A Conjecture on $S^*$-semigroups of Automata

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Introduction

In [2] Hedetniemi and Fleck define the $S^*$-semigroup of an automaton. It was conjectured that if $A$ and $A'$ are any two strong machines with the same number of states then $S^*(A)$ is isomorphically embedded in $S^*(A')$, and vice versa. In this note we prove a stronger result which settles the conjecture except in the case where exactly one of the machines is autonomous.

$S^*$-semigroups

Let $A = (S, I)$ be an automaton with states $S$ and input set $I$; if $s \in S$, $i \in I$ then we write $si$ for the successor of $s$ under input $i$. If $s, t \in S$ and $x$ is a string of symbols from $I(x \in I^*)$ such that $sx = t$ then we say that $(s, x, t)$ is a triple of $A$; if $x = i \in I$ then $(s, x, t)$ is an elementary triple. Let $U$ and $V$ be finite sets of triples of $A$; we define the product $U \circ V$ by

$$U \circ V = \{(s, x, t) \mid \exists (s, y, r) \in U, (r, z, t) \in V \text{ such that } x = yz\}.$$ 

Under the operation $\circ$ the finite sets of triples form a semigroup; we call this semigroup $S^*(A)$.

Let $A$ be a strong automaton with $n$ states and at least two inputs, $n$ and $\bar{n}$. Since $A$ is strong, for any states $s$ and $t$ of $A$ there is a string $w_{st}$ of length at least one such that $sw_{st} = t$.

Let $A'$ be any automaton with at most $n$ states, and let $\phi$ be a 1-1 map from the states of $A$ onto the states of $A'$; thus, unless $|A'| = |A|$ the domain of $\phi$ will be a proper subset $\bar{S}$ of the states of $A$. If the input set of $A'$ is $I' = \{i'_1, i'_2, ..., i'_a | a = |I'|\}$ we define a map $h: \bar{S} \times I' \rightarrow I^*$ in the following manner. Let $s \in \bar{S}$ and $i'_j \in I'$ be such that $(\phi(s), i'_j, t')$ is a triple of $A'$; choose $t \in S$ such that $\phi(t) = t'$ and define

$$h(s, i'_j) = h_s(i'_j) = n_j w_{qt},$$
where \( q = s(n_i^j) \). Clearly \( h_s \) is 1-1 for each \( s \in S \); also for each \( s \in S \), \( i_j \in I^s \), the relation \( \phi(s h_s(i_j)) = \phi(s) i_j \) holds. (In fact, the pair \((\phi, h)\) defines a generalization of the classical automata-theoretic notion of realization; this is dealt with in detail in [1]). We can also extend \( h \) to domain \( \tilde{S} \times (I')^* \) inductively by \( h_s(i_j x') = h_s(i_j) h_t(x') \), where \( x' \in (I')^* \) and \( t = s h_s(i_j) \in \tilde{S} \).

**Lemma.** The map \( h: \tilde{S} \times (I')^* \to I^* \) is 1-1 for each \( s \in \tilde{S} \).

**Proof.** Let \( j_1 \ldots j_m \) and \( k_1 \) and \( k_1 \ldots k_m \), be two strings from \((I')^*\) and let \( h_s(j_1 \ldots j_m) = h_s(k_1 \ldots k_m) = w \). Then, by definition, there are states \( r, t \in \tilde{S} \) such that \( w = h_s(j_1) h_r(j_2 \ldots j_m) = h_s(k_1) h_t(k_2 \ldots k_m) \). But there is a unique positive integer \( \ell \) such that the prefix of \( w \) having length \( \ell + 1 \) is the string \( n_\ell^c \). This uniquely determines \( j_1 = k_1 = i_\ell \), so that \( r = t = s i_\ell \). Then \( h_r(j_2 \ldots j_m) = h_r(k_2 \ldots k_m) \), and we can repeat the above process until we arrive at \( m = m' \) and \( j_\rho = k_\rho \), \( \rho = 1, 2, \ldots, m \).

Note that the lemma would not simply follow if \( h_s \) was 1-1 on symbols for each \( s \in \tilde{S} \). Using the notation of the lemma, suppose \( h_s(j_1) = a, h_s(k_1) = ab, h_r(j_2) = bc, h_t(k_2) = c \). Then \( h_s(j_1 j_2) = h_s(k_1 k_2) = abc \), but \( j_1 j_2 \neq k_1 k_2 \).

**Theorem:** Let \( A \) be a strong automaton with \( n \) states and at least two inputs, and let \( A' \) be an automaton with \( n' \leq n \) states. Then \( S^*(A') \) is isomorphic to a subsemigroup of \( S^*(A) \).
Proof: We use the maps $\phi$ and $h$ above to define the isomorphism.

Let $b' = \{(s', x', t')\}$ be a singleton in $S^*(A')$ and set

$$g(b') = (s, h_s(x'), t) | \phi(s) = s'. $$

Note that this implies that $\phi(t) = t'$. Also, since $h_s$ is 1-1 for each $s \in S$, $g(b'_1) = g(b'_2)$ if and only if $b'_1 = b'_2$; i.e., $g$ is 1-1. Let $b'_1 = \{(s'_1, x'_1, r')\}$ and $b'_2 = \{(r', x'_2, t'_2)\}$. Let $b_1 = \{(s_1, x_1, r)\} = g(b'_1)$ and $b_2 = \{(r, x_2, t_2)\} = g(b'_2)$. Then $b_1 \circ b_2 = \{(s_1, h_{s_1}(x_1' x_2'), t_2)\}$ is a singleton of $S^*(A)$ and, as $\phi(s_1) = s'_1, \phi(t_2) = t'_2, b_1 \circ b_2 = g(b'_1 \circ b'_2)$.

Thus $g(b'_1) \circ g(b'_2) = g(b'_1 \circ b'_2)$. On the other hand, if $b'_1 = \{(s'_1, x'_1, t'_1)\}$, $b'_2 = \{(s'_2, x'_2, t'_2)\}$, $g(b'_1) = \{(s_1, x_1, t_1)\}$, $g(b'_2) = \{(s_2, x_2, t_2)\}$ and $t'_1 \neq s'_2$ then $t_1 \neq s_2$, so that $b_1 \circ b_2 = \emptyset$ and $g(b'_1) \circ g(b'_2) = \emptyset$.

Now let $V'$ be any element of $S^*(A')$; $V'$ is a finite set of triples of $A'$. Extend $g$ to $g^*$ by $g^*(V') = \{g(b') | b \in V'\}$. Let $S^*_{A'}(A)$ be

$$\{g^*(V') | V' \in S^*(A')\}.$$ 

Now, if $V'_1, V'_2 \in S^*(A')$,

$$g^*(V'_1) \circ g^*(V'_2) = \left[ \bigcup_{i, j} \{g(b'_1) \circ g(d'_j) \} | b'_1 \in V'_1, d'_j \in V'_2 \right]$$

$$= \bigcup_{i, j} \{g(b'_1) \circ g(d'_j) \} | b'_1 \in V'_1, d'_j \in V'_2$$

$$= \bigcup_{i, j} \{g(b'_1 \circ d'_j) \} | b'_1 \in V'_1, d'_j \in V'_2$$

$$= \bigcup_{i, j} \{g(f'_j) \} | f'_j \in V'_1 \circ V'_2$$

$$= g^*(V'_1 \circ V'_2).$$

Thus $S^*_{A'}(A)$ is a subsemigroup of $S^*(A)$, and $g^*$ is a homomorphism.

We wish to show that $g^*$ is 1-1. Suppose $g^*(V'_1) = g^*(V'_2)$. Choose a triple $b'_1 \in V'_1$, and let $\{b\} = g(\{b'_1\})$. Then there is a triple $b'_2 \in V'_2$ such that $\{b\} = g(\{b'_2\})$. If $b = (s, w, t), b'_1 = (\phi(s), x'_1, \phi(t))$ and
\[ b'_2 = (\psi(s), x'_2, \psi(t)), \quad \text{where } w = h_s(x'_1) = h_s(x'_2). \]

Then, by the lemma, \( x'_1 = x'_2 \), so \( b'_1 = b'_2 \) and \( V'_1 \subseteq V'_2 \). The symmetric argument gives \( V'_1 = V'_2 \) so that \( g^* \) is 1-1, and hence \( g^* \) is an isomorphism between \( S^*(A') \) and \( S^*_{A'}(A) \). Q.E.D.

In particular, if \( A \) and \( A' \) are both strong \( n \)-state automata,

\[ S^*(A) \cong S^*_{A}(A') < S^*(A') \cong S^*_{A'}(A) < S^*(A), \]

where \( S_1 < S_2 \) indicates that \( S_1 \) is a proper subsemigroup \( S_2 \). To complete our partial solution to the conjecture, we need only note that any two strong, autonomous, \( n \)-state automata are isomorphic, and hence have isomorphic \( S^* \)-semigroups. To completely settle the conjecture it only remains to decide whether the \( S^* \)-semigroup of the unique autonomous, strong, \( n \)-state automaton can isomorphically contain the \( S^* \)-semigroup of every other strong, \( n \)-state automaton.

REFERENCES

