THE UNIVERSITY OF MICHIGAN
INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

A THEORETICAL STUDY OF LINEAR DYNAMIC SYSTEMS WITH PERIODIC, PIECEWISE CONSTANT PARAMETERS

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This dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan, 1956.

January, 1957

IP-200

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to the co-chairmen of his Doctoral Committee, Professors R. M. Howe and L. L. Rauch, for their encouragement, assistance, and interest throughout the inquiry. Thanks are also due the members of the committee, Professors W. Kaplan, A. B. Macnee, and C. B. Sharpe, for their comments and suggestions. The author is especially indebted to E. G. Gilbert for his constructive criticism and suggestions during the preparation of the manuscript.

Thanks are due to the Industry Program of the College of Engineering for the reproduction of the dissertation.

The author's wife typed the complete rough draft of the manuscript and offered many valuable suggestions. For this and for her encouragement, the author is deeply grateful.

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I. INTRODUCTION

The study of linear dynamic systems with time-variant parameters is of increasing interest and importance. A significant class of such systems is characterized by periodic, piecewise constant parameters. It is the purpose of this dissertation to investigate this class of systems with particular attention to the determination of response and response characteristics. The resultant theory is applicable to many physical systems including electrical networks with periodic switching, control systems with certain periodic variations, and time division multiplex systems. A wider range of problems may be attacked by approximating general periodic parameters by piecewise constant representations.

Study of response presupposes a mathematical formulation. The considered systems are described in every fundamental period in a sequence of time intervals by a corresponding sequence of constant coefficient differential equations. Additional relations are required to establish initial conditions in each interval of every fundamental period. In many systems these are nothing more than continuity requirements; in others, variables may exhibit prescribed jumps from interval to interval.[1] Essentially then, the systems are described by a linear differential equation with periodic coefficients. Before introducing the methods of analysis used, it is worthwhile to examine the techniques used previously.

With few exceptions past workers have not considered the response problem but have devoted their efforts to the homogeneous equation

[1] Such is the case when two capacitors are switched in parallel.

$$\frac{d^n V}{dt^n} + f_n(t)\frac{d^{n-1}V}{dt^{n-1}} + \cdots + f_2(t)\frac{dV}{dt} + f_1(t)\cdot V = 0 , \qquad (1.1)$$

where the $f_i(t)$ are periodic functions of t with period $\frac{2\pi}{\omega_0}$. This problem is of significance, however, since if a set of n linearly independent solutions to equation (1.1) are obtained, the method of variation of parameters yields the desired solution to the non-homogeneous problem [In. 1, p. 122]. The Floquet theory [In. 1, pp. 381-382] gives the functional form of a set of independent solutions to equation (1.1). Assuming a distinct set of λ_i, these may be written $V_i(t) = e^{\lambda_i t}\phi_i(t)$, where the $\phi_i(t)$ are periodic functions of t with a period $\frac{2\pi}{\omega_0}$. The λ_i, known as the characteristic exponents, determine the general nature of the solution. Unfortunately, these solutions are generally difficult to obtain and only Hill's equation,

$$\frac{d^2V}{dt^2} + f_1(t)V = 0 , \qquad (1.2)$$

is discussed extensively in the literature.[1]

The most widely used method of solution assumes Fourier series representation of $f_1(t)$ and $V(t) = e^{\lambda t}\sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t}$, which leads to the evaluation of an infinite order determinant [In. 1, sec. 15.72; Wh. 1, sec. 19.42]. For $f_1(t) = \alpha + \beta\cos\omega_0 t$, Hill's equation reduces to Mathieu's equation and solutions are well understood and tabulated [Mc. 1; Gra. 1]. Of particular interest is the work of Van der Pol and Strutt [Va. 1] who considered Hill's

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[1] See the book by McLachlin [Mc. 1] for a list of 226 references to the literature of Hill's equation.

equation for the case of rectangular variation of $f_1(t)$ (a periodic, piecewise constant function). They determined the stability of solutions by using a fundamental result of the Floquet theory which states that the characteristic exponents are determined if n independent solutions and their derivatives to order n-1 are known at values of t differing by the fundamental period.

More recently Pipes [Pi. 1, 2, 3, 4, 5] developed a matrix multiplication technique that is useful in determining the character of solutions to equation (1.1). The solution and its derivatives to order n-1 are obtained at multiples of the fundamental period by determining powers of an nth order matrix. Thus

$$\left.\begin{matrix} V(k\frac{2\pi}{\omega_0}) \\ V'(k\frac{2\pi}{\omega_0}) \\ \vdots \\ V^{(n-1)}(k\frac{2\pi}{\omega_0}) \end{matrix}\right] = M^k \left.\begin{matrix} V(0) \\ V'(0) \\ \vdots \\ V^{(n-1)}(0) \end{matrix}\right] . \tag{1.3}$$

The matrix M is determined by evaluating a fundamental set of solutions and their derivatives to order n-1 at $t = \frac{2\pi}{\omega_0}$. In most cases Pipes applies the method to Hill's equation where the powers of the matrix may be determined quite easily by the application of Sylvester's theorem of matrix algebra [Pi. 3, 4, 5]. The method is particularly useful for systems with periodic, piecewise constant parameters

In March, 1955, Bennett [Ben. 1] presented a method for the determination of the steady-state response for electrical networks containing periodically operated switches. For a single storage element and an input $e^{j\omega t}$, he determines a solution of the form $e^{j\omega t} \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t}$. Use of matrix notation extends the results to networks with n storage elements. The theory is considered for two switch positions in the fundamental period and requires determination of two n^2 functions of time before the solution may be formulated. Extensive computations remain to determine the Fourier coefficients.

The methods to be employed here are based on the time-variant transfer function and the time-variant impulse response.[1] The significant advantage of this approach lies in its emphasis of the response problem and in its analogy to common procedures used in the study of invariant systems. A direct application of this theory to the differential equation formulation of the problem results in complicated notation and difficulty of manipulation. For this reason the problem is better stated in terms of a vector differential equation. To further simplify the problem a change in time scale is made so that the fundamental period is unity. Manipulation then becomes independent of system order and fundamental period and important results are obtained with little complication. Appendices I and II introduce this notation and present the methods of Zadeh in vector form.

Chapter II discusses the two interval problem. Results include iteration formulas,[2] the time-variant transfer function,[3] an expression for the output spectrum, and the time-variant impulse

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[1] These functions, their relationship, and application have been investigated extensively by Zadeh [Za. 1, 2, 3, 4, 5, 6, 7, 8].

[2] For the homogeneous problem the result may be compared with the matrix method of Pipes [Pi. 1, 2, 3, 4, 5].

[3] The development is similar to that of Bennett [Ben. 1].

response.[1] Chapter III extends the results of Chapter II to the multi-interval problem. General periodic parameters are considered as a limiting case. The chapter is concluded with a summary of results and a discussion of their application. Chapter IV is devoted to system stability and response characteristics. A stability criterion is presented and characteristic roots are defined.[2] Methods are presented for simplified analysis and elementary synthesis. Chapter V applies the theory to several physical examples.

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[1] The expression for the time-variant impulse response uses the **Z** transform theory of sampled-data systems [Ra. 1; Tr. 1, Chap. 9].

[2] The characteristic roots are actually the characteristic exponents of the Floquet theory.

II. THE TWO INTERVAL PROBLEM

If a linear dynamic system is characterized by periodic, piecewise constant parameters, it is possible in every fundamental period to describe the system in a sequence of inclusive elementary time intervals by a corresponding sequence of linear differential equations with constant coefficients. In this chapter basic results will be developed for systems with two such intervals in the fundamental period. Included are the piecewise solution, the time-variant transfer function, an expression for the output spectrum, and the time-variant impulse response. The application and significance of these results will be examined but briefly in this chapter. A more detailed discussion will be made in the following chapter after derivation of similar results for the multi-interval problem.

<u>Introduction</u>

The work to follow assumes that the problem has been formulated in vector notation with a fundamental period of unity. Then the considered class of systems can be described piecewise in time by the vector differential equations

$$\frac{dy}{dt} = Ay + x , \qquad k < t < k+a \tag{2.1}$$

$$\frac{dy}{dt} = By + x , \qquad k+a < t < k+a+b = k+1 \tag{2.2}$$

where k is any integer, x and y are n dimensional column vectors with components x_i and y_i, and A and B are n by n matrices with finite components a_{ij} and b_{ij}. Equations (2.1) and (2.2) represent systems of linear differential equations with constant coefficients. Therefore, assuming that the X_i are sectionally continuous functions of time, there exist continuous solutions for equations (2.1) and (2.2) in the intervals indicated, provided that the vector initial condition for each interval is known [In. 1, pp. 71-72]. The initial condition value for each interval is determined from the solution value at the end of the previous interval; that is, $y(k+)$ is determined from $y(k-)$ and $y(k+a+)$ is determined from $y(k+a-)$.[1] The most general possible initial conditions result from matrix transformations of the end values. The transformations may be written

$$y_{Ak}^{+} = B^{*} y_{Bk-1} \tag{2.3}$$

$$y_{Bk}^{+} = A^{*} y_{Ak} , \tag{2.4}$$

where the initial values y_{Ak}^{+} and y_{Bk}^{+} and end values y_{Ak} and y_{Bk-1} are fixed by the definitions

$$\begin{aligned} y_{Ak}^{+} &= y(k+) \\ y_{Bk}^{+} &= y(k+a+) \\ y_{Ak} &= y(k+a-) \\ y_{Bk} &= y(k+1-) \end{aligned} \tag{2.5}$$

and where A^{*} and B^{*} are n by n matrices with finite components a_{ij}^{*} and b_{ij}^{*} . Hence the vector equations (2.1) and (2.2) with the initial condition equations (2.3) and (2.4) describe a time-variant system with a unique solution. The solution is continuous for all t

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1 $y(k+)$ indicates the limit of $y(t)$ as t approaches k for values $t > k$. Similarly, $y(k-)$ would indicate the limit of $y(t)$ as t approaches k for values $t < k$. The notation shall be used throughout the dissertation.

if and only if A^* and B^* equal the identity matrix. Otherwise the solution will be sectionally continuous with jumps at $t=k$ and/or $t=k+a$.

The solution at times $t=k$ and $t=k+a$ is not defined by the equations previously noted. Arbitrarily the solution will be defined

$$y(k) \equiv y(k-) = y_{Bk-1}$$

$$y(k+a) \equiv y(k+a-) = y_{Ak} \;. \tag{2.6}$$

Then equations (2.1), (2.2), (2.3), and (2.4) are described by the single equation

$$\frac{dy}{dt} = F(t)y + x \;, \tag{2.7}$$

where

$$F(t) = (B^*-I)\,\delta(t-k) + (A^*-I)\,\delta(t-k-a) + \begin{cases} A, & 0<t-k<a \\ B, & a<t-k<1. \end{cases} \tag{2.8}$$

The unit impulse functions $\delta(t-k)$ and $\delta(t-k-a)$ in $F(t)$ give the required jumps in the solution when A^* and B^* do not equal the identity matrix.

<u>The Piecewise Solution of the Problem</u>

Since the problem is described piecewise in time by the constant coefficient differential equations (2.1) and (2.2), it is feasible to obtain the solutions interior to each interval and then piece them together with the initial condition equations (2.3) and (2.4). This procedure leads to a piecewise solution and to vector iteration formulas that determine initial or end values from interval to interval. These results are useful in themselves as a solution to the problem. They are also important in the development of the time-variant transfer function.

To facilitate the piecewise solution the time scale is shifted so that zero time corresponds to the beginning of each interval. This shift defines new functions that are more conveniently manipulated to give desired results. Thus in equation (2.1) let $\tau = t - k$ and define

$$y_{Ak}(\tau) = y(\tau + k)$$

$$x_{Ak}(\tau) = x(\tau + k) \quad . \tag{2.9}$$

It is then possible to write

$$y_{Ak}(\tau) = e^{A\tau} y_{Ak}^{+} + \int_0^{\tau} e^{A\lambda} x_{Ak}(\tau - \lambda)\, d\lambda \quad .^{1} \tag{2.10}$$

In equation (2.2) let $\tau = t - k - a$ and define

$$y_{Bk}(\tau) = y(\tau + k + a)$$

$$x_{Bk}(\tau) = x(\tau + k + a) \quad . \tag{2.11}$$

Then

$$y_{Bk}(\tau) = e^{B\tau} y_{Bk}^{+} + \int_0^{\tau} e^{B\lambda} x_{Bk}(\tau - \lambda)\, d\lambda \quad . \tag{2.12}$$

Further simplification is possible by defining separately the particular integrals for the solutions (2.10) and (2.12). Thus

$$y_{Ak}^{*}(\tau) = \int_0^{\tau} e^{A\lambda} x_{Ak}(\tau - \lambda)\, d\lambda$$

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1 See Appendix 1, equation (A1.21).

$$y_{Bk}^{*}(\tau) = \int_{0}^{\tau} e^{B\lambda} x_{Bk}(\tau-\lambda)\,d\lambda\,. \tag{2.13}$$

Since $x(t)$ is assumed sectionally continuous,

$$y_{Ak}^{*}(a-) = y_{Ak}^{*}(a) \equiv y_{Ak}^{*}$$

$$y_{Bk}^{*}(b-) = y_{Bk}^{*}(b) \equiv y_{Bk}^{*} \quad ; \tag{2.14}$$

also

$$e^{Aa-} = e^{Aa}$$

$$e^{Bb-} = e^{Bb}\,. \tag{2.15}$$

The constant vectors y_{Ak}^{*} and y_{Bk}^{*} defined in equations (2.14) will be used shortly. From equations (2.14) and (2.15) it follows that

$$y_{Ak} = y(k+a-) = y_{Ak}(a-) = y_{Ak}(a)$$

$$y_{Bk} = y(k+1-) = y_{Bk}(b-) = y_{Bk}(b)\,. \tag{2.16}$$

Using equations (2.3), (2.4), (2.10), and (2.12) through (2.16) gives the initial condition values

$$y_{Ak}^{+} = B^{*}\left(e^{Bb} y_{Bk-1}^{+} + y_{Bk-1}^{*}\right) \tag{2.17}$$

$$y_{Bk}^{+} = A^{*}\left(e^{Aa} y_{Ak}^{+} + y_{Ak}^{*}\right)\,. \tag{2.18}$$

Alternately equations for the end values y_{Ak} and y_{Bk} may be written, resulting in

$$y_{Ak} = e^{Aa} B^* y_{Bk-1} + y_{Ak}^* \tag{2.19}$$

$$y_{Bk} = e^{Bb} A^* y_{Ak} + y_{Bk}^* . \tag{2.20}$$

In many cases $y(t)$ does not change significantly in any interval and the solution at $t = k$ and $t = k + a$ is sufficient. Equations (2.17) and (2.18) are vector iteration formulas that give the desired values in a step by step process. The solution at $t = k-$ and $t = k + a-$ is obtained in a similar manner by equations (2.19) and (2.20). This method is particularly useful when the vector constants y_{Ak}^* and y_{Bk}^* are zero[1] or independent of k .

The iteration formulas (2.17) and (2.18) in conjunction with equations (2.10) and (2.12) give the piecewise solution. Thus substituting for τ in equations (2.10) and (2.12) yields

$$\begin{aligned} y(t) &= e^{A(t-k)} y_{Ak}^+ + \int_0^{t-k} e^{A\lambda} x(t-\lambda)\, d\lambda , \quad 0 < t-k < a \\ &= e^{B(t-k-a)} y_{Bk}^+ + \int_0^{t-k-a} e^{B\lambda} x(t-\lambda)\, d\lambda , \quad a < t-k < 1 , \end{aligned} \tag{2.21}$$

where y_{Ak}^+ and y_{Bk}^+ are obtained from the iteration formulas.

<u>The Time-Variant Transfer Function</u>

In this section the time-variant transfer function $H(j\omega,t)$ will be derived for systems described by the vector differential equation (2.7). The function is useful in determination of system response and is required for the derivation of the output spectrum and the time-variant impulse response.

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1 The matrix method of Pipes [Pi. 1, 2, 3, 4, 5] is basically identical for this homogeneous problem.

$H(j\omega,t)$ and $H(s,t)$ are defined respectively as the Fourier and Laplace transforms of the impulse response $W(t,\tau)$. These essentially equivalent definitions do not, however, provide a practical method for obtaining the transfer function. The technique to be used here will be based on the fact that $y(t) = H(j\omega,t)e^{j\omega t}c$ is a particular integral for the input $x(t) = e^{j\omega t}c$. In Appendix II it is shown that $H(j\omega,t)$ is periodic in t if the system has periodic coefficients, the period of $H(j\omega,t)$ being identical with the period of the coefficients. In such systems it is further shown that if any particular integral $y(t) = H^*(j\omega,t)\,e^{j\omega t}\,c$ can be obtained where $H^*(j\omega,t)$ is periodic in t with period identical to the coefficient period for all $j\omega$, then $H^*(j\omega,t)$ must equal $H(j\omega,t)$. Thus the problem of finding the time-variant transfer function for the system (2.7) is reduced to finding a particular integral $y(t) = H(j\omega,t)e^{j\omega t}c$ for the input $x(t) = e^{j\omega t}c$ where it is required that $H(j\omega,t)$ be periodic with a period of unity. This may be done employing the piecewise solution of the problem.

The piecewise solution requires that $y^*_{AK}(\tau)$ and $y^*_{BK}(\tau)$ be determined for the input $x(t) = e^{j\omega t}c$. To simplify the work $j\omega$ will be replaced by s. Then since $x(t) = e^{st}c$, $x_{AK}(\tau) = e^{s(\tau+k)}c = e^{sk}e^{s\tau}c$ and $x_{BK}(\tau) = e^{s(\tau+k+a)} = e^{s(k+a)}e^{s\tau}c$.

By the first equation of equations (2.13)

$$
\begin{aligned}
y^*_{AK}(\tau) &= e^{sk}\int_0^{\tau} e^{A\lambda}\, e^{s(\tau-\lambda)}\, d\lambda \cdot c \\
&= e^{sk}\, e^{s\tau} \int_0^{\tau} e^{-(sI-A)\lambda}\, d\lambda \cdot c \\
&= e^{sk}\, e^{s\tau}\, (sI-A)^{-1}\left[I - e^{-(sI-A)\tau}\right] c\,. \qquad (2.22)
\end{aligned}
$$

For convenience this is written

$$y_{Ak}^{*}(\tau) = e^{sk} e^{s\tau} \overline{Y}_A(s,\tau)\, c \;, \tag{2.23}$$

where

$$\overline{Y}_A(s,\tau) = (sI - A)^{-1}\left[I - e^{-(sI-A)\tau}\right]. \tag{2.24}$$

Similarly,

$$y_{Bk}^{*}(\tau) = e^{s(k+a)} e^{s\tau} \overline{Y}_B(s,\tau)\, c \;, \tag{2.25}$$

where

$$\overline{Y}_B(s,\tau) = (sI - B)^{-1}\left[I - e^{-(sI-B)\tau}\right]. \tag{2.26}$$

The desired solution is of the form

$$y(t) = H(s,t)\, e^{st} c \;, \tag{2.27}$$

where $H(s,t)$ is periodic with a period of unity. Since this is true, $y(t)e^{-st} = y(t+1)\, e^{-s(t+1)}$ or $y(t) = y(t+1)\, e^{-s}$. Thus $y_{Bk-1}^{+} = y_{Bk}^{+} e^{-s}$ and $y_{Bk-1}^{*} = y_{Bk}^{*} e^{-s}$. Equations (2.17) and (2.18) may now be written

$$y_{Ak}^{+} = B^{*} e^{-s}\left(e^{Bb} y_{Bk}^{+} + y_{Bk}^{*}\right) \tag{2.28}$$

$$y_{Bk}^{+} = A^{*}\left(e^{Aa} y_{Ak}^{+} + y_{Ak}^{*}\right). \tag{2.29}$$

Solving for y_{Ak}^{+} and y_{Bk}^{+} results in

$$y_{Ak}^{+} = \left(I - e^{-s} B^{*} e^{Bb} A^{*} e^{Aa}\right)^{-1}\left[B^{*} e^{-s} y_{Bk}^{*} + B^{*} e^{Bb} A^{*} e^{-s} y_{Ak}^{*}\right] \tag{2.30}$$

$$y_{B_k}^{+} = \left(I - e^{-s}A^{*}e^{Aa}B^{*}e^{Bb}\right)^{-1}\left[A^{*}y_{A_k}^{*} + A^{*}e^{Aa}B^{*}e^{-s}y_{B_k}^{*}\right]. \tag{2.31}$$

$y_{A_k}^{*}$ and $y_{B_k}^{*}$ are obtained by evaluating equations (2.23) and (2.25) for $\tau = a$ and $\tau = b$ respectively. It is then possible to write

$$y_{A_k}^{+} = e^{sk}\,\overline{Z}_A(s)\,c \tag{2.32}$$

$$y_{B_k}^{+} = e^{s(k+a)}\,\overline{Z}_B(s)\,c\,, \tag{2.33}$$

where

$$\overline{Z}_A(s) = \left(I - e^{-s}B^{*}e^{Bb}A^{*}e^{Aa}\right)^{-1}\left[B^{*}\overline{Y}_B(s,b) + B^{*}e^{Bb}A^{*}e^{-sb}\overline{Y}_A(s,a)\right] \tag{2.34}$$

$$\overline{Z}_B(s) = \left(I - e^{-s}A^{*}e^{Aa}B^{*}e^{Bb}\right)^{-1}\left[A^{*}\overline{Y}_A(s,a) + A^{*}e^{Aa}B^{*}e^{-sa}\overline{Y}_B(s,b)\right]. \tag{2.35}$$

Substituting equations (2.23), (2.25),(2.32) and (2.33) into equations (2.10) and (2.12) results in

$$y_{A_k}(\tau) = e^{sk}\left[e^{A\tau}\overline{Z}_A(s) + e^{s\tau}\overline{Y}_A(s,\tau)\right]c \tag{2.36}$$

$$y_{B_k}(\tau) = e^{s(k+a)}\left[e^{B\tau}\overline{Z}_B(s) + e^{s\tau}\overline{Y}_B(s,\tau)\right]c. \tag{2.37}$$

Substituting the respective values of τ and arranging yields

$$\begin{aligned} y(t) &= \left[\overline{Y}_A(s,t-k) + e^{-(sI-A)(t-k)}\overline{Z}_A(s)\right]e^{st}c, \quad 0<t-k<a \\ &= \left[\overline{Y}_B(s,t-k-a) + e^{-(sI-B)(t-k-a)}\overline{Z}_B(s)\right]e^{st}c, \quad a<t-k<1. \end{aligned} \tag{2.38}$$

The solution is of the form $H(s,t)\, e^{st}c$ where $H(s,t)$ is periodic in t with period unity and is given by

$$H(s,t) = \overline{Y}_A(s,t-k) + e^{-(sI-A)(t-k)}\,\overline{\overline{Z}}_A(s), \quad 0 < t-k < a$$

$$= \overline{Y}_B(s,t-k-a) + e^{-(sI-B)(t-k-a)}\,\overline{\overline{Z}}_B(s), \quad a < t-k < 1. \tag{2.39}$$

The time-variant transfer function $H(s,t)$ is given piecewise in time by equation (2.39). Since $H(s,t)$ is periodic in t, it may also be expressed as a complex Fourier series in t with matrix coefficients $C_n(s)$ obtained by integrating $H(s,t)\, e^{-j2\pi nt}$ from $t=0$ to $t=1$. Integration and manipulation yield the result

$$H(s,t) = \sum_{n=-\infty}^{\infty} C_n(s)\, e^{j2\pi nt} \quad , \tag{2.40}$$

where

$$C_n(s) = \frac{1}{j2\pi n}\left[\overline{Y}_A(s+j2\pi n,a) - e^{-j2\pi na}\,\overline{Y}_A(s,a)\right] + \overline{Y}_A(s+j2\pi n,a)\,\overline{\overline{Z}}_A(s)$$
$$+ e^{-j2\pi na}\left\{\frac{1}{j2\pi n}\left[\overline{Y}_B(s+j2\pi n,b) + e^{-j2\pi nb}\,\overline{Y}_B(s,b)\right] + \overline{Y}_B(s+j2\pi n,b)\,\overline{\overline{Z}}_B(s)\right\}. \tag{2.41}$$

For convenience the equations for $\overline{Y}_A(s,\tau)$, $\overline{Y}_B(s,\tau)$, $\overline{\overline{Z}}_A(s)$ and $\overline{\overline{Z}}_B(s)$ are summarized below.

$$\overline{Y}_A(s,\tau) = (sI-A)^{-1}\left[I - e^{-(sI-A)\tau}\right] \tag{2.42}$$

$$\overline{Y}_B(s,\tau) = (sI-B)^{-1}\left[I - e^{-(sI-B)\tau}\right] \tag{2.43}$$

$$\overline{\overline{Z}}_A(s) = \left(I - e^{-s}B^* e^{Bb} A^* e^{Aa}\right)^{-1}\left[B^*\,\overline{Y}_B(s,b) + B^* e^{Bb} A^* e^{-sb}\,\overline{Y}_A(s,a)\right] \tag{2.44}$$

$$\bar{Z}_B(s) = (I - e^{-s}A^*e^{Aa}B^*e^{Bb})^{-1}\left[A^*\bar{Y}_A(s,a) + A^*e^{Aa}B^*e^{-sa}\bar{Y}_B(s,b)\right]. \tag{2.45}$$

It is also convenient to express $\bar{Z}_A(s)$ and $\bar{Z}_B(s)$ by

$$\bar{Z}_A(s) = \bar{Z}_{AA}(e^s)\bar{Y}_A(s,a)e^{-sb} + \bar{Z}_{AB}(e^s)\bar{Y}_B(s,b) \tag{2.46}$$

$$\bar{Z}_B(s) = \bar{Z}_{BA}(e^s)\bar{Y}_A(s,a) + \bar{Z}_{BB}(e^s)\bar{Y}_B(s,b)e^{-sa}, \tag{2.47}$$

where

$$\bar{Z}_{AA}(e^s) = \bar{Z}(e^s)\bar{B}A^*$$

$$\bar{Z}_{AB}(e^s) = \bar{Z}(e^s)B^*$$

$$\bar{Z}_{BA}(e^s) = \bar{A}\ \bar{Z}(e^s)\bar{A}^{-1}A^*$$

$$\bar{Z}_{BB}(e^s) = \bar{A}\ \bar{Z}(e^s)\bar{A}^{-1}\bar{A}B^* = \bar{A}\bar{Z}(e^s)B^*,[1] \tag{2.48}$$

if

$$\bar{Z}(e^s) = (I - e^{-s}\bar{B}\bar{A})^{-1} \tag{2.49}$$

and

$$\bar{A} = A^*e^{Aa}$$

$$\bar{B} = B^*e^{Bb} . \tag{2.50}$$

Theoretically it is possible to compute the output for any input by obtaining the inverse Fourier transform of $H(j\omega,t)\ \bar{X}(j\omega)$ or the inverse Laplace transform of $H(s,t)\ \bar{X}(s)$, where $\bar{X}(j\omega)$ and $\bar{X}(s)$ are the corresponding transforms of the input. In these

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[1] Observe that $\bar{A}\bar{Z}(e^s)\bar{A}^{-1} = (I - e^{-s}\bar{A}\bar{B})$.

calculations t is considered as a parameter. However, such operations are not practically feasible since the inverse transform of a function of e^s and s is generally necessary. Approximations of one sort or another are required; for example, a few terms of the Fourier series for $H(s,t)$ may be used and the expressions in e^s may be approximated by polynomials of s .[1] It will be seen that the functions $\overline{Z}_A(s)$ and $\overline{Z}_B(s)$ give much insight into the behavior of the system and an explicit solution for a particular input is not often required.

The Output Spectrum

Since $H(j\omega,t)$ is a function of t as well as $j\omega$, it may not be used directly to obtain the output spectrum $y(j\omega)$ for a given input spectrum $\overline{X}(j\omega)$. The desired result is given by equation (A2.23) of Appendix II. Substituting the required $C_n(j\omega)$ from equation (2.41) yields

$$\overline{y}(j\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{j2\pi n}\left[\overline{Y}_A(j\omega,a) - e^{-j2\pi na}\overline{Y}_B(j\omega-j2\pi n)\right]\overline{X}(j\omega-j2\pi n)$$

$$+\overline{Y}_A(j\omega,a)\sum_{n=-\infty}^{\infty}\overline{Z}_A(j\omega-j2\pi n)\overline{X}(j\omega-2\pi n)$$

$$+\sum_{n=-\infty}^{\infty}\frac{e^{-j2\pi na}}{j2\pi n}\left[\overline{Y}_B(j\omega,b) - e^{-j2\pi nb}\overline{Y}_B(j\omega-j2\pi n)\right]\overline{X}(j\omega-j2\pi n)$$

$$+\overline{Y}_B(j\omega,b)\sum_{n=-\infty}^{\infty}e^{-j2\pi na}\,\overline{Z}_B(j\omega-j2\pi n)\overline{X}(j\omega-j2\pi n). \qquad (2.51)$$

This may be written

[1] Similar problems arise in the study of sampled data systems [Ra. 1].

$$
\begin{aligned}
\bar{y}(j\omega) = {} & \bar{Y}_A(j\omega,a)\left[a\,\bar{X}(j\omega) + \sum_{n=-\infty}^{\infty} \bar{Z}_A(j\omega+j2\pi n)\bar{X}(j\omega+j2\pi n)\right] \\
& + \bar{Y}_B(j\omega,b)\left[b\,\bar{X}(j\omega) + \sum_{n=-\infty}^{\infty} e^{j2\pi na}\bar{Z}_B(j\omega+j2\pi n)\bar{X}(j\omega+j2\pi n)\right] \\
& + \sum_{n=-\infty}^{\infty} \alpha_n\left[e^{j2\pi na}\bar{Y}_A(j\omega+j2\pi n,a) - \bar{Y}_A(j\omega,a)\right]\bar{X}(j\omega+j2\pi n) \\
& + \sum_{n=-\infty}^{\infty} \alpha_n e^{j2\pi na}\left[e^{j2\pi nb}\bar{Y}_B(j\omega+j2\pi n,b) - \bar{Y}_B(j\omega,b)\right]\bar{X}(j\omega+j2\pi n),
\end{aligned}
\tag{2.52}
$$

where

$$
\begin{aligned}
\alpha_n &= \frac{1}{j2\pi n}, \quad n \neq 0 \\
&= 0, \quad n = 0 .
\end{aligned}
\tag{2.53}
$$

Equation (2.52) shows that new frequency components $\bar{X}(j\omega+j2\pi n)$ are created by time-variant behavior of the system. In some problems it is valid to approximate $\bar{y}(j\omega)$ by considering only the fundamental component $\bar{X}(j\omega)$. This yields

$$
\begin{aligned}
\bar{y}(j\omega) \approx {} & \bar{Y}_A(j\omega,a)\left[a + \bar{Z}_A(j\omega)\right]\bar{X}(j\omega) \\
& + \bar{Y}_B(j\omega,b)\left[b + \bar{Z}_B(j\omega)\right]\bar{X}(j\omega).
\end{aligned}
\tag{2.54}
$$

Higher order approximations readily follow, but the computation required increases accordingly. In Chapter IV the multi-interval equivalent of equation (2.52) will be used to obtain an equivalent physical system useful in analysis.

The Time-Variant Impulse Response

The time-variant impulse response $W(t,\tau)$ gives the system response at time t for an impulse applied τ units previously. The function is useful in determining system response for general inputs, and a study of its functional form leads to an understanding of response characteristics. These aspects will be discussed in the summary of Chapter III and in Chapter IV.

By definition $W(t,\tau)$ and $H(s,t)$ form a Laplace transform pair with respect to the variables τ and s. Therefore the inverse Laplace transform of $H(s,t)$ with respect to τ yields the impulse response $W(t,\tau)$. Since $H(s,t)$ is given in two forms (piecewise in time and as a complex Fourier series), it is possible to develop two forms of $W(t,\tau)$. The more compact and significant arises from transforming the terms in the piecewise in time representation given by equation (2.39). Since $H(s,t)$ is periodic in t, only the fundamental interval $0 < t \leq 1$ will be investigated ($k = 0$ in equation (2.39)). Consider first the transforms of the terms $\overline{Y}_A(s,t)$ and $\overline{Y}_A(s,t-a)$ defined in equations (2.42) and (2.43). Comparison with equations (A1.36) and (A1.37) yields

$$
\begin{aligned}
L^{-1}\left[\overline{Y}_A(s,t)\right] = Y_A(\tau,t) &= e^{A\tau} \quad , \quad 0 < \tau \leq t \\
&= 0 \quad , \quad \tau \leq 0,\ \tau > t
\end{aligned}
\tag{2.55}
$$

$$
\begin{aligned}
L^{-1}\left[\overline{Y}_B(s,t-a)\right] = Y_B(\tau,t-a) &= e^{B\tau} \quad , \quad 0 < \tau \leq t-a \\
&= 0 \quad , \quad \tau \leq 0,\ \tau > t-a.
\end{aligned}
\tag{2.56}
$$

Thus the functions $\overline{Y}_A(\tau,\alpha)$ and $\overline{Y}_B(\tau,\alpha)$ equal $e^{A\tau}$ and $e^{B\tau}$ respectively for $0<\tau\leq\alpha$ and are zero for other τ. The remaining terms to be transformed are $e^{-(sI-A)t}\overline{Z}_A(s)=e^{At}e^{-st}\overline{Z}_A(s)$ and $e^{-(sI-B)(t-a)}\overline{Z}_B(s) = e^{B(t-a)}e^{-s(t-a)}\overline{Z}_B(s)$. By denoting $Z_A(\tau)$ and $Z_B(\tau)$ as the transforms of $\overline{Z}_A(s)$ and $\overline{Z}_B(s)$, it is possible to write

$$L^{-1}\left[e^{-(sI-A)t}\,\overline{Z}_A(s)\right] = e^{At}Z_A(\tau-t) \quad , \quad \tau>t$$

$$= 0 \quad , \quad \tau\leq t \tag{2.57}$$

$$L^{-1}\left[e^{-(sI-B)(t-a)}\overline{Z}_B(s)\right] = e^{B(t-a)}Z_B[\tau-(t-a)] \quad , \quad \tau>t-a$$

$$= 0 \quad , \quad \tau\leq t-a. \tag{2.58}$$

Study of equations (2.39) and (2.55) through (2.58) shows that for particular t and τ all but one of the four equations are zero, and hence $W(t,\tau)$ is simply defined. The functional form of $W(t,\tau)$ is easily determined for all possible t and τ by considering regions in the t, τ plane as indicated in Figure 2.1. To agree with the convention established in defining $F(t)$ (see equations (2.6) and (2.8)), the regions include boundaries as follows:

The line	$\tau=t$	for	$0<t\leq a$	lies in region			A1,
"	" $\tau=t-a$	"	$a<t\leq 1$	"	"	"	B1,
"	" $t=a$	"	$0<\tau\leq a$	"	"	"	A1,
"	" $t=a$	"	$a<\tau$	"	"	"	A2,
"	" $t=1$	"	$0<\tau\leq b$	"	"	"	B1,
"	" $t=1$	"	$b<\tau$	"	"	"	B2,
"	" $\tau=0$	"	$0<t\leq 1$	"	" the region $\tau\leq 0$.		

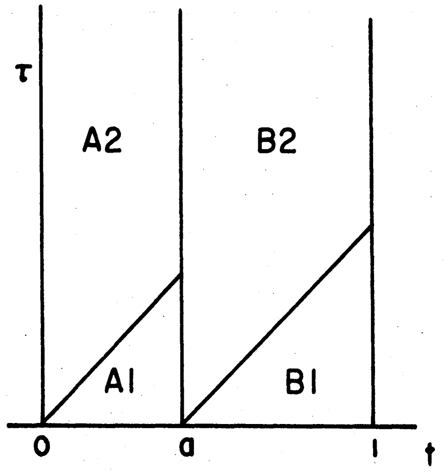

Figure 2.1 Regions in the t , τ Plane

Then it is possible to write

$$
\begin{aligned}
W(t,\tau) &= 0 , \quad \tau \leq 0 \\
&= e^{A\tau} , && \text{region A1} \\
&= e^{At} Z_A(\tau - t) , && \text{" A2} \\
&= e^{B\tau} , && \text{" B1} \\
&= e^{B(t-a)} Z_B[\tau - (t-a)] && \text{" B2}
\end{aligned}
\tag{2.59}
$$

$W(t,\tau)$ is given for other t by periodically repeating the regions defined in Figure 2.1 and replacing t by $t-k$ in equation (2.59) when $0 < t-k \leq 1$. The behavior of $W(t,\tau)$ for constant t or τ is understood by considering the appropriate vertical or horizontal line in Figure 2.1.

It is useful to obtain the impulse response $W_1(t,\lambda)$, representing the system response at time t for a vector impulse applied at time λ . This is done by replacing τ in $W(t,\tau)$ by $t-\lambda$. To facilitate the change of variable, the regions in the

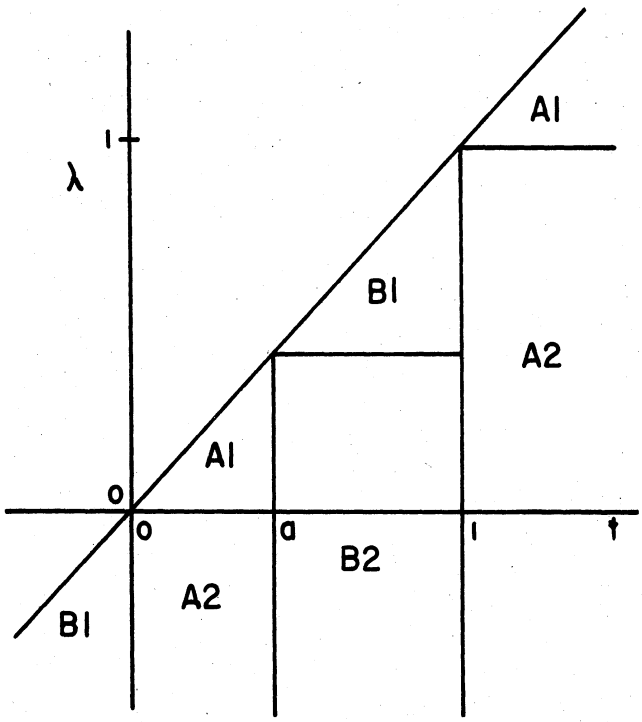

Figure 2.2 Regions in the t , λ Plane

t , τ plane are mapped into the t , λ plane of Figure 2.2.
Then, if $0 < t - k \leq 1$, $W_1(t,\lambda)$ is given by

$$\begin{aligned} W_1(t,\lambda) &= 0 , && t \leq \lambda \\ &= e^{A(t-\lambda)}, && \text{region A1} \\ &= e^{A(t-k)} Z_A(k-\lambda), && \text{" A2} \\ &= e^{B(t-\lambda)}, && \text{" B1} \\ &= e^{B(t-k-a)} Z_B(k+a-\lambda), && \text{" B2} \end{aligned} \tag{2.60}$$

As before the behavior of $W_1(t, \lambda)$ for constant t or λ is understood by considering the appropriate vertical or horizontal line in Figure 2.2. As an example suppose λ is a constant such that $0 \leq \lambda - j < a$ (j an integer). Then

$$
\begin{aligned}
W_1(t,\lambda) &= 0 , && t \leq \lambda \\
&= e^{A(t-\lambda)} , && \lambda < t < j+a \\
&= e^{B(t-k-a)} Z_B(k+a-\lambda) , && k+a < t < k+1 , \ k \geq j \\
&= e^{A(t-k)} Z_A(k-\lambda) , && k < t \leq k+a , \ k > j .
\end{aligned} \tag{2.61}
$$

It is apparent from the above expressions that $Z_A(\tau)$ and $Z_B(\tau)$ are fundamental to both forms of the impulse response, and therefore deserve considerable attention. $Z_A(\tau)$ is given by the Laplace transform of $\bar{Z}_A(s) = \bar{Z}_{AA}(e^s)\,\bar{Y}_A(s,a)e^{-sb} + \bar{Z}_{AB}(e^s)\,\bar{Y}_B(s,b)$. The function $\bar{Z}_{AA}(e^s)$ can be written

$$\bar{Z}_{AA}(e^s) = (I - e^{-s}\bar{B}\bar{A})^{-1}\bar{B}A^* = [I + e^{-s}\bar{B}\bar{A} + (e^{-s}\bar{B}\bar{A})^2 + \cdots]\bar{B}A^* \tag{2.62}$$

Postmultiplying by $\bar{Y}_A(s,a)\,e^{-sb}$ and transforming yields

$$
\begin{aligned}
&L^{-1}\left[\bar{Z}_{AA}(e^s)\,\bar{Y}_A(s,a)e^{-sb}\right] = \\
&L^{-1}\left[\sum_{n=0}^{\infty} D_n e^{-(n+b)s}\,\bar{Y}_A(s,a)\right] = \\
&\qquad\qquad \sum_{n=0}^{\infty} D_n\, Y_A(\tau-n-b,a) .
\end{aligned} \tag{2.63}
$$

The D_n may be obtained from the series expansion (2.62) or preferably from the closed form of the inverse z transform of $\bar{Z}_{AA}(z)$.[1] Define $Z_{AA}(\tau)$ as the inverse z transform of $\bar{Z}_{AA}(z)$. The components of $Z_{AA}(\tau)$ exist as elementary functions since the components of $\bar{Z}_{AA}(z)$ can be expressed as ratios of polynomials in z. The coefficients D_n are then uniquely given by $Z_{AA}(n)$, and equation (2.63) may be written

$$L^{-1}\left[\bar{Z}_{AA}(e^s)\,\bar{Y}_A(s,a)e^{-sb}\right] = \sum_{n=0}^{\infty} Z_{AA}(n)\,Y_A(\tau-n-b,a), \qquad (2.64)$$

where

$$Z_{AA}(\tau) = \text{inverse } z \text{ transform}\left[\bar{Z}_{AA}(z)\right]. \qquad (2.65)$$

Similarly,

$$L^{-1}\left[\bar{Z}_{AB}(e^s)\,\bar{Y}_B(s,b)\right] = \sum_{n=0}^{\infty} Z_{AB}(n)\,Y_B(\tau-n,b), \qquad (2.66)$$

where

$$Z_{AB}(\tau) = \text{inverse } z \text{ transform}\left[\bar{Z}_{AB}(z)\right]. \qquad (2.67)$$

---

1 Consider a continuous time function $f(t)$ defined for $t > 0$. Define $f^*(t) = \sum_{n=0}^{\infty} f(n)\,\delta(t-n)$ as the sampled $f(t)$. The Laplace transform of $f^*(t)$ is $\bar{f}^*(s) = \sum_{n=0}^{\infty} f(n)e^{-sn}$. In many cases $\bar{f}^*(s)$ is expressable as the ratio of polynomials in $e^s = z$ and is known as the z transform of $f(t)$. The inverse z transform may be obtained by tables in a manner similar to the Laplace transform technique. Note that the inverse transform is only unique for $t = n$. The techniques are readily extended to matrix functions. See the reference by Ragazzini and Zadeh [Ra. 1] for a more complete discussion of sampled-data system methods.

$Z_A(\tau)$ is given by

$$Z_A(\tau) = \sum_{n=0}^{\infty} \left[Z_{AA}(n) Y_A(\tau-n-b, a) + Z_{AB}(n) Y_B(\tau-n, b) \right]. \qquad (2.68)$$

From the definitions of $Y_A(\tau, a)$ and $Y_B(\tau, b)$ (equations (2.55) and (2.56)), it is clear that equations (2.64) and (2.66) do not overlap for any τ but instead mesh together. This is expected from the piecewise description of the system.

Transforming equation (2.47) in the same manner gives

$$Z_B(\tau) = \sum_{n=0}^{\infty} \left[Z_{BA}(n) Y_A(\tau-n, a) + Z_{BB}(n) Y_B(\tau-n-a, b) \right], \qquad (2.69)$$

where

$$Z_{BA}(\tau) = \text{inverse } z \text{ transform}\left[\bar{Z}_{BA}(z) \right] \qquad (2.70)$$

$$Z_{BB}(\tau) = \text{inverse } z \text{ transform}\left[\bar{Z}_{BB}(z) \right]. \qquad (2.71)$$

As might be expected, the functions $Z_{BA}(n)\ Y_A(\tau-n, a)$ and $Z_{BB}(n)\ Y_{BB}(\tau-n-a, b)$ do not overlap for any τ or n .

It is seen that $W(t,\tau)$ and $W_1(t,\lambda)$ are expressed piecewise as elementary functions and the output for common inputs may be determined by use of the superposition integrals (A2.5) and (A2.6). Such operations entail considerable work, even for relatively simple systems. As in non-variant systems it is possible to obtain useful results without detailed analysis for particular inputs. Techniques of this sort will be developed in Chapter IV.

III. THE MULTI-INTERVAL PROBLEM

In this chapter results obtained in Chapter II will be extended to systems with an arbitrary number of elementary intervals in the fundamental period. Systems with general periodic coefficients will be considered as a limiting case. Since the methods and discussion are basically identical to those used previously, the presentation will be somewhat abbreviated in form. The chapter will close with a summary of results and a discussion of their application.

Introduction

Consider a class of systems that can be described piecewise in time by the finite number of vector equations

$$\frac{dy}{dt} = Ay + x \quad , \quad k < t < k+a$$

$$\frac{dy}{dt} = By + x \quad , \quad k+a < t < k+a+b$$

$$\vdots$$

$$\frac{dy}{dt} = Qy + x \quad , \quad k+a+b+\cdots+p < t < k+a+\cdots+p+q = k+1 , \quad (3.1)$$

where k is any integer, x and y are n dimensional column vectors with components x_i and y_i, and $A, B, \ldots,$ and Q are n by n matrices with finite components. Assuming the sectional continuity of the x_i guarantees a unique, sectionally continuous solution of the system if the initial condition for each interval is determined by one of the equations

$$y_{A_k}^{+} = Q^{*} y_{Q_{k-1}}$$

$$y_{B_k}^{+} = A^{*} y_{A_k}$$

$$\vdots$$

$$y_{Q_k}^{+} = P^{*} y_{P_k} \qquad (3.2)$$

A^*, B^*, . . ., and Q^* are n by n non-zero matrices with finite components and the notation y_{AK}^+ , y_{QK-1} , etc. for the initial and end values is the same as defined in equations (2.5) of Chapter II. The solution is continuous for all t if and only if all the star matrices are equal to the identity matrix.

The solution at $t=k$, $k+a$, $k+a+b, \ldots, k+a+b+\cdots+p$ is not defined by the equations (3.1) and (3.2). Arbitrarily the solution will be defined

$$
\begin{aligned}
y(k) &= y(k-) = y_{QK-1} \\
y(k+a) &= y(k+a-) = y_{AK} \\
&\vdots \\
y(k+a+b+\cdots p) &= y(k+a+b+\cdots+p-) = y_{PK} .
\end{aligned}
\tag{3.3}
$$

Then equations (3.1) and (3.2) are described by the single vector equation

$$
\frac{dy}{dt} = F(t)\, y + x , \tag{3.4}
$$

where

$$
\begin{aligned}
F(t) = {} & (Q^*-I)\,\delta(t-k) + (A^*-I)\,\delta(t-k-a) + \cdots + (P^*-I)\,\delta(t-k-a-\cdots-p) \\
& + \begin{cases} A , & 0 < t-k < a \\ B , & a < t-k < a+b \\ \vdots & \\ Q , & a+b+\cdots+p < t-k < 1 . \end{cases}
\end{aligned}
\tag{3.5}
$$

<u>The Piecewise Solution of the Problem</u>

In the first equation of equations (3.1) let $\tau = t-k$; in the second let $\tau = t-k-a$; in the third let $\tau = t-k-a-b$; in the remaining continue in the same manner. Then define

$$y_{AK}(\tau) = y(\tau + k)$$

$$x_{AK}(\tau) = x(\tau + k)$$

$$y_{BK}(\tau) = y(\tau + k + a)$$

$$\vdots$$

$$x_{QK}(\tau) = x(\tau + a + b + \cdots + p) \ . \qquad (3.6)$$

The output functions of τ are given by

$$y_{AK}(\tau) = e^{A\tau} y_{AK}^{+} + \int_0^\tau e^{A\lambda} x_{AK}(\tau-\lambda)\, d\lambda$$

$$y_{BK}(\tau) = e^{B\tau} y_{BK}^{+} + \int_0^\tau e^{B\lambda} x_{BK}(\tau-\lambda)\, d\lambda$$

$$\vdots$$

$$y_{QK}(\tau) = e^{Q\tau} y_{QK}^{+} + \int_0^\tau e^{Q\tau} x_{QK}(\tau-\lambda)\, d\lambda \ . \qquad (3.7)$$

The particular integrals in these equations may be defined by $y_{AK}^*(\tau)$, $y_{BK}^*(\tau)$, --, and $y_{QK}^*(\tau)$. Evaluating these expressions for a, b, - - -, and q determines the vector constants y_{AK}^*, y_{BK}^*, - - -, and y_{QK}^*. Evaluation of equations (3.7) for the same quantities and use of equations (3.2) yield the initial condition values

$$y_{AK}^{+} = Q^*(e^{Qq} y_{QK-1}^{+} + y_{QK-1}^{*})$$

$$y_{BK}^{+} = A^*(e^{Aa} y_{AK}^{+} + y_{AK}^{*})$$

$$\vdots$$

$$y_{QK}^{+} = P^*(e^{Pp} y_{PK}^{+} + y_{PK}^{*}) \ . \qquad (3.8)$$

Alternately, expressions for the end values y_{AK}, y_{BK}, - - -, and y_{QK} may be written

$$y_{AK} = e^{Aa} Q^* y_{QK-1} + y_{AK}^*$$

$$y_{BK} = e^{Bb} A^* y_{AK} + y^*_{BK}$$

$$\vdots$$

$$y_{QK} = e^{Qq} P^* y_{PK} + y^*_{QK}. \tag{3.9}$$

Equations (3.8) and (3.9) are useful as iteration formulas and permit the piecewise solution of the problem when used with equations (3.7).

The Time-Variant Transfer Function

The time-variant transfer function $H(s,t)$ will be obtained by finding the response for the vector input $x(t) = e^{st} c$.

For this input one can write

$$y^*_{AK}(\tau) = e^{sk} e^{s\tau} \overline{Y}_A(s,\tau)$$

$$y^*_{BK}(\tau) = e^{s(k+a)} e^{s\tau} \overline{Y}_B(s,\tau)$$

$$\vdots$$

$$y^*_{QK}(\tau) = e^{s(k+a+b+\cdots+p)} e^{s\tau} \overline{Y}_Q(s,\tau), \tag{3.10}$$

where

$$\overline{Y}_A(s,\tau) = (sI-A)^{-1}\left[I - e^{-(sI-A)\tau}\right]$$

$$\overline{Y}_B(s,\tau) = (sI-B)^{-1}\left[I - e^{-(sI-B)\tau}\right]$$

$$\vdots$$

$$\overline{Y}_Q(s,\tau) = (sI-Q)^{-1}\left[I - e^{-(sI-Q)\tau}\right]. \tag{3.11}$$

Since the response can be written $y(t) = H(s,t)\, e^{st} c$ where $H(s,t)$ is periodic in t with unity period, $y^+_{QK-1} = y^+_{QK} e^{-s}$ and $y^*_{QK-1} = y^*_{QK} e^{-s}$.

Then the equations (3.8) become

$$y^+_{AK} = Q^*\left(e^{Qq} y^+_{QK} + y^*_{QK}\right) e^{-s}$$

$$y^+_{BK} = A^*\left(e^{Aa} y^+_{AK} + y^*_{AK}\right)$$

$$\vdots$$

$$y_{QK}^{+} = P^{*}\left(e^{Pp} y_{PK}^{+} + y_{PK}^{*}\right), \tag{3.12}$$

which may be solved for the initial condition vectors y_{AK}^{+} , y_{BK}^{+} , - - - , and y_{QK}^{+} in terms of y_{AK}^{*} , y_{BK}^{*} , - - - , and y_{QK}^{*} , which are given by evaluating equations (3.10) for $\tau = a$, b , - - - , and q respectively. Thus

$$\begin{aligned} y_{AK}^{+} &= e^{sk}\,\bar{Z}_A(s) \\ y_{BK}^{+} &= e^{s(k+a)}\,\bar{Z}_B(s) \\ &\;\vdots \\ y_{QK}^{+} &= e^{s(k+a+b+\cdots+p)}\,\bar{Z}_Q(s), \end{aligned} \tag{3.13}$$

where

$$\bar{Z}_A(s) = \left(I - e^{-s}Q^{*}e^{Qq}P^{*}\cdots A^{*}e^{Aa}\right)^{-1}\Big[Q^{*}\bar{Y}_Q(s,q) + e^{-sq}Q^{*}e^{Qq}P^{*}\bar{Y}_P(s,p) + e^{-s(q+p)}Q^{*}e^{Qq}P^{*}e^{Pp}O^{*}\bar{Y}_O(s,o) + \cdots + e^{-s(q+p+\cdots+b)}Q^{*}\cdots A^{*}\bar{Y}_A(s,a)\Big]$$

$$\bar{Z}_B(s) = \left(I - e^{-s}A^{*}e^{Aa}Q^{*}e^{Qq}\cdots B^{*}e^{Bb}\right)^{-1}\Big[A^{*}\bar{Y}_A(s,a) + e^{-sa}A^{*}e^{Aa}Q^{*}\bar{Y}_Q(s,q) + e^{-s(a+q)}A^{*}e^{Aa}Q^{*}e^{Qq}P^{*}\bar{Y}_P(s,p) + \cdots + e^{-s(a+q+p+\cdots+c)}A^{*}\cdots B^{*}\bar{Y}_B(s,b)\Big]$$

$$\vdots$$

$$\bar{Z}_Q(s) = \left(I - e^{-s}P^{*}e^{Pp}\cdots e^{Aa}Q^{*}e^{Qq}\right)^{-1}\Big[P^{*}\bar{Y}_P(s,p) + e^{-sp}P^{*}e^{Pp}O^{*}\bar{Y}_O(s,o) + \cdots + e^{-s(p+o+\cdots+a)}P^{*}\cdots e^{Aa}Q^{*}\bar{Y}_Q(s,q)\Big]. \tag{3.14}$$

Equations (3.7) may now be written

$$y_{AK}(\tau) = e^{sk}\left[e^{A\tau}\bar{Z}_A(s) + e^{s\tau}\bar{Y}_A(s,\tau)\right]c$$

$$y_{Bk}(\tau) = e^{s(k+a)}\left[e^{B\tau}\bar{Z}_B(s) + e^{s\tau}\bar{Y}_B(s,\tau)\right]c$$

$$\vdots$$

$$y_{Qk}(\tau) = e^{s(k+a+b+\cdots+p)}\left[e^{Q\tau}\bar{Z}_Q(s) + e^{s\tau}\bar{Y}_Q(s,\tau)\right]c. \tag{3.15}$$

Substituting the respective values of τ and arranging gives $y(t) = H(s,t)e^{st}c$, where

$$\begin{aligned} H(s,t) &= \bar{Y}_A(s,t-k) + e^{-(sI-A)(t-k)}\bar{Z}_A(s) \qquad , \quad 0 < t-k \leq a \\ &= \bar{Y}_B(s,t-k-a) + e^{-(sI-B)(t-k-a)}\bar{Z}_B(s) \qquad , \quad a < t-k \leq a+b \\ &\;\;\vdots \\ &= \bar{Y}_Q(s,t-k-a-b-\cdots-p) + e^{-(sI-Q)(t-k-a-b-\cdots-p)}\bar{Z}_Q(s) , \\ &\qquad\qquad a+b+\cdots+p < t-k \leq 1 . \end{aligned} \tag{3.16}$$

$H(s,t)$ is given piecewise in time by equation (3.16), but since $H(s,t)$ is periodic in t, it may also be expressed as a complex Fourier series with matrix coefficients. These results are summarized below. The piecewise representation is written

$$H(s,t) = \bar{Y}_G(s,t-k-\bar{g}) + e^{-(sI-G)(t-k-\bar{g})}\bar{Z}_G(s), \; \bar{g} < t-k < \bar{g}+g,$$

$$G = A, B, \text{---}, \text{ and } Q \;, \tag{3.17}$$ [1]

where

$$\bar{g} = a + b + \cdots + f. \tag{3.18}$$

The complex Fourier series form is

---

[1] Replacing G by another letter also indicates that $\bar{g}$ and g are replaced by the corresponding letters. This notation will be used henceforth.

$$H(s,t) = \sum_{n=-\infty}^{\infty} C_n(s)\, e^{j2\pi nt} \quad , \tag{3.19}$$

where

$$C_n(s) = \sum_{G=A}^{Q} \left\{ \frac{e^{-j2\pi n\bar{g}}}{j2\pi n} \left[\bar{Y}_G(s+j2\pi n, g) - e^{-j2\pi ng} \bar{Y}_G(s,g) \right] + \right.$$

$$\left. + \; \bar{Y}_G(s+j2\pi n, g)\, \bar{Z}_G(s) \right\} . \tag{3.20}$$

$\bar{Y}_G(s,\tau)$ and $\bar{Z}_G(s)$ are given by

$$\bar{Y}_G(s,\tau) = (sI - G)^{-1}\left[I - e^{-(sI-G)\tau} \right] \tag{3.21}$$

and

$$\bar{Z}_G(s) = \left(I - e^{-s}F^* e^{Ff} \cdots e^{Aa} Q^* \cdots G^* e^{Gg}\right)^{-1} \left[F^* \bar{Y}_F(s,f) \; + \right.$$

$$e^{-sf} F^* e^{Ff} E^* \bar{Y}_E(s,e) + \cdots + e^{-s(f+e+\cdots+a)} F^* \cdots e^{Aa} Q^* \bar{Y}_Q(s,q) \; +$$

$$\left. \cdots + e^{s(1-g)} F^* \cdots e^{Aa} Q^* \cdots e^{Hh} G^* \bar{Y}_G(s,g) \right] . \tag{3.22}$$

The expression $\bar{Z}_G(s)$ for $G = A$ to Q may be simplified by considering the solution to the homogeneous matrix equation and condition

$$\frac{dY}{dt} = F(t)Y \quad , \quad Y(0+) = I \; , \tag{3.23}$$

where $F(t)$ is given by equation (3.5). The solution will be of interest for the fundamental interval $0 < t \leq 1$. Considering the piecewise constant behavior of $F(t)$ in equation (3.23), it is easy to see that

$$Q^* e^{Qq} P^* \cdots A^* e^{Aa} = Y(1+) . \tag{3.24}$$

It follows that

$$F^* e^{Ff} E^* \cdots e^{Aa} Q^* \cdots G^* e^{Gg} = Y(\bar{g}+) Y(1+) Y^{-1}(\bar{g}+)^{1} \tag{3.25}$$

and

$$(I - e^{-s} F^* e^{Ff} \cdots e^{Aa} Q^* \cdots e^{Gg})^{-1} = Y(\bar{g}+)[I - e^{-s} Y(1+)]^{-1} Y^{-1}(\bar{g}+) . \tag{3.26}$$

Using similar relationships and changing the order of terms yields

$$\begin{aligned} \bar{Z}_G(s) = Y(\bar{g}+)[I - e^{-s} Y(1+)]^{-1} Y^{-1}(\bar{g}+) \Big[& Y(\bar{g}+) Y^{-1}(\bar{b}+) A^* \bar{Y}_A(s,a) e^{-s(\bar{g}-\bar{b})} + \\ & Y(\bar{g}+) Y^{-1}(\bar{c}+) B^* \bar{Y}_B(s,b) e^{-s(\bar{g}-\bar{c})} + \cdots + F^* \bar{Y}_F(s,f) + \\ & Y(\bar{g}+) Y(1+) Y^{-1}(\bar{h}+) G^* \bar{Y}_G(s,g) e^{-s(\bar{g}+1-\bar{h})} + \cdots + Y(\bar{g}+) Q^* \bar{Y}_Q(s,q) e^{-s\bar{g}} \Big], \end{aligned} \tag{3.27}$$

which may be written

$$\begin{aligned} \bar{Z}_G(s) = {} & \bar{Z}_{GA}(e^s) \bar{Y}_A(s,a) e^{-s(\bar{g}-\bar{b})} + \bar{Z}_{GB}(e^s) \bar{Y}_B(s,b) e^{-s(\bar{g}-\bar{c})} + \\ & \cdots + \bar{Z}_{GF}(e^s) \bar{Y}_F(s,f) + \bar{Z}_{GG}(e^s) \bar{Y}_G(s,g) e^{-s(\bar{g}+1-\bar{h})} + \\ & \cdots + \bar{Z}_{GQ}(e^s) \bar{Y}_Q(s,q) e^{-s\bar{g}} , \end{aligned} \tag{3.28}$$

where

$$\bar{Z}_{GA}(e^s) = Y(\bar{g}+) \bar{Z}(e^s) Y^{-1}(\bar{b}+) A^*$$

$$\bar{Z}_{GB}(e^s) = Y(\bar{g}+) \bar{Z}(e^s) Y^{-1}(\bar{c}+) B^*$$

$$\vdots$$

---

1 $Y^{-1}(t)$ exists for all finite t [Bel. 1, Theorem 2, p. 10].

$$\bar{Z}_{GF}(e^s) = Y(\bar{g}+)\bar{Z}(e^s)Y^{-1}(\bar{g}+)F^*$$

$$\bar{Z}_{GG}(e^s) = Y(\bar{g}+)\bar{Z}(e^s)Y(1+)Y^{-1}(\bar{h}+)G^*$$

$$\vdots$$

$$\bar{Z}_{GQ}(e^s) = Y(\bar{g}+)\bar{Z}(e^s)Q^* \tag{3.29}$$

and

$$\bar{Z}(e^s) = \left[I - e^{-s}Y(1+)\right]^{-1}. \tag{3.30}$$

The Output Spectrum

The output spectrum $\bar{y}(j\omega)$ is given by substituting equation (3.20) for $C_n(j\omega)$ in equation (A2.23). After manipulation

$$\bar{y}(j\omega) = \sum_{G=A}^{Q}\Bigg\{\bar{Y}_G(j\omega,g)\Big[g\bar{x}(j\omega) + \sum_{n=-\infty}^{\infty} e^{j2\pi n\bar{g}}\bar{Z}_G(j\omega+j2\pi n)\bar{x}(j\omega+j2\pi n)\Big]$$
$$+\sum_{n=-\infty}^{\infty} e^{j2\pi n\bar{g}}\alpha_n\Big[e^{j2\pi n g}\bar{Y}_G(j\omega+j2\pi n,g) - \bar{Y}_G(j\omega,g)\Big]\bar{x}(j\omega+j2\pi n)\Bigg\} \tag{3.31}$$

where the α_n are given by equation (2.53). If only the fundamental component $\bar{x}(j\omega)$ is significant, the spectrum is approximated by

$$\bar{y}(j\omega) \approx \sum_{G=A}^{Q}\bar{Y}_G(j\omega,g)\Big[g + \bar{Z}_G(j\omega)\Big]\bar{x}(j\omega). \tag{3.32}$$

The Time-Variant Impulse Response

Inverse Laplace transforming $H(s,t)$ with respect to τ determines the time-variant impulse response $W(t,\tau)$. Applying the method to the individual terms of the piecewise in time representation given by the equation (3.17) yields

$$L^{-1}\Big[\bar{Y}_G(s,t-k-\bar{g})\Big] = Y_G(\tau,t-k-\bar{g}) = e^{G\tau},\ 0<\tau\le t-k-\bar{g}$$
$$= 0,\ \tau\le 0,\ \tau>t-k-\bar{g} \tag{3.33}$$

and

$$L^{-1}\left[e^{(sI-G)(t-k-\bar{g})}\bar{Z}_G(s)\right] = e^{G(t-k-\bar{g})} Z_G\left[\tau-(t-k-\bar{g})\right], \quad \tau > t-k-\bar{g}$$
$$= 0 \quad , \quad \tau \leq t-k-\bar{g} \tag{3.34}$$

for $G = A$ to Q. Only one relation so defined is non-zero for particular t and τ. The appropriate functional form is easily seen by specifying regions in the t, τ plane of Figure 3.1. Boundaries are included in regions in accordance with the discussion of Chapter II. $W(t,\tau)$ is then given by

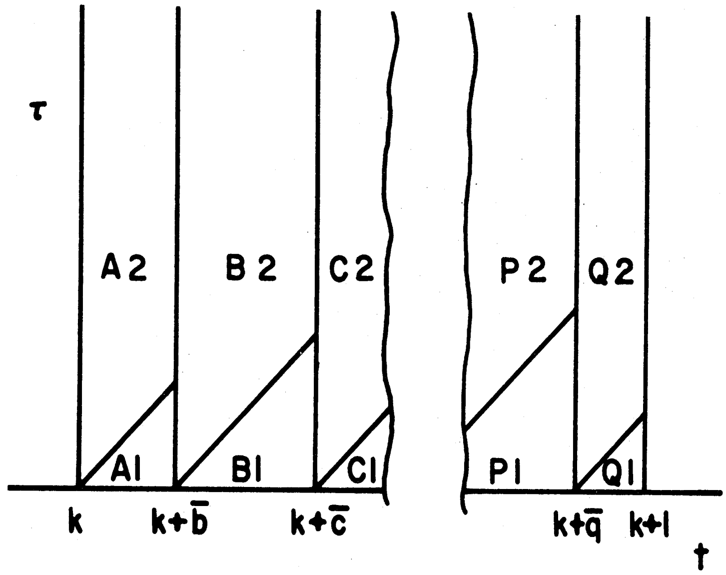

Figure 3.1 Regions in the t, τ Plane

$$W(t,\tau) = 0 \quad , \quad \tau \leq 0$$
$$= e^{G\tau} \quad , \quad \text{region G1}$$
$$= e^{G(t-k-\bar{g})} Z_G\left[\tau-(t-k-\bar{g})\right], \quad \text{region G2} \tag{3.35}$$

for $G = A$ to Q.

$W_1(t,\lambda)$ is obtained by replacing τ by $t-\lambda$ in $W(t,\tau)$ and considering the regions defined as they appear in the t, λ plane of Figure 3.2. The result is

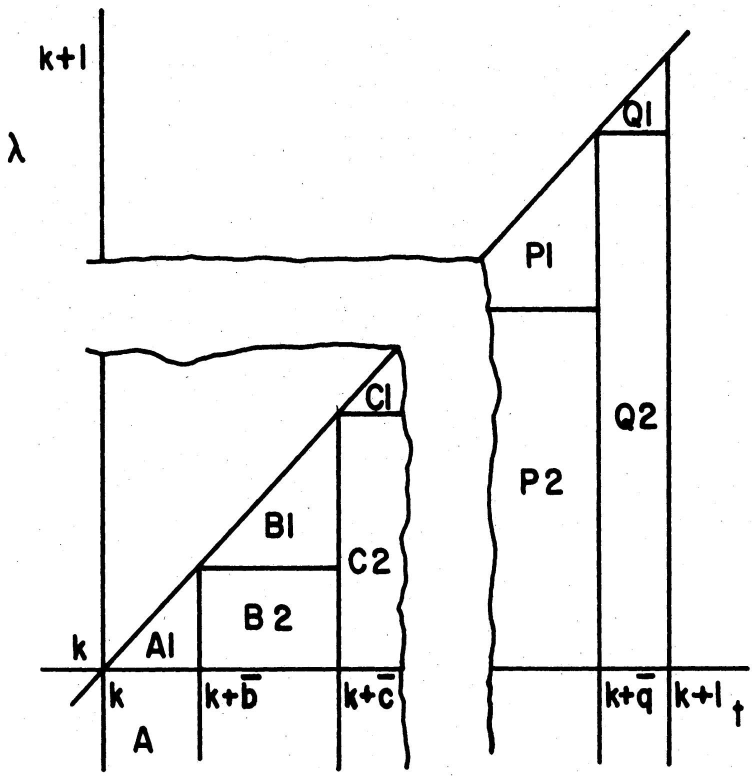

Figure 3.2 Regions in the t , λ Plane

$$W_1(t,\lambda) = 0 \quad , \quad t \leq \lambda$$

$$= e^{G(t-\lambda)} \quad , \qquad \text{region G1}$$

$$= e^{G(t-k-\bar{g})} Z_G(k+\bar{g}-\lambda) \quad , \qquad \text{region G2} \tag{3.36}$$

for $G = A$ to Q .

In the manner previously indicated, it is possible to write an equation for $Z_G(\tau)$. Thus

$$Z_G(\tau) = \sum_{n=0}^{\infty} \Big\{ Z_{GA}(n) Y_A[\tau-(n+\bar{g}-\bar{b}), a] + Z_{GB}(n) Y_B[\tau-(n+\bar{g}-\bar{c}), b]$$

$$+ \cdots + Z_{GF}(n) Y_F[\tau-n, f] + Z_{GG}(n) Y_G[\tau-(n+\bar{g}+1-\bar{h}), g]$$

$$+ \cdots + Z_{GQ}(n) Y_Q[\tau-(n+\bar{g}), q] \Big\} \tag{3.37}$$

for $G = A$ to Q, where

$$Z_{GF}(\tau) = \text{inverse } z \text{ transform}\left[\bar{Z}_{GF}(z)\right] \tag{3.38}$$

for $G, F = A$ to Q. Examination of equation (3.37) shows that only one term is non-zero for any particular value of τ. This means that $W(t,\tau)$ may be expressed simply for certain values of t and τ. For example, when $t-k = \bar{g}+$ and $\tau = n+$, it is noted that $W(t,\tau) = Z_G(n+) = Z_{GF}(n)$. From equations (3.29) it is then seen that

$$W(t,\tau) = Y(\bar{g}+)Z(n)Y^{-1}(\bar{g}+)F^{*}, \tag{3.39}$$

where

$$Z(\tau) = \text{inverse } z \text{ transform}\left[\bar{Z}(z)\right]. \tag{3.40}$$

<u>Systems with General Periodic Coefficients</u>

By consideration of limiting forms it is possible to extend results to systems with general periodic coefficients. Such systems are described by the vector differential equation

$$\frac{dy}{dt} = \bar{F}(t)y + x, \tag{3.41}$$

where $\bar{F}(t)$ is periodic with unity period and is assumed to be sectionally continuous. If the $x_i(t)$ are sectionally continuous, y is a continuous function of time and equation (3.41) may be approximated to any degree of accuracy by a system with coefficients constant in an arbitrarily large but finite number of equal intervals of the fundamental period. Thus $a = b = c = \cdots = q = \Delta t$. The approximating equation may be written

$$\frac{dy}{dt} = F_A(t)\,y + x \;, \tag{3.42}$$

where $F_A(t)$ is given by

$$F_A(t) = \bar{F}(r\Delta t)\,,\quad (r-1)\Delta t < t-k \leq r\Delta t\;,\; r = 1 \text{ to } \frac{1}{\Delta t}\,. \tag{3.43}$$

The Time-Variant Transfer Function

The limiting form of equation (3.17) defines the desired time-variant transfer function. Consider the first term of the equation for all G . The argument $t-k-\bar{g}$ will always be less than Δt . Thus $\bar{Y}_G(s, t-k-\bar{g})$ will be negligible in comparison with the second term which includes a summation of $\frac{1}{\Delta t}$ such terms with argument Δt (see equation (3.22)). It is also seen that the coefficient $e^{-(sI-G)(t-k-\bar{g})}$ of the second term approaches I for Δt arbitrarily small. To any required accuracy

$$H(s,t) = \bar{Z}_G(s) \quad , \quad \bar{g} < t-k,\ \bar{g}+\Delta t \tag{3.44}$$

for all G . The expression for $\bar{Z}_G(s)$ may be simplified since for all G , $\bar{Y}_G(s,\Delta t)$ may be approximated by

$$\bar{Y}_G(s,\Delta t) = \frac{1-e^{-s\Delta t}}{s}\,I \;, \tag{3.45}$$

since

$$\begin{aligned} Y_G(\tau,\Delta t) &= I \quad , \quad 0 < \tau \leq \Delta t \\ &= 0 \quad , \quad \tau \leq 0 \;,\; \tau > \Delta t \end{aligned} \tag{3.46}$$

for Δt arbitrarily small. Substituting equation (3.45) in equation (3.27) for all values of G and noting that the star matrices equal the identity matrix (Y is continuous) yields

$$\bar{Z}_G(s) = Y(\bar{g})\left[I - e^{-s}Y(1)\right]^{-1} Y^{-1}(\bar{g}) \Big\{ Y(\bar{g})Y^{-1}(\Delta t)e^{-s(\bar{g}-\Delta t)} + Y(\bar{g})Y^{-1}(2\Delta t)e^{-s(\bar{g}-2\Delta t)} + \cdots + Y(\bar{g})Y^{-1}(\bar{g}) + Y(\bar{g})Y(1)Y^{-1}(\bar{g}+\Delta t)\, e^{-s[\bar{g}+1-(\bar{g}+\Delta t)]} + \cdots + Y(\bar{g})Y(1)Y^{-1}(1)\, e^{-s(\bar{g}+1-1)} \Big\} \frac{1-e^{-s\Delta t}}{s} \quad (3.47)$$

[1]

After manipulation

$$\bar{Z}_G(s) = \frac{1}{s} Y(\bar{g})\left[I - e^{-s}Y(1)\right]^{-1} Y^{-1}(\bar{g}) \Big\{ \left[I - e^{-s}Y(\bar{g})Y(1)Y^{-1}(\bar{g}+\Delta t)\right] + Y(\bar{g})e^{-s\bar{g}} \cdot \Big[\{I - Y^{-1}(\Delta t)\} + \{Y^{-1}(\Delta t) - Y^{-1}(2\Delta t)\}e^{s\Delta t} + \cdots + \{Y^{-1}(\bar{g}-\Delta t) - Y^{-1}(\bar{g})\}e^{s(\bar{g}-\Delta t)}\Big] + Y(\bar{g})Y(1)e^{-s(\bar{g}+1)}\Big[\{Y^{-1}(\bar{g}+\Delta t) - Y^{-1}(\bar{g}+2\Delta t)\}e^{s(\bar{g}+\Delta t)} + \cdots + \{Y^{-1}(1-\Delta t) - Y^{-1}(1)\}e^{s(1+\Delta t)}\Big]\Big\}. \quad (3.48)$$

$H(s,t)$ is given in the limiting case by replacing $\bar{g}$ with $t-k$ and the sums with appropriate integrals. Thus

$$H(s,t) = \frac{1}{s}I - \frac{e^{-s(t-k)}}{s} Y(t-k)\left[I - e^{-s}Y(1)\right]^{-1} \left\{ \int_0^{t-k} e^{s\lambda}\, dY^{-1}(\lambda) + Y(1)e^{-s}\int_{t-k}^{1} e^{s\lambda}\, dY^{-1}(\lambda) \right\},$$

$$0 < t-k \leq 1, \quad (3.49)$$

where $Y(t)$ is the solution of the equation and condition

$$\frac{dY}{dt} = \bar{F}(t)Y \quad , \quad Y(0+) = I. \quad (3.50)$$

It should be noted that $Y(t)$ has an inverse [Bel. 1, Theorem 2, p. 10] so that the integrals indicated do exist. In most cases $H(s,t)$

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1 $Y(\alpha+)$ for all α is replaced by $Y(\alpha)$ since $Y(t)$ is continuous for $F(t) = F_A(t)$ in equation (3.23).

cannot be obtained in terms of elementary functions since $Y(t)$ is usually transcendental.

The Time-Variant Impulse Response

The time-variant impulse response is given by the inverse Laplace transform of equation (3.49) with respect to τ. This equation may be written

$$H(s,t) = Y(t-k)\left[I - e^{-s}Y(1)\right]^{-1}\left\{\frac{\left[I - e^{-s}Y(1)\right]Y^{-1}(t-k)}{s}\right.$$

$$\left. - \int_0^{t-k} \frac{e^{s[\lambda-(t-k)]}}{s}\, dY^{-1}(\lambda) - Y(1)\int_{t-k}^{1} \frac{e^{s[\lambda-(1+t-k)]}}{s}\, dY^{-1}(\lambda)\right\}. \quad (3.51)$$

Let $u(\tau)$ be the unit step at $\tau = 0$. The inverse transform of the three terms in the last bracket of equation (3.50) then gives

$$L^{-1}\left[\frac{\left[I - e^{-s}Y(1)\right]Y^{-1}(t-k)}{s}\right] = 0 \quad , \quad \tau < 0$$

$$= Y^{-1}(t-k) \, , \quad 0 < \tau < 1$$

$$= Y^{-1}(t-k) - Y(1)Y^{-1}(t-k) \, , \tau > 1 \quad (3.52)$$

$$L^{-1}\left[\int_0^{t-k} \frac{e^{s[\lambda-(t-k)]}}{s}\, dY^{-1}(\lambda)\right] =$$

$$\int_0^{t-k} u[\lambda - (t-k-\tau)]\, dY^{-1}(\lambda) = 0 \quad , \quad \tau < 0$$

$$= Y^{-1}(t-k) - Y(t-k-\tau) \, , \; 0 < \tau < t-k$$

$$= Y^{-1}(t-k) - I \, , \quad \tau > t-k \quad (3.53)$$

$$L^{-1}\left[Y(1)\int_{t-k}^{1}\frac{e^{s[\lambda-(1+t-k)]}}{s}\,dY^{-1}(\lambda)\right] =$$

$$Y(1)\int_{t-k}^{1}u[\lambda-(1+t-k-\tau)]\,dY^{-1}(\lambda) = 0 \quad , \tau < t-k$$

$$= I - Y(1)Y^{-1}(1+t-k-\tau),\ t-k<\tau<1$$

$$= I - Y(1)Y^{-1}(t-k),\ \tau>1 . \tag{3.54}$$

Adding these three terms with the correct signs yields the transform of the bracket.

$$L^{-1}\left[\left\{\qquad\right\}\right] = 0 \quad , \tau < 0$$

$$= Y^{-1}(t-k-\tau) \ , \quad 0<\tau<t-k$$

$$= Y(1)Y^{-1}(1+t-k-\tau), \quad t-k<\tau<1$$

$$= 0 \quad , \quad \tau>1. \tag{3.55}$$

The expression for $W(t,\tau)$ readily follows.

$$W(t,\tau) = Y(t-k)\sum_{n=0}^{\infty} Z(n)\,Y^{-1}[(t-k)-(\tau-n)] \quad , \ 0<\tau-n<t-k$$

$$= Y(t-k)\sum_{n=0}^{\infty} Z(n)\,Y(1)Y^{-1}[(1+t-k)-(\tau-n)] \ , \ t-k<\tau-n<1 \tag{3.56}$$

for $0<t-k<1$, where

$$Z(\tau) = \text{inverse } z \text{ transform}\left[\bar{Z}(z)\right]. \tag{3.57}$$

Summary

For convenient reference the general results determined in this chapter will be summarized. The results of Chapter II are included as a special case. The summary is followed by a discussion of the significance and application of the results as they apply to the response problem.

System Description

The considered systems have periodic, piecewise constant parameters and are described in every fundamental period in a sequence of time intervals by a corresponding sequence of vector differential equations with constant coefficients. Thus any fundamental period, which is taken as unity, can be divided into the intervals $k<t<k+a$, $k+a<t<k+a+b$, ..., and $k+a+b+\cdots p<t<k+a+b+\cdots+p+q=k+1$, where k is any integer. Defining $\bar{g}=a+b+\cdots+f$, a typical interval becomes $\bar{g}<t-k<\bar{g}+g$. All intervals are defined by considering $\bar{g}$ values from $\bar{a}$ to $\bar{q}$. The system is then described by the vector differential equations

$$\frac{dy}{dt}=Gy+x \quad , \quad \bar{g}<t-k<\bar{g}+g \tag{3.1}$$

for $G=A$ to Q,[1] where x and y are nth order column vectors and the G are n by n matrices. Initial conditions for every interval are determined by the equations

$$y_{Gk}^{+}=F^{*}y_{Fk} \quad , \quad G\neq A$$

$$y_{Ak}^{+}=Q^{*}y_{Qk-1} \quad , \tag{3.2}$$

[1] Replacing G by another letter also indicates that $\bar{g}$ and g are replaced by the corresponding letters.

where $y_{Gk}^{+} = y(k+\bar{g}+)$ and $y_{Fk} = y(k+\bar{g}-) = y(k+\bar{f}+f-)$. The initial condition matrices G^{*} must all equal the identity matrix if the solution is to be continuous for sectionally continuous X . If the $y(k+\bar{g})$ are defined equal to $y(k+\bar{g}-) = y_{FK}$, it is possible to describe the system by the single equation

$$\frac{dy}{dt} = F(t)y + x , \tag{3.4}$$

where

$$F(t) = (G^{*}-I)\delta(t-k-\bar{g}-g) + G , \quad \bar{g}+<t-k<\bar{g}+g+ \tag{3.5}$$

for $G = A$ to Q .

The Piecewise Solution

The solution in any interval can be obtained by substituting for τ in equations (3.7) and using the definitions (3.6). Thus

$$y(t) = e^{G(t-k-\bar{g})} y_{Gk}^{+} + \int_{0}^{t-k-\bar{g}} e^{G\lambda} x(t-\lambda)d\lambda, \quad \bar{g}<t-k<\bar{g}+g, \tag{3.58}$$

where the matrix functions $e^{G\tau}$, defined in Appendix II, are given by

$$e^{G\tau} = L^{-1}\left[(sI-G)^{-1}\right]. \tag{3.59}$$

The values y_{Gk}^{+} are obtained from the previous interval by the iteration formulas

$$y_{Gk}^{+} = F^{*}\left(e^{Ff} y_{Fk}^{+} + y_{Fk}^{*}\right), \quad G \neq A$$

$$y_{Ak}^{+} = Q^{*}\left(e^{Qq} y_{Qk-1}^{+} + y_{Qk-1}^{*}\right). \tag{3.8}$$

The constant vectors y_{GK}^{*} are given by

$$y_{GK}^{*} = \int_0^{g} e^{G\lambda} x(t-k-\bar{g}-\lambda)\, d\lambda \,. \tag{3.60}$$

It is also possible to write iteration formulas for the y_{GK}.

Thus

$$y_{GK} = e^{Gg} F^{*} y_{FK} + y_{GK}^{*} \,, \quad G \neq A$$

$$y_{AK} = e^{Aa} Q^{*} y_{QK-1} + y_{AK}^{*} \,. \tag{3.9}$$

The Time-Variant Transfer Function

The piecewise representation of $H(s,t)$ is given by

$$H(s,t) = \bar{Y}_G(s, t-k-\bar{g}) + e^{-(sI-G)(t-k-\bar{g})} \bar{Z}_G(s) \tag{3.17}$$

$$\bar{g} < t-k < \bar{g}+g .$$

The complex Fourier series form is

$$H(s,t) = \sum_{n=-\infty}^{\infty} C_n(s)\, e^{j2\pi n t} , \tag{3.19}$$

where

$$C_n(s) = \sum_{G=A}^{Q} \left\{ \frac{e^{-j2\pi n \bar{g}}}{j2\pi n} \left[\bar{Y}_G(s+j2\pi n, g) - e^{-j2\pi n g} \bar{Y}_G(s,g) \right] \right.$$

$$\left. + \bar{Y}_G(s+j2\pi n, g)\, \bar{Z}_G(s) \right\} . \tag{3.20}$$

The functions $\bar{Y}_G(s,\tau)$ and $\bar{Z}_G(s)$ are defined by

$$\bar{Y}_G(s,\tau) = (sI-G)^{-1} \left[I - e^{-(sI-G)\tau} \right] \tag{3.21}$$

and

$$\bar{Z}_G(s) = \bar{Z}_{GA}(e^s) \bar{Y}_A(s,a) e^{-s(\bar{g}-b)} + \bar{Z}_{GB}(e^s) \bar{Y}_B(e^s) e^{-s(\bar{g}-c)} + \ldots$$

$$+ \bar{Z}_{GF}(e^s) \bar{Y}_F(s,f) + \bar{Z}_{GG}(e^s) \bar{Y}_G(s,g) e^{-s(\bar{g}+1-\bar{h})}$$

$$+ \cdots + \bar{Z}_{GQ}(e^s)\,\bar{Y}_Q(s,q)\,e^{-s\bar{g}}, \tag{3.28}$$

where

$$\bar{Z}_{GA}(e^s) = Y(\bar{g}+)\,\bar{Z}(e^s)\,Y^{-1}(\bar{b}+)A^*$$

$$\vdots$$

$$\bar{Z}_{GF}(e^s) = Y(\bar{g}+)\,\bar{Z}(e^s)\,Y^{-1}(\bar{g}+)F^*$$

$$\bar{Z}_{GG}(e^s) = Y(\bar{g}+)\,\bar{Z}(e^s)\,Y(1+)\,Y^{-1}(h+)\,G^*$$

$$\vdots$$

$$\bar{Z}_{GQ}(e^s) = Y(\bar{g}+)\,\bar{Z}(e^s)\,Q^*, \tag{3.29}$$

and

$$\bar{Z}(e^s) = \left[I - e^{-s}Y(1+)\right]^{-1}. \tag{3.30}$$

The function $Y(t)$ is the solution to the matrix equation and condition

$$\frac{dY}{dt} = F(t)Y, \quad Y(0+) = I. \tag{3.23}$$

The Output Spectrum

The output spectrum $\bar{y}(j\omega)$ is determined by the equation

$$\bar{y}(j\omega) = \sum_{G=A}^{Q} \Big\{ \bar{Y}_G(j\omega, g)\Big[g\,\bar{x}(j\omega) + \sum_{n=-\infty}^{\infty} e^{j2\pi n\bar{g}}\,\bar{Z}_G(j\omega + j2\pi n)\,\bar{x}(j\omega + j2\pi n)$$

$$+ \sum_{n=-\infty}^{\infty} e^{j2\pi n\bar{g}}\,\alpha_n \Big[e^{j2\pi n g}\,\bar{Y}_G(j\omega + j2\pi n, g) - \bar{Y}_G(j\omega, g)\Big]\bar{x}(j\omega + j2\pi n)\Big\}, \tag{3.31}$$

where

$$\alpha_n = \frac{1}{j2\pi n} \quad , \quad n \neq 0$$
$$= 0 \quad , \quad n = 0. \tag{2.53}$$

The Time-Variant Impulse Response

The impulse response $W(t,\tau)$ is given by

$$W(t,\tau) = 0 \quad , \quad \tau \leq 0$$
$$= e^{G\tau} \quad , \qquad \text{region G1}$$
$$= e^{G(t-k-\bar{g})} Z_6[\tau-(t-k-\bar{g})], \quad \text{region G2.} \tag{3.35}$$

The regions G1 and G2 are defined in Figure 3.1. The impulse response $W_1(t,\lambda)$ is expressed

$$W_1(t,\lambda) = 0 \quad , \quad t \leq \lambda$$
$$= e^{G(t-\lambda)} \quad , \qquad \text{region G1}$$
$$= e^{G(t-k-\bar{g})} Z_6(k+\bar{g}-\lambda), \quad \text{region G2.} \tag{3.36}$$

In this case the regions are defined in Figure 3.2. The $Z_6(\tau)$ can be written

$$Z_6(\tau) = \sum_{n=0}^{\infty} \Big\{ Z_{6A}(n) Y_A[\tau-(n+\bar{g}-\bar{b}),a] + Z_{GB}(n) Y_B[\tau-(n+\bar{g}-\bar{c}),b]$$
$$+ \cdots + Z_{GF}(n) Y_F[\tau-n, f] + Z_{GG}(n) Y_G[\tau-(n+\bar{g}+l-h), g]$$
$$+ \cdots + Z_{GQ}(n) Y_Q[\tau-(n+\bar{g}), q] \Big\}, \tag{3.37}$$

where the

$$Z_{GF}(\tau) = \text{inverse } Z \text{ transform} \left[\bar{Z}_{GF}(z)\right]. \qquad (3.38)$$

These functions may be expressed in terms of the inverse Z transform of $\bar{Z}(z)$ using the relations (3.39).

Systems with General Periodic Coefficients

The time-variant transfer function of systems with general periodic coefficients is

$$H(s,t) = \frac{1}{s}I - \frac{e^{-s(t-k)}}{s} Y(t-k)\left[I - e^{-s}Y(1)\right]^{-1} \left\{ \int_0^{t-k} e^{s\lambda} dY^{-1}(\lambda) + Y(1)e^{-s} \int_{t-k}^{1} e^{s\lambda} dY^{-1}(\lambda) \right\},$$

$$0 < t-k \leq 1, \qquad (3.49)$$

where $Y(t)$ is the solution of the equation and the initial condition (3.23), where $F(t)$ is now a general periodic matrix coefficient. The time-variant impulse response is

$$\begin{aligned} W(t,\tau) &= Y(t-k)\sum_{n=0}^{\infty} Z(n)\, Y^{-1}[(t-k)-(\tau-n)], \quad 0 < \tau-n < t-k \\ &= Y(t-k)\sum_{n=0}^{\infty} Z(n)\, Y(1)\, Y^{-1}[(1+t-k)-(\tau-n)], \quad t-k < \tau-n < 1 \end{aligned} \qquad (3.56)$$

for $0 < t-k < 1$, where

$$Z(\tau) = \text{inverse } Z \text{ transform} \left[\bar{Z}(z)\right]. \qquad (3.57)$$

Discussion

Before considering the individual results and their application, a few general comments are in order. First, the determination of response for the considered time-variant systems is inherently much more complex than for time-invariant systems. It is therefore to be expected that the solution of the response problem will be beset with computational difficulties, especially when the system order is high. Second, varied methods of attack exist. In particular problems one technique may be more meaningful and manageable than another. This is especially true when an approximate solution is acceptable. Finally, computational difficulties do not detract from the physical understanding that is obtained by examination of the different solution forms.

Equation (3.58) gives the solution interior to any interval. It does not give insight into the general behavior of the system. Such insight is obtained from the iteration formulas (3.8) and (3.9). These formulas give the response at the interval boundaries. The main difficulty in applying these formulas is computational in nature. This is due to the many matrix multiplications required, especially when there are many elementary intervals in the fundamental period. The work is greatly simplified when the y^*_{GK} are zero.[1] In this case $y^+_{AK} = Y(1+)^K y^+_{A0}$, where $Y(1+) = Q^* e^{Qq} \cdots A^* e^{Aa}$. The powers of $Y(1+)$ are conveniently obtained using Z notation. Thus $Y(1+)^K = Z(K)$ where $Z(\tau)$ = inverse z transform $[I - z^{-1} Y(1+)]^{-1}$. This is seen by writing $[I - z^{-1} Y(1+)]^{-1} = I + z^{-1} Y(1+) + z^{-2} Y(1+)^2 + \cdots$ and applying the inverse Z transform to individual terms.

System response, at least theoretically, is obtained from the inverse Fourier transform of $H(j\omega, t)\, \bar{X}(j\omega)$ or the inverse Laplace transform of $H(s,t)\, \bar{X}(s)$, where $\bar{X}(j\omega)$ and $\bar{X}(s)$ are the transforms of the input and t is considered as a parameter.

1 This unforced problem is solved by Pipes [Pi. 1, 2, 3, 4, 5], but not using the Z notation.

Practical difficulties arise because the transforms involve rational functions of s and e^s. Tables of such transforms are not available. In some instances a valid approximation is obtained by considering a few terms in the Fourier series for $H(s,t)$ and approximating the function of e^s by polynomials in s. The Fourier series terms to be used depend upon the application and the accuracy required. In many problems the fundamental term $C_0(s)$ is the most important. In modulating circuits, it is probable that the term $C_1(s)$ would be more significant.

The time-variant transfer function is particularly useful in the sense that it determines the response for an exponential input. For example, suppose that $x(t) = e^{-\alpha t}c$ for $t > 0$ and is zero for $t < 0$, and that the system is initially at rest. A particular integral for $t > 0$ is $H(-\alpha, t)e^{-\alpha t}c$. In general this particular integral is not zero for $t = 0+$ and is therefore not the required solution. The solution may be made equal to zero at $t = 0+$ by adding the correct amount of a solution to the homogeneous equation. Such a solution is the impulse response $W_1(t, \lambda)d$. Since $W_1(0+, 0) = I$, it is seen that the desired solution is

$$y(t) = H(-\alpha, t)e^{-\alpha t}c \quad - W_1(t, 0)H(-\alpha, 0+)c, \quad t > 0 \quad .$$

Sinusoidal response for the system is determined by taking the real part of the solution for the exponential input $e^{j\omega t}c$. The Fourier series form of this solution, $y(t) = \sum_{n=-\infty}^{\infty} C_n(j\omega)\, e^{j(2\pi n+\omega)t}$, has therefore considerable physical significance. First, the periodic character of the system generates the new frequencies $\omega + 2\pi n$. Second, each new frequency component appears in the output to an extent determined by the matrix functions $C_n(j\omega)$, which particularize a given system.

The output spectrum $\bar{Y}(j\omega)$ is given by equation (3.31). In agreement with the remarks of the preceding paragraph, it is observed that the output contains new frequency components $\bar{X}(j\omega + j2\pi n)$. The given expression is primarily useful when the spectrum is required per se. This is because application of the inverse Fourier transform to equation (3.31) to obtain $y(t)$ involves the same complications present in transforming $H(j\omega,t)\,\bar{X}(j\omega)$. However, the spectrum does give a clear understanding of the approximations previously considered. For example, approximating $H(j\omega,t)$ by $C_1(j\omega)\,e^{j2\pi t}$ corresponds to neglecting all terms in equation (3.31) except the term for $n=1$.

The time-variant impulse response is defined in two forms. $W(t,\tau)$ is the response at time t for an impulse applied τ units previously; $W_1(t,\lambda)$ is the response at time t for an impulse applied at time λ.

The form $W_1(t,\lambda)$ is particularly useful as a solution to the homogeneous equation. By definition $W_1(t,\lambda)c$ is a solution to the unforced system for any c and for $t>\lambda$. Since $W_1(\lambda+,\lambda) = I$, the solution for the initial condition $y(\lambda)$ is $W_1(t,\lambda)\,y(\lambda)$.

The form $W(t,\tau)$ is more useful for determining system response. The superposition integral (A2.6) gives the desired solution for any input $x(t)$. Examination of this integral and the expressions for $W(t,\tau)$ and $Z_G(\tau)$ (equations (3.35) and (3.37)) shows that integrals of the form $\int e^{G(\tau-\alpha)}x(t-\tau)\,d\tau$ are required. The value of α depends on n (the index of summation in equation

(3.37)) and the time at which the response is desired. The integral is always evaluated over a finite interval. Since the components of the $e^{G\tau}$ are expressed as a linear combination of products of polynomials and exponentials in τ , the integrations are frequently straightforward, but often formidable.

From the above discussion, it is quite obvious that a solution to the response problem is possible. However, it is also clear that the computational difficulties are substantial. In some applications the required effort may be warranted; in many others it is not. The following chapter will introduce more manageable techniques. These techniques will not give the response for particular inputs but will give an understanding of the response characteristics.

IV. SYSTEM STABILITY AND RESPONSE CHARACTERISTICS

Application of the methods and results of the previous chapters permit the determination of system output for particular inputs. Unfortunately, the effort involved in these computations is considerable, even for relatively simple systems. This practical infeasibility indicates the need for less general but more facile techniques. Therefore, it is the purpose of this chapter to investigate more easily determined system properties that give insight into the behavior of system response. Considered will be (1) the definition and condition for system stability and (2) the system characteristic roots and their influence on response.

An Equivalent System

The subject of stability and response characteristics is conveniently introduced by considering an equivalent system in which the time-variant operations are simple and isolated. It is necessary to define these operations mathematically and schematically. Define first the sampling function

$$S_\tau(t) = \sum_{n=-\infty}^{\infty} \delta(t-n-\tau) \ , \qquad 0 \leq \tau \leq 1 \ . \tag{4.1}$$

The operation of sampling occurs when a function is multiplied by $S_\tau(t)$. Thus

$$g(t) = \sum_{n=-\infty}^{\infty} f(n+\tau)\,\delta(t-n-\tau) \tag{4.2}$$

is the sampled $f(t)$, where $f(t)$ and $g(t)$ can be scalar, vector, or matrix time functions. Equation (4.2) may be written in different form by replacing $S_\tau(t)$ by its Fourier series representation, giving

$$g(t) = \sum_{n=-\infty}^{\infty} e^{-j2\pi n\tau} e^{j2\pi nt} f(t) . \tag{4.3}$$

The spectrum of the sampled function is then

$$\bar{g}(j\omega) = \sum_{n=-\infty}^{\infty} e^{-j2\pi n\tau} \bar{f}(j\omega - j2\pi n) = \sum_{n=-\infty}^{\infty} e^{j2\pi n\tau} \bar{f}(j\omega + j2\pi n) . \tag{4.4}$$

Define next the scanning function. This periodic function is the saw tooth wave

$$\bar{S}_\tau(t) = -(t-n-\tau) + \frac{1}{2} \quad , \quad \tau < t-n < \tau+1$$

$$= \int_0^t S_\tau(t)\, dt - (t-\tau) + \frac{1}{2} . \tag{4.5}$$

The operation of scanning occurs when a function is multiplied by $\bar{S}_\tau(t)$. If $\bar{S}_\tau(t)$ is replaced by its Fourier series representation, this yields

$$h(t) = \sum_{n=-\infty}^{\infty} \frac{e^{-j2\pi n\tau}}{j2\pi n} e^{j2\pi nt} f(t) , \quad n \neq 0 \tag{4.6}$$

the scanned $f(t)$. The spectrum of the scanned function is then

$$\bar{h}(j\omega) = \sum_{n=-\infty}^{\infty} \alpha_n e^{-j2\pi n\tau} \bar{f}(j\omega - j2\pi n) = -\sum_{n=-\infty}^{\infty} \alpha_n e^{j2\pi n\tau} \bar{f}(j\omega + j2\pi n), \tag{4.7}$$

where the α_n are given by equation (2.53). These operations are shown schematically in Figure 4.1. Examination of equation (3.31) shows that the output spectrum is expressed in terms similar to those of equations (4.4) and (4.7) This leads to the block diagram of Figure 4.2, which is an exact representation of the considered systems. The only time variant operations in this equivalent system are the sampling and scanning operations.

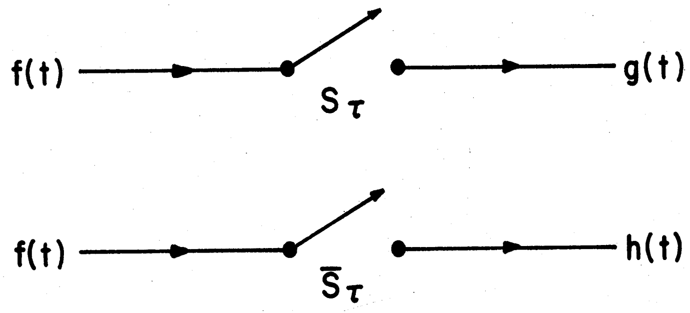

Figure 4.1 Sampling and Scanning Operations

The block diagram is heuristically useful in determining a stability criterion. If the time-invariant operators $\overline{Z}_G(s)$ are removed from the block diagram, the system is stable since the components of the functions $\overline{Y}_G(s,q)$, defined by equation (3.21), have no poles for all s. Stability is then assured if all the poles of the components of the $\overline{Z}_G(s)$ lie in the left half of the s plane. Equations (3.28) and (3.29) indicate that such poles are common to the components of $\overline{Z}(e^s)$. The poles of $\overline{Z}(e^s)$ are determined by the zeros of the determinant $|I - e^{-s}Y(1+)|$, where $Y(1+)$ is given by equation (3.24). Thus the system is stable if the equation $|I - e^{-s}Y(1+)| = 0$ has every root in the left half of the s plane. Equivalently, the corresponding roots in $z = e^s$ must be less than one in magnitude or lie within the unit circle of the z plane.

It seems likely that these s and z roots are fundamental to the character of the system response. Thus if some roots were just left of the imaginary axis, the response would be highly oscillatory with little damping; if the roots were far left of the imaginary axis,

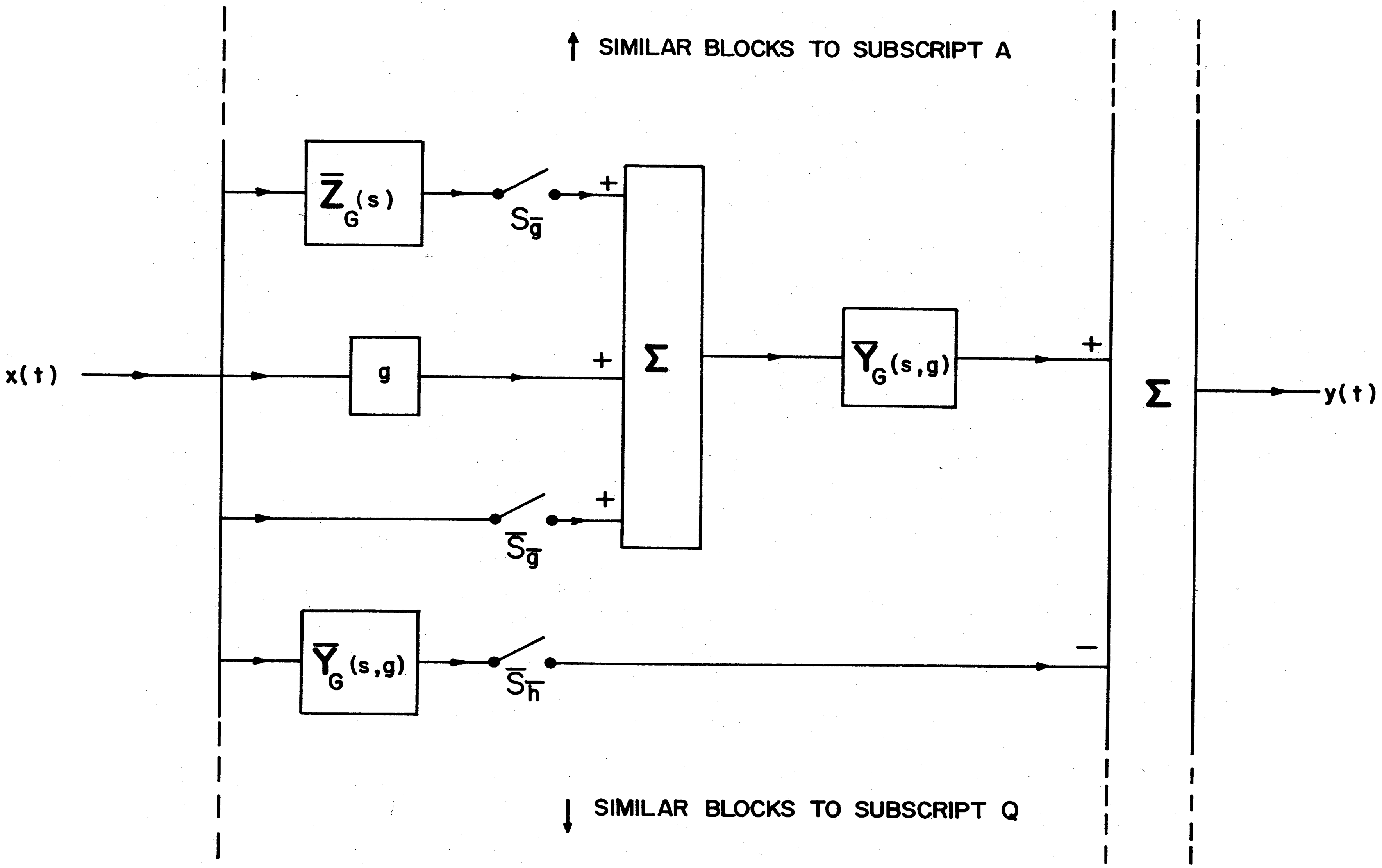

Figure 4.2 An Equivalent System

the response would be quickly damped. For this reason the s values in the strip $\pm j\pi$ and the z values, such that $z = e^s$ satisfies the equation $|zI - Y(1+)| = 0$, will be known as the characteristic roots of the time-variant system. The ideas introduced here will be more fully developed in the following section.

Stability

A rigorous concept of stability is given by the stability definition. - Systems described by equations (3.1) and (3.2) are stable if and only if every bounded vector input produced a bounded vector output. If some bounded vector input produced an unbounded vector output, the system is unstable.[1,2]

This definition agrees with the engineering concept of stability except that it is occasionally somewhat more restrictive than desired. For example, suppose a system has an unbounded output for some bounded input if and only if the vector component x_1 is non-zero. Suppose further that the formulation of the original problem is such that x_1 is identically zero. By the above definition the system would be unstable; in the usual sense it would not. This sort of behavior indicates a lack of coupling between different parts of the system and seldom occurs in practice.

Returning to the stability criterion mentioned earlier, it is possible to state more rigorously the stability theorem. - Systems described by equations (3.1) and (3.2) are stable if and only if every root of the equation

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1 A vector $x(t)$ is considered bounded if $\|x\| = \sum_{i=1}^{n} |x_i|$ is bounded for all t.

2 This definition of stability is an extension of that proposed by James, Nicholls, and Phillips [Ja. 1, p. 38].

$$\left| e^{s} I - Y(1+) \right| = 0 \tag{4.8}$$

lies in the left half of the S plane, where $Y(1+)$ is given by the equation (3.24).

Proof. The characteristic roots of the matrix $Y(1+)$ are the values $Z = e^{s}$ satisfying equation (4.8). Since the transformation $Z = e^{S}$ transforms the left half S plane into the unit circle of the Z plane, it is possible to replace the theorem by the following equivalent: Systems described by equations (3.1) and (3.2) are stable if and only if the magnitude of every characteristic root of $Y(1+)$ is less than one. The sufficiency of this statement is proved by examining the superposition integral for bounded inputs. Thus by equation (3.35)

$$\begin{aligned} y(t) &= \int_0^{\infty} W(t,\tau)\, d\tau \\ &= \int_0^{t-k-\bar{g}} e^{G\tau} x(t-\tau)\, d\tau + \int_{t-k-\bar{g}}^{\infty} e^{G(t-k-\bar{g})} Z_G[\tau-(t-k-\bar{g})]\, x(t-\tau)\, d\tau, \\ & \qquad 0 < (t-k-\bar{g}) < g. \end{aligned} \tag{4.9}$$

Define the notation $\|C\| = \sum_{i,j=1}^{n} |c_{ij}|$ for an nth order matrix C with components c_{ij}, and the notation $\|d\| = \sum_{i=1}^{n} |d_i|$ for an nth order vector d. Then it is seen that

$$\begin{aligned} \|y(t)\| \leq & \int_0^{t-k-\bar{g}} \|e^{G\tau}\| \cdot \|x(t-\tau)\|\, d\tau \\ & + \|e^{G(t-k-\bar{g})}\| \int_0^{\infty} \|Z_G(\lambda)\| \cdot \|x(k+\bar{g}-\lambda)\|\, d\lambda, \end{aligned}$$

$$0 < t-k-\bar{g} < g$$

$$\leq \alpha_1 \alpha_2 \left[1 + \int_0^\infty \| Z_G(\lambda) \| \, d\lambda \right] \quad , \quad 0 < t-k-\bar{g} < g, \tag{4.10}$$

where $\alpha_1 = \| C_1 \|$ and $\alpha_2 > \| X(t) \|$ for all t . C_1 is the matrix whose components equal the largest magnitude of the respective components of $e^{G\tau}$ for all G and for $0 < \tau < 1$. α_1 and α_2 are finite. Examination of equations (3.37) and (3.29) shows that

$$\| Z_G(\lambda) \| \leq \sum_{n=0}^{\infty} \| Y(\bar{g}+) \| \cdot \| Z(n) \| \Big\{ \| Y^{-1}(\bar{b}+) A^* \| + \| Y^{-1}(\bar{c}+) B^* \|$$
$$+ \cdots + \| Y^{-1}(\bar{g}+) F^* \| + \| Y(1+) Y^{-1}(\bar{h}+) G^* \| + \cdots + \| Q^* \| \Big\} \alpha_1(\tau - n), \tag{4.11}$$

where

$$\begin{aligned} \alpha_1(\tau) &= \alpha_1 \quad , \quad 0 < \tau < 1 \\ &= 0 \quad , \quad \tau < 0 \; , \; \tau > 1. \end{aligned} \tag{4.12}$$

Hence

$$\| Z_G(\lambda) \| \leq \alpha_3 \sum_{n=0}^{\infty} \| Z(n) \| \alpha_1(\tau - n) , \tag{4.13}$$

where α_3 is finite. Then

$$\| y(t) \| \leq \alpha_1 \alpha_2 \left(1 + \alpha_1 \alpha_3 \sum_{n=0}^{\infty} \| Z(n) \| \right), \; 0 < t-k-\bar{g} \leq g. \tag{4.14}$$

Similar bounds on $y(t)$ can be found for other G . It remains to be shown that $\sum_{n=0}^{\infty} \| Z(n) \|$ is bounded. This will be done assuming that the characteristic roots are distinct, although this is not necessary.

On page 48 it was shown that $Z(m) = Y(1+)^m$. But $Y(1+)^m = T\,D^m\,T^{-1}$, where D is a diagonal matrix with components equal to the n distinct characteristic roots z_i and T is a non-singular matrix with finite components. Thus

$$\|Z(m)\| = \|Y(1+)^m\| \leq \|T\| \sum_{i=1}^{n} |z_i| \; \|T^{-1}\| . \tag{4.15}$$

If ρ is the largest magnitude of any of the characteristic roots

$$\| Z(m)\| \leq \alpha_4 \, n \rho^m , \tag{4.16}$$

where α_4 is finite. Then

$$\sum_{m=0}^{\infty} \| Z(m)\| \leq \alpha_4 \, n \sum_{m=0}^{\infty} \rho^m . \tag{4.17}$$

The right hand side of the inequality converges since ρ must be less than one. The sufficiency is proven. To prove the necessity of the theorem, it is sufficient to show that if any characteristic z root has a magnitude of one or greater, then there is some bounded input which produces an unbounded output. Consider the output at $t = 0+$. Suppose that $x[(0+)-\tau] = 0$ for $n+q < \tau < n+1$, where $n = 0, 1, 2, --, \infty$. Then by the superposition integral and equations (3.35), (3.37) and (3.29)

$$y(0+) = \int_0^{\infty} Z_A(\tau)\, x[(0+)-\tau]\, d\tau$$

$$= \int_0^{\infty} \sum_{n=0}^{\infty} Z_{AQ}(n)\, Y_Q(\tau - n, q)\, x(-\tau)\, d\tau$$

$$= \int_0^\infty \sum_{n=0}^{\infty} Z(n) Q^* Y_Q(\tau - n, q) x(-\tau) d\tau. \tag{4.18}$$

The matrix $Q^* Y_Q(\tau-n,q)$ is non-singular so for $n < \tau < n+q$ the vector equation

$$Q^* Y_Q(\tau-n,q) x(-\tau) = d \tag{4.19}$$

has a bounded solution $x(-\tau)$ for any vector constant d. Thus

$$y(0+) = q\left[\sum_{n=0}^{\infty} Z(n)\right] d. \tag{4.20}$$

The geometric matrix series $R = \sum_{n=0}^{\infty} Z(n) = \sum_{n=0}^{\infty} Y(1+)^n$ converges if and only if the magnitude of every characteristic root of $Y(1+)$ is less than unity [Ma. 1, Theorem 49, p. 98]. If the series does not converge at least one component of R, say r_{ij}, does not approach a finite limit. The vector d is then chosen so that $d_k = 0,\ k \neq j$ and $d_j = 1$. $y_i(0+)$ is then obviously unbounded and the bounded input given by equation (4.19) produces an unbounded output. This completes the proof.

The first step in determining system stability is to compute the matrix $Y(1+)$ given by equation (3.24). Although the calculations may be extensive, they are straightforward. The second step is to determine the magnitude of the roots $z = e^s$ of the nth degree polynomial with real coefficients[1] given by equation (4.8). This may be done without determining the values of the characteristic roots by applying Routh's criterion [Ga. 1, pp. 197-201] to a transformed

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1 The coefficients are real since every component of $Y(1+)$ is real.

equation.[1] The transformation $z = \frac{w+1}{w-1}$ maps the inside of the unit circle in the z plane into the left half of the w plane. Substituting this expression for $z = e^s$ into equation (4.8) yields an nth degree polynomial equation in w. This equation must have no roots in the right half of the w plane. Applying Routh's criterion to the equation in w determines system stability. As an example suppose that the characteristic equation is

$$z^2 + C_1 z + C_2 = 0. \tag{4.21}$$

The equation in w is

$$(w+1)^2 + (w+1)(w-1)C_1 + (w-1)^2 C_2 = 0, \tag{4.22}$$

which may be written

$$(1 + C_1 + C_2)w^2 + 2(1 - C_2)w + (1 - C_1 + C_2) = 0. \tag{4.23}$$

The Routh array becomes

$$\begin{array}{ll} (1 + C_1 + C_2) & (1 - C_1 + C_2) \\ 2(1 - C_2) & \\ (1 - C_1 + C_2) & \end{array} \tag{4.24}$$

If the system is stable, all terms in the first column must be greater than zero. Slight manipulation yields the conditions

$$\begin{aligned} |C_2| &< 1 \\ |C_1| &< 1 + C_2 \end{aligned} \tag{4.25}$$

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[1] This method is used by Truxal [Tr. 1, p. 523] but was developed independently by the author.

The technique is perfectly general. For convenience equations in Z and W to the fifth degree are tabulated in Table I. Also shown are general stability conditions to degree three.

Response Characteristics

In linear time-invariant systems the characteristic roots[1] determine the time behavior of the impulse response $W(t)$ and thus the general response character of the system; that is, roots just left of the imaginary axis indicate an oscillatory $W(t)$ and thus oscillatory response characteristics while roots far left of the imaginary axis indicate a quickly damped $W(t)$ and thus quickly damped response characteristics. In linear time-variant systems described by equation (3.4) it is possible to define analogous characteristic roots that determine the time behavior of the impulse response $W_1(t, \lambda)$ at intervals of unity. This result is particularly indicative of the general response character of the system in many practical problems where the response changes little in one unit of the independent variable.

Examination of equations (3.36) and (3.37) shows that the functions $Z_{GF}(m)$ for all G and F are of principal interest in evaluating $W_1(t, \lambda)$ at intervals of unity. The components of the $Z_{GF}(m)$ are given by the inverse Z transform of the components of the $\bar{Z}_{GF}(Z)$. These components are ratios of polynomials in Z^{-1} where the denominators are the common polynomial $|I - Z^{-1}Y(1+)|$. Thus the components of the $Z_{GF}(m)$ are all of the form $\sum_{j=0}^{n-1} \beta_j \,\theta(m-j)$, where the scalar function

1 If the system is described by a vector differential equation, the characteristic roots of the system are given by the equation $|\lambda I - A| = 0$, where A is the constant matrix of the system.

TABLE I

POLYNOMIAL EQUATIONS IN z AND w, AND STABILITY CONDITIONS

Degree	Polynomials in z and w	Stability Conditions
1	$z + C_1 = 0$ $(1 + C_1)w + (1 - C_1) = 0$	$\lvert C_1 \rvert < 1$
2	$z^2 + C_1 z + C_2 = 0$ $(1 + C_1 + C_2)w^2 + 2(1 - C_2)w + (1 - C_1 + C_2) = 0$	$\lvert C_2 \rvert < 1$ $\lvert C_1 \rvert < 1 + C_2$
3	$z^3 + C_1 z^2 + C_2 z + C_3 = 0$ $(1 + C_1 + C_2 + C_3)w^3 + (3 + C_1 - C_2 - 3C_3)w^2 + (3 - C_1 - C_2 + 3C_3)w + (1 - C_1 + C_2 - C_3) = 0$	$1 + C_2 > \lvert C_1 + C_3 \rvert$ $1 - C_3 > 1/3(C_2 - C_1)$ $1 - C_2 > C_3(C_3 - C_1)$
4	$z^4 + C_1 z^3 + C_2 z^2 + C_3 z + C_4 = 0$ $(1 + C_1 + C_2 + C_3 + C_4)w^4 + 2(2 + C_1 - C_3 - 2C_4)w^3 + 2(3 - C_2 + 3C_4)w^2$ $+ 2(2 - C_1 + C_3 - 2C_4)w + (1 - C_1 + C_2 - C_3 + C_4) = 0$	
5	$z^5 + C_1 z^4 + C_2 z^3 + C_3 z^2 + C_4 z + C_5 = 0$ $(1 + C_1 + C_2 + C_3 + C_4 + C_5)w^5 + (5 + 3C_1 + C_2 - C_3 - 3C_4 - 5C_5)w^4 + 2(5 + C_1 - C_2 - C_3 + C_4 + 5C_5)w^3 + 2(5 - C_1 - C_2 + C_3 + C_4 - 5C_5)w^2 + (5 - 3C_1 + C_2 + C_3 - 3C_4 + 5C_5)w + (1 - C_1 + C_2 - C_3 + C_4 - C_5) = 0$	

$$\theta(\tau) = \text{inverse } z \text{ transform}\left[\frac{1}{\left|I - z^{-1}Y(1+)\right|}\right] \tag{4.26}$$

and where $\sum_{j=0}^{n-1} \beta_j z^{-j}$ is the numerator of the particular component. Consequently, $\theta(m)$ governs the intrinsic behavior of $W_1(t,\lambda)$. Therefore if $\theta(\tau)$ is oscillatory, the system has oscillatory response; if $\theta(\tau)$ is quickly damped, the system has quickly damped response.

The function $\theta(\tau)$ can be expressed as a linear combination of products of polynomials and exponentials in τ by equation (4.26). The values of the exponential coefficients are given by the location of the roots of $\left|e^s I - Y(1+)\right| = 0$ in the fundamental strip $\pm j\pi$ of the s plane. If these characteristic roots of the system are just left of the imaginary axis in the s plane, $\theta(\tau)$ is highly oscillatory; if the roots are far left of the imaginary axis, $\theta(\tau)$ is quickly damped. Thus the characteristic roots determine the response characteristics of the system.

If the system is first order, the characteristic root is determined by the equation

$$z + C_1 = 0 . \tag{4.27}$$

The s value of the characteristic root is

$$s = \ln(-C_1) . \tag{4.28}$$

If $-1 < C_1 < 0$, the root

$$s = \ln(-C_1) = -\frac{1}{T} \tag{4.29}$$

is located on the negative real axis and $\Theta(\mathcal{T})$ exhibits damped exponential character with a time constant $\overline{\mathcal{T}}$. If $1 > C_1 > 0$, the two roots

$$s = \ln(-C_1) = -\frac{1}{\overline{\mathcal{T}}} \pm j\pi \tag{4.30}$$

are determined and $\Theta(\mathcal{T})$ exhibits an exponentially damped oscillation with a time constant $\overline{\mathcal{T}}$ and a frequency of π rad/sec. Thus a first order system with time-variant coefficients can have an oscillatory response.

It is useful to compare such a set of complex conjugate roots to a similar set of roots for a second order linear time-invariant system. The roots for such a second order system are of the form

$$s = -\omega_N\left(\zeta \pm j\sqrt{1-\zeta^2}\right), \tag{4.31}$$

where ω_N is the undamped natural frequency and ζ is the damping ratio. The damped natural frequency is defined

$$\omega_D = \sqrt{1-\zeta^2}\ \omega_N . \tag{4.32}$$

Figure 4.3 specifies these quantities geometrically in the S plane. In this terminology the roots of equation (4.30) have a damped frequency of π rad/sec and a damping ratio

$$\zeta = \frac{\ln|C_1|}{\sqrt{\pi^2 + \ln^2|C_1|}} . \tag{4.33}$$

If the system is second order, the characteristic roots are determined by the equation

$$z^2 + C_1 z + C_2 = 0 \tag{4.34}$$

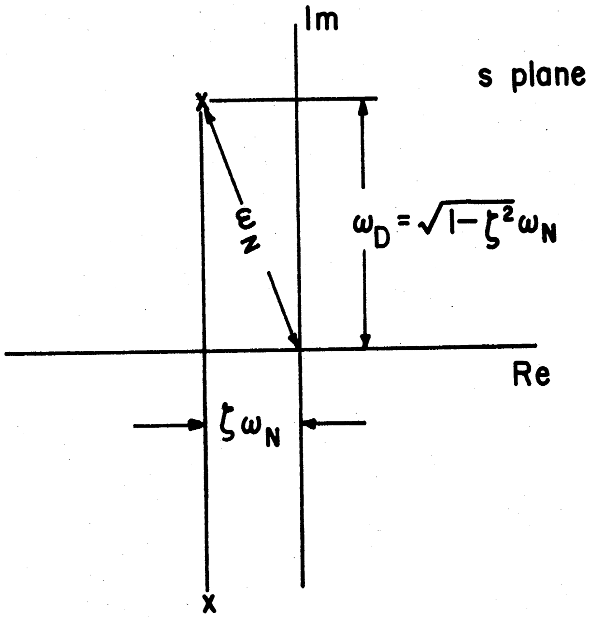

Figure 4.3 Complex Conjugate Roots in the S Plane

The characteristic roots are

$$Z = -\frac{C_1}{2} \pm \sqrt{\left(\frac{C_1}{2}\right)^2 - C_2} \quad . \tag{4.35}$$

The corresponding S values are given by the natural logarithms of the Z values. Various C_1 and C_2 established different classes of such roots. Consider the regions in the C_1, C_2 plane of Figure 4.4. The S values then have the form

$$\begin{aligned} S &= -\alpha \pm j\beta && \text{in region A} \\ &= -\alpha_1 \ , \ -\alpha_2 && \text{in region B} \\ &= -\alpha_1 \pm j\pi \ , \ -\alpha_2 \pm j\pi && \text{in region C} \\ &= -\alpha_1 \ , \ -\alpha_2 \pm j\pi && \text{in region D,} \end{aligned} \tag{4.36}$$

where α , α_1 , α_2 , and β are positive real values. For C_1 and C_2 values outside the triangle the real part of the S values

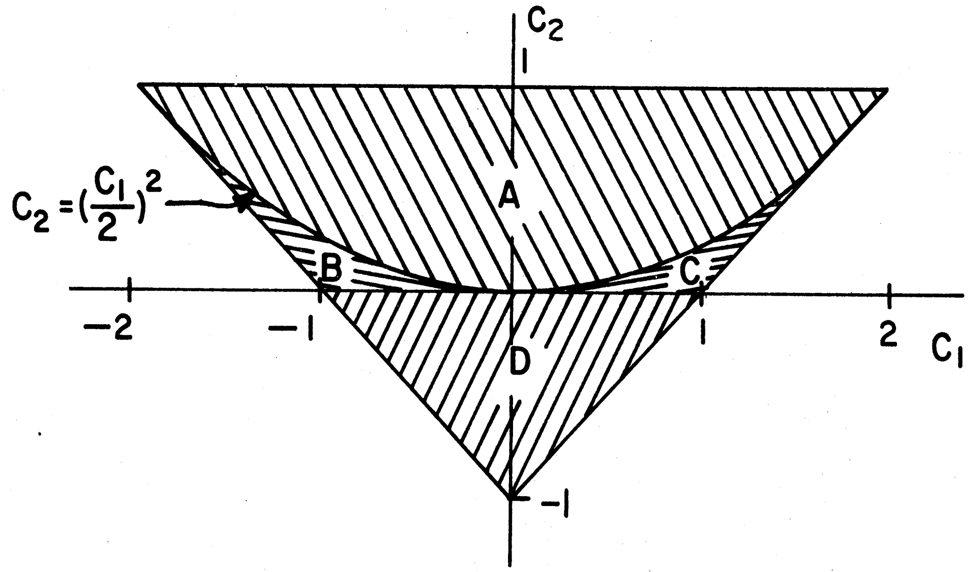

Figure 4.4 Regions in the C_1, C_2 Plane

is greater than zero and the system is unstable. The α , α_1 , and α_2 values are exponential damping coefficients for $\Theta(\tau)$ and β and π , when they apply, are the oscillatory frequencies of $\Theta(\tau)$.

α , α_1 , α_2 , and β values are conveniently obtained for particular C_1 and C_2 values by plotting contours for constant α , α_1 , α_2 , and β in the C_1 , C_2 plane. This is done in Figure 4.5. It is not really necessary to plot contours for constant α_1 and α_2 (regions B , C , and D of Figure 4.4) since equation (4.34) may then be factored into two first degree terms with real coefficients and the previous discussion of first order systems applies. Contours for constant $\zeta < 1$ and $f_N = \frac{\omega_N}{2\pi}$ are more useful in most problems. This establishes the general approach in Figure 4.6. In region A of this plot, $f = f_N$. In region B , $2\pi f = \alpha_1$, the most important root since $\alpha_1 < \alpha_2$; the ζ curves do not appear in the region. In region C , $f_D = \frac{\omega_D}{2\pi} = .5$ and the ζ

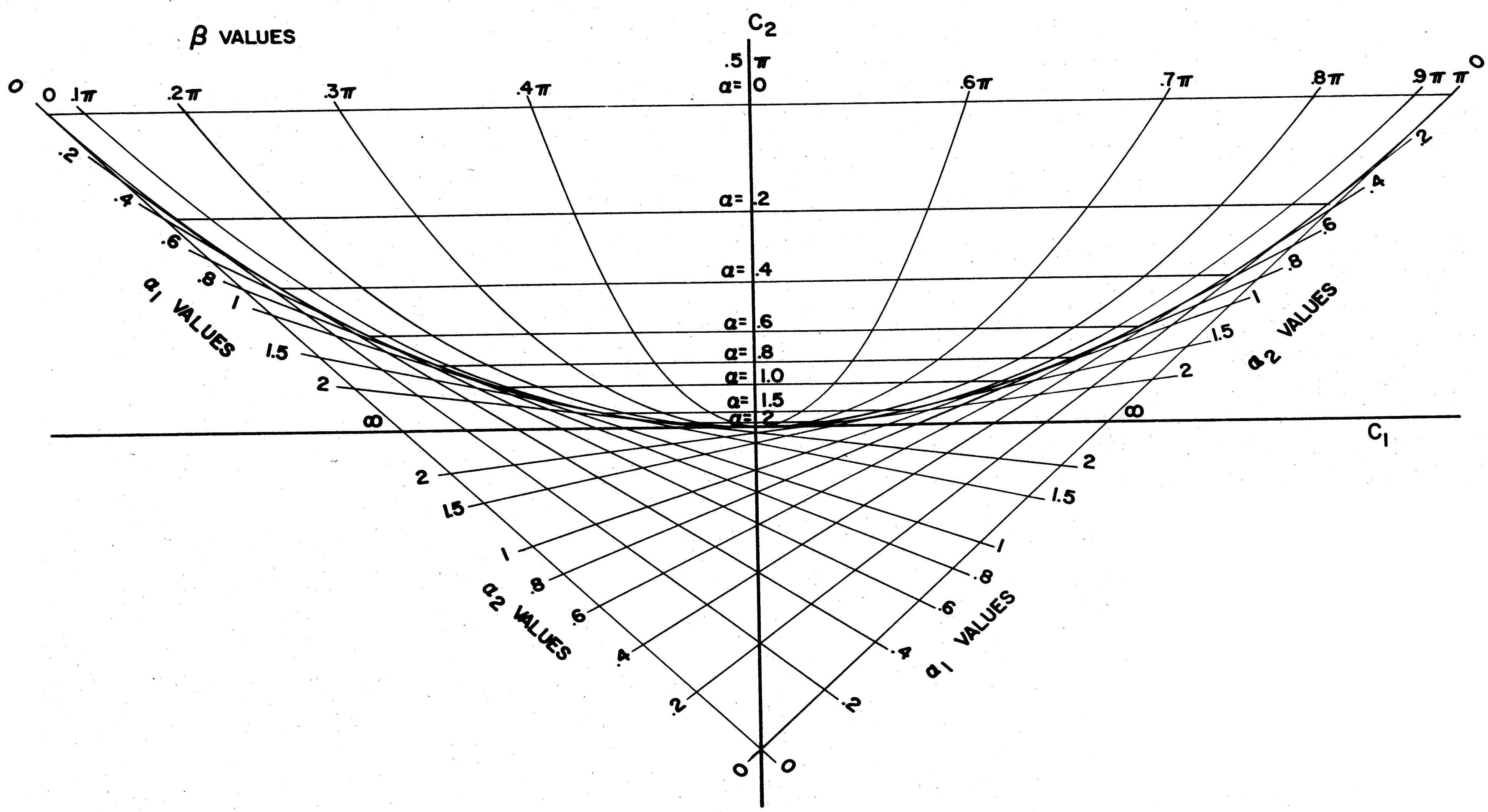

Figure 4.5 Contours of Constant α , α_1 , α_2 , and β in the C_1 , C_2 Plane

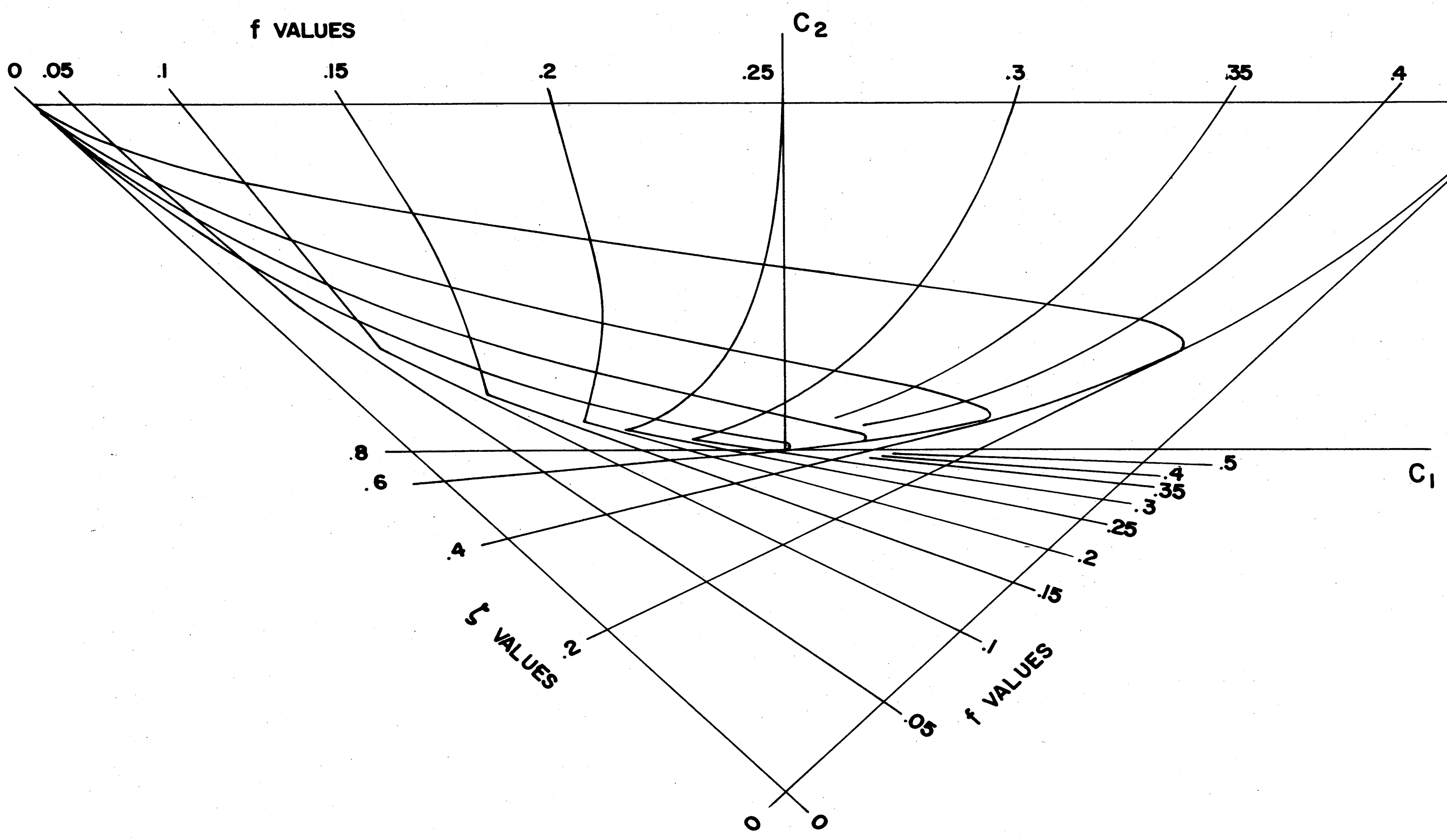

Figure 4.6 Contours of Constant ζ and f in the C_1, C_2 Plane

curves give the damping ratio of the least damped set of the two sets of complex roots. In region D , $2\pi f = \alpha_1$, the real characteristic root and ζ gives the damping ratio of the complex roots which have $f_D = .5$.

Systems with C_1 and C_2 such that ζ is constant exhibit approximately the damping characteristics of a corresponding linear invariant second order system. Systems with C_1 and C_2 such that f is constant exhibit approximately the time of response characterized by linear invariant systems with the constant bandwidth f . Figure 4.6 is therefore very useful in the design of second order time-variant systems for particular response characteristics. For example, suppose that one parameter in the time-variant system is allowed to vary defining a curve in the C_1 , C_2 plane. A plot of this curve on Figure 4.6 would then give the required parameter value.

If a system is third or higher order, it is possible to fix response characteristics by factoring the characteristic equation into first and second degree terms. The S values of the characteristic roots are then easily obtained from equation (4.28) for the first degree terms and from Figure 4.5 for second degree terms. These S values yield the general behavior of $\theta(\tau)$ and hence the response characteristics of the system.

Application to Systems with General Periodic Coefficients

The methods of this chapter have assumed that the considered time-variant system could be described by equation (3.4), where the coefficients are piecewise constant in time. If the number of intervals in which the coefficients are constant is increased, it is possible to closely approximate a system with continuously variable periodic coefficients. For this reason it seems more than likely that the methods to determine stability and response characteristics are extendable to systems with general periodic coefficients.

In this general case the fundamental matrix $Y(1+)$ would be obtained by evaluating at $t = 1+$ the solution to equation (3.23) with the piecewise constant $F(t)$ replaced by the continuous period matrix $\bar{F}(t)$. The only difficulty is that this solution does not usually exist in closed form. Two alternatives are possible. First, the value of $Y(1+)$ could be approximated by considering a system with piecewise constant coefficients; second, the value of $Y(1+)$ could be obtained from an analog computer solution of the equations represented by the matrix equation (3.23). From this point on the application of the methods of this chapter would be straightforward.

V. EXAMPLES

This chapter will consider several examples illustrating the methods of Chapter IV. It will be seen that the determination of particular response by the more involved techniques of earlier chapters is not often required or justified. To minimize the complexity of computation the examples are of the second order, two interval type. This does not indicate a lack of generality since the techniques are basically independent of the number of intervals and of the system order.

A General Second Order System

Consider a general second order system with two elementary time intervals in the fundamental period, which is taken as unity. The equations describing the system are

$$\ddot{v} + a_2\dot{v} + a_1 v = u \quad , \quad k < t < k+a$$

$$\ddot{v} + b_2\dot{v} + b_1 v = u \quad , \quad k+a < t < k+a+b = k+1, \tag{5.1}$$

where it will be assumed that v and $\dot{v}$ are continuous. The equations could represent a mass-spring-damper system with time-variant damping and spring parameters or an RLC circuit with time-variant resistance and capacitance.

Suppose that the stability conditions are required for the system. The first step is to formulate the problem in vector notation. This is done by defining $y_1 = v$, $y_2 = \dot{v}$, and $x_2 = u$. Then

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -a_1 y_1 - a_2 y_2 + x_2 \quad , \quad k < t < k+a$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -b_1 y_1 - b_2 y_2 + x_2, \quad k+a<t<k+1. \tag{5.2}$$

Equations (5.2) are now in the form of equations (2.1) and (2.2) where A and B are given by

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ -b_1 & -b_2 \end{bmatrix}. \tag{5.3}$$

The matrices A^* and B^* equal the identity matrix since y_1 and y_2 are continuous. Determination of the matrix $(sI-A)^{-1}$ and application of the Laplace transform yields

$$e^{At} = \begin{bmatrix} \left(\cosh \bar{a}t + \frac{a_2}{2}\frac{\sinh \bar{a}t}{\bar{a}}\right) & \left(\frac{\sinh \bar{a}t}{\bar{a}}\right) \\ \left(-a_1\frac{\sinh \bar{a}t}{\bar{a}}\right) & \left(\cosh \bar{a}t - \frac{a_2}{2}\frac{\sinh \bar{a}t}{\bar{a}}\right) \end{bmatrix}, \tag{5.4}$$

where

$$\bar{a} = \sqrt{\left(\frac{a_2}{2}\right)^2 - a_1} \quad . \tag{5.5}$$

The matrix e^{Bt} is obtained by replacing the "a" values by the corresponding "b" values. The matrix $Y(1+)$ is given by the product

$e^{Bb}\,e^{Aa}$, and finally, the characteristic equation is determined by computation of the determinant $|zI - Y(1+)|$. After manipulation this yields

$$z^2 + C_1 z + C_2 = 0 \ , \tag{5.6}$$

where

$$C_1 = -e^{-\frac{a_2 a + b_2 b}{2}} \left[\left(\frac{a_2 b_2}{2} - a_1 - b_1 \right) \frac{\sinh \bar{a} a}{\bar{a}} \cdot \frac{\sinh \bar{b} b}{\bar{b}} + 2 \cosh \bar{a} a \cdot \cosh \bar{b} b \right] \tag{5.7}$$

and

$$C_2 = e^{-(a_2 a + b_2 b)} \ . \tag{5.8}$$

The conditions for stability are given by the inequalities (4.25). For this system the conditions may be stated

$$a_2 a + b_2 b > 0$$

$$2 \cosh \frac{a_2 a + b_2 b}{2} > \left| \left(\frac{a_2 b_2}{2} - a_1 - b_1 \right) \frac{\sinh \bar{a} a}{\bar{a}} \cdot \frac{\sinh \bar{b} b}{\bar{b}} + 2 \cosh \bar{a} a \cdot \cosh \bar{b} b \right| . \tag{5.9}$$

A necessary condition for stability is that the average damping over the fundamental period must be greater than zero. However, the more complicated second relation must also be met to assure stability.

If a_2 and b_2 are zero, and a equals b , and the system is unforced, equations (5.1) reduce to the Hill-Meissener equation. Van der Pol and Strutt [Va. 1] determined the conditions for bounded solutions of this equation. It is interesting to note that the result is the same as indicated by the second inequality of (5.9) if $a_2 = b_2 = 0$ and $a = b$.

An Electrical Network with Switching

One of the most important applications of the theory is to electrical networks with periodic switching. Such networks possess interesting properties that do not occur in the usual time-invariant networks.

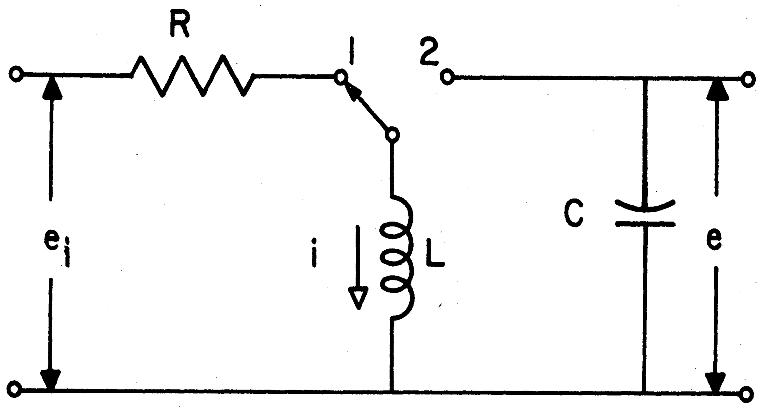

Figure 5.1 An Electrical Switching Network.

Consider the circuit shown in Figure 5.1. The switch is in position one for $kT < t < (k+a)T$ and in position two for $(k+a)T < t < (k+a+b)T = (k+1)T$. It is possible to write circuit equations for each switch position. The dependent variables will be taken as the current through the inductance and the voltage across the capacitance. In position one the equations are

$$L \frac{di}{dt} = -iR + e_i$$

$$C \frac{de}{dt} = 0 \qquad (5.10)$$

In position two they are

$$L \frac{di}{dt} = e$$

$$C \frac{de}{dt} = -i \quad . \tag{5.11}$$

A change of time scale $\tau = \frac{t}{T}$ puts the equations into the desired vector form. Thus

$$\frac{dy}{d\tau} = Ay + x \quad , \quad k < \tau < k+a$$

$$\frac{dy}{d\tau} = By + x \quad , \quad k+a < t < k+a+b = k+1, \tag{5.12}$$

where

$$y_1 = i(\tau)$$

$$y_2 = e(\tau)$$

$$x_1 = \begin{cases} \frac{T}{L} e_i(\tau) & , \; k < t < k+a \\ 0 & , \; k+a < t < k+1 \end{cases} \tag{5.13}$$

$$x_2 = 0$$

and

$$A = \begin{bmatrix} -\frac{RT}{L} & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \frac{T}{L} \\ -\frac{T}{C} & 0 \end{bmatrix} \quad . \tag{5.14}$$

The switching is assumed instantaneous so e and i are continuous and hence $A^* = B^* = I$. Then $Y(1+) = e^{Bb}e^{Aa}$ and the determinant $|zI - Y(1+)|$ gives the characteristic equation $z^2 + C_1 z + C_2$, where

$$C_1 = -\left(1 + e^{-\frac{R}{L}Ta}\right)\cos\frac{T}{\sqrt{LC}}b \tag{5.15}$$

and

$$C_2 = e^{-\frac{R}{L}Ta}. \tag{5.16}$$

Consideration of the stability conditions shows that the system is stable if R, L, and C are greater than zero. This is expected since only passive components are present in the circuit.

Response characteristics are obtained by plotting values of C_1 and C_2 in the C_1, C_2 planes of Figure 4.5 and Figure 4.6. A little thought indicates that the characteristic oscillatory frequency is changed by varying the ratio of a to b. This is easily seen by assuming that $R = 0$. The characteristic equation is then

$$z^2 - \left(2\cos\frac{T}{\sqrt{LC}}b\right)z + 1 = 0 \tag{5.17}$$

Solving for z yields $z = e^{\pm j\frac{T}{\sqrt{LC}}b}$. The s values of the characteristic roots are then $s = \pm j\frac{T}{\sqrt{LC}}b$. The system may be considered "neutrally stable"[1] and possess an undamped oscillatory response

---

1 The system is unstable in the defined sense, but only barely so. Thus systems with roots on the imaginary axis will be defined as "neutrally stable". Actually, if R is very small but not quite zero, the system will be stable in the defined sense and exhibit slowly damped response characteristics.

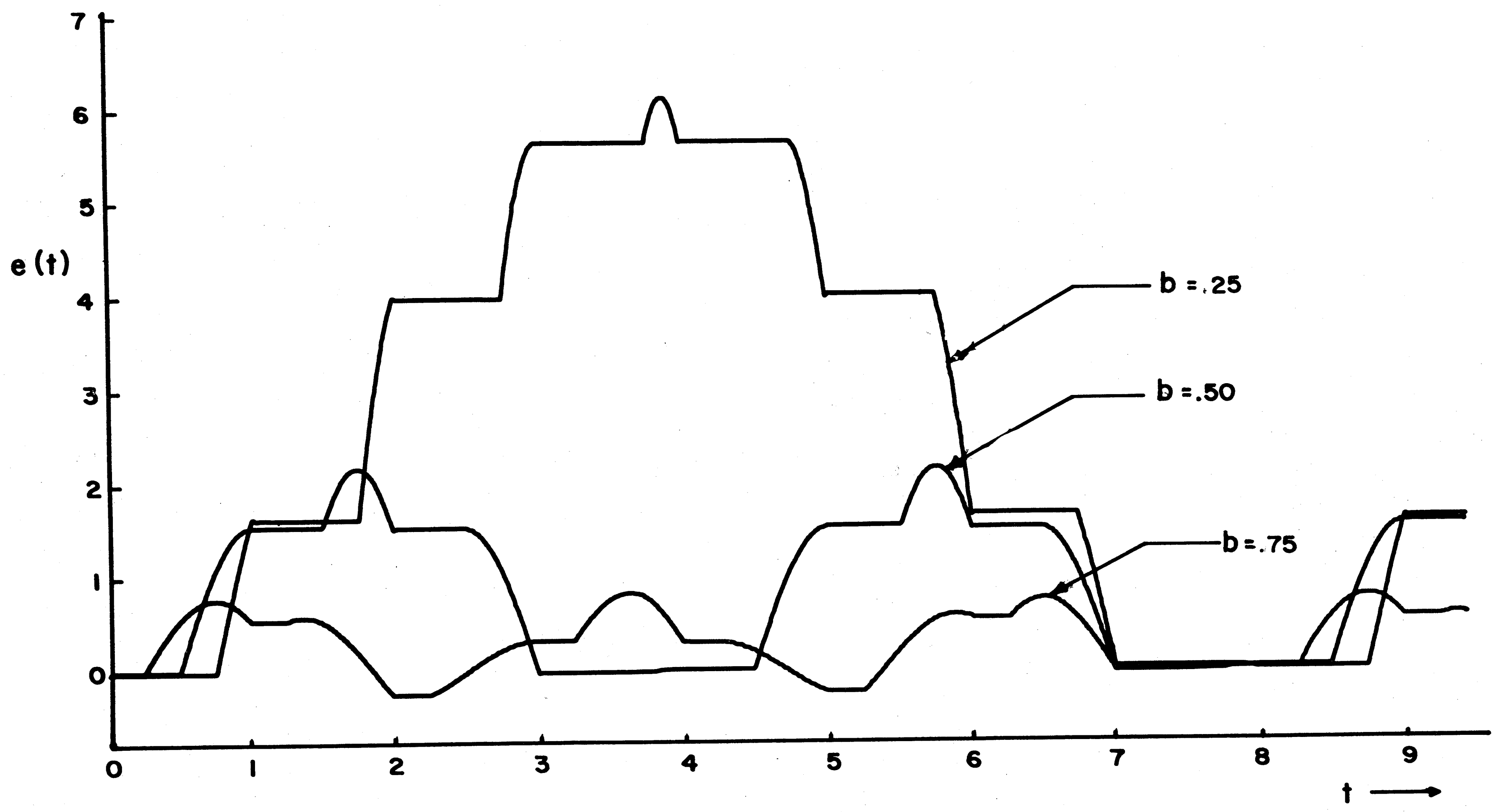

Figure 5.2 Unit Step Responses for the Electrical Network with Switching

character with a frequency $\frac{T}{\sqrt{LC}} b$ rad/sec. It is apparent that this frequency may be changed by varying b or more precisely the ratio of a to b. Figure 5.2 shows differential analyzer solutions of the system equations for a unit step input applied at time $t = 0$ when $T = 1$, $L = .5$, $C = \frac{2}{\pi^2}$, $R = 0$, and $b = \frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. The responses are sustained oscillations with the predicted frequencies of 0.375, 0.25, and 0.125 cps. It is to be noted that the characteristic roots do not predict the waveform or the magnitude of oscillation, but only the frequency of oscillation. The more involved techniques of Chapter II would be required to determine these factors.

A Control System with a Continuous and Clamped Error Signal

A final example will consider a control system with a continuous and clamped error signal. A block diagram of the system is shown in Figure 5.3. The continuous and clamped error signal $\bar{\epsilon}$ is given by the equation

$$\begin{aligned} \bar{\epsilon}(t) &= \epsilon(t) = \theta_i(t) - \theta_o(t), \quad k < t < k+a \\ &= \epsilon(k+a) = \theta_i(k+a) - \theta_o(k+a), \quad k+a < t < k+a+b = k+1. \end{aligned} \tag{5.18}$$

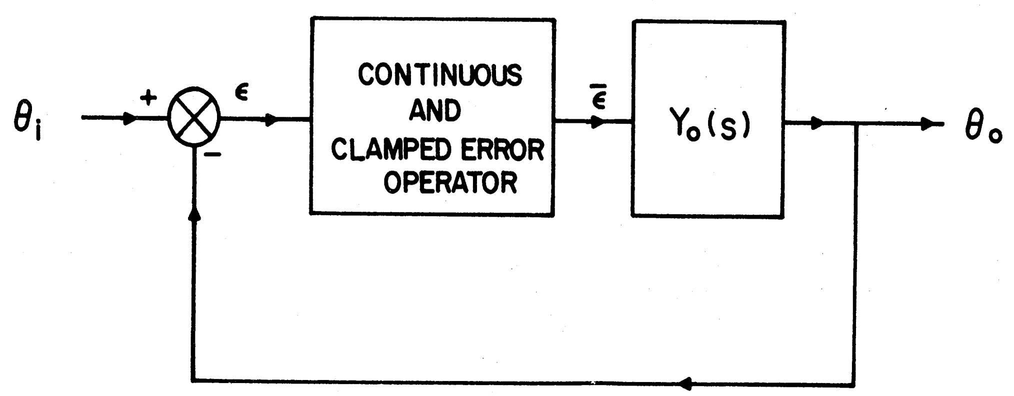

Figure 5.3 Control System Block Diagram

The transfer function $Y_0(s)$ is given by

$$Y_0(s) = \frac{K}{s(\tau s + 1)} \quad . \tag{5.19}$$

In order to hold the values $\theta_0(k+a)$ it is necessary to add an additional coordinate to the system. For $k < t < k+a$, let $y_1 = \theta_0(t)$, $y_2 = \dot{\theta}_0(t)$, $y_3 = 0$, and $x_2 = \frac{K}{\tau}\theta_i(\tau)$. Then

$$\begin{aligned} \frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= -\frac{1}{\tau}y_1 - \frac{K}{\tau}y_2 + x_2 \\ \frac{dy_3}{dt} &= 0 \qquad , \quad k<t<k+a \end{aligned} \tag{5.20}$$

For $k+a<t<k+a+b = k+1$ let $y_1 = \theta_0(t)$, $y_2 = \dot{\theta}_0(t)$, $y_3 = \theta_0(k+a)$, and $x_2 = \frac{K}{\tau}\theta_i(k+a)$. Then

$$\begin{aligned} \frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= -\frac{1}{T_1}y_1 - \frac{K}{\tau}y_3 + x_2 \\ \frac{dy_3}{dt} &= 0 \qquad , \quad k+a<t<k+1 . \end{aligned} \tag{5.21}$$

Since $\bar{\epsilon}$ is piecewise continuous, y_1 and y_2 are continuous. This and the required initial values of y_3 determine the matrices

$$B^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(5.22)

and

$$A^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

(5.23)

System characteristics are fixed by the characteristic equation $|zI - Y(1+)| = 0$, where $Y(1+) = B^* e^{Bb} A^* e^{Aa}$. The result is of the form

$$z\left(z^2 + C_1 z + C_2\right) = 0 . \qquad (5.24)$$

The root $Z = 0$ is introduced by the clamping requirement and is not significant. The values of the remaining two roots are determined by

$$C_1 = e^{-\frac{a}{2\tau}} \left\{ \left[(1 - e^{-\frac{b}{\tau}})(1 - 3K\tau) - K\tau \frac{b}{\tau} \right] \frac{\sinh \sqrt{1-4K\tau}\,\frac{a}{2\tau}}{\sqrt{1-4K\tau}} \right.$$

$$+\left[1+e^{-\frac{b}{\tau}}+K\tau\left(1-\frac{b}{\tau}+e^{-\frac{b}{\tau}}\right)\right]\cosh\sqrt{1-4K\tau}\,\frac{a}{2\tau}\Big\}$$

(5.25)

and

$$C_2 = e^{-\frac{a}{\tau}}\left[-\left\{e^{-\frac{b}{\tau}}+K\tau\left[1-e^{-\frac{b}{\tau}}\left(5+\frac{b}{\tau}\right)\right]+4(K\tau)^2\left[e^{-\frac{b}{\tau}}\left(1+\frac{b}{\tau}\right)-1\right]\right\}\cdot\right.$$

$$\left.\left\{\frac{\sinh\sqrt{1-4K\tau}\,\frac{a}{2\tau}}{\sqrt{1-4K\tau}}\right\}^2+\left\{e^{-\frac{b}{\tau}}+K\tau\left[1-e^{-\frac{b}{\tau}}\left(1+\frac{b}{\tau}\right)\right]\right\}\cosh\sqrt{1-4K\tau}\,\frac{a}{2\tau}\right] .$$

(5.26)

A study of system response is conveniently introduced by considering two limiting forms. First, let $a \to 1$ and $b \to 0$ reducing the system to a conventional control system with a continuous error signal. The values C_1 and C_2 become

$$C_1 = -2\,e^{-\frac{1}{2\tau}}\cosh\sqrt{1-4K\tau}\,\frac{1}{2\tau}$$

$$C_2 = e^{-\frac{1}{\tau}} \quad .$$

(5.27)

Solving for the s values of the characteristic roots yields the well-known result

$$s = -\frac{1}{2\tau} \pm \sqrt{\left(\frac{1}{2\tau}\right)^2 - \frac{K}{\tau}} \quad . \tag{5.28}$$

Second, let $a \to 0$ and $b \to 1$ reducing the system to a closed-loop sampled-data system. The values of C_1 and C_2 then become

$$C_1 = -\left[1+e^{-\frac{1}{\tau}}+K\tau\left(1-\frac{1}{\tau}-e^{-\frac{1}{\tau}}\right)\right]$$

$$C_2 = \left\{ e^{-\frac{1}{T}} + KT\left[1 - e^{-\frac{1}{T}}\left(1 + \frac{1}{T}\right)\right]\right\}. \tag{5.29}$$

This result is readily obtained by application of sampled-data theory to the limiting form of the control system [Ra. 1].

A frequent design problem is to determine K for favorable response when T and a are fixed. This may be done by plotting C_2 vs. C_1 on tracing paper for different K values and overlaying the result on Figure 4.6.

Equations (5.27) and (5.29) show that these contours are straight lines for the continuous system and the sampled-data system. This is not true for the more general case, however. Figure 5.4 shows such contours for variable K when $T = .5$ and $a = 0+, .5$, and 1. The values of K for neutral stability are determined by the intersection of the contours and the stability triangle. In a control system it is desirable that the response be as rapid as possible without being unduly oscillatory. This criterion will be met by requiring that $\zeta \geqslant .4$ in Figure 4.6. Overlay of Figure 5.4 on Figure 4.6 determines values of K and $f_D = \frac{f}{\sqrt{1-\zeta^2}}$ for $a = 0$, $.5$ and 1 when $T = .5$. These values and the values for neutral stability are tabulated in Table II. Figure 5.5 shows differential analyzer solutions for a unit step input and the indicated K values. Percent overshoot and damped frequencies are measured and compared with theoretical values in Table II. It is obvious that the approximate theoretical method yields excellent results for this system.

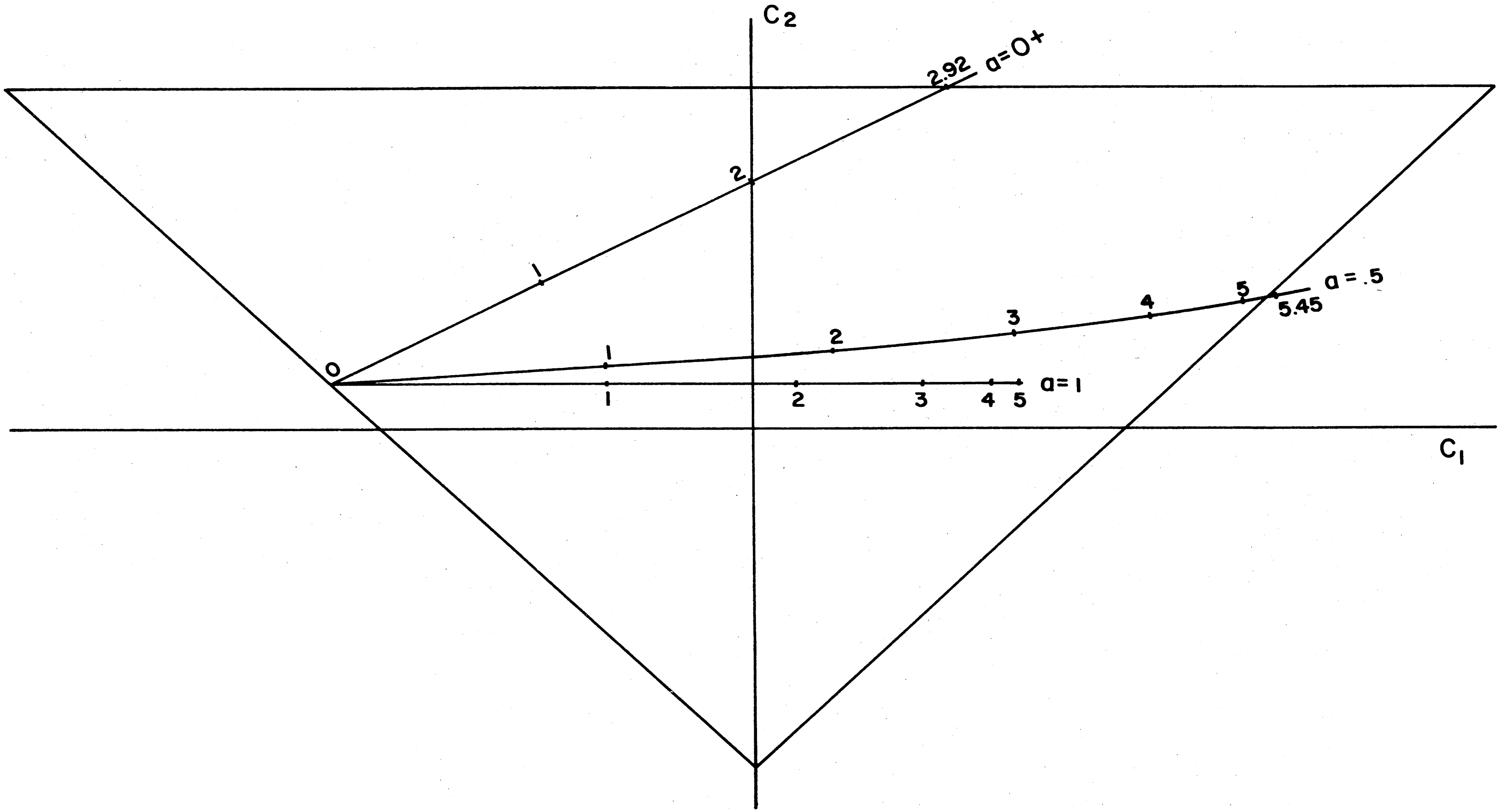

Figure 5.4 Contours for Variable K when $\tau = .5$ and $a = 0+, .5$, and 1

Table II

STEP RESPONSE DATA

a*	γ *	K*	f_D *	f_D **	% overshoot**
0.0	0.0	2.92	-	-	-
0.5	0.0	5.35	.5	-	-
1.0	0.0	∞	∞	-	-
0.0	.4	.89	.168	.170	23.8
0.5	.4	1.84	.273	.269	25.9
1.0	.4	3.125	.367	.367	25.5***

* Values obtained from overlay of Figure 5.4 on Figure 4.6.

** Values obtained from Figure 5.5.

*** In this case percent overshoot = 25.4% by theory.

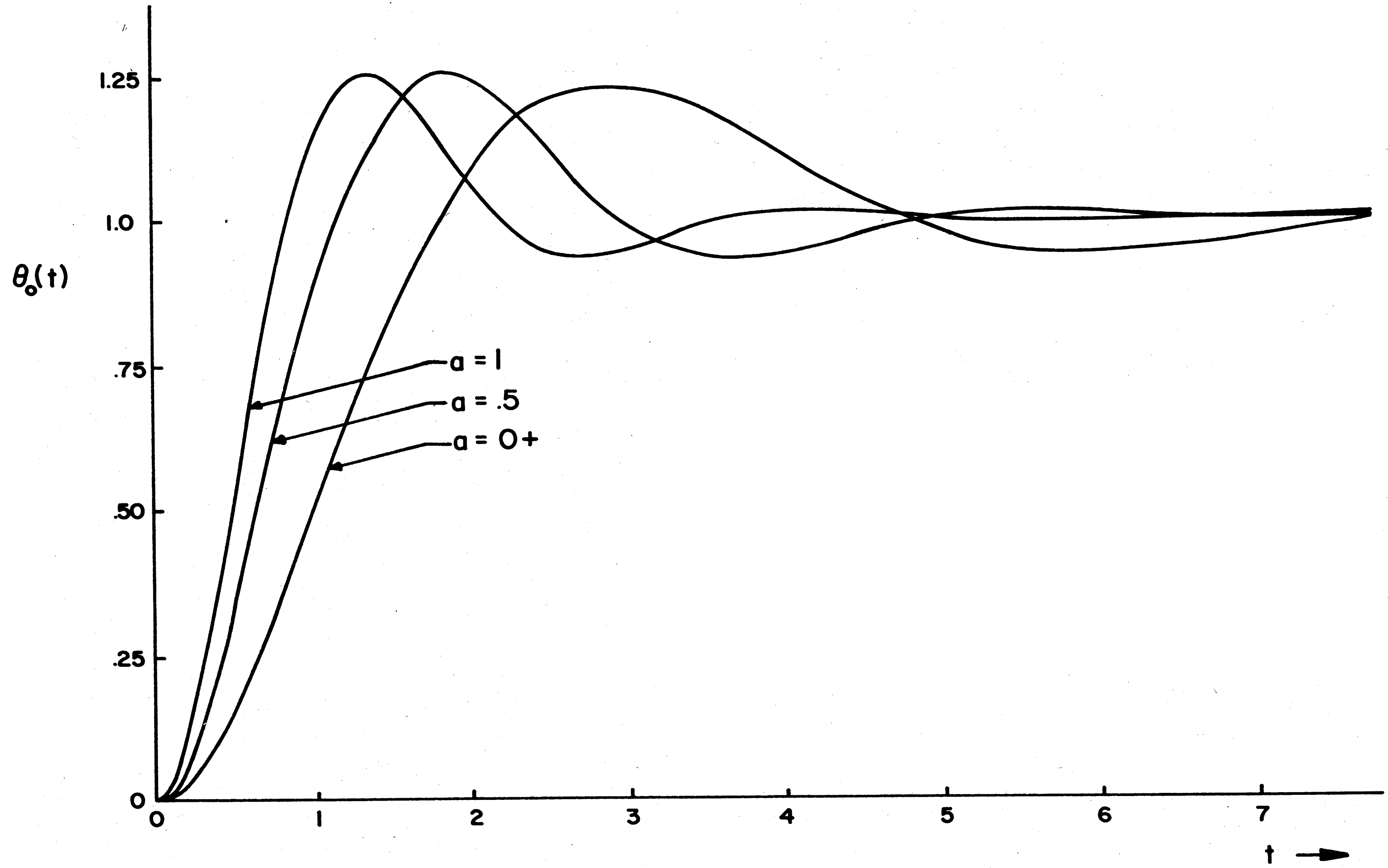

Figure 5.5 Unit Step Responses for the Control System with Continuous and Clamped Error Signal

VI. CONCLUSION

This dissertation has investigated the response of linear dynamic systems with periodic, piecewise constant parameters. The methods of analysis have been based on the time-variant transfer function and on the time-variant impulse response. The resulting theory has the significant advantage of being directly analogous to the familiar operational methods employed in the analysis of linear time-invariant systems.

The results obtained provide a variety of solutions to the previously unsolved general response problem. The piecewise solution with the iteration formulas yields a step by step solution of the problem. Fourier and Laplace transform procedures involving the time-variant transfer function offer another solution. Usually, simplifying approximations are required to obtain the inverse transforms. The time-variant transfer function is particularly useful in that it gives directly an exact solution for the important exponential input. The time-variant impulse response permits a direct solution to the homogeneous problem and through the superposition integral gives the response for an arbitrary input. These varied methods furnish a systemized and straightforward approach to the response problem. Their main disadvantage is the excessive length of the required computations. This is not so much a fault of the methods as it is an indication of the problem complexity.

Fortunately, much can be learned about system response without determining the response for particular inputs. The methods are based on the characteristic roots given by the determinant equation

$|zI - Y(1+)| = 0$, where $z = e^s$. The stability of the system response is assured if the s values of the characteristic roots lie in the left half of the s plane. Stability can be obtained directly without determining the s values by examination of the characteristic equation in z . The response characteristics of the system are determined to a great extent by the location of the roots in the s plane. A graphical method is presented for determining these locations from the characteristic equation in z . The technique is valuable in the synthesis of second order systems for desired response characteristics.

It is hoped that the application of the theory will prove fruitful. A study of electrical networks with switching offers particularly interesting possibilities. The electrical network considered in Chapter V is an illustrative example. The steady state response for b = .25 is obviously greater than unity. Thus the network exhibits a voltage gain. As was observed, it is also possible to change the characteristic frequency of the system by changing the ratio of a to b . These qualities have obvious practical application.

APPENDIX I. THE LINEAR VECTOR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

The purpose of this appendix is to investigate the vector equation $\frac{dy}{dt} = Ay + x$ and thereby obtain the solution and certain properties of the solution pertinent to the text of the dissertation. With this in mind the treatment shall be as concise as possible consistent with an understandable development.[1]

<u>Solution of the Vector Equation $\frac{dy}{dt} = Ay + x$</u>

The following system of linear differential equations with constant coefficients is specified:

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} y_j + x_i \quad , \quad i = 1, 2, \ldots, n . \tag{A1.1}$$

This system is described more compactly by the vector equation

$$\frac{dy}{dt} = Ay + x , \tag{A1.2}$$

where x and y are nth order column vectors and A is an n by n matrix with components a_{ij}. It will be assumed that the x_i are sectionally continuous in time. The problem is completely specified by imposing the initial conditions

$$y_i(0+) \equiv y_i^+ \quad , \qquad i = 1, 2, \ldots, n \tag{A1.3}$$

on the equations (A1.1) or the equivalent vector condition

$$y(0+) = y^+ \tag{A1.4}$$

1 No attempt will be made to give completely rigorous proof of some of the results. The reader is referred to the book by Belman [Bel. 1, pp. 1-31] for a more complete presentation.

on equation (A1.2). Under the conditions stated there exists a unique set of continuous solutions of the system (A1.1) which assume the values y_i^+ at $t = 0+$ or equivalently there exists a unique continuous vector solution of equation (A1.2) which assumes the value y^+ at $t = 0+$ [In. 1, pp. 71-72].

These solutions are conveniently obtained by applying the Laplace transform method to equations (A1.1) and (A1.2). Denoting transformed quantities by bars it is possible to write

$$s\,\bar{y}_i(s) - y_i^+ = \sum_{j=1}^{n} a_{ij}\,\bar{y}_j(s) + \bar{x}_i(s)\,, \quad i = 1, 2, \ldots, n \tag{A1.5}$$

or in vector notation

$$s\,\bar{y}(s) - y^+ = A\,\bar{y}(s) + \bar{x}(s). \tag{A1.6}$$

The set of linear algebraic equations (A1.5) may be solved by conventional techniques; however, manipulation of the equivalent vector equation (A1.6) is simpler and yields the same result. Solving equation (A1.6) for $\bar{y}(s)$ gives

$$\bar{y}(s) = (sI - A)^{-1} y^+ + (sI - A)^{-1}\bar{x}(s). \tag{A1.7}$$

Obtaining the inverse transform gives the desired solution

$$y(t) = W(t)\,y^+ + \int_0^t W(\lambda)\,x(t-\lambda)\,d\lambda\,, \tag{A1.8}$$

where the matrix function $W(t)$ is determined by

$$W(t) = L^{-1}\left[(sI - A)^{-1}\right]. \tag{A1.9}$$

Equations (A1.8) and (A1.9) allow a straightforward approach in obtaining the solution although the effort involved increases rapidly as n becomes large.

<u>Solution of the Matrix Equation $\frac{dY}{dt} = AY$</u>

Certain properties of $W(t)$ are not apparent from equation (A1.9) but may be uncovered by investigating the homogeneous matrix equation

$$\frac{dY}{dt} = AY \tag{A1.10}$$

with the initial condition

$$Y(0+) = I \; , \tag{A1.11}$$

where $Y(t)$ is an n by n matrix with components $y_{ij}(t)$. Equation (A1.10) is a matrix representation of the system of linear differential equations with constant coefficients

$$\frac{dy_{ij}}{dt} = \sum_{k=1}^{n} a_{ik} y_{kj} \quad , \quad i,j = 1,2,\ldots,n \tag{A1.12}$$

with the initial conditions

$$\begin{aligned} y_{ij} &= 1 \quad , \quad i = j \\ &= 0 \quad , \quad i \neq j \, . \end{aligned} \tag{A1.13}$$

Under the conditions stated there exists a unique continuous matrix solution of the equation (A1.10) which assumes the value I at $t = 0+$ [In.1, pp. 71-72].

Laplace transforming the matrix equation (A1.10) results in

$$s\overline{Y}(s) - I = A\overline{Y}(s) \; , \tag{A1.14}$$

which solved for $\overline{Y}(s)$ gives

$$\overline{Y}(s) = (sI - A)^{-1}. \qquad \text{(A1.15)}$$

Applying the inverse transform results in

$$Y(t) = L^{-1}\left[(sI-A)^{-1}\right]. \qquad \text{(A1.16)}$$

It is seen that $W(t)$ and $Y(t)$ are identical; therefore, properties of $W(t)$ may be obtained by investigating the solution of equation (A1.10) with the initial condition (A1.11).

The matrix $(sI-A)^{-1}$ may be written $\frac{1}{s}\left(I - \frac{1}{s}A\right)^{-1}$, and assuming convergence, it is possible to write formally

$$\left(I - \frac{1}{s}A\right)^{-1} = I + \frac{1}{s}A + \frac{1}{s^2}A^2 + \cdots \; .[1] \qquad \text{(A1.17)}$$

Then $Y(t)$ can be written

$$\begin{aligned} Y(t) &= L^{-1}\left[\frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots\right] \\ &= I + At + A^2\frac{t^2}{2!} + \cdots \; . \end{aligned} \qquad \text{(A1.18)}$$

This matrix power series in t is analogous to the scalar power series for the exponential function; hence the definition

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \qquad \text{(A1.19)}$$

where A^0 is understood to be the identity matrix. That the matrix function e^{At} is indeed the solution of the equation (A1.10) and the initial condition (A1.11) can be verified by differentiating e^{At} to give

[1] Power series of matrices are discussed by MacDuffee [Ma. 1, pp. 97-99].

$$\frac{d}{dt} e^{At} = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A e^{At} ,$$

(A1.20)

substituting in equation (A1.10), and noting that $e^{At}\big|_{t=0} = I$.
The operations indicated are permissible because of the uniform convergence and continuity of the series (A1.19) for finite A and t .[1]
Henceforth $W(t)$ will be written e^{At} where it is understood that is given in closed form by equation (A1.9). Thus the solution of the vector equation may be written

$$y(t) = e^{At} y^{+} + \int_0^t e^{A\lambda} x(t-\lambda)\, d\lambda . \tag{A1.21}$$

Some Properties of the Matrix Function e^{At}

The importance of the matrix series e^{At} lies in its ease of manipulation independent of the system order. It will be seen that, for the most part, these manipulative properties are directly analogous with those for the scalar exponential function.

The Convergence of the Matrix Series e^{At}

To assure the convergence of the matrix series e^{At} it must be shown that the components of e^{At} defined by the n^2 series occurring on the right side of equation (A1.19) converge. Consider as a measure of the magnitude of A the summation

$$\|A\| = \sum_{i,j=1}^{n} |a_{ij}| \tag{A1.22}$$

It is not difficult to prove that $\|A^k\| \leq \|A\|^k$. Then each of the n^2 series is majorized by the series $\sum_{k=0}^{\infty} \|A\|^k \frac{t^k}{k!}$, and they are therefore uniformly convergent and continuous for finite A and t .

[1] See the following section.

Differentiation and Integration of e^{At}

Since the matrix series is uniformly convergent and continuous for finite A and t, it is possible to differentiate e^{At} term by term obtaining equation (A1.20). Similarly it is possible to integrate e^{At} from 0 to t obtaining

$$\int_0^t e^{At}dt = \sum_{k=0}^{\infty} A^k \frac{t^{k+1}}{(k+1)!} = A^{-1}(e^{At} - I). \tag{A1.23}$$

The Inverse and Nonsingularity of e^{At}

Suppose the matrix equation

$$\frac{dZ}{dt} = -ZA \tag{A1.24}$$

with the initial condition

$$Z(0+) = I \tag{A1.25}$$

is given. By substitution it is seen that the solution is

$$Z(t) = e^{-At} = \sum_{k=0}^{\infty} \frac{(-At)^k}{k!}, \tag{A1.26}$$

where $Z(t)$ exists for finite A and t. Premultiplying equation (A1.10) by Z and postmultiplying equation (A1.24) by Y and adding gives

$$Z\frac{dY}{dt} + \frac{dZ}{dt}Y = 0. \tag{A1.27}$$

Integrating and using the conditions (A1.11) and (A1.25) results in

$$ZY = I. \tag{A1.28}$$

It follows that Z and Y are inverses of one another; that is,

$$\left(e^{At}\right)^{-1} = e^{-At}. \tag{A1.29}$$

Since inverses exist, e^{At} and e^{-At} are nonsingular for finite A and t.

Product Functions

Using series representation it can be shown that $e^{At} \cdot e^{A\tau} = e^{A(t+\tau)}$; viz.,

$$\begin{aligned} e^{At} \cdot e^{A\tau} &= \left(\sum_{k=0}^{\infty} A^k \frac{t^k}{k!}\right) \cdot \left(\sum_{m=0}^{\infty} A^m \frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty} A^m \sum_{k=0}^{m} \frac{t^{m-k}\tau^m}{(m-k)!\,k!} \\ &= \sum_{m=0}^{\infty} A^m \frac{(t+\tau)^m}{m!} = e^{A(t+\tau)}. \end{aligned} \tag{A1.30}$$

The operations indicated are permissible because of the convergence of the series for e^{At} and $e^{A\tau}$. By manipulation of the series for e^{At} and $e^{B\tau}$ it is possible to show that $e^{At} \cdot e^{B\tau} = e^{At+B\tau}$ if and only if $AB = BA$. As an example consider $e^{At} \cdot e^{st}$ where e^{st} is a scalar. The expression may be written $e^{At}(Ie^{st})$, but $Ie^{st} = e^{Ist}$ by the definition (A1.19). Thus

$$e^{At} \cdot e^{st} = e^{At} \cdot e^{Ist}. \tag{A1.31}$$

Since $IA = AI$ it is possible to write

$$e^{At} \cdot e^{st} = e^{(A+sI)t}. \tag{A1.32}$$

The Laplace Transform of e^{At}

The Laplace transform of e^{At} is given by

$$L\left[e^{At}\right] = \int_0^{\infty} e^{At} \cdot e^{-st}\, dt, \tag{A1.33}$$

which may be written

$$L[e^{At}] = \int_0^\infty e^{-(sI-A)t}\,dt$$
$$= \lim_{t\to\infty} (sI-A)^{-1}\left[I - e^{-(sI-A)t}\right]. \qquad \text{(A1.34)}$$

For Re$[s]$ large enough the second term in the bracket vanishes as the limit is taken[1] giving

$$L[e^{At}] = (sI-A)^{-1}. \qquad \text{(A1.35)}$$

Suppose the function

$$\begin{aligned} Y_A(t,a) &= e^{At} \quad , \quad 0 < t \leq a \\ &= 0 \quad , \quad t \leq 0 \,,\; t > a \end{aligned} \qquad \text{(A1.36)}$$

is given. The Laplace transform of $Y_A(t,a)$ is then

$$\bar{Y}_A(s,a) = (sI-A)^{-1}\left[I - e^{-(sI-A)a}\right]. \qquad \text{(A1.37)}$$

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1 If λ_i are the roots of the equation $|\lambda I - A| = 0$, then Re$[s]$ must be greater than Re$[\lambda_i]$ for all i.

APPENDIX II. THE TIME-VARIANT IMPULSE RESPONSE AND THE TIME-VARIANT TRANSFER FUNCTION

The first problem in investigating a physical system is to represent the system mathematically. The systems considered here will be described by the linear time-variant vector differential equation

$$\frac{dy}{dt} = F(t)y + x , \tag{A2.1}$$

where x and y are nth order column vectors and $F(t)$ is an n by n matrix time function. Such an equation conveniently describes a set of simultaneous first order differential equations with time-variant coefficients. The second and main problem is to find some functional characterization of the system response. For linear time-variant systems one such functional characterization involves the time-variant impulse response or weighing function; another involves the time-variant transfer function. It is the purpose of this appendix to study the properties and relationships of these two functions as they apply to the solution of equation (A2.1).[1]

<u>The Time-Variant Impulse Response</u>

In the work to follow, unless particularly noted, it will be assumed that the system described by equation (A2.1) is unexcited or in a relaxed state; that is, there is no output if there is no input. If the input is

$$x(t) = \delta(t - \lambda)\, c , \tag{A2.2}$$

[1] The reader is referred to papers by Zadeh and Miller for a general treatment of the time-variant impulse response and transfer function [Za. 1, 2, 8, Mi. 1].

where $\delta(t-\lambda)$ is the unit impulse function at time λ and $\underline{c}$ is an arbitrary constant vector, the corresponding response may be written

$$y(t) = W(t, t-\lambda)c, \tag{A2.3}$$

where $W(t,t-\lambda)$ is an n by n matrix function of t and λ. If the system is time-invariant ($F(t)$ is then a constant matrix), equation (A2.3) reduces to the familiar equation

$$y(t) = W(t-\lambda)c. \tag{A2.4}$$

The function $W(t,\tau)$ defined by equation (A2.3) is the time-variant impulse response, where τ is the so-called age variable. That is, $W(t,\tau)$ represents the system response at time t to an impulse vector applied τ units previously. As seen in equation (A2.4) the impulse response for invariant systems reduces to the familiar function of a single variable, the age variable τ. It is also interesting to note that $W(t,\tau)$ is periodic in t if $F(t)$ is periodic in t.

The principle of superposition permits the determination of $y(t)$ for any $x(t)$. Thus

$$y(t) = \int_{-\infty}^{t} W(t, t-\lambda)\, x(\lambda)\, d\lambda \tag{A2.5}$$

or by change of variable

$$y(t) = \int_{0}^{\infty} W(t,\tau)\, x(t-\tau)\, d\tau. \tag{A2.6}$$

Alternately, equation (A2.3) could be written

$$y(t) = \underline{W}_1(t, \lambda)\, c. \tag{A2.7}$$

Thus $W_1(t,\lambda)$ represents system response at time t for an impulse occurring at time λ. It is seen that $W_1(t,\lambda) = 0$ for $t < \lambda$ and that $W(t,\tau) = W_1(t, t-\tau)$. $W(t,\tau)$ and $W_1(t,\lambda)$ are essentially equivalent and both appear in the literature of time-variant systems. $W(t,\tau)$ is similar to the impulse response for invariant systems and maintains similar forms in relations yet to be derived.

$W_1(t,\lambda)$ is more closely associated with classical mathematical techniques. If $x(t) = 0$ for $t < 0$, equation (A2.5) may be written

$$y(t) = \int_0^t W_1(t,\lambda)x(\lambda)\,d\lambda . \tag{A2.8}$$

From the purely mathematical point of view [Bel. 1, p. 12], the solution of equation (A2.1) may be written

$$y(t) = \int_0^t Y^{-1}(t)\,Y(\lambda)\,x(\lambda)\,d\lambda \tag{A2.9}$$

if the system is initially at rest, if $x(t) = 0$ for $t < 0$, and if $Y(t)$ is the solution of the matrix equation

$$\frac{dY}{dt} = F(t)Y \tag{A2.10}$$

and the initial condition $Y(0+) = I$. $Y^{-1}(t)\,Y(\lambda)$ is essentially the one-sided Green's function [Mi. 1] for the vector equation (A1.1). Equations (A2.8) and (A2.9) are similar in form except that $Y^{-1}(t)\,Y(\lambda)$ is not necessarily zero for $t < \lambda$.[1] Thus

$$\begin{aligned} W_1(t,\lambda) &= Y^{-1}(t)\,Y(\lambda) \quad &, \ t > \lambda \\ &= 0 &, \ t < \lambda . \end{aligned} \tag{A2.11}$$

[1] Note that $Y^{-1}(t)\,Y(\lambda)$ is not actually required for $t < \lambda$.

The Time-Variant Transfer Function

Assuming that the Fourier transform of $x(t)$ exists and that the order of integration may be interchanged, the superposition integral (A2.6) may be written

$$y(t) = \int_0^\infty W(t,\tau) \frac{1}{2\pi} \int_{-\infty}^\infty \bar{x}(j\omega)\, e^{j\omega(t-\tau)}\, d\omega\, d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_0^\infty W(t,\tau)\, e^{-j\omega\tau} d\tau \right] \bar{x}(j\omega) e^{j\omega t} d\omega, \qquad \text{(A2.12)}$$

where $\bar{x}(j\omega)$ indicates the transform of $x(t)$. Suppose that the integral in the bracket exists. It is then possible to define the time-variant transfer function

$$H(j\omega,t) = \int_0^\infty W(t,\tau)\, e^{-j\omega\tau} d\tau. \qquad \text{(A2.13)}$$

The matrix $H(j\omega,t)$ is the Fourier transform of the time-variant impulse response matrix with respect to the age variable τ. It is seen that $H(j\omega,t)$ and $W(t,\tau)$ form a transform pair where t is considered a parameter. Thus

$$W(t,\tau) = \frac{1}{2\pi} \int_{-\infty}^\infty H(j\omega,t)\, e^{j\omega\tau} d\omega. \qquad \text{(A2.14)}$$

Substituting the definition (A2.13) into equation (A2.12) gives

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^\infty H(j\omega,t)\, \bar{x}(j\omega)\, e^{j\omega t} d\omega. \qquad \text{(A2.15)}$$

Consider next the vector input $x(t) = e^{j\omega t} c$ which has existed for all t. By the superposition integral

$$y(t) = \int_0^\infty W(t,\tau)\, e^{j\omega(t-\tau)}\, d\tau \cdot c$$

$$= \left[\int_0^\infty W(t,\tau)\, e^{-j\omega\tau}\, d\tau \right] e^{j\omega t} c \tag{A2.16}$$

$$= H(j\omega, t)\, e^{j\omega t} c \; .$$

It is seen that $H(j\omega,t)\, e^{j\omega t} c$ is a particular integral of equation (A2.1) for the input $x(t) = e^{j\omega t} c$.

In some problems it is possible to determine a particular integral for the input $x(t) = e^{j\omega t} c$. Suppose that this particular integral may be written

$$y(t) = H^*(j\omega,t)\, e^{j\omega t} c \; . \tag{A2.17}$$

Under what conditions will this particular integral agree with the particular integral given by equation (A2.16) or equivalently, when will $H^*(j\omega,t) = H(j\omega,t)$? This question will not be answered in general but will be answered only for the case when $F(t)$ in equation (A2.1) is periodic with period T . Then a necessary and sufficient condition that $H^*(j\omega,t) = H(j\omega,t)$ is that $H^*(j\omega,t)$ be periodic in t with period T for all $j\omega$. The proof follows. The two particular integrals can differ only by a solution of the homogeneous equation corresponding to equation (A2.1). The most general solution of this homogeneous equation is $Y(t)\, d$ where $Y(t)$ is the solution of equation (A2.10) and the condition $Y(0+) = I$ and d is any vector not a function of t . It follows that

$$H^*(j\omega,t)\, e^{j\omega t} c = H(j\omega,t) e^{j\omega t} c + Y(t)\, G(j\omega)\, c \; , \tag{A2.18}$$

where $G(j\omega)$ is some matrix function of $j\omega$. Then

$$H^*(j\omega,t) = H(j\omega,t) + Y(t)\,G(j\omega)\,e^{-j\omega t}. \tag{A2.19}$$

Since $F(t)$ is periodic in t , $W(t,\tau)$ is periodic in t . Then by equation (A2.13) it is seen that $H(j\omega,t)$ is periodic in t with period T for all $j\omega$. The term $Y(t)\,G(j\omega)\,e^{j\omega t}$ cannot be periodic in t with period T for all $j\omega$ and hence must be zero if $H^*(j\omega,t)$ is periodic in t with period T for all $j\omega$. This requires that the matrix $G(j\omega)$ be zero. The proof is completed.

Since $H(j\omega,t)\ \bar{X}(j\omega)$ is a function of time it cannot represent the frequency spectrum of $y(t)$. If $H(j\omega,t)$ is periodic in t with a fundamental frequency ω_0 , it is possible to obtain the spectrum in the following manner. Being periodic, $H(j\omega,t)$ may be written as the complex Fourier series

$$H(j\omega,t) = \sum_{n=-\infty}^{\infty} C_n(j\omega)\, e^{jn\omega_0 t} . \tag{A2.20}$$

Substituting equation (A2.20) in equation (A2.15) and manipulating gives

$$y(t) = \sum_{n=-\infty}^{\infty} F_n(t)\, e^{jn\omega_0 t} , \tag{A2.21}$$

where

$$F_n(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_n(j\omega)\, \bar{X}(j\omega)\, e^{j\omega t} d\omega. \tag{A2.22}$$

Finally, the spectrum of $y(t)$ is given by the Fourier transform of equation (A2.21); viz.,

$$\bar{y}(j\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F_n(t)\, e^{-j(\omega - n\omega_0)t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} C_n(j\omega - jn\omega_0)\, \bar{x}(j\omega - jn\omega_0). \qquad \text{(A2.23)}$$

By assuming $x(t) = 0$ for $t < 0$ and replacing $j\omega$ by s, it is possible to obtain Laplace transform equivalents of the above expressions and assure convergence for a wider class of functions. For example,

$$y(t) = L^{-1}\left[H(s,t)\, \bar{x}(s)\right], \qquad \text{(A2.24)}$$

where t is again considered a parameter. $H(s,t)$ and $W(t,\tau)$ form a Laplace transform pair with respect to the variables τ and s. Similarly, $H(s,t)e^{st}c$ is a particular integral for $x(t) = e^{st}c$ where s is any complex number.

Initial Condition Problems

In the above work it has been assumed that the system was initially unexcited. In other words, non-zero initial conditions in equation (A2.1) were not permitted. Suppose that $x(t)$ is sectionally continuous and equal to zero for $t < \lambda$. It is then apparent that $y(\lambda+) = 0$ if the system is initially unexcited. A non-zero initial condition at $t = \lambda+$ will then add a solution of the homogeneous equation to that given by the superposition integral. Such a solution is the impulse response for $\tau > 0$. Thus

$$y(t) = W(t, t-\lambda)\, y(\lambda+) + \int_{\lambda}^{t} W(t, t-\tau)\, x(\tau)\, d\tau \qquad \text{(A2.25)}$$

is the solution to the problem for the initial condition $y(\lambda+)$ since $W(\lambda+, 0+) = I$. An equivalent expression for $y(t)$ is

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega,t)\left[\bar{x}(j\omega) + e^{-j\omega\lambda}y(\lambda+)\right]e^{j\omega t}\,d\omega. \qquad \text{(A2.26)}$$

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