The $n$-th Root of a Digraph

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Abstract

For any digraph $D$ and any $n \geq 2$, necessary and sufficient conditions are given for there to be a digraph $E$ such that $E^n \leq D$. The absolute $n$-th power is defined, and a characterization of digraphs which can be expressed as the absolute $n$-th power of another digraph is also given.

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Graphs and digraphs having at least one square root have been characterized in [1] and [2]. In this note we extend the results of those papers and, for any \( n \geq 2 \), given necessary and sufficient conditions for a digraph to have an \( n \)-th root.

Let \( D = (V,X) \) be a digraph. We represent the adjacency relation on \( D \) by \( r_D \); thus \( r_D(u) = \{v : uv \in E D \} \) and \( r_D^{-1}(u) = \{v : vu \in E D \} \). The \( n \)-th power \( D^n \) of \( D \) is a digraph with \( V(D^n) = V(D) \) and \( uv \in E D^n \) if and only if there is a walk of length at most \( n \) from \( u \) to \( v \) in \( D \).

Let \( S_{1}, S_{2}, \ldots, S_{2n-1} \) be sets, not necessarily disjoint, subject to the constraints that \( S_{n} \neq \emptyset \) and if \( S_{n-j} = \emptyset (S_{n+j} = \emptyset) \) then for all \( k > j \) \( S_{n-k} = \emptyset (S_{n+k} = \emptyset) \). Let \( K = K_n(S_{1}, \ldots, S_{2n-1}) \) be the digraph with

\[
V(K) = \bigcup_{j=1}^{2n-1} S_j \quad \text{and} \quad X(K) = \bigcup_{j=1}^{n-1} S_j \times \bigcup_{j=n}^{2n-2} S_j \times \bigcup_{j=n}^{2n-1} S_j.
\]

Theorem 1: Let \( D \) be a digraph and let \( n \geq 2 \). There exists a digraph \( E \) such that \( E\) = \( D \) if and only if there is a collection of subdigraphs

\[ K_i = K_n(S_{i,1}, \ldots, S_{i,2n-1}) \]

of \( D \) associated with the points \( u_i \) of \( D \) such that

1. \( S_{i,n} = \{u_i\} \)
2. \( X(D) = \bigcup X(K_i) \)
3. \( u_i \in S_{j,n-1} \) if and only if \( u_j \in S_{i,n+1} \)
4. for any \( 0 < r < n-1 \) and \( s = r+1 \): \( u_k \in S_{i,n-s} \) if and only if there is a \( u_j \in S_{i,n-r} \) such that \( u_k \in S_{j,n-1} \); \( u_k \in S_{i,n+s} \) if and only if there is a \( u_j \in S_{i,n+r} \) such that \( u_k \in S_{j,n+1} \).

Proof: Let \( E \) be a digraph. For each \( u_i \in E \) define \( S_{i,j} \) to be \( r_{E}^{j-n}(u_i) \); in particular \( S_{i,n} = \{u_i\} \). In \( E^n \), for each \( 1 \leq j \leq n-1 \), each point of \( S_{i,j} \)
is adjacent to each point of $S_{i,n}$, $S_{i,n+1}, \ldots, S_{i,n+j}$, and if $n s j s n-2$, each point of $S_{i,j}$ is adjacent to each point of $S_{i,j+1}, \ldots, S_{i,2n-1}$.

Each arc $u_iu_j$ of $E^n$ is determined by a path $u_iu_k \ldots u_j$ of length $rsn$, so that $u_i \in S_{k,n-1}$ and $u_j \in S_{k,r}$, and hence $u_iu_j \in E_k$. Conditions (3) and (4) follow from the properties of $E_k$.

Conversely, let $D$ have such a collection. Define a digraph $E$ by setting $V(E) = V(D)$ and $u_iu_j \in E$ just when $u_i \in S_{j,n-1}$. We show that $E^n = D$ by demonstrating that, for each $u_i$ and each $1 s k s 2n-1$, $S_{i,k} = r^{k-n}(u_i)$.

It will then follow that $u_iu_j \in E^n$ if and only if $u_iu_j \in D$. For, if $u_iu_j \in D$, $u_iu_j$ is in some $E_k$: thus for some $r$ and $s$, $u_i \in S_{k,r}$ and $u_j \in S_{k,s}$, where $r s n$ and $s s r+n$. Then there is a $t_1 \in E^r(u_k)$ such that $u_i \in S_{t_1,n-1}$, so that $u_iu_t^1 \in E$, and $u_t^1 \in S_{k,r+1}$. We then find $t_2$ such that $u_t^1u_t^2 \in E$ and $u_t^2 \in S_{k,r+2}$; we continue until we find $t_b$ such that $u_t^1u_t^2 \in E$ and $u_t^b \in S_{k,n-1}$, so that $u_t^b \in E$. Similarly we find a sequence $t_d^1, t_d^2, \ldots, t_d^b$, where $d = s - r - 1$, such that $u_ku_t^1u_t^2 \in E$ and $t_d^b$ all arcs of $D$. This defines a walk of length $s - r - 1$ between $u_i$ and $u_j$ in $E$, thus $u_iu_j \in E^n$. If $u_iu_j \in E^n$ then $u_i$ and $u_j$ are joined by a walk $u_i\ldots u_j$ of length at most $n$. Now, $u_i \in E_{s-1}(u_t^1) = S_{t_1,n-1}$ and $u_j \in E_{r}(u_t^1) = S_{t_1,n+k}$. Since $k < n$, $n+k < n+(n-1)$; thus $u_iu_j \in D$.

We proceed to show that $S_{i,k} = r^{k-n}(u_i)$. If $k = n-1$ then $r^{k-n}(u_i) = r^{n-1}(u_i) = \{u_j | u_ju_i \in E\} = \{u_j | u_j \in S_{i,n-1}\} = S_{i,k}$. If $k = n+1$, $r^{k-n}(u_i) = r^{j}(u_i) = \{u_j | u_ju_i \in E\} = \{u_j | u_j \in S_{i,n-1}\}$. But $u_i \in S_{i,n-1}$ just when $u_j \in S_{i,n+1}$; thus $r^{j}(u_i) = S_{i,n-1}$.

Suppose the result holds for $n-r < k < n+r$, and let $s = r+n$. Then $r^{(n-s)}(u_i) = r^{s-n}(u_i) = r^{s-n}(r^{r}(u_i)) = r^{r}(S_{i,n-r}) = \bigcup \{r^{r}(u_j) | u_j \in S_{i,n-r}\}$. For any $u_j$, $r^{r}(u_j) = S_{j,n-1}$. Thus if $u_i \in r^{r}(u_j)$ and $u_j \in S_{i,n-r}$ then $u_i \in S_{i,n-r-1} = S_{i,n-s}$. On the other hand,
if \( u_p \in S_i, n-r-1 \) then there is a \( u_j \) such that \( u_p \in S_j, n-1 \) and \( u_j \in S_i, n-r \), 
so that \( u_p \in E^{-1}(u_j) \subseteq E^{-1}(S_i, n-r) = r^{-1}(r^{-r-1}(u_j)) = r^{-S}(u_j) \). Similarly, 
we can show that \( S_i, n+S \) establishes the theorem.

The theorems in [1] and [2] follow as corollaries for the case 
\( n=2 \). The absolute 2-th power \( D^{n=2} \) of \( D \) has \( V(D^{n=2}) = V(D) \), and \( uv \in D^{n=2} \)
if there is a walk of length exactly \( n \) joining \( u \) and \( v \) in \( D \). Let 
\( H = H_n(S_1, \ldots, S_{2n-1}) \) be the digraph with 
\( V(H) = \bigcup_{j=1}^{2n-1} S_j \) and 
\( X(H) = \bigcup_{j=1}^{n-1} [S_j \times S_{j+1}] \). The proof of the next theorem is virtually identical 
to that of Theorem 1.

**Theorem 2:** Let \( D \) be a digraph and \( n \geq 2 \). There is a digraph \( E \) such that 
\( E^n = D \) if and only if there is a collection \( H_i = H_n(S_1, \ldots, S_{2n-1}) \) associated 
with the points \( u_i \) of \( D \) which satisfies conditions (1) - (4) of Theorem 1.

**Corollary:** A digraph \( D \) has an absolute square root if and only if there 
are sets \( S_i \) and \( T_i \) associated with the points \( u_i \) of \( D \) such that (1) each 
point of \( S_i \) is adjacent to each point of \( T_i \); (2) for each arc \( x \) of \( D \) 
there is some \( u_i \) for which \( x \) joins \( S_i \) and \( T_i \); and (3) \( u_i \in S_j \) if and only 
if \( u_j \notin T_i \).

**Corollary:** A graph \( G \) has an absolute square root if and only if there is a 
collection of complete subgraphs \( K_i \) associated with the points \( u_i \) of 
\( G \) such that \( G = \bigcup K_i \) and \( u_i \in K_j \) if and only if \( u_j \in K_i \).

**References**

