mapsto R^+$ is continuous. ∎

[2] This is due to the finiteness of $\mathcal{F}(T)$.

5 Existence of Forecast Horizons.

In this section we prove the existence of Forecast Horizons for the class of dynamic optimization problems considered by exploiting the monotonicity properties of optimal solutions. It is worth emphasizing that these monotonicity properties are a a pervasive feature of many applications of the class of problems we study. To reader is referred to Heyman and Sobel's (1984) chapter 8.

Let us now state and prove the most important result :

Theorem 2: (Forecast Horizon Existence) Under Assumption 1 and assuming that action sets are finite and that there exists a doubly indexed collection :

$$\{a^{T,p} \in A^*(T,p)\}_{T\in\mathcal{N},p\in\mathcal{F}}$$

such that first period actions are monotonically increasing in $p \in \mathcal{F}(T)$, i.e :

$$p \succeq q \Longrightarrow a_0^{T,p} \geq a_0^{T,q}$$

then there exists an infinite horizon optimal solution $a^p \in A^*(p)$ and a finite planning horizon $\bar{T}$ such that for $T \geq \bar{T}$ and for every $q \in \mathcal{F}$ such that $p_t = q_t$, for $0 \leq t \leq \bar{T}$ we have :

$$a_0^{T,q} = a_0^p$$

Such horizon is called a "forecast horizon".

Proof: Let us construct a sequence of forecasts based on $p \in \mathcal{F}$ as follows; we append $\bar{p}_T$ to the $T-$truncation of $p \in \mathcal{F}$ we obtain the forecast :

$$u(T) = (p_0, p_1, \ldots, p_{T-1}, \bar{p}_T, \bar{p}_{T+1,\ldots})$$

where clearly we have :

$$u(T) \rightarrow p \text{ as} T \rightarrow \infty$$

But by the Solution Horizon Existence theorem, we know that :

$$a_0^{T,p} \rightarrow a_0^p \text{ as} T \rightarrow \infty \quad (1)$$

and by Corollary 1 the sequence $\{a_0^{u(T)}\}$ is monotonically decreasing in T, i.e :

$$a_0^{u(T+1)} \leq a_0^{u(T)}$$

By compactness of the first period action set and continuity of the map $a_0^p : \mathcal{F} \mapsto R^+$:

$$a_0^{u(T)} \rightarrow a_0^p \; as \; T \rightarrow \infty \quad (2)$$

So the results (1) and (2) ensure the existence of a large enough horizon such that $\bar{T}$ such that for $T \geq \bar{T}$:

$$a_0^{T,p} = a_0^{u(T)} = a_0^p$$

Now let us consider $q \in \mathcal{F}$ such that $p_t = q_t$, for $0 \leq t \leq \bar{T}$, by monotonicity it follows that and the choice of $\bar{T}$, for every $T \geq \bar{T}$:

$$a_0^{u(T)} \geq a_0^q \geq a_0^{T,p}$$

Hence, $a_0^p = a_0^q$; in words, infinite horizon first period optimal solution a_0^p is insensitive to parameter changes after time period $\bar{T}$. ∎

6 Detection of Minimal Forecast Horizons.

The proof of existence of a forecast horizon provides a few clues on how to effectively compute it, by means of a stopping rule. For the sake of concreteness we shall illustrate the suggested stopping rule in the context of production planning with convex production and inventory holding costs.

6.1 Application to Production Planning with Convex Costs.

In Smith and Zhang (1997) a closed form formula is developed for the production planning problem with convex production and inventory holding costs, by exploiting the monotonicity properties of production plans with respect to demand, a result due to Veinnot(1964).
As an application of the theory above presented, we propose a stopping rule for the same problem structure. Moreover, we show that it detects the *minimal* forecast horizon and the reader should note that it requires less restrictive monotonicity properties than those required in Smith and Zhang's paper in that it is only assumed that *first* and not *all* periods optimal decisions are monotone.

Nonetheless, the *minimality* of the Forecast Horizon detected through the Stopping Rule suggested is not evident. The reason being that in the construction carried out in the Solution Horizon Existence Theorem of an infinite horizon optimal solution whose first period action was monotone we *selected* for the finite horizon approximates *any* optimal monotone solution. This is not enough to ensure that the Forecast Horizon effectively computed through the above procedure is *minimal*. For that purpose, we need to select for the finite horizon approximates the *smallest* monotone optimal solution. Here we digress a bit from Veinnott's result in that it does not ensure the existence of such object. However, Topkis(1978) and more recently Milgrom and Shannon(1994) have developed a general monotonicity theory of optimal solutions using lattice programming techniques that not surprisingly apply for the production planning problem with convex costs. This theory ensures the existence of a *smallest* and a *largest* optimal solutions that are monotone.

6.2 The Stopping Rule

Assuming costs are uniformly bounded as follows :

$$\sup_t C_t(\cdot) \leq \bar{C}(\cdot) \quad \sup_t h_t(\cdot) \leq \bar{h}(\cdot)$$

One can construct a *pessimistic* scenario, in which demand, production and inventory holding costs are at their maximal levels, namely :

$$\begin{array}{lll} \min & \limsup_{N\to\infty} \sum_{t=0}^{N-1} \alpha^t(\bar{C}(x_t) + \bar{h}(I_{t+1})) & \\ s.t & I_{t+1} = I_t + x_t - \bar{d} & \\ & M \geq x_t \geq 0, I_t \geq 0 & \\ & x_t, I_t \text{ integer} & t = 0,1,2,... \end{array}$$

The above problem is very easy to solve by means of the functional equation :

$$(DP) \quad V(I) = \min_{M\geq x\geq 0}\{\bar{C}(x) + \bar{h}(x + I - \bar{d}) + \alpha V(x + I - \bar{d})\}$$

Let us now consider the *quasi-nonstationary* infinite horizon production planning problem:

$$\begin{array}{lll} \min & \limsup_{N\to\infty} \sum_{t=0}^{N-1} \alpha^t(C_t(x_t) + h_t(I_{t+1})) & \\ s.t & I_{t+1} = I_t + x_t - d_t & \\ & d_t = \bar{d} & t \geq T \\ & M \geq x_t \geq 0, I_t \geq 0 \text{ integer} & t = 0,1,2,... \end{array}$$

Which can be solved by the next simpler finite dimensional problem :

$$(\bar{P}_T) \quad \begin{array}{lll} \min & \sum_{t=0}^{T} \beta^t(C(x_t) + h(I_{t+1})) + \beta^{T+1}\cdot V(I_T) & \\ s.t & I_{t+1} = I_t + x_t - d_t & \\ & x_t \geq 0, I_t \geq 0 \text{ integer} & t = 0,1,...,T-1 \end{array}$$

By the Corollary to Solution Horizon Existence Theorem there exists a optimal solution to the problem $(\bar{P}_T)$ such that its first period action is monotone in the demand parameters, say a_0^T and $\bar{a}_0^T \geq \bar{a}_0^{T+1}$. Moreover, by the *smallest monotone* selection rule that we discussed above, we know that $\bar{a}_0^T$ is the *smallest optimal solution* to the above stated problem.

Similarly, if we solve:

$$(\underline{P}_T) \quad \begin{array}{lll} \min & \sum_{t=0}^{T} \beta^t(C_t(x_t) + h_t(I_{t+1})) & \\ s.t & I_{t+1} = I_t + x_t - d_t & \\ & M \geq x_t \geq 0, I_t \geq 0 \text{ integer} & t = 0,1,...,T-1 \end{array}$$

we know that there exist and optimal solution such that its first period action say, $\underline{a}_0^T$ is monotonically increasing in T, i.e :

$$\underline{a}_0^{T+1} \geq \underline{a}_0^T$$

Let us set $\underline{a}_0^T$ to be the *largest optimal solution.* By the Forecast Horizon Existence Theorem, we know that these sequences must meet, in other words the algorithm we are to describe below must stop after a finite number of steps.

Step 1. Solve Functional Equation (DP). T=1
Step 2. Solve $(\bar{P}_T)$ and $(\underline{P}_T)$ for $\bar{a}_0^T$ and $\underline{a}_0^T$
Step 3. If $\bar{a}_0^T = \underline{a}_0^T$ then Stop.
Else T=T+1; Go to Step 2.

Proposition 1 Let T^* be the Forecast Horizon detected by the above procedure, T^* is also the *minimal* Forecast Horizon.

Proof: By contradiction, let us assume there exists $T < T^*$ such that T is the *minimal* Forecast Horizon. By hypothesis:

$$\bar{a}_0^T > \underline{a}_0^T$$

But since $\bar{a}_0^T$ is the first period action of the *smallest* optimal solution to problem $(\bar{P}_T)$ and $\underline{a}_0^T$ is the first period action of the *largest* optimal solution to problem $(\underline{P}_T)$, this implies that the above inequality is valid for any chosen pair of optimal solutions to the problems $(\bar{P}_T)$ and $(\underline{P}_T)$, but this contradicts T being a Forecast Horizon. ∎

7 Conclusion.

We have presented strong existence and computational results for forecast horizons in a large class of dynamic optimization problems. These results depend critically upon the monotonicity properties of optimal solutions, which is a rather natural and pervasive feature of these models. We are currently exploring the extension of this approach to infinite horizon stochastic dynamic optimization problems which also posess monotonicity properties of optimal *early* decisions.

References

[1] Bès C. and Sethi S. "Concepts of forecast and decision horizons: applications to dynamic stochastic optimization problems" *Mathematics of Operations Research* 13 pp 295-310

[2] Flåm, S and Fougères A. "Infinite horizon programs : convergence of approximate solutions" *Annals of Operations Research* 29 (1991) pp 333-350

[3] Haurie A. and Sethi S. "Decisions and forecast horizons, agreeable plans and the maximum principle for infinite horizon control problems" *Operations Research Letters* Vol 3 (5) (1984) pp 261-266

[4] Heyman D. and Sobel M. "Stochastic Models in Operations Research" Vol II" *McGraw Hill N.Y* ,1984

[5] Milgrom P. and Shannon C. "Monotone comparative statics" *Econometrica* 62 (1994) pp 157-180

[6] Morton, T. "The nonstationary infinite horizon inventory problem" *Management Science* Vol 24 No. 14 1978

[7] Schochetman I. and Smith R.L "Infinite horizon optimization" *Mathematics of Operations Research* Vol 14 No. 3 1989 pp 559-554

[8] Schochetman I. and Smith R. L. "Finite dimensional approximation in infinite dimensional mathematical programming". *Mathematical Programming* 54(1992) 307-333

[9] Smith R.L.and Zhang R. "Infinite horizon production planning in time varying systems with convex production and inventory Costs" *Management Science* (in press)

[10] Topkis D. "Minimizing a submodular function on a lattice" *Operations Research* Vol. 26 No.2 1978. p.p 305-321

[11] Veinnott A. "Production planning with convex costs : a parametric study" *Management Science* 10 441-460 (1964)