ADAPTIVE METHODS TO SOLVE FREE BOUNDARY PROBLEMS OF FLOW THROUGH POROUS MEDIA

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SUMMARY

Grid adaptive methods combined with domain adaptation are discussed for two-dimensional seepage flow problems with free boundaries through porous media. Examples of grid and domain adaptive methods are presented to demonstrate several ways to predict grids and shapes of free boundaries using an iterative scheme. Finally, the combined adaptive methods are applied to obtain smooth non-oscillatory shape of a free boundary of seepage flow through non-homogeneous porous media.

INTRODUCTION

Although several fixed domain methods have been introduced to solve free boundary problems of flow through porous media, it is still popular to apply domain adaptive methods which adjust the flow domain and the free boundary at each iteration so that the free boundary condition can be satisfied at the last moment of the convergent iterative procedure. Among such adaptive methods to solve free boundary problems, the works by Taylor and Brown,[1] Finn[2] and Neuman and Witherspoon[3] are fundamental. Especially, Neuman and Witherspoon provided a sophisticated two-step iterative procedure which could solve many difficult free boundary problems. After their work, major contributions in this area were introduction of various fixed domain methods which need not define new finite element grids during iteration to adjust the flow domain and the free boundary. The flow domain itself was a part of the solution to be obtained. Fixed domain methods introduced so far can be classified into two groups; the extended pressure head method and the variational inequality method. The first method is based on the concept that the pressure head P, which is related to the head ϕ and the height y of a point from the datum line by $P = \gamma(\phi - y)$, where γ is specific weight of fluid, is extended to the unsaturated area 'smoothly' from the flow domain Ω. This extension of the pressure head to the entire porous media yields the extended coefficient of permeability of the medium:

$$K_{\text{ext}} = K + \frac{1}{\varepsilon}(1 - \chi_\Omega) \tag{1}$$

where χ_Ω is the characteristic function of the flow domain Ω defined by

$$\chi_\Omega = 1 \quad \text{if} \quad x \in \Omega, \qquad \chi_\Omega = 0 \quad \text{if} \quad x \notin \Omega$$

and ε is a regularity parameter which is expected to be a very small positive real number, for example, $\varepsilon = (10^{-3}–10^{-5})K_{\text{min}}$ and K_{min} is the minimum of the K in the flow domain. Mathematically, this extended pressure head method was introduced by Alt[4,5] and was also studied by Brezis *et al.*[6] Application of these mathematicians' work to solve free boundary

problems of flow through porous media was done by Le Tallec[7] and Oden and Kikuchi.[8] In an engineering context this method was introduced by Desai[9,10] independently of the above mathematicians' studies; he solved a number of practical problems together with experimental verification of his residual flow procedure.

The variational inequality method was first introduced by Baiocchi[11] and was applied to solve free boundary problems by Baiocchi *et al.*[12] Some extension of their works and extensive study on numerical methods were performed by Bruch[13] and Oden and Kikuchi.[8] This method, however, suffers a serious restriction; although it possesses a beautiful mathematical structure for its theory, application of the method to general irregular domains is almost impossible. More precisely, the method requires a kind of regular domain such as a rectangular domain.

In the present paper we shall return to domain adaptive methods, and shall provide new insight to these more or less classical methods by applying studies of grid adaptive finite element methods based on mathematical error analysis. Grid adaptive methods are regarded as schemes to improve the quality of finite element approximations by introducing refinement of grids in the area where the approximation error is large. That is, we shall consider a combination of grid and domain adaptive methods to solve free boundary problems of flow through porous media.

FLOW THROUGH POROUS MEDIA WITH FREE BOUNDARIES

A standard description of finite element methods is given in this section for a two-dimensional irrotational incompressible seepage flow through a porous dam with free boundaries. Let a flow field be denoted by Ω, and let it be Lipschitz as shown in Figure 1. The boundary Γ of the flow domain Ω consists of three mutually disjoint segments Γ_0, Γ_1 and Γ_2. The free boundary Γ_0 is *a priori* unknown and satisfies the free boundary conditions. On Γ_1 the head is prescribed, while the flux is given on Γ_2.

The boundary value problem is formulated as

$$-\nabla \cdot (K \nabla \phi) = 0 \qquad \text{in } \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \tag{2}$$

$$\phi = H \qquad \text{on AB} \tag{3}$$

$$\phi = h \qquad \text{on CD} \tag{4}$$

$$-K\nabla \phi \cdot \mathbf{n} = 0 \qquad \text{on AC} \tag{5}$$

$$P = \gamma(\phi - y) = \text{constant} \qquad \text{on BE and DE} \tag{6}$$

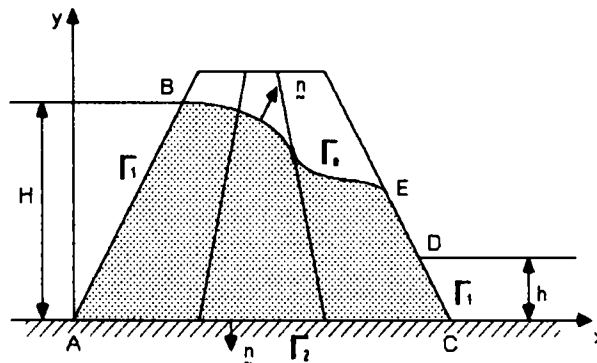$$-K\nabla \phi \cdot \mathbf{n} = 0 \qquad \text{on BE} \tag{7}$$

Figure 1. Example of non-homogeneous dam

Here, $\Gamma_1 = AB \cup CD$, $\Gamma_2 = AC$, $\Gamma_0 = BE$, H and h are prescribed heads, γ is the specific weight of water, and $\mathbf{K}$ is permeability matrix of dam material. Equation (5) is referred from Darcy's law as the no-flux condition across the boundary Γ_2, that is, Γ_2 is an impermeable boundary. For simplicity, by taking zero pressure P, hydraulic head ϕ is equal to the vertical height y along the free boundary Γ_0 and seepage surface DE. The free boundary conditions (6) and (7) must be satisfied on the free boundary whose position is also unknown *a priori* as well as the head ϕ.

The weak form of the local description (2) is obtained by multiplying both sides of (2) by an arbitrary C^0 continuous function $\bar{\phi}$ and integrating over the domain Ω, applying the divergence theorem, and substitution of the boundary condition (5) and $\bar{\phi} = 0$ on Γ_1:

$$\int_\Omega (\nabla \bar{\phi})^\mathrm{T} \mathbf{K} \, \nabla \phi \, d\Omega = 0 \qquad \text{for all } \bar{\phi} \text{ such that } \bar{\phi} = 0 \text{ on AB, CD and DE} \tag{8}$$

with boundary conditions

$$\phi = H \qquad \text{on AB} \tag{9}$$

$$\phi = h \qquad \text{on CD} \tag{10}$$

and the remaining free boundary conditions

$$\phi = y \qquad \text{on BE and DE,} \tag{11}$$

$$\mathbf{K} \, \nabla \phi \cdot \mathbf{n} = 0 \qquad \text{on BE} \tag{12}$$

while (5) has been applied to obtain the form (8). It is noted that the weak form (8) is nonlinear since the flow domain Ω is unknown, i.e. the position of the free boundary Γ_0 and the seepage point E are also unknown, although (8) is apparently linear in ϕ. Now finite element discretization can proceed using the shape functions of four-node quadrilateral isoparametric element:

$$\phi = \sum_i \phi_i N_i = \mathbf{N}^\mathrm{T} \phi \tag{13}$$

$$\bar{\phi} = \sum_i \bar{\phi}_i N_i \qquad i = 1, \ldots, n \tag{14}$$

where n is the total number of nodes. Thus, we have a typical statement giving

$$\sum_{e=1}^{E} \int_{\Omega_e} (\nabla N_i)^\mathrm{T} \mathbf{K} \, \nabla N_j d\Omega \phi_j = 0 \qquad i = 1, \ldots, n$$

$$\text{in } \Omega \text{ except on AB, CD and DE, where } \bar{\phi} = 0 \tag{15}$$

for which $\phi = H$ on AB and $\phi = h$ on CD are prescribed. Here Ω_e denotes a discretized element and E is the total number of elements. The derivatives of shape functions with respect to the global co-ordinates is obtained from the isoparametric relation between an element Ω_e in the global co-ordinate system (x_1, x_2) and the square master element in the normalized co-ordinate system (ξ_1, ξ_2). That is,

$$\frac{\partial N_i}{\partial x_j} = \sum_{q=1}^{2} \left[J_{jq}^{-1} \frac{\partial \hat{N}_i}{\partial \xi_q}(\xi) \right] \tag{16}$$

where $\hat{N}_i$ is the shape functions corresponding to the four corner nodes of the master element in (ξ_1, ξ_2), and J^{-1} is the inverse of the transpose of the Jacobian matrix. The stiffness matrix is then

obtained by a numerical integration such as 2×2 Gaussian rule:

$$H_{ij} = \int_{\Omega_e} (\nabla N_i)^\mathrm{T} \mathbf{K} \nabla N_j \, \mathrm{d}\Omega \tag{17}$$

GRID ADAPTIVE METHODS

Although the application of finite element methods to solve shape design problems is standard because of their simplicity and generality, there are few works which thoroughly discuss the quality of finite element approximations, especially the nature of dependency of finite element solutions on the form of the finite element grids. It is certain that convergence and stability of finite element approximations have been established mathematically by, for example, Ciarlet[14] and Oden and Reddy.[15] However, quantitative and qualitative analyses of finite element approximate solutions have not been discussed in the past. Recently, grid adaptive finite element methods are introduced in order to grasp both quantitative and qualitative behaviour of the error involved in finite element approximations (see, for example, Shephard[16] and Babuska et al.[17,18]). While there are many adaptive methods to define grids, typical adaptive methods can be classified into three categories: p-method, h-method, and r-method. The first two methods increase the degrees-of-freedom by the increase of the order of polynomials and by the subdivision of elements for grid refinement, respectively. The r-method optimally distributes the error into the entire domain by relocating the nodes and thereby changing the shape of elements.

In this section we shall discuss briefly the main aspect of the r- and h-methods, which serve to eliminate oscillatory behaviour of design boundaries when those adaptations are combined with shape optimization algorithm and used in the iteration process. The minimization of the maximum value of error in r- and h-methods is based on two observations:

1. In the finite element model with uniform grid the error measure of approximation is biggest near the singular points and decays rapidly far away from these points.
2. The total amount of strain energy or potential energy is almost the same regardless of the location of nodes inside the domain.

Details can be found in the paper by Diaz et al.[19]

The grid design problem may be defined as follows. Let E_e be an error measure of the eth finite element Ω_e, and E represents the total number of elements. Then the min-max design problem of grid design for the r-method is defined by

$$\underset{\substack{\text{node} \\ \text{relocation}}}{\text{Min}} \left\{ \underset{1 \leqslant e \leqslant E}{\text{Max}} E_e \right\} \tag{18}$$

The min-max problem

$$\underset{\text{refinement}}{\text{Min}} \left\{ \underset{1 \leqslant e \leqslant E}{\text{Max}} E_e \right\} \tag{19}$$

characteristics the h-method. For the above problems a necessary condition for the optimality is that the error measure of each finite element must be constant all over the finite element mode, i.e.

$$E_e = \text{constant} \qquad e = 1, \ldots, E \tag{20}$$

Note that we are minimizing the maximum error instead of total amount of errors. However, we

can reduce the total amount of error as well by minimizing the maximum error which bounds the total amount of error e_h defined as

$$e_h = \left[\sum_{i=1}^{E} E_e^2 \right]^{1/2} \tag{21}$$

i.e. we have

$$e_h \leqslant E^{1/2} \left(\underset{1 \leqslant e \leqslant E}{\text{Max}} E_e \right) \tag{22}$$

where the equality holds when the optimality condition (20) is satisfied.

For the choice of quantitative measure of error in finite element approximations, we shall take the idea that finite element error is bounded by interpolation error. Let us define the error measure E_e of seepage flow as

$$E_e = \left[c \int_{-1}^{1} \int_{-1}^{1} \sum K_{ij} \hat{J}_{mi}^{-1} \hat{J}_{nj}^{-1} |\nabla_\xi^2 \phi| |\nabla_\xi^2 \phi| J \, d\xi_1 \, d\xi_2 \right]^{1/2} \tag{23}$$

(sum in m, n)

in the normalized co-ordinate system (ξ_1, ξ_2) for a positive constant c. Here, $\hat{J}^{-1}$ is the inverse of the transpose of the Jacobian matrix in the isoparametric relation, ϕ is the solution, and J is the Jacobian. Equation (23) is obtained as an upper bound of the interpolation error using

$$\left[\frac{\partial \phi}{\partial \xi_j} - \frac{\partial \phi_h}{\partial \xi_j} \right] (\xi_1, \xi_2) \leqslant c |\nabla_\xi^2 \phi|, \qquad j = 1 \text{ and } 2 \tag{24}$$

for a positive constant $c \geqslant 0$ independent of ϕ, where

$$\nabla_\xi^2 = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + 2 \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \tag{25}$$

and ϕ_h is the interpolation of ϕ. Details of interpolation for finite element approximation error can be found in, for example, Reference 19.

The next step is the approximation of the error measure E_e by an error $\hat{E}_e$ which can be calculated from the finite element solution $\hat{\phi}_h$ because the unknown ϕ cannot be used to compute E_e. To do this the 'second derivatives' of $\hat{\phi}_h$ are obtained by taking derivatives of the continuous first derivatives of $\hat{\phi}_h$ computed by the least squares method from the discontinuous ones in the four node quadrilateral isoparametric elements. Using the above approximations, an approximated necessary condition (20) to the grid design problem (18) and (19) is defined by

$$\hat{E}_e = \text{constant}, \qquad e = 1, \ldots, E \tag{26}$$

where

$$\hat{E}_e = \left[\int_{-1}^{1} \int_{-1}^{1} \sum K_{ij} \hat{J}_{mi}^{-1} \hat{J}_{nj}^{-1} |\nabla_\xi^2 \hat{\phi}_h| |\nabla_\xi^2 \hat{\phi}_h| J \, d\xi_1 \, d\xi_2 \right]^{1/2} \tag{27}$$

(sum in m, n)

Similarly, if the interpolation error on the head, i.e. $\phi - \phi_h$, is the basis for the error measure to the grid design problem, E_e may be defined by (refer to Reference 20)

$$E_e = \left[\int_{\Omega e} \nabla \phi \cdot \nabla \phi \, d\Omega \right]^{1/2} \tag{28}$$

which leads to the approximation

$$\hat{E}_e = \left[\int_{\Omega_e} \nabla \hat{\phi}_h \cdot \nabla \hat{\phi}_h \, d\Omega \right]^{1/2} \tag{29}$$

It must be emphasized that grids are adapted in different ways for different error measures, and that *the error measure must be selected carefully to characterize the nature of the given problem.*

The major advantage of using the error indicator E_e defined in the above is that the existing preprocessor and finite element codes can be used to introduce adaptive schemes of finite element grids. There are many ways to relocate nodes or subdivide elements until the optimality condition '$\hat{E}_e$ = constant' is satisfied. For the r-method, one of the relocation schemes is based on

$$x_n = \sum_e x_e (\hat{E}_e / A_e) \bigg/ \sum_e (\hat{E}_e / A_e) \tag{30}$$

where x_n is the new location of the nth node, x_e is the geometric centroid of the element Ω_e, A_e is the area of Ω_e, $\hat{E}_e$ is the error indicator of Ω_e, and the summation is taken over the elements related to the nth node. In the h-method, elements whose error indicator is larger than some constant times the average error indicator in the domain, are subdivided. Usually, this constant is bigger than 1·0 and its value is set according to the user's choice.

An example is presented to show the differences in the solutions with and without using the grid adaptation. In Figures 2–5 a seepage flow in a foundation with two unsymmetric impermeable walls is considered. Obviously, the velocity of seepage flow becomes infinite at the tips of the walls. Starting from the uniform grid, the adaptive grid are obtained by the r- and h-method. In both adaptive grids, elements are accumulated near the tips of the walls. The very large velocities are computed at these tips, while the uniform grid gives a not so large velocity there. However, the
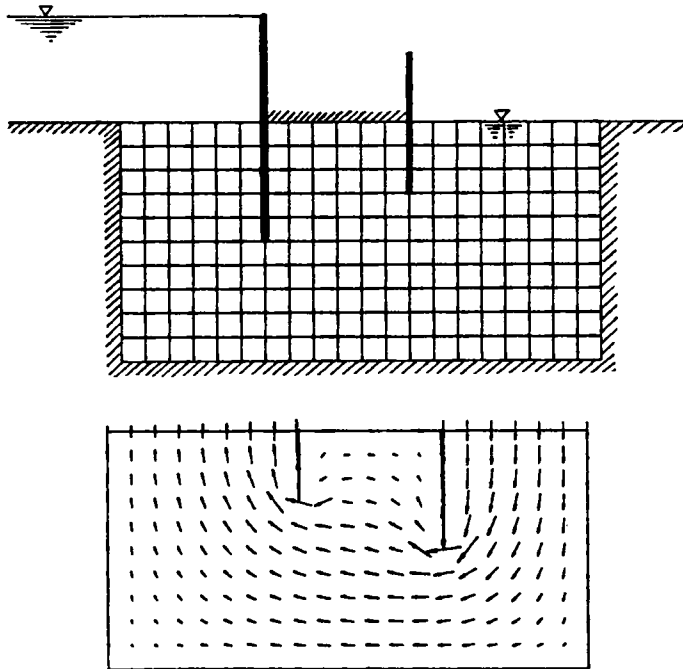

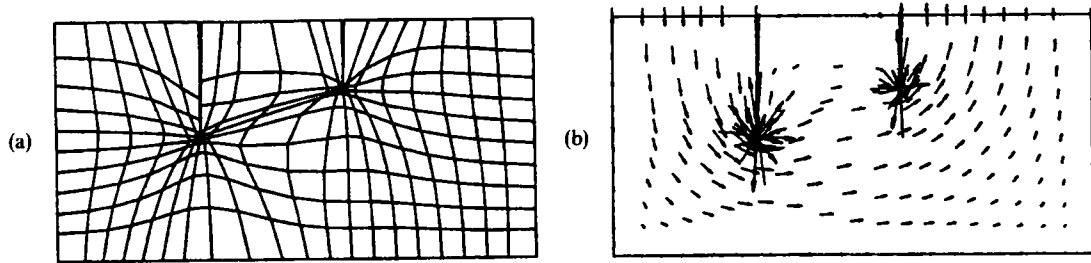Figure 2. (a) Seepage flow model with a uniform grid; (b) velocity field

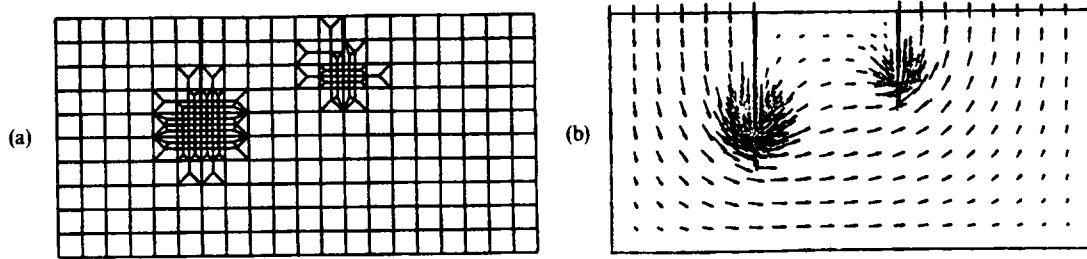Figure 3. (a) Grid adapted by *r*-method; (b) velocity field

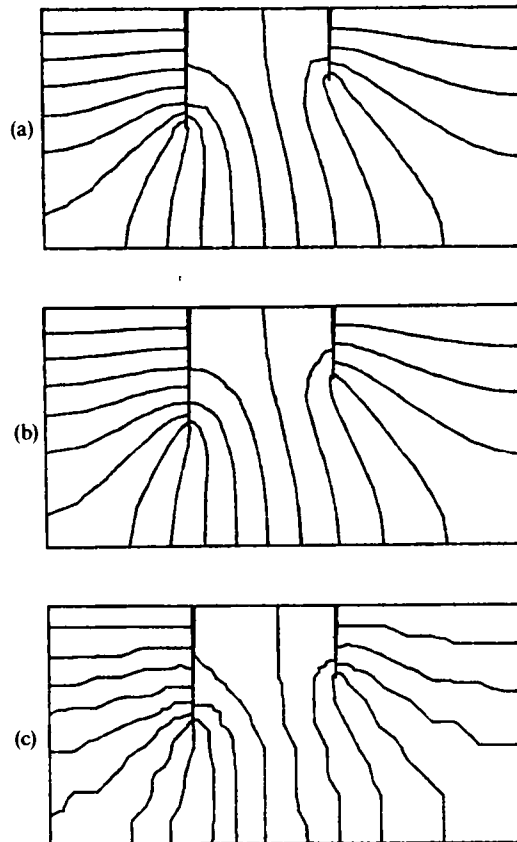Figure 4. (a) Grid adapted by *h*-method; (b) velocity field

Figure 5. Hydraulic head contours: (a) uniform grid; (b) *r*-adapted grid; (c) *h*-adapted grid
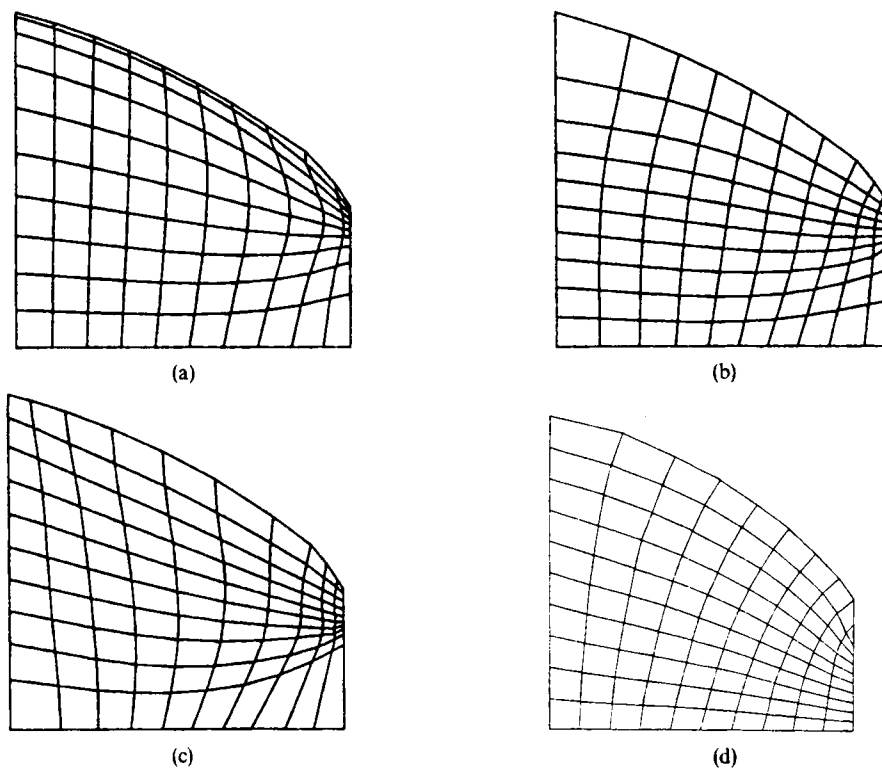
Figure 6. Adapted grid by r-method with error measures: (a) inverse of pressure; (b) derivative of velocity component; (c) Adapted grid by r-method with derivative of head as an error measure; (d) by elliptic generator

distribution of hydraulic head contours has not been changed significantly after applying the adaptive methods. These results prove the initial observation, i.e. the total amount of potential energy is almost the same regardless of the grid design.

Figure 6(a–d) demonstrate the dependency of grid adaptation on error measures. In Figure 6(b), the first derivatives of velocity components were used to calculate the error measure, while the inverse of pressure was used in the case of Figure 6(a). The adaptive grid in Figure 6(c) is obtained using the derivatives of the head as an error measure, i.e. equation (29). Since the rectangular dam is homogeneous, the first derivatives of velocity components are equivalent to the second derivatives of the head ϕ, i.e. the error measure $\hat{E}_e$ defined by (27) is applied in Figure 6(b). As shown in Figure 6(d), all the adaptive grids are different from the one obtained by the numerical grid generation method using a system of elliptic differential equation described in Thompson,[21] although both the head and flow velocity are not affected by the grid in this particular example problem.

Another way to move nodes to have an optimal grid is that introduced by Taylor in 1972.[22] In this case, using, for example, 8-node elements, mid-nodes are adjusted so that 'singular' behaviour of the solution can be well simulated; see also Bathe and Sussman.[23]

DOMAIN ADAPTIVE METHODS TO SOLVE FREE BOUNDARY PROBLEMS

Domain adaptive methods have widely been used in many kinds of free boundary problems. In seepage flow problems, Neuman and Witherspoon,[3] Oden and Kikuchi[8] and Askew and

Thatcher[24] have developed their own methods. In most cases, the idea of domain adaptive methods is the two-step iteration algorithm introduced in Reference 3 for the purpose of satisfying the free boundary conditions, which usually come out to be constant values of some quantities. Consequently, the first step is the calculation of some quantities under the assumption that the free boundaries are fixed, and the second step is the movement of nodes on the free boundaries in the ratio of differences between the calculated quantities and the known constants at the nodes of a finite element model.

In seepage flow problems, two free boundary conditions are given as (6) and (7). Those are: (a) pressure is equal to the atmospheric pressure, in other terms, the hydraulic head at nodes on the free boundary is equal to the vertical height of the nodes; (b) the normal component of velocity is zero along the free boundary. Domain adaptive methods can be used to satisfy the above two conditions or either one of those conditions as long as the final shape of free boundaries is converged so that both conditions are satisfied at the end stage. In finite element methods which solve for the hydraulic head, it is more convenient to manage the first condition than the second one because the accuracy of the hydraulic head is higher than the accuracy of the flow velocity, since the latter is obtained from the derivatives of hydraulic head according to Darcy's law.

In this section, we shall discuss a new but simple method of moving the nodes in the specified direction.

The expression for the iterative scheme to find new co-ordinates of nodes on the free boundaries is

$$x_i^{k+1} = x_i^k + \Delta x_i^k \qquad i = 1, \ldots, N \qquad (31)$$

in kth iteration, where N is the total number of nodes on the free boundary. Here, Δx_i^k is the movement of nodes in the specified direction and this value can be obtained from the movement in the normal direction:

$$\Delta x_i = \frac{\Delta x_{n_i}}{\cos \theta_i} \qquad (32)$$

in the kth iteration. θ_i is the angle between the unit normal vector n_i and the unit direction vector α_i at the ith node in the specified direction (Figure 7).
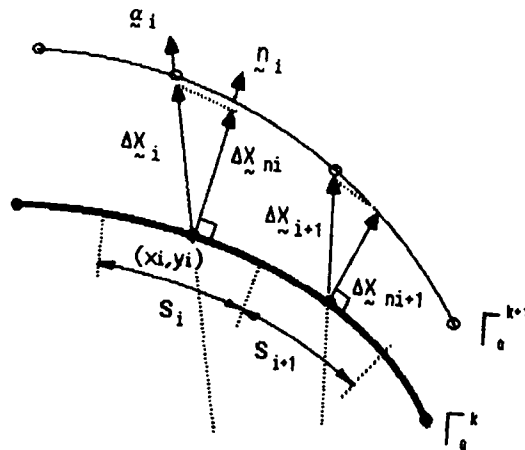

Figure 7. Movement of nodes on the free boundary Γ_0^k at the kth iteration

For simplicity, iteration superscript k is omitted in the following. The normal movement x_{n_i} at the ith node in the kth iteration is obtained from

$$\Delta x_{n_i} = \frac{\Delta A_i}{S_i} \tag{33}$$

$$S_i = \tfrac{1}{2}[l_{i+1} + l_i] \tag{34}$$

where ΔA_i is the area alloted at the ith node and l_i is the length of the ith element on the boundary. The alloted area ΔA_i at the ith node is obtained from the ratio between the hydraulic head ϕ_i calculated in the previous step of the kth iteration and the y-co-ordinate y_i:

$$\Delta A_i = \Delta\text{Area} \left[\frac{\dfrac{\phi_i - y_i}{y_i} \cdot S_i}{\displaystyle\sum_{i=1}^{N} \left[\dfrac{\phi_i - y_i}{y_i} \right] \cdot S_i} \right] \tag{35}$$

ϕ_i is the value ϕ at node i, and ΔArea is the area between the free boundary Γ_0^{k+1} and Γ_0^k. The value of ΔArea must be large enough at the beginning and diminish as the iteration goes on in order to have convergence. For this purpose we define a percent deviation from the optimum as

$$\text{DTP} = \left[\frac{1}{N} \sum_{i=1}^{N} \left[S_i \cdot \frac{\phi_i - y_i}{y_i} \right]^2 \right]^{1/2} \cdot \frac{1}{\bar{S}} \tag{36}$$

using L^2 norm, where $\bar{S}$ is the average length of elements of the free boundary

$$\bar{S} = \frac{1}{N-1} \left[\sum_{i=1}^{N} S_i \right]. \tag{37}$$

The value for ΔArea is given by

$$\Delta\text{Area} = (\text{Total area of domain}) \times C \times \text{DTP} \tag{38}$$

From our experience, the value of C is between zero and $1\cdot0$, although the proper value must be adjusted according to the problem and the speed of convergence. That is, the bigger the C value, the faster the converging speed. However, faster convergence is sometimes accompanied by oscillations of the free boundaries, and possibly by oscillations of the percent deviation as well, as the iterations proceed. On the other hand, a small value of C may result in very slow convergence without oscillations.

An example of the domain adaptive method is presented in Figure 8 (a–d). A free boundary of a saturated unconfined incompressible two-dimensional flow through a vertical homogeneous dam, is obtained by 10 iterations. With given upstream height 10 and downstream 3 and unit permeability, the calculated flow rate was $4\cdot59$ which agrees with exact solution $4\cdot55$ from Polubarinova-Kochina[25] and Liggett's result.[26] The final results show $0\cdot1$ per cent deviation, which means almost zero pressure along the design boundary, and velocity field in Figure 8(c) shows zero normal component on the free boundary and impermeable boundary. The hydraulic head contours in Figure 8(d) are normal to the free boundary and the bottom boundary, which are stream lines. It should be mentioned that the penalty method[27] was used in the finite element analysis to satisfy the boundary conditions of given values of hydraulic heads on Γ_0, e.g. on the upstream face, the downstream face and the seepage face.
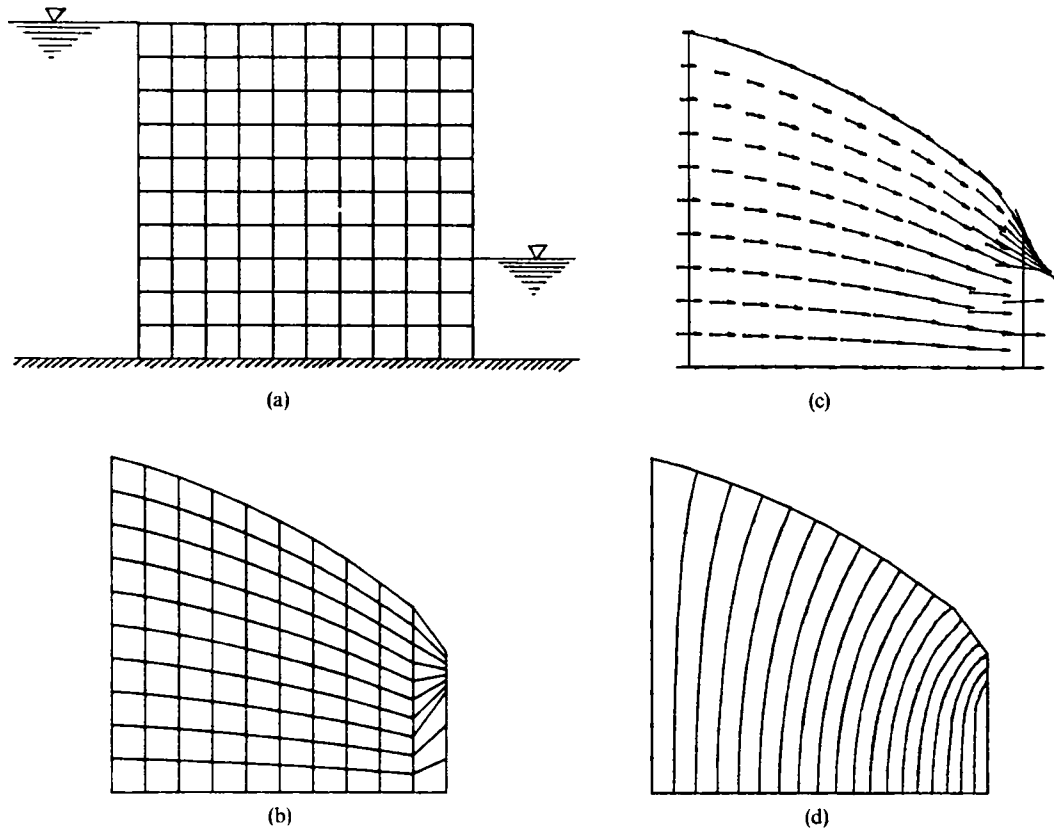
Figure 8. Shape design of vertical homogeneous dam: (a) initial model; (b) converged shape; (c) velocity field in the converged shape; (d) hydraulic head contours;

COMBINATION OF GRID AND DOMAIN ADAPTIVE METHODS

As we have discussed, the quality of finite element approximations is strongly affected by finite element grids. In free boundary problems, the accuracy of finite element solutions is very important for the determination of free boundaries by approaching to the final shape iteratively. Finite element grids which are not well designed for singularities of the drastic change of the velocity field frequently lead to boundary shapes with unreasonable oscillation of the boundary shape. In seepage flow problems, oscillations or big jumps of free boundaries usually occur in non-homogeneous domains of very different permeabilities (see Fig. A.2 in Reference 8).

In this sense, the combination of grid and domain adaptive methods may be necessary not only to avoid oscillations of free boundaries, but also to obtain the best possible computed results. Figure 9 shows an algorithm for the combination of both grid and domain adaptation to solve free boundary problems.

Seepage flow through a non-homogeneous dam is chosen for an example, as shown in Figure 1. Permeabilities are $K_1 = 1$, $K_2 = 0.05$, $K_3 = 1$, respectively, and the height of the upstream and of the downstream are $H = 16$ and $h = 2$. The interface of material 2 and 3 is also taken to be a part of the free stream line. Starting with the uniform grid in Figure 10(a), the shape converged after 10 iterations of the domain adaptive method and contains the oscillation near the interface, see Figure 10(b). Clearly, this shape cannot be the shape of free boundary because of its non-zero

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┌──────────────────────────────────────┐
│   Generate an Initial Finite Element Grid   │
└──────────────────────────────────────┘
              │
┌──────────────────────────────┐
│   Solve Boundary Value Problem   │
└──────────────────────────────┘
              │
          ◇ R-Adaptation? ◇ ──── yes ────┐
              │ no
┌──────────────────────────────────────┐
│  Domain Adaptive (moving Nodes) so that  │
│  Pressure = Const. on the Free Boundaries │
└──────────────────────────────────────┘
              │
   no ──── ◇ Converged? ◇
              │ yes
          ◇ Is R-Adaptation
            Necessary? ◇ ──── no ──── │ End │
              │ yes
┌──────────────────────────────┐
│   Compute Ê_e, e=1, . . . , E   │
└──────────────────────────────┘
              │
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│  R-Adaptation (Node Relocation) so that  │
│  Ê_e = Constant is more Realistic         │
└──────────────────────────────────────┘
              │
   yes ──── ◇ Further Relocation ◇
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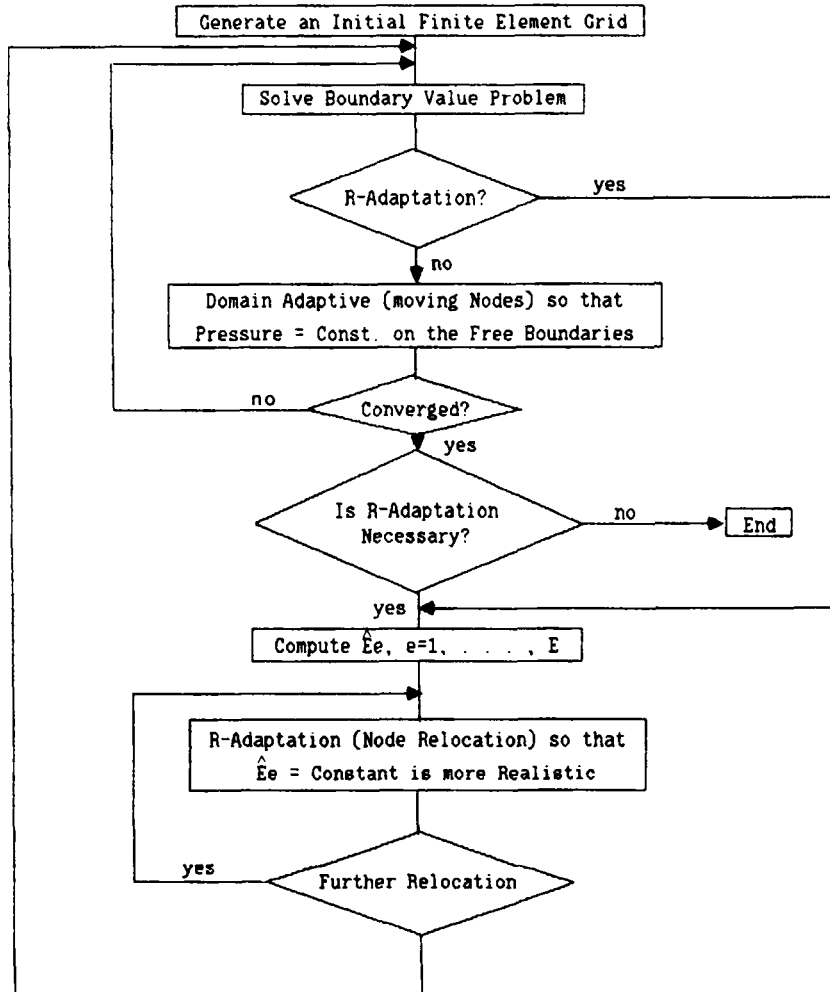
Figure 9. Algorithm for combination of domain and grid adaptive method

normal velocities (Figure 10c), although the pressure reached almost zero on the boundary. However, restarting with the remeshed grid using the r-method (Figure 11a), the final shape is obtained after 5 more iterations of domain adaptive scheme and it shows no oscillation (Figure 11b) and the normal velocity is completely vanished on the free boundary (Figure 11c). It is noted that in this example the inverse of absolute value of pressure is used to define the error measure in the grid adaptation.

CONCLUSIONS

In this paper, we have discussed grid and domain adaptive methods. It has been shown that the combination of grid and domain adaptive methods is useful to solve oscillatory behaviour or unreasonable shape of free boundaries, as well as to provide more consistent velocity fields in the flow through porous media.
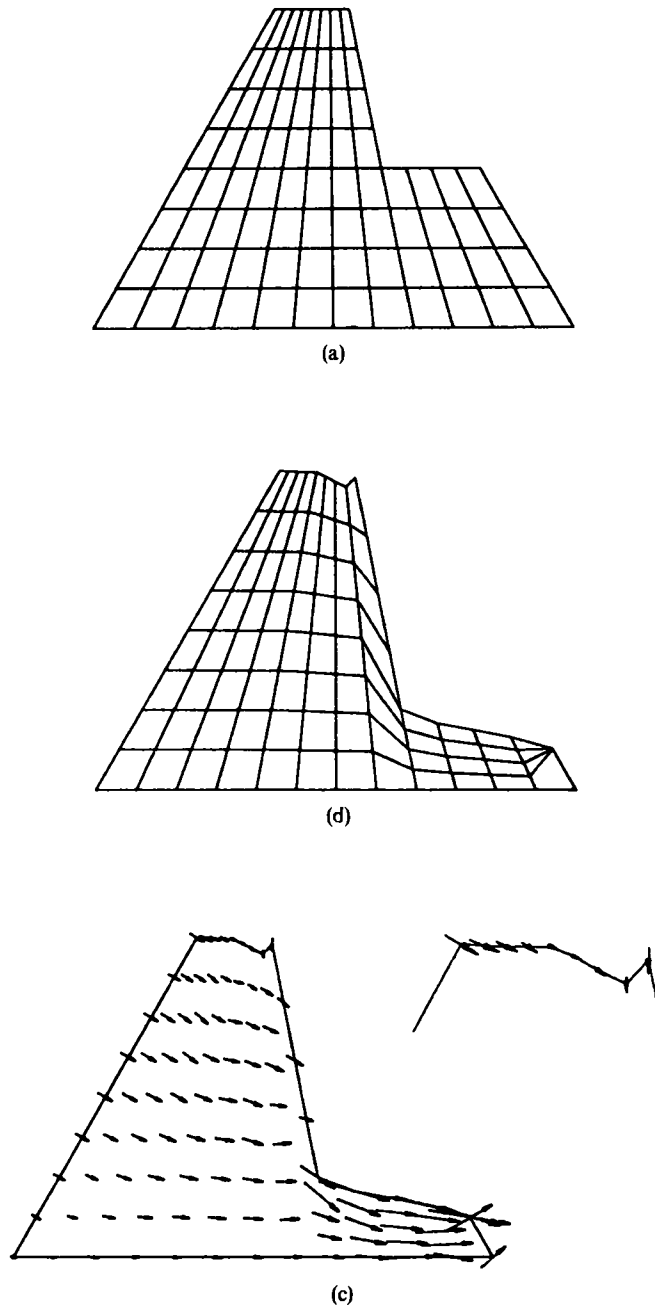
Figure 10. Non-homogeneous dam with uniform grid: (a) initial model; (b) converged shape before applying e-method; (c) velocity field in the shape converged before applying r-method
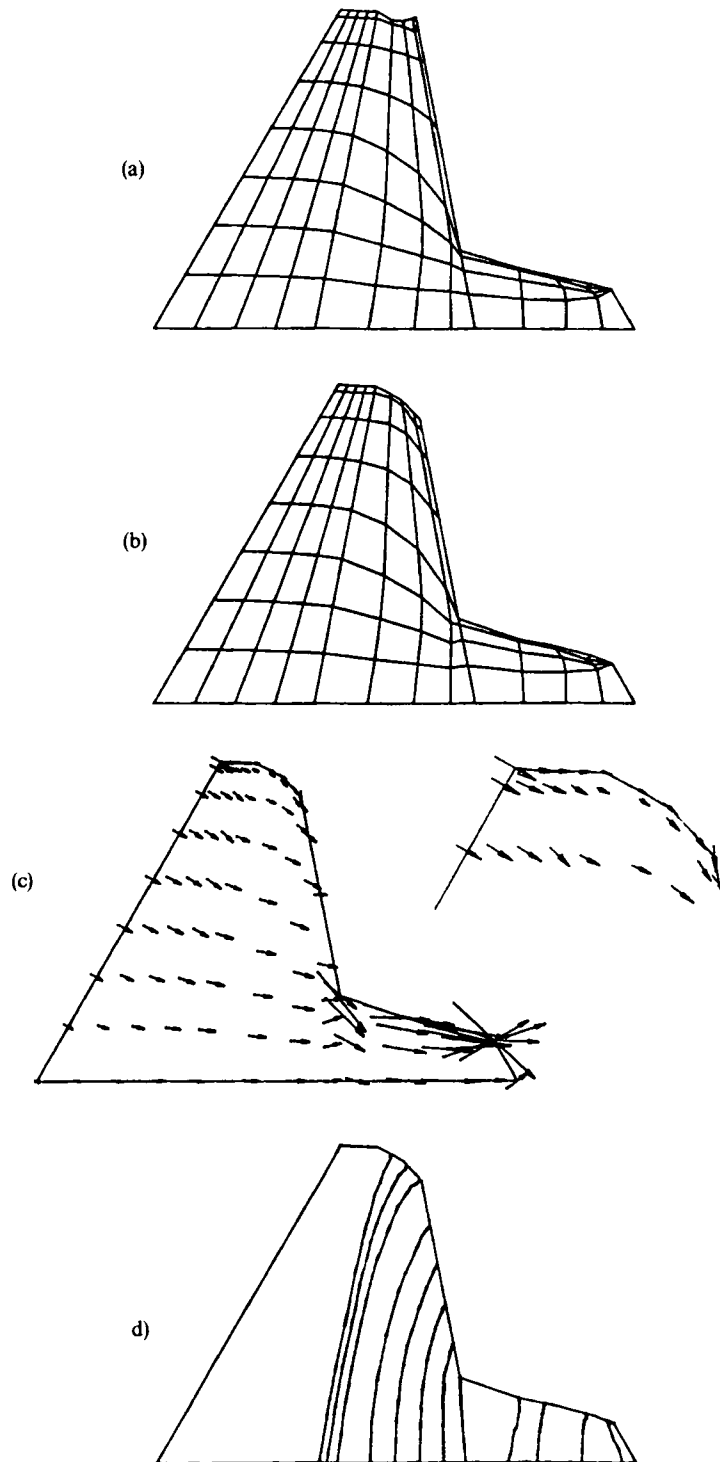
Figure 11. Shape design with remeshed model: (a) restarting shape adapted by r-method; (b) final shape converged; (c) velocity field in the final shape; (d) hydraulic head contours in the final shape

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