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Technical Report

How to Color the Lines of a Bigraph

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ABSTRACT

It is shown that the lines of any bigraph G can be colored from $\{\beta_1,\ldots,\beta_\Delta \mid \Delta$ is the maximum of the degrees of the points of $G\}$ in such a way that all lines colored from $\{\beta_j \mid j \geq d\}$ are incident with points of degree at least d.

König [2] showed that the lines of any bigraph[1] G can be colored from a set $\{\beta_1,\ldots,\beta_\Delta\}$ in such a way that no two lines which are assigned the same color are adjacent, where $\Delta = \Delta(G)$ is the maximum of the degrees of the points in G. It is then natural to conjecture that a bigraph G should have a line-coloring in which the only lines colored β_Δ are incident with points of degree Δ. This is indeed true; in this paper we prove the following stronger result:

<u>Theorem 6:</u> Let G be a bigraph with degree sequence $d_1<d_2<\ldots<d_r = \Delta$. Then G has a line-coloring from $\{\beta_1,\ldots,\beta_\Delta\}$ such that all lines colored

$$\beta_{d_i+1},\ldots,\beta_{d_{i+1}}$$

are incident to points of degree at least d_{i+1}.

Before we begin proving the theorem, we present some convenient notation and definitions. If in some line-coloring of G a line x = uv is colored with α we say that x is an α-*line*; we also say that α *appears at* both u and v. If α and β are two colors used in a line-coloring of G then by $G|_{\alpha,\beta}$ we mean that subgraph of G induced by the lines colored α or β. In any bigraph G, we will let V_1 and V_2 stand for the blocks of the partition of the point set of G.

<u>Theorem 1:</u> Let G be a bigraph such that all points of maximum degree are in V_2. Then G can be line colored from $\{\beta_1,\ldots,\beta_\Delta\}$ such that the only lines colored β_Δ are incident with points of maximum degree.

<u>Proof:</u> We suppose the result to be true for bigraphs with q-1 lines.

[1]Definitions not given here can be found in [1].

Let G have q lines and let all points of maximum degree be in V_2; let $x = uv$ be incident with one such point v. If $\Delta(G-x) < \Delta(G) = \Delta$ then v was the only point of maximum degree in G. So any line-coloring of G-x from $\{\beta_1,\ldots,\beta_{\Delta-1}\}$ extends to a line-coloring of G in which only x is colored β_Δ.

If $\Delta(G-x) = \Delta(G) = \Delta$ then we can color G-x with Δ colors so that all lines colored β_Δ are incident with points of maximum degree; in particular, no line colored β_Δ is incident with v. If there is no β_Δ-line at u, then x can be colored β_Δ in G. Otherwise, there is a β_Δ-line uv_1 [where $\deg v_1 = \Delta$]. Since $\deg u < \Delta$ there is some color α which does not appear at u. Clearly however, there is a line v_1u_1 colored α. Thus we get a sequence $\langle u = u_0, v_1, u_1, v_2, \ldots \rangle$ such that each v_i has maximum degree, each u_jv_{j+1} is colored β_Δ and each v_ju_j is colored α. Since $\deg v_j = \Delta$, the process cannot stop with a v_j, so it must stop at some u_j, at which there is no β_Δ-line. We have thus defined a component of the subgraph $G-x|_{\alpha,\beta_\Delta}$, and can interchange the colors α and β_Δ in this subgraph, preserving the validity of the coloring. But now β_Δ does not appear at u, so x can be colored with β_Δ.

Theorem 2: In a bigraph G, suppose $\max\{\deg u | u \in V_1\} = n$ and that there are at least two degrees greater than or equal to n realized by points of V_2, the two largest being $n \leq \Delta' < \Delta$. Then there is a line-coloring of G from $\{\beta_1,\ldots,\beta_\Delta\}$ such that all lines colored $\beta_{\Delta'+1},\ldots,\beta_\Delta$ are incident with points of maximum degree.

Proof: Clearly the result is true whenever n=1. Suppose it to be true for n-1, and suppose that in G, $\max\{\deg u | u \in V_1\} = n$. Note that if $\Delta - \Delta' = 1$ then the result holds by Theorem 1.

Suppose now that the result is true for $\Delta-\Delta' = t-1$, and that in G, $\Delta-\Delta' = t$, where $v_1,\ldots,v_r$ are the points of degree Δ. We remove an independent set X of r lines, one adjacent to each of $v_1,\ldots,v_r$, to get $G':\Delta(G') = \Delta(G)-1$. If, in G', max $\{\deg u | u \in V_1\} = n$ then G' can be colored with $\Delta(G)-1$ colors in the prescribed manner: the only lines colored $\beta_{\Delta'+1},\ldots,\beta_{\Delta-1}$ are incident with the v_r. Now, the lines of X can be colored β_Δ.

Otherwise, max $\{\deg u | u \in V_1\} = n-1$. By induction on n, a line-coloring of G' with the desired properties can be achieved, and this coloring uses only $\Delta-1$ colors, as above. Again, the lines of X can be colored β_Δ.

Theorem 3: Let G be a bigraph in which max $\{\deg u | u \in V_1\} = n$, $\Delta(G) = \Delta > n$, and suppose there are no points of degree $n+1,\ldots,\Delta-1$. Then G has a line-coloring from $\{\beta_1,\ldots,\beta_\Delta\}$ such that all lines colored $\beta_{n+1},\ldots,\beta_\Delta$ are incident with points of maximum degree.

Proof: By Theorem 1 we know the result is true, for any n, if $\Delta = n+1$. Also, the result is trivially true whenever $n=1$. Suppose that the result holds when max $\{\deg u | u \in V_1\} = n-1$, and let G have max $\{\deg u | u \in V_1\} = n$. Since we know the result holds for $\Delta = n+1$ suppose that it holds for $\Delta = n+k-1$, and let G have $\Delta(G) = \Delta = n+k$. Suppose the points of degree Δ are $v_1,\ldots,v_r$. Remove an independent set X of lines which covers $\{v_1,\ldots,v_r\}$, and let $G-X = G'$. If in G', max $\{\deg u | u \in V_1\} = n$ then the resulting graph satisfies the conditions of the theorem with $\Delta = n+k-1$, so there is a line-coloring where all lines colored $\beta_{n+1},\ldots,\beta_{n+k-1}$ are incident with the v_i. Then the lines in X can be colored β_{n+k} and the result holds.

Otherwise, in G', max $\{\deg u | u \in V_1\} = n-1$. Then the result holds for G' by induction unless there were points of degree n in V_2. If so, G' satisfies the conditions of Theorem 2 with $\Delta'(G') = n$, $\Delta(G') = \Delta-1$, and so there is a line-coloring of G' from $\{\beta_0,\ldots,\beta_{\Delta-1}\}$ such that all lines colored $\beta_{\Delta'+1} = \beta_{n+1},\ldots,\beta_{\Delta-1}$ are incident with the v_i. By coloring the lines of X with β_Δ the result holds. If V_2 had no points of degree n, then by the inductive hypothesis [induction on n], in the line-coloring of G' all lines colored $\beta_n,\beta_{n+1},\ldots,\beta_{\Delta-1}$ are incident with the v_i. We can again color the lines of X with β_Δ, proving the theorem.

Theorem 4: Let G be a bigraph in which $\max\{\deg u \mid u \in V_1\} = n = d_0$, and suppose that the degrees greater than n which are realized in V_2 are $n<d_1<d_2<\ldots<d_r = \Delta$. Then there is a line coloring of G from $\{\beta_1,\ldots,\beta_\Delta\}$ such that all lines colored

$$\beta_{d_i+1},\ldots,\beta_{d_{i+1}}$$

are incident to points of degree greater than d_i, for $i = 0,\ldots,r-1$.

Proof: The result is trivial for n=1. Also, by Theorem 3, it holds whenever r=1, so we can assume it true for bigraphs in which $\max\{\deg u \mid u \in V_1\} \le n-1$ and also for bigraphs in which $\max\{\deg u \mid u \in V_1\} = n$ and r-1 degrees greater than n are realized in V_2. Let G have $\max\{\deg u \mid u \in V_1\} = n$ and let degrees $n<d_1<\ldots<d_r = \Delta$ be realized in V_2. We first prove the result in the case $d_r - d_{r-1} = 1$. Suppose we remove an independent set X covering the points of degree d_r, to get a graph G'. If the maximum degree of the points in V_1 is reduced to n-1, then G' has a line-coloring of the desired type; in particular, all lines colored $\beta_{d_{r-2}},\ldots,\beta_{d_{r-1}}$ are incident with points of degree d_{r-1} in G', those being the points of degree d_{r-1} or d_r in G. Then by coloring the lines of X with β_Δ, the desired coloring results. If in G' the maximum degree of the points in V_1 is n, then the inductive hypothesis on r guarantees the desired coloring for G', and again we can color the lines of X with β_Δ.

Now, suppose the result holds for $d_r - d_{r-1} = t-1$ and suppose that, in G, $d_r - d_{r-1} = t$. Let $v_1,\ldots,v_s$ be the points of degree $d_r = \Delta$. We can remove an independent set of lines X which covers $\{v_1,\ldots,v_s\}$, giving a graph G' with maximum degree $\Delta-1$, and next largest degree d_{r-1}; note that $(\Delta-1) - d_{r-1} = t-1$. If, in G', $\max\{\deg u | u \in V_1\} = n$ we get a line coloring of G' from $\{\beta_1,\ldots,\beta_{\Delta-1}\}$ with the desired properties by the inductive hypothesis on t. If not, we get a line-coloring by the inductive hypothesis on n. Either way, we can color the lines of X with β_Δ.

Theorem 5: Suppose G is a bigraph and that there are points of degree $\Delta(G) = \Delta$ in both V_1 and V_2. Then G has a coloring from $\{\beta_1,\ldots,\beta_\Delta\}$ such that all lines colored β_Δ are incident with points of degree Δ.

Proof: Assume that the result holds for bigraphs with at most q-1 lines and let G have q lines. Suppose $u \in V_1$ has $\deg u = \Delta$ and that v is a point with $\deg v < \Delta$ such that $x = uv$ is a line of G. Let $G' = G-x$.

Clearly $\Delta(G') = \Delta$, for G had points of degree Δ in both V_1 and V_2. There are then two cases to consider.

Case I: In G' there are points of degree Δ in both V_1 and V_2. We apply the inductive hypothesis to get a line coloring of G' such that each β_Δ-line is incident to a point of degree Δ. If neither u nor v is incident with a β_Δ-line, we can color x with β_Δ in G. Suppose first there is a β_Δ-line vu_1 at v. We choose a color α which does not appear at v and form the sequence $\langle v_0 = v, u_1, v_1, u_2, v_2, \ldots \rangle$ where each line $v_i u_{i+1}$ is colored β_Δ and each line $u_i v_i$ is colored α. The process defines a component of $G'|_{\alpha,\beta_\Delta}$ and we can interchange α and β_Δ so that there is now no β_Δ-line at v. Note that this procedure cannot introduce a β_Δ line at u. If u is a point u_i

then there is already a line $v_{i-1}u$ colored β_Δ: it may however happen that as a result of the process there is no β_Δ-line at u. In any event, if there is now no β_Δ-line at u we can color x with β_Δ in G. So suppose uv_1 is a β_Δ-line at u. Since in G' deg u = Δ-1 there is a color δ which does not appear at u. If it is also true that δ does not appear at v then by the same procedure we can replace the β_Δ-line at u with a δ-line at u, without reintroducing a β_Δ-line at v, and then x can be colored with β_Δ in G. Otherwise, if δ appears at v, since in G' deg v $\leq \Delta$-2 there is some color $\gamma \neq \beta_\Delta$ which does not appear at v, and δ can be replaced by γ at v. This replacement does not alter the set of colors at u. We now have the situation where color δ appears at neither u nor v. Then as noted, we can recolor the β_Δ-line at u by δ, and then, in G, x can be colored with β_Δ.

Case II: In G' there are no points of degree Δ in V_1. We can now apply Theorem 1 and color the lines of G' in such a way that all lines colored β_Δ are incident to points of maximum degree. Since there are no points with degree Δ in V_1 we know that there is no β_Δ-line at v. If there is also no β_Δ-line at u, we can color x with β_Δ in G. Otherwise, as above, we choose a color α which does not appear at u and interchange β_Δ with α. Then x can be colored with β_Δ.

It may, however, have been the case that in G, every point of degree Δ was adjacent only to other points of degree Δ. But then, for each point u with degree Δ, the component of G containing u is the complete bigraph $K_{\Delta,\Delta}$ for which the result certainly holds.

Suppose that G is a bigraph such that there are points of maximum degree in both V_1 and V_2. Color the lines of G in the manner prescribed by the theorem, and let X_1 be the set of lines colored β_Δ. If $G_1 = G-X_1$, then $\Delta(G_1) = \Delta-1$, there are points of degree $\Delta-1$ in both V_1 and V_2, and G_1 has a line-coloring from $\{\beta_1,\ldots,\beta_{\Delta-1}\}$ such that all lines colored $\beta_{\Delta-1}$ are incident with points of degree $\Delta-1$. The line-coloring of G_1 clearly extends to G; since X_1 was an independent set of lines, all the lines in X_1 can be colored β_Δ. Thus G has a line-coloring from $\{\beta_1,\ldots,\beta_\Delta\}$ such that all lines colored β_Δ are incident with points of degree Δ and all lines colored $\beta_{\Delta-1}$ are incident with points of degree $\Delta-1$ or Δ. But the procedure we have outlined applies to G_1: we can remove the set X_2 of lines colored $\beta_{\Delta-1}$ resulting in a bigraph G_2 in which points of maximum degree appear in both V_1 and V_2 giving rise to a line-coloring of G in which all lines colored $\beta_{\Delta-i}$ are incident to points of degree at least $\Delta-i$ for $i = 0,1,2$. The process clearly extends further. We have thus proved.

Corollary: Let G be a bigraph in which there are points of degree $d_1<d_2<\ldots<d_r = \Delta$, and such that both V_1 and V_2 have points of degree Δ. Then G has a line coloring from $\{\beta_1,\ldots,\beta_\Delta\}$ such that all lines colored $\beta_k, k>d_j$, are incident to points of degree at least d_{j+1}, for $j = 1,2,\ldots,r-1$.

We can now apply the procedure used to prove the Corollary to bigraphs in which points of maximum degree appear only in, say, V_2. At each stage we remove an independent set of lines and lower the maximum degree by 1. If at some step we arrive at a bigraph in which there are points of maximum degree in both V_1 and V_2, we apply the Corollary. Otherwise, the procedure terminates when all points in V_1 have degree one or zero, we can then apply Theorem 4. We have thus proved:

Theorem 6: Let G be a bigraph in which there are points of degree $d_1<d_2<\ldots<d_r = \Delta$. Then G has a line-coloring from $\{\beta_1,\ldots,\beta_\Delta\}$ such that

all lines colored

$$\beta_{d_{i+1}},\ldots,\beta_{d_{i+1}}$$

are incident with points of degree at least d_{i+1}.

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