The n-th Root of a Digraph

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Abstract

For any digraph D and any $n \geq 2$, necessary and sufficient conditions are given for there to be a digraph E such that $E^n = D$. The absolute n-th power is defined, and a characterization of digraphs which can be expressed as the absolute n-th power of another digraph is also given.

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Graphs and digraphs having at least one square root have been characterized in [1] and [2]. In this note we extend the results of those papers and, for any $n \geq 2$, given necessary and sufficient conditions for a digraph to have an n-th root.

Let $D = (V,X)$ be a digraph. We represent the adjacency relation on D by Γ_D; thus $\Gamma_D(u) = \{v | uv \varepsilon D\}$ and $\Gamma_D^{-1}(u) = \{v | vu \varepsilon D\}$. The <u>n-th power</u> D^n <u>of D</u> is a digraph with $V(D^n) = V(D)$ and $uv \varepsilon D^n$ if and only if there is a walk of length at most n from u to v in D.

Let $S_1, S_2, \ldots, S_{2n-1}$ be sets, not necessarily disjoint, subject to the constraints that $S_n \neq \emptyset$ and if $S_{n-j} = \emptyset (S_{n+j} = \emptyset)$ then for all $k > j$ $S_{n-k} = \emptyset (S_{n+k} = \emptyset)$. Let $K = K_n(S_1, \ldots, S_{2n-1})$ be the digraph with

$$V(K) = \bigcup_{j=1}^{2n-1} S_j \text{ and } X(K) = [\bigcup_{j=1}^{n-1} S_j x (S_n \cup \ldots \cup S_{n+j})] \cup [\bigcup_{j=n}^{2n-2} S_j x (S_{j+1} \cup \ldots \cup S_{2n-1})].$$

<u>Theorem 1</u>: Let D be a digraph and let $n \geq 2$. There exists a digraph E such that $E^n = D$ if and only if there is a collection of subdigraphs $K_i = K_n(S_{i,1}, \ldots, S_{i,2n-1})$ of D associated with the points u_i of D such that

(1) $S_{i,n} = \{u_i\}$

(2) $X(D) = \bigcup X(K_i)$

(3) $u_i \varepsilon S_{j,n-1}$ if and only if $u_j \varepsilon S_{i,n+1}$

(4) for any $0<r<n-1$ and $s=r+1$: $u_k \varepsilon S_{i,n-s}$ if and only if there is a $u_j \varepsilon S_{i,n-r}$ such that $u_k \varepsilon S_{j,n-1}$; $u_k \varepsilon S_{i,n+s}$ if and only if there is a $u_j \varepsilon S_{i,n+r}$ such that $u_k \varepsilon S_{j,n+1}$.

<u>Proof</u>: Let E be a digraph. For each $u_i \varepsilon E$ define $S_{i,j}$ to be $\Gamma_E^{j-n}(u_i)$; in particular $S_{i,n} = \{u_i\}$. In E^n, for each $1 \leq j \leq n-1$, each point of $S_{i,j}$

is adjacent to each point of $S_{i,n}$, $S_{i,n+1},\ldots,S_{i,n+j}$, and if $n \le j \le 2n-2$, each point of $S_{i,j}$ is adjacent to each point of $S_{i,j+1},\ldots,S_{i,2n-1}$. Each arc u_iu_j of E^n is determined by a path $u_iu_k\ldots u_j$ of length $r \le n$, so that $u_i \varepsilon S_{k,n-1}$ and $u_j \varepsilon S_{k,n+r}$, and hence $u_iu_j \varepsilon K_k$. Conditions (3) and (4) follow from the properties of Γ_E.

Conversely, let D have such a collection. Define a digraph E by setting $V(E) = V(D)$ and $u_iu_j \varepsilon E$ just when $u_i \varepsilon S_{j,n-1}$. We show that $E^n = D$ by demonstrating that, for each u_i and each $1 \le k \le 2n-1$, $S_{i,k} = \Gamma_E^{k-n}(u_i)$. It will then follow that $u_iu_j \varepsilon E^n$ if and only if $u_iu_j \varepsilon D$. For, if $u_iu_j \varepsilon D$, u_iu_j is in some K_k; thus for some r and s, $u_i \varepsilon S_{k,r}$ and $u_j \varepsilon S_{k,s}$, where $r<n<s$ and $s \le r+n$. Then there is a $u_{t_1} \varepsilon \Gamma_E^{(r+1)-n}(u_k)$ such that $u_i \varepsilon S_{t_1,n-1}$, so that $u_iu_{t_1} \varepsilon E$, and $u_{t_1} \varepsilon S_{k,r+1}$. We then find t_2 such that $u_{t_1}u_{t_2} \varepsilon E$ and $u_{t_2} \varepsilon S_{k,r+2}$; we continue until we find t_b such that $u_{t_{b-1}}u_{t_b} \varepsilon E$ and $u_{t_b} \varepsilon S_{k,n-1}$, so that $u_{t_b}u_k \varepsilon E$. Similarly we find a sequence $t_d, t_{d-1},\ldots,t_{b+2}$, where $d = s-r-1$, such that $u_k\, u_{t_{b+2}}, u_{t_{b+2}}u_{t_{b+2}},\ldots,u_{t_d}u_j$ are all arcs of D. This defines a walk of length $s-r \le n$ between u_i and u_j in E; thus $u_iu_j \varepsilon E^n$. If $u_iu_j \varepsilon E^n$ then u_i and u_j are joined by a walk $u_iu_{t_1}\ldots u_{t_k}u_j$ of length at most n. Now, $u_i \varepsilon \Gamma_E^{-1}(u_{t_1}) = S_{t_1,n-1}$ and $u_j \varepsilon \Gamma^k(u_{t_1}) = S_{t_1,n+k}$. Since $k<n$, $n+k<n+(n-1)$; thus $u_iu_j \varepsilon D$.

We proceed to show that $S_{i,k} = \Gamma_E^{k-n}(u_i)$. If $k = n-1$ then $\Gamma_E^{k-n}(u_i) = \Gamma_E^{-1}(u_i) = \{u_j | u_ju_i \varepsilon E\} = \{u_j | u_j \varepsilon S_{i,n-1}\} = S_{i,k}$. If $k = n+1$, $\Gamma^{k-n}(u_i) = \Gamma_E(u_i) = \{u_j | u_iu_j \varepsilon E\} = \{u_j | u_i \varepsilon S_{j,n-1}\}$. But $u_i \varepsilon S_{j,n-1}$ just when $u_j \varepsilon S_{i,n+1}$; thus $\Gamma_E(u_i) = S_{i,n+1}$.

Suppose the result holds for $n-r \le k \le n+r$, and let $s=r+1$. Then $\Gamma_E^{(n-s)-n}(u_i) = \Gamma_E^{-s}(u_i) = \Gamma_E^{-1}(\Gamma_E^{-r}(u_i)) = \Gamma_E^{-1}(S_{i,n-r}) = \bigcup \{\Gamma_E^{-1}(u_j) | u_j \varepsilon S_{i,n-r}\}$. For any u_j, $\Gamma_E^{-1}(u_j) = S_{j,n-1}$. Thus if $u_p \varepsilon \Gamma_E^{-1}(u_j)$ and $u_j \varepsilon S_{i,n-r}$ then $u_p \varepsilon S_{i,n-r-1} = S_{i,n-s}$. On the other hand, if $u_p \varepsilon S_{i,n-r-1}$ then there is a u_j such that $u_p \varepsilon S_{j,n-1}$ and $u_j \varepsilon S_{i,n-r}$, so that $u_p \varepsilon \Gamma_E^{-1}(u_j) \subset \Gamma_E^{-1}(S_{i,n-r}) = \Gamma_E^{-1}(\Gamma_E^{-r}(u_i)) = \Gamma_E^{-s}(u_i)$. Similarly, we can show that $S_{i,n+s} = \Gamma^s(u_i)$ establishing the theorem.

The theorems in [1] and [2] follow as corollaries for the case $n=2$. The <u>absolute</u> <u>n-th</u> <u>power</u> $D^{<n>}$ <u>of</u> <u>D</u> has $V(D^{<n>}) = V(D)$, and $uv \varepsilon D^{<n>}$ if there is a walk of length exactly n joining u and v in D. Let $H = H_n(S_1,\ldots,S_{2n-1})$ be the digraph with $V(H) = \bigcup_{j=1}^{2n-1} S_j$ and $X(H) = \bigcup_{j=1}^{n-1}[S_j \times S_{j+n}]$. The proof of the next theorem is virtually identical to that of Theorem 1.

<u>Theorem 2</u>: Let D be a digraph and $n \geq 2$. There is a digraph E such that $E^n = D$ if and only if there is a collection $H_i = H_n(S_1,\ldots,S_{2n-1})$ associated with the points u_i of D which satisfies conditions (1) - (4) of Theorem 1.

<u>Corollary</u>: A digraph D has an absolute square root if and only if there are sets S_i and T_i associated with the points u_i of D such that (1) each point of S_i is adjacent to each point of T_i; (2) for each arc x of D there is some u_i for which x joins S_i and T_i; and (3) $u_i \varepsilon S_j$ if and only if $u_j \varepsilon T_i$.

<u>Corollary</u>: A graph G has an absolute square root if and only if there is a collection of complete subgraphs K_i associated with the points u_i of G such that $G = \bigcup K_i$ and $u_i \varepsilon K_j$ if and only if $u_j \varepsilon K_i$.

References

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