

THE UNIVERSITY OF MICHIGAN
INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

AN APPROXIMATE METHOD FOR ANALYTICALLY EVALUATING
THE RESPONSE OF TIME-VARIABLE LINEAR SYSTEMS

Elmer G. Gilbert

July, 1960

IP-446

Engw
UMR
15&1

TABLE OF CONTENTS

	<u>Page</u>
LIST OF FIGURES.....	iii
ABSTRACT.....	iv
INTRODUCTION.....	1
RESPONSE DETERMINATION FOR T.V.L.S.....	1
THE APPROXIMATE REPRESENTATION.....	3
METHODS FOR EVALUATION OF THE APPROXIMATE REPRESENTATION....	5
ORTHONORMAL APPROXIMATING FUNCTIONS.....	10
EVALUATION OF THE $a_i(t)$	13
THE APPROXIMATE REPRESENTATION IN TERMS OF EXPONENTIAL FUNCTIONS.....	14
RESPONSE TO STOCHASTIC INPUTS.....	15
Examples.....	19
CONCLUSION.....	24
ACKNOWLEDGEMENT.....	24
REFERENCES.....	35

LIST OF FIGURES

<u>Figure</u>		<u>Page</u>
1	Physical Interpretation of Approximation to T.V.L.S.	25
2	Feedback Control System with Finite Width Sampling..	26
3	Periodic Functions of Normal Solution.....	27
4	Step Responses for Exact and Approximate Systems....	28
5	Form of $h(t,\tau)$ for $t \gg 1$	29
6	$h(t,\tau)$ and $h^*(t,\tau)$ for $t = -2, -1, 0, 1, 2$	30-34

ABSTRACT

For all but the most elementary time-variable linear systems solutions of the response problem must be particular solutions obtained by machine computation. Thus the insight into system response characteristics which is gained from analytic procedures is lost. The paper proposes an approximate representation of the time-variable weighting function $h(t, \tau)$ (system response at time t to a unit impulse at time $t - \tau$) which permits a straightforward analytic evaluation of response for both deterministic and stochastic inputs. Although the evaluation of the approximate representation may require machine computation, once the approximate representation is obtained, no further machine calculations are necessary.

The approximation has the form $h^*(t, \tau) = \sum_{i=1}^N a_i(t) \phi_i(\tau)$ where the $\phi_i(\tau)$ are arbitrary approximating functions. Methods are given for evaluating the $a_i(t)$. Several families of orthonormal approximating functions are presented which simplify both the evaluation and utilization of the approximation. For example, when $\phi_i(\tau)$ have rational Laplace transforms the $a_i(t)$ are computed directly from the solutions of certain time-variable linear differential equations. For stationary stochastic inputs simple series expressions for the output correlation function are derived and it is shown that propitious choice of the $\phi_i(\tau)$ greatly reduce the number of integrations which must be made in evaluating the series. Two example problems illustrate the feasibility of the proposed procedures.

INTRODUCTION

Methods for response determination of time-variable linear systems (T.V.L.S. for short) are well developed and have been described by many authors.^(1,2) However, except in rather special situations, application of these methods has been restricted, mainly because of the very great computational difficulties. Consequently, for most practical systems, solutions of the response problem must be particular solutions obtained by machine computation, and the insight into system response characteristics which is gained from analytic procedures is lost.

This paper investigates the nature of the computational difficulties and proposes an approximate representation for T.V.L.S. which admits analytic determination of response for both deterministic and stochastic inputs. Although the evaluation of the approximate representation may require machine computation, once the representation is obtained, no further computer calculations are required for a large class of inputs.

To begin, some general matters concerning the response of T.V.L.S. are reviewed and the approximate representation is presented. Then, methods for evaluation and utilization of the representation are considered. Finally two illustrative examples are described.

RESPONSE DETERMINATION FOR T.V.L.S.

The T.V.L.S. considered are represented by

$$L_{\alpha}(p, t) x(t) = L_{\beta}(p, t) f(t) \quad (1)$$

where $x(t)$ is the response, $f(t)$ is the input, $L_{\alpha}(p,t) = \alpha_n(t)p^n + \dots + \alpha_0(t)$, $L_{\beta}(p,t) = \beta_m(t)p^m + \dots + \beta_0(t)$, and $p = d/dt$. Straightforward generalizations, such as making x and f vectors, are possible but are avoided for conciseness.

If it is assumed that the T.V.L.S. is initially at rest so that the response is due entirely to $f(t)$, $x(t)$ can be given by the integral

$$x(t) = \int_{-\infty}^{\infty} h(t,\tau) f(t-\tau) d\tau \quad (2)$$

where $h(t,\tau)$ is the time-variable weighting function or system response at time t to a unit impulse at time $t-\tau$. Initial conditions, if present, can be introduced by a suitable equivalent input.⁽³⁾ Familiar restrictions on system characteristics have a one-to-one correspondence to simple conditions on $h(t,\tau)$, for example: non-prediction and $h=0$, $\tau < 0$; time invariance and $h(t,\tau) = h(t-\tau)$; periodicity with period T and $h(t,\tau) = h(t+T,\tau)$; stability and $\int_{-\infty}^{\infty} |h(t,\tau)| d\tau < M < \infty$, $-\infty < t < \infty$.⁽⁴⁾ The weighting function can be determined from the Green's function⁽⁵⁾ and for $m < n$ takes the form

$$\begin{aligned} h(t,\tau) &= 0, \quad \tau < 0 \\ &= \sum_{i=1}^n x_i(t) C_i(t-\tau), \quad \tau \geq 0 \end{aligned} \quad (3)$$

$x_1(t), \dots, x_n(t)$ are a set of linearly independent solutions of the homogeneous equation $L_{\alpha}x = 0$ and C_1, \dots, C_n are obtained from x_1, \dots, x_n in a complicated way. Even for the relatively few equations of low order where the $x_i(t)$ are known and tabulated, substitution of (3) in (2) generally yields impossibly difficult integrations.

Various transform methods have been proposed for the solution of (1),^(6,7) the most familiar being that of Zadeh⁽⁸⁾ which evaluates response as a superposition of exponential responses. Zadeh's approach is to define the time-variable transfer function

$$H(s, t) = \left. \frac{\chi(t)}{f(t)} \right|_{f=e^{st}} = \int_{-\infty}^{\infty} h(t, \tau) e^{-s\tau} d\tau \quad (4)$$

and show that

$$\chi(t) = \frac{1}{2\pi j} \int_C H(s, t) F(s) e^{st} ds \quad (5)$$

where $F(s)$ is the Laplace transform of $f(t)$ * and C is a contour properly chosen in the s -plane. The method has the advantage that it maintains the same basic philosophy which is employed in the Laplace transform solution of time-invariant linear systems, including the use of residues in the evaluation of (5); it has the disadvantage that it is rarely possible to express $H(s, t)$ in terms of known functions. Thus any computational advantage implied by (5) is of limited practical value.

THE APPROXIMATE REPRESENTATION

The preceding paragraphs substantiate the fact that the basic difficulty in the analysis of T.V.L.S. is not theoretical but computational. Since lengthy computation does appear to be unavoidable, it would seem better to direct the computational effort toward results of

* Upper-case-letter functions denote the Laplace transform of corresponding lower-case-letter functions.

a general, rather than a specific nature. To be more explicit, it would be better to obtain a representation for $h(t, \tau)$ which would permit simple integration of (2) for a large class of inputs, than to obtain particular solutions of (1) for several different forcing functions. While it has only been possible to find such representations as approximations to $h(t, \tau)$, accuracy of the representations generally can be made good enough for most engineering applications.

The approximation of $h(t, \tau)$ which has been found most useful is

$$h^*(t, \tau) = \sum_{i=1}^N a_i(t) \phi_i(\tau) \quad (6)$$

It has the advantage that the approximate response is given simply as

$$\begin{aligned} x^*(t) &= \int_{-\infty}^{\infty} h^*(t, \tau) f(t-\tau) d\tau \\ &= \sum_{i=1}^N a_i(t) \int_{-\infty}^{\infty} \phi_i(\tau) f(t-\tau) d\tau \end{aligned} \quad (7)$$

The approximating functions $\phi_i(\tau)$ are purposely chosen to make the N integrations straightforward, the complexity of the problem being embodied in the $a_i(t)$, which if necessary can be presented in tables or graphs. An alternative approach to the evaluation of $x^*(t)$ is to observe that

$$H^*(s, t) = \sum_{i=1}^N a_i(t) \Phi_i(s) \quad (8)$$

so that

$$x^*(t) = \sum_{i=1}^N a_i(t) \frac{1}{2\pi j} \int_C \Phi_i(s) F(s) e^{st} ds \quad (9)$$

Often, the complex integral can be evaluated by means of residues or tables of inverse Laplace transforms.

Another advantage of (6) is that it has a simple physical interpretation. Noting that each integral in (7) corresponds to the response of a time-invariant linear filter gives the realization shown in Figure 1, a parallel set of time-invariant linear filters followed by a set of time-variable gains. This realization has led Cruz and Van Valkenburg⁽⁹⁾ to employ (6) and similar representations in a procedure for the synthesis of T.V.L.S. For purpose of analysis the physical interpretation helps in attaining a good qualitative understanding of general response characteristics. At any given instant of time the response $x^*(t)$ consists of a simple linear combination of responses of N time-invariant linear filters. Furthermore, a translation in time of an input function $f(t)$ produces only a corresponding translation in time of the outputs of the N time-invariant linear filters. Thus the effect of applying the same input at different times is easily appraised.

The details of applying (7) and (9) when $f(t)$ is deterministic are obvious and will not be developed further. When stochastic inputs are considered the choice of the $\phi_i(\tau)$ is crucial; therefore, discussion of such inputs will follow a description of several useful families of ϕ_i . First, methods for evaluating the approximation will be presented.

METHODS FOR EVALUATION OF THE APPROXIMATE REPRESENTATION

Usually, the functions $\phi_n(\tau)$ are chosen quite arbitrarily before the evaluation of the approximate representation begins. However,

when L_α and L_β in (1) are periodic with period T , there is a natural choice for the $\phi_i(\tau)$ which results from the Fourier series expansion of certain periodic functions having their origin in the Floquet theory. (1)

The Floquet theory for periodic linear differential equations shows that the linearly independent solutions of the homogeneous equation $L_\alpha x = 0$ may be written as

$$\chi_i(t) = q_i(t) e^{\rho_i t}, \quad i=1, \dots, n \quad (10)$$

where the $q_i(t)$ are periodic with period T . This result presumes that the characteristic exponents ρ_i are distinct, an assumption which causes no great loss of generality. If the solutions (10) are substituted in (3), $h(t, \tau)$ takes the form

$$\begin{aligned} h(t, \tau) &= \sum_{i=1}^n q_i(t) e^{\rho_i \tau} r_i(t-\tau), \quad \tau \geq 0 \\ &= 0, \quad \tau < 0 \end{aligned} \quad (11)$$

where the $r_i(t)$ are periodic with period T .

Since under very reasonable conditions on L_α and L_β , $r_i(t)$ is continuous and has a sectionally continuous derivative, it is possible to express $r_i(t)$ as the complex Fourier series

$$r_i(t) = \sum_{k=-\infty}^{\infty} r_{ik} e^{j \frac{2\pi k t}{T}} \quad (12)$$

Substitution of (12) in (11) then gives

$$\begin{aligned} h(t, \tau) &= \sum_{i=1}^n \sum_{k=-\infty}^{\infty} r_{ik} q_i(t) e^{j \frac{2\pi k t}{T}} e^{(\rho_i - j \frac{2\pi k}{T}) \tau}, \quad \tau \geq 0 \\ &= 0, \quad \tau < 0 \end{aligned} \quad (13)$$

which, except for the change in summing notation, is the same as (6). The time-variable gains are the periodic functions $r_{ik}q_i(t)e^{j\frac{2\pi kt}{T}}$; the time-invariant filters have impulse responses $e^{(\rho_i - j\frac{2\pi k}{T})\tau}$.

Note that the time-invariant filters have rational Laplace transforms $(s - \rho_i + j\frac{2\pi k}{T})^{-1}$ and are stable if the T.V.L.S. is stable (the $\rho_i < 0$, $i=1, \dots, n$).

In terminating the infinite series (13) the relative size of the various terms must be considered. Certainly, when k gets sufficiently large the r_{ik} become small. Also, many of the $q_i(t)e^{\rho_i T}r_i(t-\tau)$ may be negligibly small so that response is essentially determined by a few "dominant modes." When the $\alpha_i(t)$ and $\beta_i(t)$ consist of a large constant part plus a small variable part, the $\rho_i \approx \lambda_i$ the characteristic roots of the invariant system formed by setting the small variable parts of the α_i and β_i to zero. In addition, the fluctuations of the $r_i(t-\tau)$ become so small that all but the $k=0$ terms in (13) may be neglected.

While the above procedure leads to a reasonable choice for the $\phi_i(\tau)$ and provides a direct method for computing the $a_i(t)$, it is restricted to periodic T.V.L.S. and the reduction of h to normal form (11) may be difficult. Moreover, even for periodic systems, it may be convenient for computational purposes to specify the $\phi_i(\tau)$ in different form. If the $\phi_i(\tau)$ are specified arbitrarily, there are a number of possible procedures available for choosing the $a_i(t)$. For the purposes considered here it seems most convenient to minimize the integral-square error

$$E(t) = \int_{-\infty}^{\infty} [h(t, \tau) - h^*(t, \tau)]^2 d\tau \quad (14)$$

Although it is not necessary or always desirable to assume that the $\phi_i(\tau)$ are orthogonal, it will be done to simplify the presentation. Later, a number of useful orthogonal approximating functions will be described. Since orthogonal function approximations are well known and have been used successfully for the approximation of both time-invariant linear systems^(10,11) and T.V.L.S.⁽⁹⁾ the presentation which follows serves mainly to introduce notation.

For convenience it is assumed that the $\phi_i(\tau)$ are orthonormal, i.e.,

$$\int_{-\infty}^{\infty} \phi_i(\tau) \phi_k(\tau) d\tau = 1, \quad i = k$$
$$= 0, \quad i \neq k \quad (15)$$

Minimization of $E(t)$ then gives

$$a_i(t) = \int_{-\infty}^{\infty} h(t, \tau) \phi_i(\tau) d\tau \quad (16)$$

and

$$E_{\min} = \int_{-\infty}^{\infty} h^2(t, \tau) d\tau - \sum_{i=1}^N a_i^2(t) \quad (17)$$

It must be assumed that $\int_{-\infty}^{\infty} h^2(t, \tau) d\tau$ exists, a requirement which is satisfied if the approximated system is stable ($\int_{-\infty}^{\infty} |h(t, \tau)| d\tau < M_S < \infty$) and has a uniformly bounded weighting function ($|h(t, \tau)| < M_h < \infty$).

These conditions are satisfied in most practical problems, at least for the range of t which must be considered. Evaluation of $a_i(t)$ is usually difficult and will be examined in a subsequent section. One important fact is obvious - if h is periodic with period T then the a_i are periodic with period T .

The usual extensions of the above results are possible. For example, the weighted-integral-square error

$$E_w(t) = \int_{-\infty}^{\infty} W(\tau) [h(t, \tau) - h^*(t, \tau)]^2 d\tau \quad (18)$$

can be minimized. The problem of finding suitable functions $\phi_i(\tau)$ orthonormal with respect to $W(\tau)$ then exists. Actually, the approximation obtained by terminating (13) is a minimum weighted-integral-square-error approximation where the weight $e^{-2\rho_i\tau}$ is applied to the approximation of the i -th mode. This results from the minimum integral-square-error property of the Fourier series. Another extension is that of the constrained approximation. A constrained approximation permits certain properties of (1) to be represented exactly in $h^*(t, \tau)$. An example of a useful constraint would be

$$\int_{-\infty}^{\infty} h(t, \tau) d\tau = \int_{-\infty}^{\infty} h^*(t, \tau) d\tau \quad (19)$$

This constraint would assure $x(t) = x^*(t)$ when $f(t) = \text{constant}$. It is also possible to represent the functions $a_i(t)$ in a series of functions $\delta_k(t)$ so that

$$h^*(t, \tau) = \sum_{i, k=1}^{N, M} a_{ik} \delta_k(t) \phi_i(\tau) ; \quad (20)$$

but, such a representation seems to have more value in synthesis than in analysis. (9)

Little can be said in a general way about the error in response $|x - x^*|$ and how it is influenced by error $|h - h^*|$ in the representation. However, if the input is bounded by M_f and both h and h^* are stable, then $|x - x^*|$ is bounded. This can be seen from

$$\begin{aligned}
 |x - x^*| &= \left| \int_{-\infty}^{\infty} h(t, \tau) f(t - \tau) d\tau - \int_{-\infty}^{\infty} h^*(t, \tau) f(t - \tau) d\tau \right| \\
 &\leq \int_{-\infty}^{\infty} |h(t, \tau) - h^*(t, \tau)| |f(t - \tau)| d\tau \\
 &\leq M_f \int_{-\infty}^{\infty} |h(t, \tau) - h^*(t, \tau)| d\tau \\
 &\leq M_f \int_{-\infty}^{\infty} |h(t, \tau)| d\tau + M_f \int_{-\infty}^{\infty} |h^*(t, \tau)| d\tau \\
 &\leq M_f (M_s + M_s^*) < \infty \quad (21)
 \end{aligned}$$

If either h or h^* is unstable then it can be shown that $|x - x^*|$ can become unbounded if the bounded input is chosen properly. Thus stability is a crucial factor in representability. (12)

ORTHONORMAL APPROXIMATING FUNCTIONS

Three primary considerations should be observed in choosing the orthonormal approximating functions $\phi_i(\tau)$: 1) the complexity of the computations for the $a_i(t)$, 2) the complexity of the integrations in (7), 3) the number N of terms in the approximation for a given accuracy. Usually, it is not possible to make all computations simple and N small, and still achieve good approximation accuracy. Where the compromises are made depends on the system being considered and on the intended applications of the approximate representation. A few of the more useful orthonormal sets are given below. A great variety of other sets which have been used in signal theory and synthesis are also applicable. (13,14)

The simplest orthogonal functions are those which are non-overlapping functions of τ , e.g., the set

$$\phi_i(\tau) = u_0(\tau - \tau_i), \quad i=1, \dots, N \quad (22)$$

where $u_0(\tau)$ is the unit impulse at $\tau = 0$ and $\tau_1 < \tau_2 < \dots < \tau_N$. This is the set proposed by Cruz.^(9,15) He lets the coefficients $a_i(t) = h(t, \tau_i)$ and realizes the $\phi_i(\tau)$ by a tapped delay line. The representation is not a minimum-integral-square-error approximation because the $\phi_i(\tau)$ cannot be normalized. An orthonormal set which is non-overlapping is

$$\begin{aligned} \phi_i(\tau) &= 0, \quad \tau \leq (i-1)\Delta\tau, \quad \tau > i\Delta\tau \\ &= \frac{1}{\sqrt{\Delta\tau}}, \quad (i-1)\Delta\tau < \tau \leq i\Delta\tau \end{aligned} \quad (23)$$

Although computation of the $a_i(t)$ is now more difficult, a stepwise rather than an impulse representation is obtained.

The set

$$\begin{aligned} \Phi_i(j\omega) &= 0, \quad \omega \leq (i-1)2\pi\Delta f, \quad \omega > i2\pi\Delta f \\ &= \frac{1}{\sqrt{\Delta f}}, \quad (i-1)2\pi\Delta f < \omega \leq i2\pi\Delta f \end{aligned} \quad (24)$$

is orthonormal because the functions are non-overlapping in the frequency domain. Since the $\phi_i(\tau)$ corresponding to (24) are complex and non-zero for negative τ , the functions are non-realizable, a limitation which does not limit their usefulness in analysis.

Usually, the non-overlapping functions given by (23) and (24) lead to a fairly large N (good accuracy requires small $\Delta\tau$ or Δf) and

difficult computations for the $a_i(t)$. However, because the functions are non-overlapping, the influence of the $a_i(t)$ on response is easily interpreted.

From the standpoint of general simplicity, functions $\Phi_i(s)$ which are rational in s are most advantageous. If it is assumed that $\Phi_i(s)$ has poles at s_1, s_2, \dots, s_i , orthogonalization in the frequency domain yields. (11)

$$\Phi_i(s) = \frac{(s+s_1)(s+s_2) \cdots (s+s_{i-1})}{(s-s_1)(s-s_2) \cdots (s-s_{i-1})} \frac{\sqrt{-2s_i}}{s-s_i} \quad (25)$$

$$\Phi_i(s) = \frac{(s+s_1)(s+s_2) \cdots (s+s_{i-1})}{(s+s_1)(s+s_2) \cdots (s+s_{i-1})} \frac{2|s_i| \sqrt{\Re(-s_i)}}{s^2 - (2\Re s_i)s + |s_i|^2} \quad (26)$$

$$\Phi_{i+1}(s) = \frac{(s+s_1)(s+s_2) \cdots (s+s_{i-1})}{(s-s_1)(s-s_2) \cdots (s-s_{i-1})} \frac{2\sqrt{\Re(-s_i)} s}{s^2 - (2\Re s_i)s + |s_i|^2} \quad (27)$$

where $\Re s_i < 0$. Formula (25) is used when s_i is real; formula (26) and (27) are used when s_i and s_{i+1} are complex conjugates.

In several special cases the functions $\phi_i(\tau)$ corresponding to (25) may be expressed in terms of familiar functions. For example, when all $s_i = -1$

$$\begin{aligned} \Phi_i(\tau) &= \sqrt{2} e^{-\tau} L_{i-1}(2\tau), \quad \tau \geq 0 \\ &= 0, \quad \tau < 0 \end{aligned} \quad (28)$$

where $L_i(y)$ are the Laguerre polynomials of order i . If $s_i = i$

$$\begin{aligned} \Phi_i(\tau) &= (-1)^i (2)^{\frac{1}{2}} (i+1)^{\frac{3}{2}} e^{-\tau} J_i(2, 2|e^{-\tau}|), \quad \tau \geq 0 \\ &= 0, \quad \tau < 0 \end{aligned} \quad (29)$$

where $J_i(2,2|y)$ are Jacobi polynomials of order i .⁽¹⁶⁾ In any case, applying the inverse Laplace transform to $\Phi_i(s)$ gives (providing none of the s_i are equal)

$$\begin{aligned}\Phi_i(\tau) &= \sum_{k=1}^i d_{ik} e^{s_k \tau}, \quad \tau \geq 0 \\ &= 0, \quad \tau < 0\end{aligned}\tag{30}$$

The constants d_{ik} have been tabulated by Kautz⁽¹⁰⁾ for a number of different choices of s_i .

When the $\Phi_i(s)$ are rational, both the evaluation of the $a_i(t)$ and the integrals in (7) and (9) become relatively easy. Also, freedom of choice of the s_i often provides good approximation accuracy with small N .

EVALUATION OF THE $a_i(t)$

The evaluation of the $a_i(t)$ given by (16) is difficult for many of the same reasons that cause difficulties in the response evaluation of T.V.L.S. Therefore machine computation is usually required. Evaluation of $a_i(t)$ by a digital computer could be carried out in two steps: 1) numerical solution of the adjoint differential equation⁽⁵⁾ for $h(t, \tau)$ as a function of τ . 2) numerical integration of (16). Differential analyzer evaluation of $a_i(t)$ using techniques similar to those proposed by Gilbert⁽¹¹⁾ for the approximation of time-invariant systems appears promising and is presently being investigated.

When the $\Phi_i(s)$ are rational functions of s a simplification occurs. Substituting (30) in (16) gives

$$\begin{aligned}
 a_i(t) &= \sum_{k=1}^i d_{ik} \int_0^{\infty} h(t, \tau) e^{s_k \tau} d\tau \\
 &= \sum_{k=1}^i d_{ik} H(-s_k, t)
 \end{aligned} \tag{31}$$

But as Zadeh shows $H(s, t)$ satisfies the differential equation

$$L_{\alpha}(p+s, t) H(s, t) = L_{\beta}(s, t) \tag{32}$$

If the T.V.L.S. described by $h(t, \tau)$ is stable and $\text{Re } s_i < 0$, $i=1, \dots, N$, $H(-s_i, t)$ exists and is given as the solution of (32) with $p^k H(-s_i, t) = 0$ for $k=0, 1, \dots, n-1$ at $t = t_1$. The time t_1 must be taken in the distant past from the values of t for which $H(-s_i, t)$ is desired. To compute all the $a_i(t)$, N equations like (32) must be solved. If many values of t are to be considered this is considerably less work than solving the adjoint equation for $h(t, \tau)$ for every value of t .

The main limitation on effective use of (31) is an accuracy requirement. For large i , $a_i(t)$ tends to be much smaller in magnitude than the individual terms of the sum. Thus small errors in the $H(-s_i, t)$ may cause large errors in the $a_i(t)$

THE APPROXIMATE REPRESENTATION IN TERMS OF EXPONENTIAL FUNCTIONS

If the $\Phi_n(s)$ are rational in s (30) can be substituted into (6) yielding

$$\begin{aligned}
 h^*(t, \tau) &= \sum_{i=1, k=1}^{N, i} d_{ik} a_i(t) e^{s_k \tau} \\
 &= \sum_{i=1}^N b_i(t) e^{s_i \tau}, \quad \tau \geq 0 \\
 &= 0, \quad \tau < 0
 \end{aligned} \tag{33}$$

as a sum of exponentials rather than a sum of orthonormal functions.

Thus (7) and (9) become

$$x^*(t) = \sum_{i=1}^N b_i(t) \int_0^{\infty} e^{s_i \tau} f(t-\tau) d\tau \tag{34}$$

and

$$x^*(t) = \sum_{i=1}^N b_i(t) \frac{1}{2\pi j} \int_C \frac{F(s)}{s-s_i} e^{st} ds \tag{35}$$

The resulting simplification in response evaluation is often worthwhile. From (33) and (31) it is clear that

$$b_i(t) = \sum_{k=i}^N a_k(t) d_{ki} \tag{36}$$

$b_i(t)$ could also be derived by substituting the non-orthogonal series (33) in (14) and choosing the $b_i(t)$ to minimize $E(t)$.

RESPONSE TO STOCHASTIC INPUTS

It is clear how the approximate representation (6) is utilized in evaluating response to deterministic inputs. The determination of response characteristics, e.g., the correlation function

of $x^*(t)$, when the input is a random process requires elaboration. As in the case of deterministic inputs, it will be seen that the functional form of (6) is advantageous.

Let ensemble averages be represented by a bar. Then by using (7) the correlation function of x^* can be written as

$$\begin{aligned} \Psi_{xx}(t_1, t_2) &= \overline{x^*(t_1) x^*(t_2)} \\ &= \sum_{i=1, k=1}^N a_i(t_1) a_k(t_2) \iint_{-\infty}^{\infty} \phi_i(\tau) \phi_k(\tau_2) \overline{f(t_1-\tau) f(t_2-\tau_2)} d\tau d\tau_2 \end{aligned} \quad (37)$$

$$= \sum_{i=1, k=1}^N a_i(t_1) a_k(t_2) \iint_{-\infty}^{\infty} \phi_i(\tau) \phi_k(\tau_2) \Psi_{ff}(t_1-\tau, t_2-\tau_2) d\tau d\tau_2 \quad (38)$$

where Ψ_{ff} is the correlation function of the input:

$$\Psi_{ff}(t_1, t_2) = \overline{f(t_1) f(t_2)} \quad (39)$$

Evaluation of the integrals in (38) is usually a difficult matter. Freedom in choosing the $\phi_i(\tau)$ is sometimes helpful, particularly when $\Psi_{ff}(t_1-\tau_1, t_2-\tau_2)$ can be written in the form $\eta(t_1, t_2)\theta(\tau_1, \tau_2)$

If the input process is stationary $\Psi_{ff}(t_1, t_2) = \Psi_{ff}(t_2-t_1)$ and (38) simplifies to

$$\Psi_{xx}(t_1, t_2) = \sum_{i=1, k=1}^N a_i(t_1) a_k(t_2) \iint_{-\infty}^{\infty} \phi_i(\tau) \phi_k(\tau_2) \Psi_{ff}(t_1-\tau-t_2-\tau_2) d\tau d\tau_2 \quad (40)$$

By letting $\tau_3 = \tau_2 - \tau_1$ and by defining

$$r_{ik}(\tau_3) = \int_{-\infty}^{\infty} \phi_i(\tau) \phi_k(\tau_3 + \tau) d\tau \quad (41)$$

(40) can be written as

$$Y_{xx}(t_1, t_2) = \sum_{i,k=1}^N a_i(t_1) a_k(t_2) \int_{-\infty}^{\infty} \Psi_{ff}(t_2 - t_1 - \tau_3) r_{ik}(\tau_3) d\tau_3 \quad (42)$$

Equations (41) and (42) have frequency domain equivalents

$$\Gamma_{ik}(s) = \Phi_i(-s) \Phi_k(s) \quad (43)$$

and

$$Y_{xx}(t_1, t_2) = \sum_{i,k=1}^N a_i(t_1) a_k(t_2) \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_{ff}(j\omega) \Gamma_{ik}(j\omega) e^{j\omega(t_2 - t_1)} d\omega \quad (44)$$

where $\Psi_{ff}(j\omega)$ is the spectral density of the input process. Certainly, if $\Psi_{ff}(j\omega)$ and $\Phi_i(j\omega)$ are rational in $j\omega$ there should be no trouble in evaluating the integrals in (44) by means of residues. Because $\Gamma_{ik}(j\omega) \Psi_{ff}(j\omega) = \Gamma_{ki}(-j\omega) \Psi_{ff}(-j\omega)$, only the sign of $(t_2 - t_1)$ is changed in the integral when k and i are interchanged. Thus the total number of integrations required in (44) is $\frac{N(N+1)}{2}$ and not N^2 .

The number of integrations can be reduced still further by propitious choice of the $\Phi_i(j\omega)$. The Laguarre functions

$$\Phi_i(s) = \sqrt{-2\bar{s}} \frac{(s + \bar{s})^{i-1}}{(s - \bar{s})^i} \quad (45)$$

illustrate the point since for them

$$\begin{aligned}\Gamma_{ik}(s) &= \Phi_i(-s) \Phi_k(s) \\ &= 2\bar{s} \frac{(s+\bar{s})^{k-i-1}}{(s+\bar{s})^{k-i+1}}\end{aligned}\quad (46)$$

Hence, there are only $2N-1$ functions $\Gamma_{ik}(s)$ as i and k take on all values between 1 and N . Because of the result noted in the previous paragraph, this leads to N separate integrations in the evaluation of (44). The advantage is offset somewhat by the fact that the Laguarre functions have multiple poles, making computation of the residues of $\Psi_{ff}(s) \Gamma_{ik}(s)$ more difficult.

The non-overlapping functions given in (23) provide an even greater simplification. In these functions

$$\Gamma_{ik}(j\omega) = \Delta T \frac{(\sin \frac{\omega \Delta T}{2})^2}{(\frac{\omega \Delta T}{2})^2} e^{-j\omega(k-i)\Delta T}\quad (47)$$

Substitution of (47) in the integrals of (44) gives

$$\Psi_{xx}(t_1, t_2) = \sum_{i=1, k=1}^N a_i(t_1) a_k(t_2) \mu(t_2 - t_1 + i\Delta T - k\Delta T)\quad (48)$$

where

$$\mu(\gamma) = \frac{\Delta T}{2\pi} \int_{-\infty}^{\infty} \Psi_{ff}(j\omega) \frac{(\sin \frac{\omega \Delta T}{2})^2}{(\frac{\omega \Delta T}{2})^2} e^{j\omega\gamma} d\omega\quad (49)$$

Thus only one integration is needed.

Expressions for other correlations can be derived in a manner similar to that outlined above. This has been done for the cross-correlation of $x^*(t)$ and $g(t)$ (a member of a random process) and the cross correlation of $x^*(t)$ and $y^*(t)$ where $y^*(t)$ is the output a filter having a weighting function $\sum_{i=1}^N b_i(t) \phi_i(\tau)$ and an input $g(t)$. Since these results do not contribute any new knowledge of a general nature and are lengthy, they are not given here.

Examples

The first example is the second-order control system shown in Figure 2. This example has been treated previously by Farmanfarma (17) and Gilbert. (18) The variable gain constant is given by

$$\begin{aligned} K(t) &= 6, & 0 \leq t-k < .6 \\ &= 0, & .6 \leq t-k < 1. \end{aligned} \quad (50)$$

$k = \text{integer}$

and represents finite width sampling. Writing the differential equation between the input f_1 and the output x_0 gives

$$\ddot{x}_0 + 5 \dot{x}_0 + K(t) x_0 = K(t) f_1(t) \quad (51)$$

Since the coefficients in (51) are periodic ($T=1$) the approximation (13) is applicable. However, the multiplication of $f_1(t)$ by $K(t)$ yields functions $r_1(t)$ and $r_2(t)$ which are zero for $.6 \leq t-k < 1.$, and have jump discontinuities at $t-k = .6, 1$. This means that a fairly large number of terms must be included in (13) for good accuracy.

This difficulty can be avoided, at least for inputs which do not vary too rapidly, by writing (51) in terms of the error, $x = f_1 - x_0$:

$$\ddot{x} + 5\dot{x} + k(t)x = \ddot{f}_1 + 5\dot{f}_1 = f(t) \quad (52)$$

In what follows the error x and the equivalent input $f(t) = \ddot{f}_1(t) + 5\dot{f}_1(t)$ are the $x(t)$ and $f(t)$ in (1).

Reducing the weighting function for (52) to normal form (2) yields

$$h(t, \gamma) = 1.283 \left[r_2(t) e^{-.916\gamma} r_1(t-\gamma) - r_1(t) e^{-4.084\gamma} r_2(t-\gamma) \right] \quad (53)$$

where $r_1(t)$ and $r_2(t)$ are functions with period $T=1$ given by

$$\begin{aligned} r_2(t) &= e^{-1.084t} - .634e^{-2.084t}, \quad 0 < t < .6 \\ &= .139 \left[e^{.916t} + 8.34e^{-4.084t} \right], \quad .6 < t < 1 \end{aligned} \quad (54)$$

$$\begin{aligned} r_1(t) &= e^{1.084t} - .348e^{2.084t}, \quad 0 < t < .6 \\ &= 1.150 \left[e^{-.916t} + .0028e^{4.084t} \right], \quad .6 < t < 1 \end{aligned} \quad (55)$$

In this problem $q_1(t) = r_2(t)$ and $q_2(t) = r_1(t)$.

As Figure 3 indicates $r_1(t)$ and $r_2(t)$ are functions whose variations are fairly small compared to their average value. Thus a reasonable approximation can be obtained by neglecting all but the $k=0$ terms in (13):

$$h^*(t, \gamma) = .883 r_2(t) e^{-.916\gamma} - .462 r_1(t) e^{-4.084\gamma} \quad (56)$$

Since the variations in $r_1(t)$ and $r_2(t)$ are nearly sinusoidal with period $T=1$, addition of the $k=+1, -1$ terms should give an excellent approximation. Actually the $k=+1, -1$ terms are not too important if the input $f(t)$ has most of its energy well below $\omega = 2\pi$.

If $f(t) = u_{-1}(t)$ the unit step function at $t=0$, $f(t) = u_1(t) + 5u_0(t)$. Although $f(t)$ contains the doublet $u_1(t)$, the approximate response $x^*(t)$ is still quite accurate as can be seen from Figure 4. Even better results are obtained for smoother inputs $f_1(t)$ such as the ramp.

For the second example consider the differential equation

$$\ddot{x} + (2-t)\dot{x} - tx = f(t) \quad (57)$$

It can be verified easily that for this equation

$$h(t, \tau) = e^{\frac{t^2}{2}} e^{-\tau} \int_0^t e^{-\frac{\lambda^2}{2}} d\lambda, \quad \tau \geq 0 \quad (58)$$

$$= 0, \quad \tau < 0$$

and

$$H(s, t) = \frac{f(s+1-t)}{s+1} \quad (59)$$

with

$$f(\lambda) = e^{\frac{\lambda^2}{2}} \left[\sqrt{\pi/2} - \int_0^\lambda e^{-\frac{\lambda^2}{2}} d\lambda \right] \quad (60)$$

That h and H are expressible in terms of elementary functions indicates that Equation (57) was chosen with care.

Before discussing the approximation of $h(t, \tau)$ several properties of $h(t, \tau)$ should be noted. First,

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t, \tau)| d\tau &\leq e^{\frac{t^2}{2}} \int_0^{\infty} e^{-\tau} \max\left(\int_{t-\tau}^t e^{-\frac{\lambda^2}{2}} d\lambda\right) d\tau \\ &\leq e^{\frac{t^2}{2}} \int_0^{\infty} e^{-\tau} \sqrt{2\pi} d\tau \\ &= e^{\frac{t^2}{2}} \sqrt{2\pi} < \infty, \quad t \text{ finite} \end{aligned} \quad (61)$$

This coupled with the boundedness of h assures $\int_0^{\infty} |h(t, \tau)|^2 d\tau < \infty$, t finite. Because (61) holds only for t finite, stability of (57) is not guaranteed. In fact, solutions of (57) which diverge as $t \rightarrow +\infty$ can be exhibited. Still, for finite t Equation (21) holds. Thus as long as t remains finite no fundamental difficulties in approximating h should arise.

Second, for $|t|$ large $h(t, \tau)$ has simplified asymptotic representations. As $t \rightarrow +\infty$

$$\begin{aligned} h(t, \tau) &\cong e^{\frac{t^2}{2}} e^{-\tau} \left[\sqrt{\pi/2} - \int_0^{t-\tau} e^{-\frac{\lambda^2}{2}} d\lambda \right] \\ &\cong e^{\frac{t^2}{2} - t} \eta(\tau - t) \end{aligned} \quad (62)$$

where

$$\eta(\lambda) = e^{-\lambda} \left[\sqrt{\pi/2} + \int_0^{\lambda} e^{-\frac{\lambda^2}{2}} d\lambda \right] \quad (63)$$

The function $\eta(\lambda)$ is shown in Figure 5. Thus for large t , $h(t, \tau)$ is a pulse like function centered near $\tau = t$. For $t \ll -1$, $\tau \gg 1/-t$,

the asymptotic expansions of $\int_0^t e^{-\frac{\lambda^2}{2}} d\lambda$ and $\int_0^{t-\tau} e^{-\frac{\lambda^2}{2}} d\lambda$ give

$$h(t, \tau) \cong \frac{e^{-\tau}}{-t}, \quad \tau \geq 0$$

$$= 0, \quad \tau < 0 \quad (64)$$

Thus for large negative t the T.V.L.S. acts like a first order time-variant system followed by a gain $(-t^{-1})$.

Because $h(t, \tau)$ gives a delay-like behavior for large t , it is impossible to select a small number of approximating functions which will give good approximation accuracy for all t . However, if $t < -2$, $h(t, \tau)$ is well behaved and most any reasonable set of approximating functions will give good accuracy. For convenience the functions $\phi_1(s)$ in (25) are taken with $s_1 = ic$. By taking $c = 1/2$ the "length" of the approximating functions is made approximately equal to the "length" of $h(t, \tau)$. Application of (31) gives (for d_{ik} and tables of $\phi_1(\tau)$ see Lanning and Battin⁽¹⁹⁾).

$$a_1(t) = \frac{2}{3} f(1.5-t)$$

$$a_2(t) = -\frac{4\sqrt{2}}{3} f(1.5-t) + \frac{3\sqrt{2}}{2} f(2-t)$$

$$a_3(t) = 2\sqrt{3} f(1.5-t) - 6\sqrt{3} f(2-t) + 4\sqrt{3} f(2.5-t) \quad (65)$$

$$a_4(t) = -\frac{16}{3} f(1.5-t) + 30 f(2-t) - 48 f(2.5-t) + \frac{70}{3} f(3-t)$$

For positive arguments the function $\xi(\lambda)$ has been tabulated⁽²⁰⁾ which aids in the computation of the $a_i(t)$. Figure 6 shows h and h^* for $t = -2, -1, 0, 1, 2$ and $N = 4$. Excellent approximation accuracy is apparent.

As would be expected approximation accuracy steadily deteriorates as t increases beyond 2. However, since $\xi(\lambda) \rightarrow -\frac{1}{\lambda}$ as $\lambda \rightarrow \infty$,

$$a_1 \rightarrow -\frac{2}{3}t^{-1}; \quad a_2 \rightarrow -\frac{\sqrt{2}}{6}t^{-1}; \quad a_i \rightarrow 0, \quad i > 2$$

as $t \rightarrow -\infty$. Thus, by means of (30) it can be shown that

$$h^*(t, \tau) \rightarrow h(t, \tau)$$

as $t \rightarrow -\infty$.

CONCLUSION

Methods, which are adaptable to machine computation, have been developed for approximating the weighting function $h(t, \tau)$ of a T.V.L.S. Since the form of the approximation is simpler than that given for $h(t, \tau)$ by classical theory, analytic solution of the response problem is possible for both deterministic and stochastic inputs. Freedom in the choice of the approximating functions $\phi_i(\tau)$ often leads to simplified evaluation or utilization of the approximation. While the amount of effort required to evaluate the response of a T.V.L.S. is still considerable, the introduction of workable analytic procedures should prove valuable in a number of important problems.

ACKNOWLEDGEMENT

The author gratefully acknowledges the support of the Institute of Science and Technology, The University of Michigan.

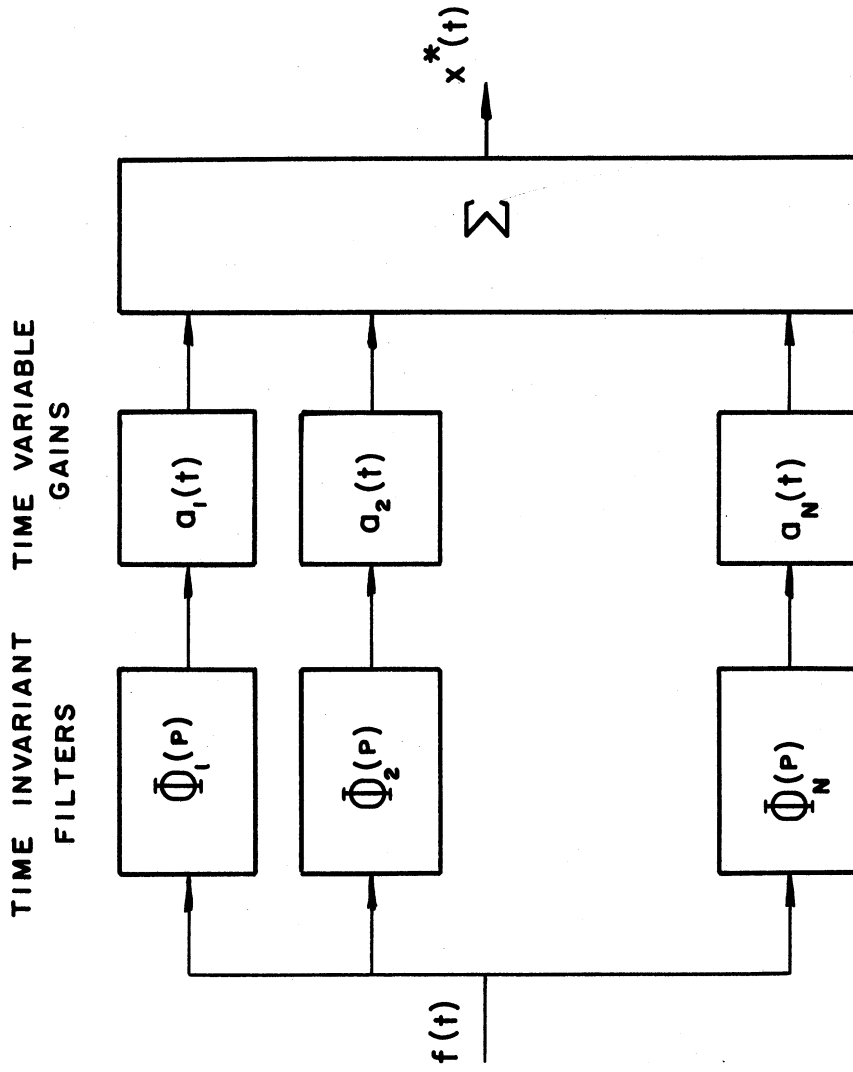


Figure 1. Physical Interpretation of Approximation to T.V.I.S.

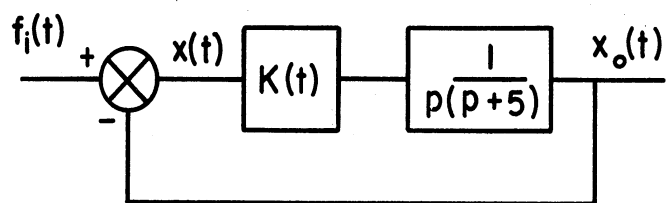


Figure 2. Feedback Control System with Finite Width Sampling.

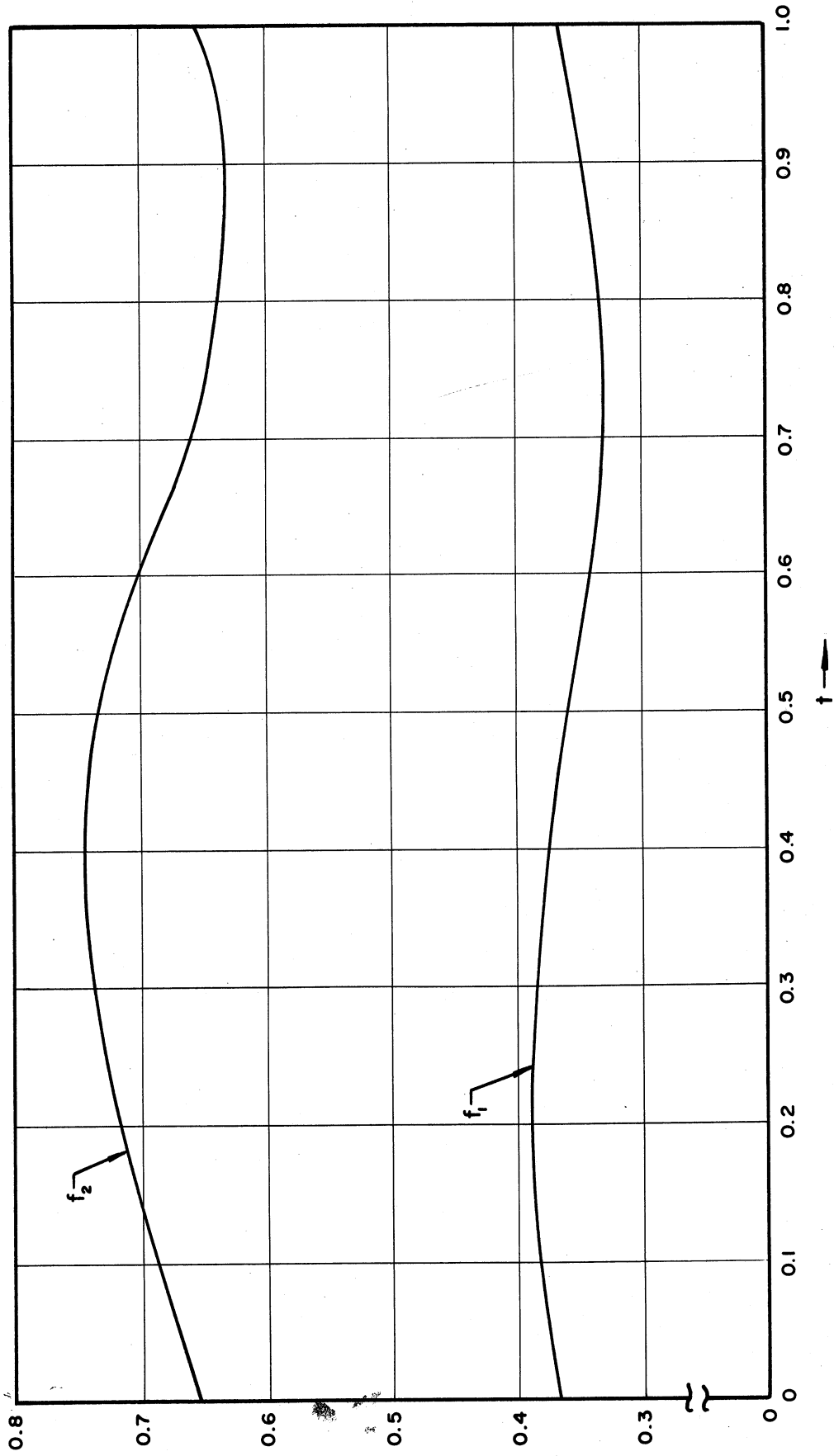


Figure 3. Periodic Functions of Normal Solution.

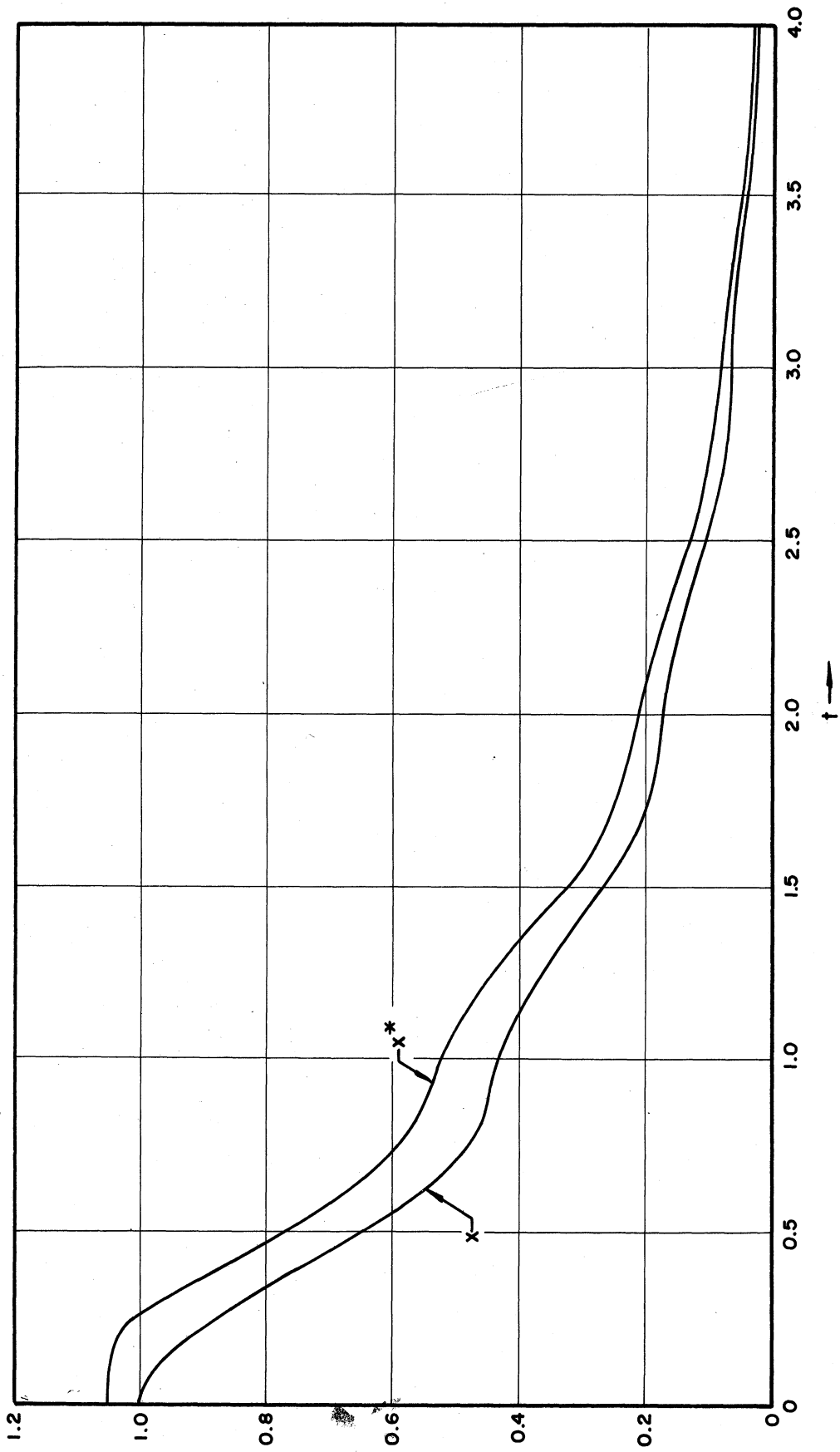


Figure 4. Step Responses for Exact and Approximate Systems.

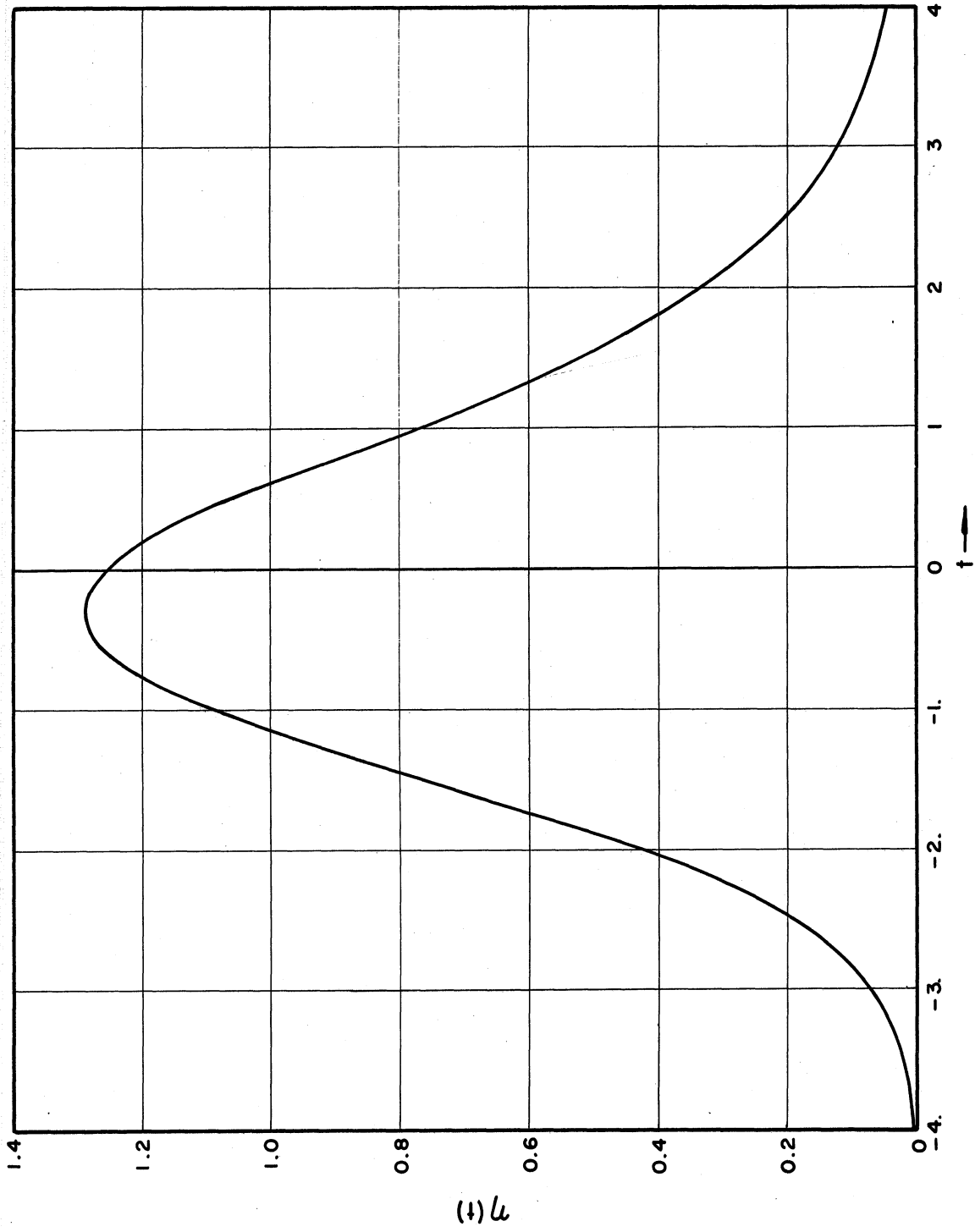


Figure 5. Form of $h(t, \tau)$ for $t \gg 1$.

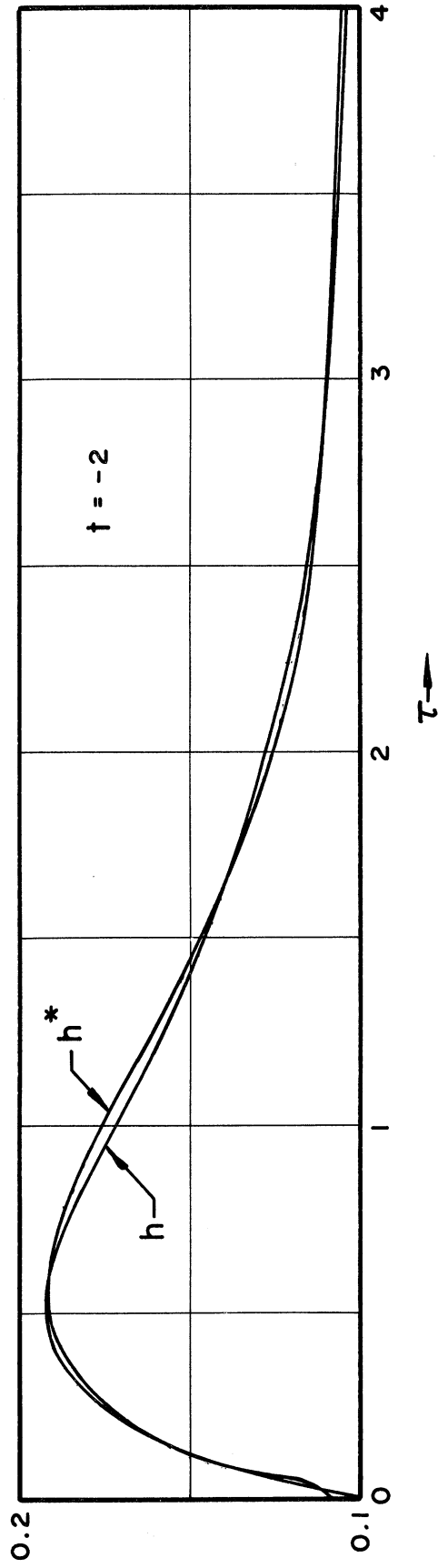


Figure 6a. $h(t, \tau)$ and $h^*(t, \tau)$ for $t = -2$.

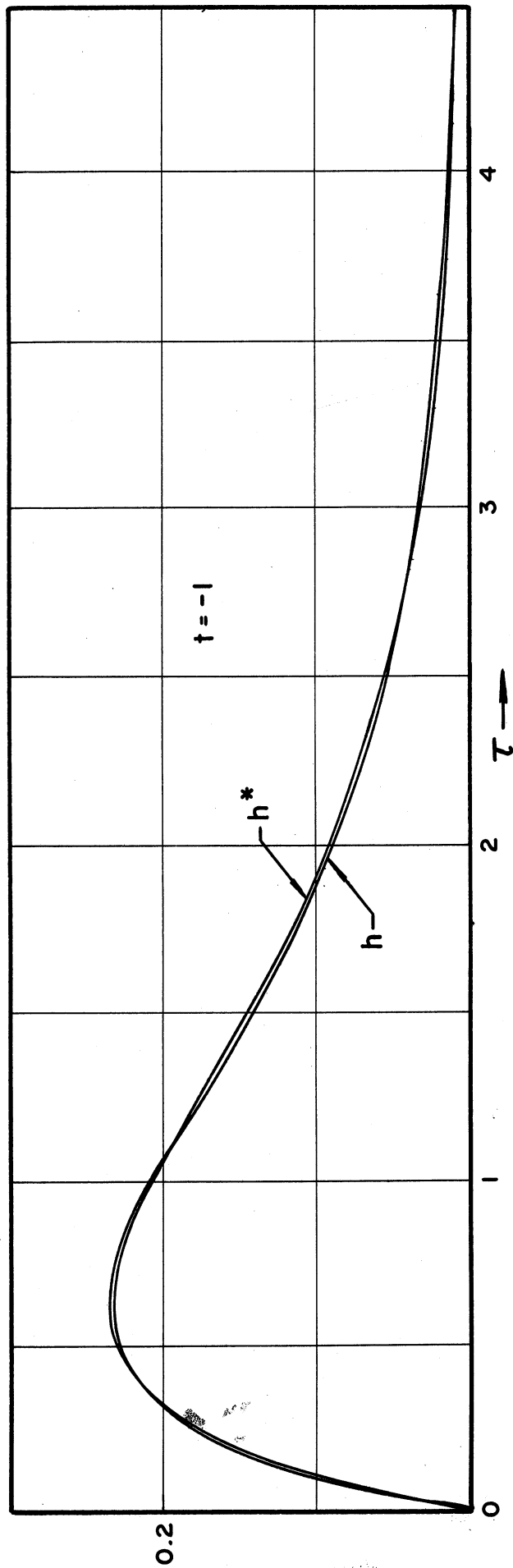


Figure 6b. $h(t,\tau)$ and $h^*(t,\tau)$ for $t = -1$.

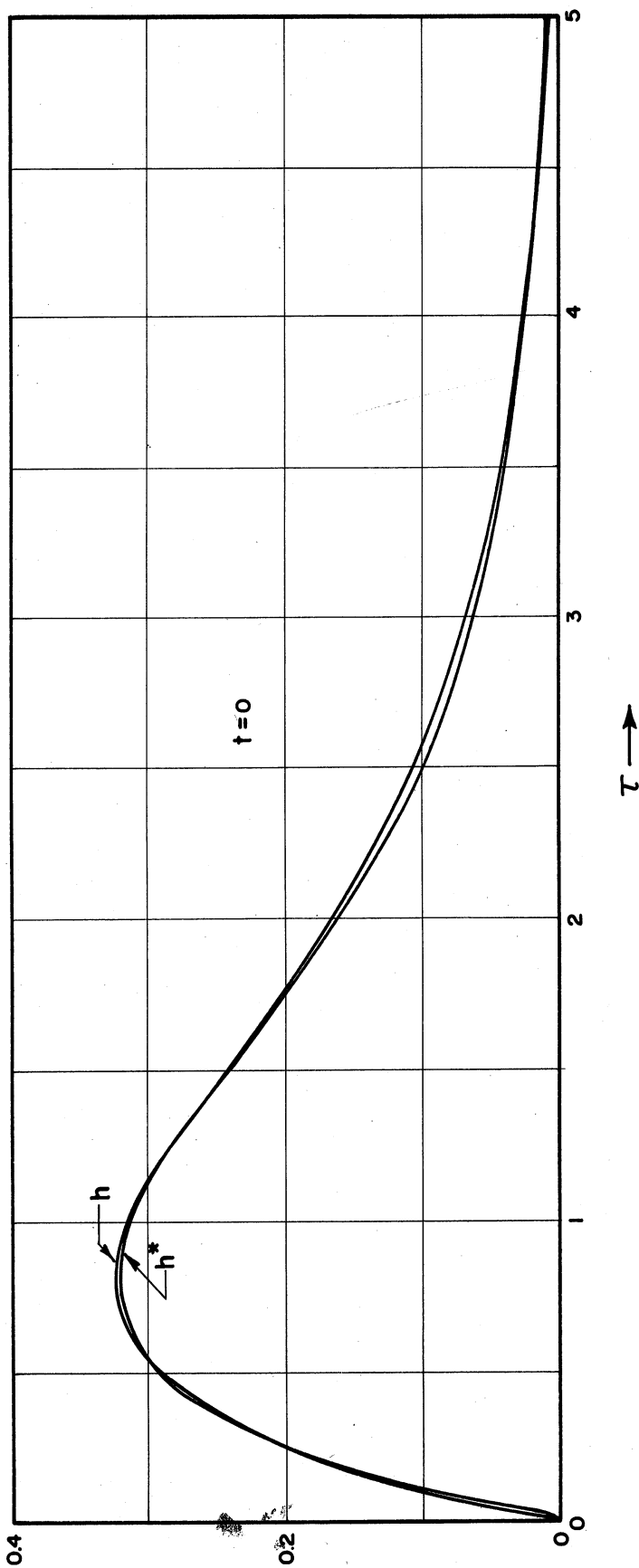


Figure 6c. $h(t, \tau)$ and $h^*(t, \tau)$ for $t = 0$.

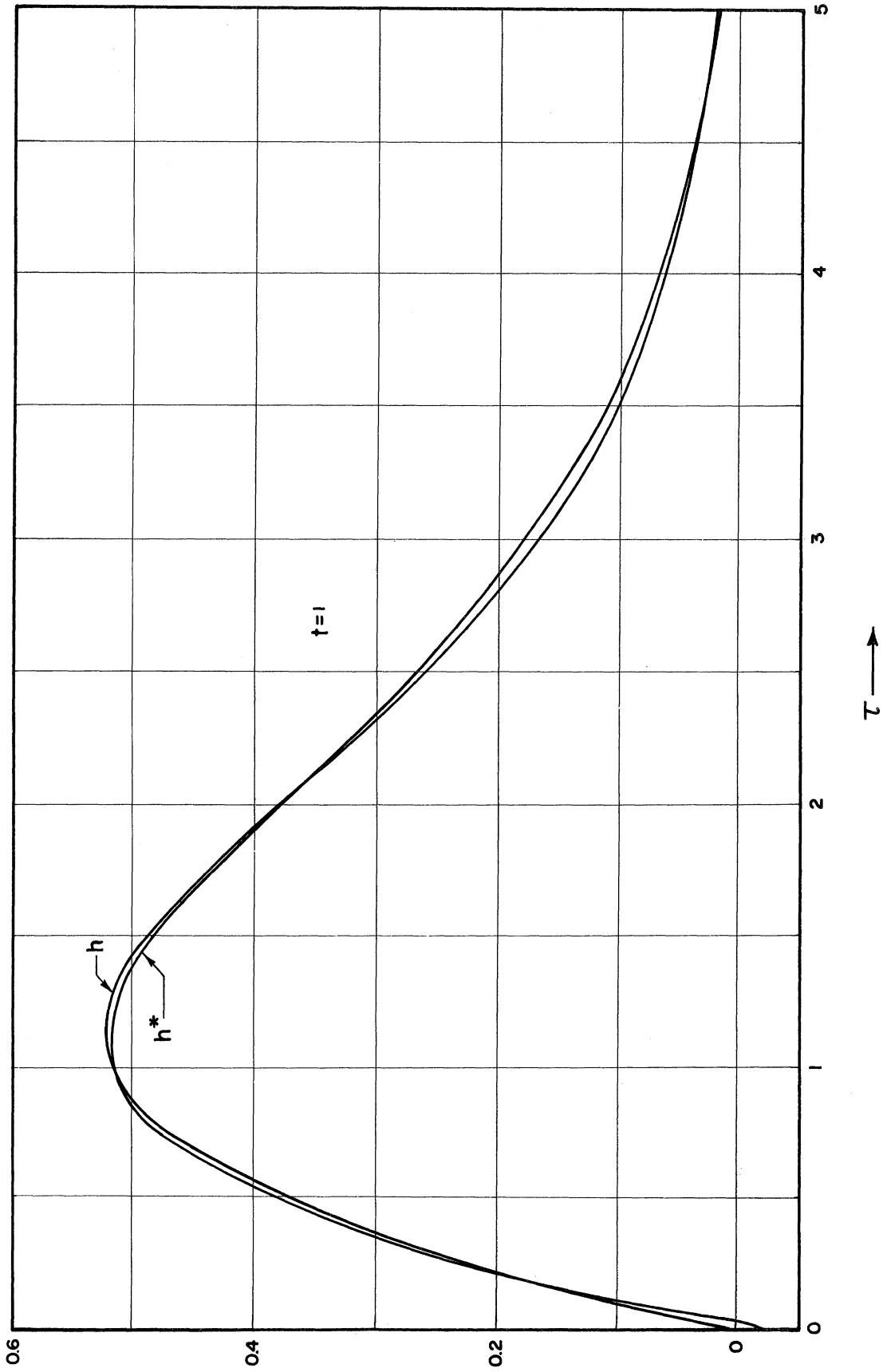


Figure 6d. $h(t, \tau)$ and $h^*(t, \tau)$ for $t = 1$.

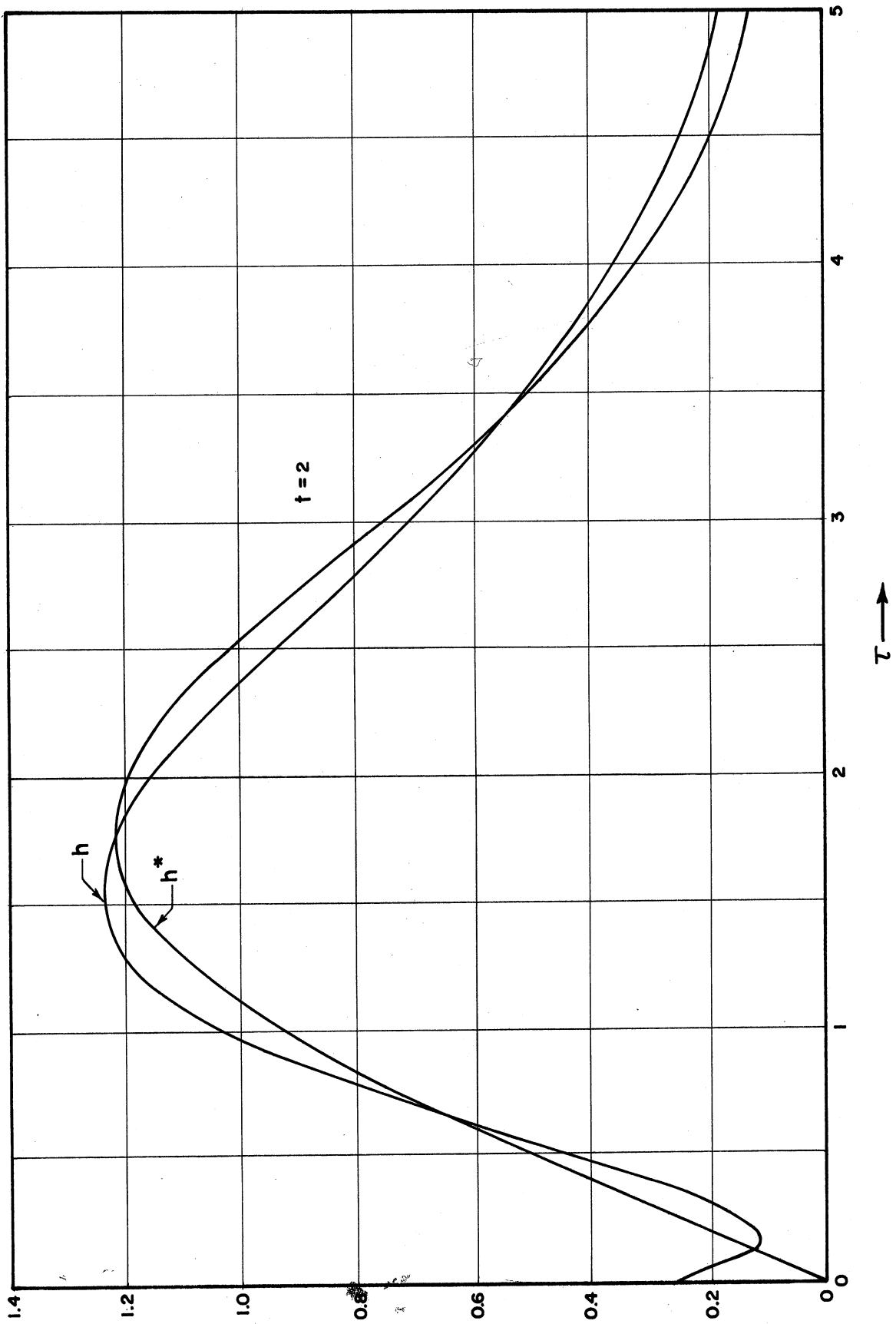


Figure 6e. $h(t, \tau)$ and $h^*(t, \tau)$ for $t = 2$.

REFERENCES

1. Ince, E. L. Ordinary Differential Equations. New York: Dover Publications, Inc., 1926.
2. Coddington, E. A. and Levinson, N. Theory of Ordinary Differential Equations. New York: McGraw-Hill Book Company, Inc., 1955.
3. Zadeh, L. A. "Initial Conditions in Linear Varying Parameter Systems." Journ. of Applied Physics, 22, (July, 1951), 782-786.
4. Zadeh, L. A. "On Stability of Linear Varying Parameter Systems." Journ. of Applied Physics, 22, (July, 1951), 402-405.
5. Miller, K. S. "Properties of Impulsive Responses and Green's Functions." IRE Transactions on Circuit Theory, CT-2, (March, 1955), 26-32.
6. Aseltine, J. A. "Transform Method for Linear Time-Varying Systems." Journ. of Applied Physics, 25, (June, 1954), 761-764.
7. Gerlach, A. A. "A Time-Variable Transform and Its Application to Spectral Analysis." IRE Transactions on Circuit Theory, CT-2, (March, 1955), 22-26.
8. Zadeh, L. A. "Frequency Analysis of Variable Networks." Proc. IRE, 38, (March, 1950), 291-299.
9. Cruz, J. B., Jr. and Van Valkenburg, M. E. "The Synthesis of Models for Time-Varying Linear Systems." Proc. of the Symposium on Active Networks and Feedback Systems, Polytechnic Inst. of Brooklyn, April, 1960.
10. Kautz, W. H. "Transient Synthesis in the Time Domain." IRE Transactions on Circuit Theory, CT-1, (Sept., 1954), 29-39.
11. Gilbert, E. G. "Linear System Approximation by Differential Analyzer Simulation of Orthonormal Approximation Functions." IRE Transactions on Electronic Computers, EC-8, (June, 1959), 204-209.
12. Brodin, J. "Analysis of Time-Dependent Networks." IRE Transactions on Circuit Theory, CT-2, (March, 1955), 12-17.
13. Lerner, R. M. "The Representation of Signals." IRE Transactions on Circuit Theory, CT-6, Special Supplement, (May, 1959), 197-216.
14. Gilbert, E. G. and Otterman, J. "The Synthesis of Linear Filters with Real or Imaginary Transfer Functions." AIEE Transactions, Part I, 79, (1960),

15. Cruz, J. B. "A Generalization of the Impulse Train Approximation for Time-Varying Linear System Synthesis in the Time Domain." IRE Transactions on Circuit Theory, CT-6, (Dec., 1959), 393.
16. Armstrong, H. L. "On the Representation of Transients by Series of Orthogonal Functions." IRE Transactions on Circuit Theory, CT-6, (Dec., 1959), 351-354.
17. Farmanfarma, G. "General Analysis and Stability Study Finite Pulsed Feedback Systems." AIEE Transactions, Part II, 77, (July, 1958), 148-162.
18. Gilbert, E. O. "Analysis of Linear Periodic Feedback Systems." AIEE Transactions, Part II, 79, (Jan., 1960), 512-523.
19. Lanning, J. H., Jr. and Battin, R. H. Random Processes in Automatic Control. New York: McGraw-Hill Book Company, Inc., 1956.
20. Mills, J. P. "Table of the Ratio: Area to Bounding Ordinate for any Portion of the Normal Curve." Biometrika, 18, (1926), 395-400.