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THE SYNTHESIS OF LINEAR FILTERS WITH REAL OR IMAGINARY TRANSFER FUNCTIONS

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INTRODUCTION

The synthesis of linear filters with symmetrical (even-function) or antisymmetrical (odd-function) impulse responses has both theoretical interest and practical value. Such filters which can be called time-symmetric or time-antisymmetric, following the terminology of Storey and Grierson,[1] are characterized by transfer functions which are purely real or purely imaginary. Alternate names, proposed and employed in this paper, are real or imaginary filters. Applications of real or imaginary filters include compensating systems which must have no phase distortion, such as those often used in the field of acoustics and in the compensation of signals generated by picture scanning devices.[2,3]

In the past little work of a general nature has been done. Although Kalmann,[4] Wiener and Lee,[5,6] and others [7,8] consider delay line transversal filters which have real or imaginary transfer functions, their emphasis is on transfer functions periodic in frequency. An excellent paper by Levy [9] describes some of the properties and the usefulness of real and imaginary transfer functions; however, it limits synthesis to a lumped element delay line filter with many experimentally adjusted reflections. Storey and Grierson [1] have used a tape recorder scheme which involves storing the input signal on magnetic tape and processing the signal through reverse playback. Since these past contributions are restricted to limited types of transfer functions and physical realizations, their significance has not generally been appreciated.

It is the purpose of this paper to extend previous work by developing fully the conditions under which real and imaginary filters can be realized and by utilizing an orthogonal function approach for solving the approximation problem. Filter realizations presented in this paper do require a delay element. And although it is true that real or imaginary filters can be realized with arbitrarily small error by means of conventional lumped-element filters, it is felt that the filters presented here can often be more simply and economically realized, particularly when phase accuracy requirements are high.

STATEMENT OF THE PROBLEM

The problem of synthesis is to realize physically and as closely as practicable a prescribed mathematical operation on an input function of time. For a linear, time invariant system the prescribed operation may be expressed by the superposition integral

$$f_o(t) = \int_0^\infty h(\tau) f_i(t - \tau)\, d\tau \qquad (1)$$

where t is time, f_i is the input function, f_o is the output function, and h(t) is the response of the prescribed system to the unit impulse $u_o(t)$ occurring at t = 0.

The operation may also be expressed in terms of the Fourier transforms of f_i, f_o, and h by*

$$F_o(j\omega) = H(j\omega)\ F_i(j\omega). \tag{2}$$

$H(j\omega)$ is the sinusoidal response function or the transfer function. The Fourier transform is used rather than the Laplace transform because impulse responses h(t) will be considered which are non-zero for negative t. Synthesis is assumed complete when a physical system is determined which has an impulse response h*(t) which approximates h(t) sufficiently well for a given purpose.

An approximation generally results because the physical system imposes the realizability condition, $h^*(t) = 0$ for $t < 0$, and possible practical conditions, such as being realized by a lumped-element system ($H(j\omega)$ rational in $j\omega$). In the present case either H_R or H_I (both real functions) in

$$H = H_R + jH_I \tag{3}$$

is zero. To approximate as closely as possible such an H, we will develop the conditions under which it is possible to realize an

$$H^* = H^*_R + j\ H^*_I \tag{4}$$

(H^*_R and H^*_I are real functions) where either $H^*_I = 0$ or $H^*_R = 0$, i.e., where the realized filter is either real or imaginary.

CONDITIONS FOR REALIZABILITY

From the Fourier transform we have

$$H^*_R = \int_{-\infty}^{\infty} h^*(t) \cos \omega t\ dt,$$

$$H^*_I = - \int_{-\infty}^{\infty} h^*(t) \sin \omega t\ dt. \tag{5}$$

Thus it is clear that a symmetric impulse response,

$$h^*(t) = h^*(-t), \tag{6}$$

yields a real filter ($H^*_I = 0$) and an antisymmetric impulse response

$$h^*(t) = -h^*(-t) \tag{7}$$

yields an imaginary filter ($H^*_R = 0$).

*The upper-case-letter functions denote the transforms of the lower-case-letter functions; i.e., $H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\ h(t)\ dt$. It is tacitly assumed that the transforms exist when employed.

Since realizability requires $h^*(t) = 0$ for $t < 0$, it is clear from (6) and (7) that realizable real and imaginary filters require $h^*(t) = 0$ except at $t = 0$. The only functions which satisfy this condition are the unit impulse u_o and its derivatives, u_1, $u_2, \ldots$. But the impulse derivatives have no value as approximating functions and are not realizable by physical systems having finite gain at infinite frequency. Hence real and imaginary filters cannot be realized directly.

By introducing a time delay T, certain real and imaginary filters can be realized. The time delay allows $h^*(t)$ to be symmetic or antisymmetic about $t = T$ instead of $t = 0$. Realizability then means that

$$h^*(t) = 0, \qquad t < 0, \qquad t > 2T \tag{8}$$

$$h^*(t + T) = h^*(-t + T), \qquad 0 \leq t \leq T \tag{9}$$
(a delayed real filter)

$$h^*(t + T) = -h^*(-t + T) \qquad 0 \leq t \leq T \tag{10}$$
(a delayed imaginary filter)

Fortunately, in most practical applications the added delay T does not limit the usefulness of real and imaginary filters.

Since lumped-element systems have impulse responses of the form

$$h^*(t) = \sum_{n=1}^{N} c_n t^{k_n} e^{\alpha_n t} + c_o u_o(t), \quad t \geq 0 \tag{11}$$
$$= 0, \quad t < 0$$

(k_n = positive integer, α_n = real, imaginary, or complex number) and have infinite memory, it is obvious that they cannot satisfy (8) which states that $h^*(t)$ has a finite memory of 2T. Only through the introduction of distributed elements may (8) and (9) or (8) and (10) be satisfied.

PHYSICAL REALIZATION

Figure 1 shows how a single delay-element transfer function $e^{-j\omega 2T}$ and two lumped-element transfer functions, $H_1(j\omega)$ and $H_2(j\omega)$, may be used to realize $H^*(j\omega)$. For $t \leq 2T$ the impulse response is simply $h_1(t)$; for $t > 2T$ the impulse response is $h_1(t) - h_2(t - 2T)$. To make $h^*(t) = 0$ for $t > 2T$ it is necessary to make the responses of the two paths cancel for $t > 2T$, i.e., to make $h_1(t) = h_2(t - 2T)$ for $t > 2T$. Thus

$$\begin{aligned} h^*(t) &= 0, && t < 0 \\ &= h_1(t), && 0 \leq t \leq 2T \\ &= 0, && t > 2T \end{aligned} \tag{12}$$

and

$$h_2(t) = 0, \qquad t \leq 0$$
$$= h_1(t + 2T), \qquad t > 0 \qquad (13)$$

Since h_1 must symmetric or antisymmetric about $t = T$ in $(0, 2T)$ it is clear that h_1 can have no terms with an exponential function of time as a factor. That is, the terms in h_1 must be of the following type:

Type of functions	Form of realization
$1, t, t^2, -- t^k$	k integrators
$\cos \omega_1 t, \sin \omega_1 t, \cos \omega_2 t, \sin \omega_2 t,$ ---	undamped filter with poles at $j\omega = \pm j\omega_1, \pm j\omega_2,$ ---
$\cos \omega_1 t, \sin \omega_1 t, t \cos \omega_1 t, t \sin \omega_1 t,$ ---, $t^k \cos \omega_1 t, t^k \sin \omega_1 t.$	undamped filter with $(k + 1)$ order poles at $j\omega = \pm j\omega_1$

In every case the poles of H_1 (and also the poles of H_2 since by (13) the poles of H_2 must also be the poles of H_1) are on the imaginary axis in the complex p plane. In a later section approximating series based on the first two types of terms will be described.

Since passive circuit realization of H_1 or H_2 demands lossless inductors and capacitors, active circuitry is preferred for high accuracy realization. Actually, active circuit realization is mandatory where H_1 or H_2 are voltages ratios and have poles at $p = 0$.

Other forms for $h^*(t)$ can be obtained if more delay elements are introduced. Figure 2 shows an example. Here

$$h^*(t) = h_T(t) + h_{2T}(t) \qquad (14)$$

where

$$h_T(t) = 0, \qquad t < 0, \qquad t > T$$
$$h_{2T}(t) = 0, \qquad t < T, \qquad t > 2T \qquad (15)$$

and

$$h_{2T}(t + T) = h_T(-t + T), \qquad 0 \leq t \leq T \qquad (16)$$
(a delayed real filter)

or

$$h_{2T}(t + T) = -h_T(-t + T), \qquad 0 \leq t \leq T \qquad (17)$$
(a delayed imaginary filter)

The functions h_T and h_{2T} are realized as $h^*(t)$ was realized in Figure 1. In addition to requiring more than one delay element, the realization possesses practical limitations. If either h_T or h_{2T} contains a term with a decaying exponential factor, then by the symmetry conditions (16) or (17) the other contains a term with a corresponding growing exponential factor which leads to an active unstable lumped element filter in the overall realization. Since the matching of gains in the unstable lumped element filters cannot be exact, it is impossible to make $|h^*(t)| < \infty$ as $t \to \infty$. When the lumped element filters in h_T and h_{2T} have all their poles on the imaginary axis in the p plane a gain matching problem is still present, though it is not nearly so severe. Further remarks on accuracy of realization are given in the section titled "Illustrative Examples."

Many other types of responses are available from filters which contain both lumped elements and delay elements. Various authors have discussed limited classes of such filters.[10,11] For the realization of real and imaginary filters it does not seem worth-while to use more than a single delay element unless several delay elements will allow exact or nearly exact synthesis of the prescribed impulse response (at least within the time delay T). Certainly, when h(t) consists of sections of rather simple functions this is the case. Another example occurs when

$$h^*(t) = \sum_{n=1}^{N} a_n u_o(t - t_n) \tag{18}$$

$$t_n = \text{real number}$$

$$a_n = \text{real number}$$

Such an $h^*(t)$ can be realized with a tapped delay line and a summing amplifier and leads to the periodic transfer functions described by Kalmann,[4] Wiener and Lee,[5,6] White and Ruvin,[7] and Urkowitz.[8]

THE APPROXIMATION PROBLEM

Approximation involves the representation of a prescribed system function ($h(t)$ or $H(j\omega)$) in terms of a physically realizable system function ($h^*(t)$ or $H^*(j\omega)$). If the physical system is lumped, then $H^*(j\omega)$ is rational in $j\omega$; if the physical system has a delayed real or imaginary transfer function, then $h^*(t)$ and $H^*(j\omega)$ assume the forms discussed in the previous section. Of the many approximating techniques available, this paper will present a minimum integral-square-error approach based on orthogonal approximating functions. Orthogonal function approximation has the following advantages: 1) the integral-square-error is minimized and easily computed, 2) the approximation may be computed readily in either the time or frequency domains, 3) approximation error tends to be small throughout the interval of expansion (as opposed to a Taylor expansion where the approximation tends to be good only at one point), 4) orthogonal functions suitable for realization of real and imaginary filters are well known and tabulated, 5) additional terms may be added to an approximating series to further reduce error without forcing a new calculation of the terms already determined. Orthogonal functions have been employed by many authors for the synthesis of lumped-element systems.[12,13,14,15] However, because of the objectives of this paper, different types of approximating functions must be considered here. Whereas the orthogonal time functions which appear in the series for $h^*(t)$ for a lumped-element filter consist of exponential functions defined in the time interval $(0,\infty)$, the functions used in real or imaginary filter approximation must be orthogonal in $(0, 2T)$ and be symmetric or antisymmetric about $t = T$.

ORTHOGONAL APPROXIMATING FUNCTIONS

Legendre Polynomials

When $h^*(t)$ is a polynomial in t, as is the case when a real or imaginary filter is realized by a delay line and a number of integrators, Legendre polynomials are employed to form an orthogonal set $\Theta_1(t)$, $\Theta_2(t)$, ... Since Legendre polynomials are orthogonal in $(-1, 1)$ it is necessary to time scale the prescribed response so $h(t)$ is satisfactorily represented in $(-1, 1)$. Once the approximation has been completed, $h^*(t)$, which is in $(-1, 1)$, may be delayed and time scaled to $(0, 2T)$ for physical realization. A later example will illustrate how these delaying and scaling steps are carried out.

Following the usual definition of the Legendre polynomials the Θ_n are given by 16

$$\begin{aligned} \Theta_n &= 0, && |t| > 1,\ n = 0, 1, 2, \ldots \\ &= 1, && -1 \le t \le 1,\ n = 0 \\ &= \frac{1}{2^n n!}\,\frac{d^n(t^2-1)^n}{dt^n}, && -1 \le t \le 1,\ n = 1, 2, 3, \end{aligned} \tag{19}$$

This is know as Rodrigues' formula. Several of the Θ_n are tabulated and plotted in Table I. If higher order functions are required they can be obtained most conveniently from the recursion relation 16

$$n\Theta_n = (2n - 1)\,t\,\Theta_{n-1} - (n - 1)\,\Theta_{n-2} \tag{20}$$

The approximating series assumes the form

$$h^*(t) = \sum_{n=0}^{N} c_n\Theta_n(t) \tag{21}$$

where

$$c_n = (n + \tfrac{1}{2}) \int_{-1}^{1} h\Theta_n dt = (n + \tfrac{1}{2}) \int_{-\infty}^{\infty} h\Theta_n dt. \tag{22}$$

The factor $(n + \frac{1}{2})$ occurs in (22), because the Θ_n are not normalized, i.e., $\int_{-1}^{1}\Theta_n^2 dt = (n + \frac{1}{2})^{-1} \neq 1$. Since N is finite (filter complexity grows directly with N) the integral-square error,

$$E = \int_{-\infty}^{\infty} (h-h^*)^2 dt = \int_{-\infty}^{\infty} h^2 dt - \sum_{n=0}^{N} \frac{c_n^2}{n+\frac{1}{2}}, \tag{23}$$

is in general not zero. Approximation accuracy requirements set an upper bound on E which in turn establishes the minimum N.

From Table I it is seen that the θ_n are symmetric functions for even n and antisymmetric functions for odd n. Thus the even terms are used for a real filter approximation and the odd terms for an imaginary filter approximation. This can also be seen from (22) which gives $c_n = 0$ for n odd when h is a symmetric function and $c_n = 0$ for n even when h is an antisymmetric function.

In the frequency domain (21), (22), and (23) become respectively

$$H^*(j\omega) = \sum_{n=0}^{N} c_n \Theta_n(j\omega), \tag{24}$$

$$c_n = \frac{1}{2\pi}\left(n + \frac{1}{2}\right) \int_{-\infty}^{\infty} H\bar{\Theta}_n \, d\omega, \tag{25}$$

and

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H|^2 d\omega - \sum_{n=0}^{N} \frac{c_n^2}{n+\frac{1}{2}}, \tag{26}$$

where $\Theta_n(j\omega)$ is the Fourier transform of $\theta_n(t)$ and $\bar{\Theta}_n$ is the complex conjugate of Θ_n. Expressions (25) and (26) are derived from (22) and (23) by application of Parseval's theorem.[17]

The formulas for Θ_0 and Θ_1 are

$$\Theta_o = \int_{-\infty}^{\infty} e^{-j\omega t} \theta_o \, dt = \int_{-1}^{1} e^{-j\omega t} \, dt = \frac{2 \sin\omega}{\omega} \tag{27}$$

and

$$\Theta_1 = \int_{-\infty}^{\infty} e^{-j\omega t} \theta_1 \, dt = \int_{-1}^{1} e^{-j\omega t} t \, dt = j2\left[\frac{\cos\omega}{\omega} - \frac{\sin\omega}{\omega^2}\right] \tag{28}$$

Although the higher order functions can be obtained in the same way, a much more rapid procedure is to use the recursion formula

$$\Theta_n = \frac{2n-1}{j\omega} \Theta_{n-1} + \Theta_{n-2} \tag{29}$$

which is developed in the Appendix. The Θ_n are also tabulated and plotted in Table I. As should be expected, the Θ_n are real for even n and imaginary for odd n.

Table I shows that the approximation H* contains terms of the type

$$j\,\frac{\sin\omega}{(j\omega)^n} = \frac{e^{j\omega} - e^{-j\omega}}{2(j\omega)^n}$$

and

$$\frac{\cos\omega}{(j\omega)^n} = \frac{e^{j\omega} + e^{-j\omega}}{2(j\omega)^n}$$

When h*(t) is delayed by T = 1 to the interval (0, 2) for realization (time scaling to (0, 2T) will not be considered here) functions of the type

$$\frac{1 - e^{-j\omega 2}}{2(j\omega)^n}$$

and

$$\frac{1 + e^{-j\omega 2}}{2(j\omega)^n}$$

are obtained. Thus H* assumes the form

$$H^* = \sum_{n=0}^{N} q_n \frac{1}{(j\omega)^n} + e^{-j\omega 2} \sum_{n=0}^{N} r_n \frac{1}{(j\omega)^n} \qquad (30)$$

where the q_n and r_n are determined from the c_n and the expressions for the Θ_n. Figure 3 shows the realization.

Sine and Cosine Functions in the Time Domain

If a real or imaginary filter is realized with a single delay line and an undamped lumped element filter, h* consists of sections of sine and cosine functions and the most obvious orthogonal series is a Fourier series in the time domain.* For convenience we will consider a Fourier series with the interval of expansion (-1, 1). Then the previous remarks which were made concerning time scale still hold.

*It is possible to generate other orthogonal series from sine and cosine functions. For example the sine and cosine functions would not have to be harmonic.

Since the period of expansion is 2 the Fourier series can be written as

$$h^* = \frac{a_o}{2} + \sum_{n=1}^{N} (a_n \cos \pi nt + b_n \sin \pi nt), \quad -1 \leq t \leq 1$$

$$= 0, \quad |t| > 1 \tag{31}$$

Because two types of approximating functions are involved for each n (sin πnt and cos πnt) the functions

$$\alpha_n(t) = \cos \pi nt, \qquad -1 \leq t \leq 1$$

$$= 0, \qquad |t| > 1 \tag{32}$$

$$\beta_n(t) = \sin \pi nt, \qquad -1 \leq t \leq 1$$

$$= 0, \qquad |t| > 1 \tag{33}$$

are defined. The Fourier series (31) then becomes

$$h^*(t) = \frac{a_o}{2} \alpha_o + \sum_{n=1}^{N} a_n \alpha_n + \sum_{n=1}^{N} b_n \beta_n \tag{34}$$

where the coeffiients are given by

$$a_n = \int_{-\infty}^{\infty} h \, \alpha_n \, dt = \int_{-1}^{1} h \cos \pi nt \, dt \tag{35}$$

and

$$b_n = \int_{-\infty}^{\infty} h \, \beta_n \, dt = \int_{-1}^{1} h \sin \pi nt \, dt \tag{36}$$

For a real filter $b_n = 0$ and for an imaginary filter $a_n = 0$. Thus the α_n form real filter approximations and the β_n form imaginary filter approximations.

In the frequency domain the real-filter functions are given by the Fourier transform of the α_n

$$A_n = \int_{-\infty}^{\infty} e^{-j\omega t} \alpha_n \, dt = \int_{-1}^{1} e^{-j\omega t} \cos \pi nt \, dt$$

$$= \frac{j\omega(-1)^n \, (e^{j\omega} - e^{-j\omega})}{(j\omega)^2 + (\pi n)^2}$$

$$= -\frac{2\omega(-1)^n \sin\omega}{(j\omega)^2 + (\pi n)^2} \tag{37}$$

Similarly for the imaginary-filter functions

$$B_n = \int_{-\infty}^{\infty} e^{-j\omega t} \beta_n \, dt = \int_{-1}^{1} e^{-j\omega t} \sin \pi nt \, dt$$

$$= \frac{\pi n(-1)^n \, (e^{j\omega} - e^{-j\omega})}{(j\omega)^2 + (\pi n)^2}$$

$$= j \, \frac{2\pi n(-1)^n \sin\omega}{(j\omega)^2 + (\pi n)^2} \qquad (38)$$

Table II summarizes these results and shows plots of the functions α_n, β_n, A_n, and B_n.*

The approximation coefficients can be evaluated in the frequency domain by the following equations (obtained by applying Parseval's theorem to (35) and (36)

$$a_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} H \, \bar{A}_n \, d\omega \qquad (39)$$

$$b_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} H \, \bar{B}_n \, d\omega \qquad (40)$$

The integral-square-error is

$$E = \int_{-1}^{1} h^2 dt - \frac{a_o^2}{2} - \sum_{n=1}^{N} (a_n^2 + b_n^2) = \frac{1}{2\pi} \int_{-1}^{+1} |H|^2 d\omega - \frac{a_o^2}{2} - \sum_{n=1}^{N} (a_n^2 + b_n^2). \qquad (41)$$

When h* is delayed to the interval (0, 2) for realization the functions A_n and B_n become

$$A_n \, e^{-j\omega} = \frac{j\omega(-1)^n \, (1 - e^{-j\omega 2})}{(j\omega)^2 + (\pi n)^2} \qquad (42)$$

*A brief tabulation of the A_n and B_n, and also the Θ_n is given in Report S-58 of the Radiation Laboratory, M.I.T., August 1945.

and

$$B_n\, e^{-j\omega} = \frac{\pi n(-1)^n\,(1 - e^{-j\omega 2})}{(j\omega)^2 + (\pi n)^2} \qquad (43)$$

Thus H* assumes the form

$$H* = (1 - e^{-j\omega 2})H_3 \qquad (44)$$

and may be realized as shown in Figure 4. For a real filter

$$H_3 = \frac{a_o}{2j\omega} + \sum_{n=1}^{N} \frac{a_n(j\omega)(-1)^n}{(j\omega)^2 + (\pi n)^2}\,; \qquad (45)$$

for an imaginary filter

$$H_3 = \sum_{n=1}^{N} \frac{b_n \pi n(-1)^n}{(j\omega)^2 + (\pi n)^2}\,. \qquad (46)$$

H_3 can be realized as a voltage ratio with lossless passive circuits except for the $\frac{a_o}{2j\omega}$ term.

ILLUSTRATIVE EXAMPLES

For the first example a real, ideal low pass filter is approximated in the time interval (-1, 1) by sine and cosine functions. If the ideal filter has a cutoff frequency ω_c, then the transfer function is:

$$H = H_R = 1, \qquad -\omega_c \leq \omega \leq \omega_c$$
$$= 0, \qquad |\omega| > \omega_c \qquad (47)$$

and

$$h = \frac{\sin\omega_c t}{\pi t} \qquad (48)$$

The coefficients a_n (the $b_n = 0$) are determined most easily from (35) with h given by (48):

$$a_n = \int_{-1}^{1} \frac{\sin\omega_c t}{\pi t} \cos \pi n t\, dt = 2\int_0^1 \frac{\sin\omega_c t}{\pi t} \cos \pi n t\, dt$$

$$= \frac{1}{\pi}\int_0^1 \frac{\sin(\omega_c + \pi n)t}{t}\,dt + \frac{1}{\pi}\int_0^1 \frac{\sin(\omega_c - \pi n)t}{t}\,dt$$

$$= \frac{1}{\pi}\,[S_i(\omega_c + \pi n) + S_i(\omega_c - \pi n)]. \tag{49}$$

$S_i(x) = \int_0^x \frac{\sin y}{y}\,dy$, the sine integral function, for which tables are available. The integral-square error is given by (41) as

$$E = \frac{\omega_c}{\pi} - \frac{a_o^2}{2} - \sum_{n=1}^{N} a_n^2 \tag{50}$$

A measure of the fractional approximation error is the relative mean square error

$$E_{rel} = \frac{E}{\int_{-\infty}^{\infty} h^2 dt} \tag{51}$$

which in this case is

$$E_{rel} = 1 - \frac{\pi}{\omega_c}\left(\frac{a_o^2}{2} + \sum_{n=1}^{N} a_n^2\right) \tag{52}$$

Since E_{rel} varies with ω_c (the a_n depend on ω_c) it is natural to ask what cutoff frequency yields the best approximation. Table III gives some results for N = 3,4,5. For N = 3 (four non-zero terms in h*) the smallest E_{rel} is obtained for $\omega_c = 3.5\pi$. Figure 5 gives the a_n and shows h*(t) and H*(jω) for $\omega_c = 3.5\pi$ and N = 3. It is not surprising that $\omega_c = 3.5\pi$ gives the best approximation since $3.5\pi \cong 3\pi$, the highest frequency component in h*(t) for N = 3.

Inasmuch as the $\omega_c = 3.5\pi$ design gives the smallest E_{rel} it should serve as the prototype for other cutoff frequencies. Thus, where a cutoff frequency of 3.5π yields T = 1, a cutoff frequency of ω_c yields $T = \frac{3.5\pi}{\omega_c}$. The transfer function is

$$H^* = 2\left(\frac{3.5\pi\omega}{\omega_c}\right)\sin\frac{3.5\pi\omega}{\omega_c}\left[.5025\left(\frac{\omega_c}{3.5\pi\omega}\right)^2 + \frac{-.9937}{\pi^2 - \left(\frac{3.5\pi\omega}{\omega_c}\right)^2} + \frac{1.0130}{4\pi^2 - \left(\frac{3.5\pi\omega}{\omega_c}\right)^2} + \frac{-.9356}{9\pi^2 - \left(\frac{3.5\pi\omega}{\omega_c}\right)^2}\right] \tag{53}$$

and can be realized in the form of Figure 4 with an added delay $T = \frac{3.5\pi}{\omega_c}$.

Proceeding in a similar fashion the real, ideal low pass filter can be approximated by Legendre polynomials. The c_n are given by (22) with h given by (48) and θ_n by Table I. Of course, $c_n = 0$ for odd n. Equation (22) involves integrals of the form $\int_0^1 t^k \sin \omega_c t \, dt$ (k = -1, 1, 3, 5...) which are readily evaluated. From (23) and (51)

$$E_{rel} = 1 - \frac{\pi}{\omega_c} \sum_{n=0}^{N} \frac{1}{n + \frac{1}{2}} c_n^2 \tag{54}$$

In this case the smallest E_{rel} is obtained for $\omega_c = 2\pi$ when N = 6 (four non-zero terms in h*). Figure 6 gives the c_n and shows $h^*(t)$ and $H^*(j\omega)$ for $\omega_c = 2\pi$ and N = 6. Note that $E_{rel} = .0506$ which is somewhat higher than obtained for the same number of terms in the cosine function representation. Using the above design for an N = 6 prototype yields $T = \frac{2\pi}{\omega_c}$ and the transfer function

$$H^* = \sin \frac{2\pi\omega}{\omega_c} [-.1988 \frac{\omega_c}{2\pi\omega} + 115.6 (\frac{\omega_c}{2\pi\omega})^3 - 5,195. (\frac{\omega_c}{2\pi\omega})^5 + 12,140. (\frac{\omega_c}{2\pi\omega})^7]$$

$$+ \cos \frac{2\pi\omega}{\omega_c} [- 2.747 (\frac{\omega_c}{2\pi\omega})^2 + 1,148. (\frac{\omega_c}{2\pi\omega})^4 - 12,140. (\frac{\omega_c}{2\pi\omega})^6] \tag{55}$$

Realization in the form of Figure 3 requires an additional delay $T = \frac{2\pi}{\omega_c}$

The last example is a sine and cosine function approximation of the imaginary, ideal low pass filter characteristic

$$\begin{aligned} H = jH_I &= +j, & 0 < \omega \leq \omega_c \\ &= -j, & -\omega_c \leq \omega < 0 \\ &= 0, & \omega = 0, \ |\omega| > \omega_c \end{aligned} \tag{56}$$

and

$$h = \frac{\cos \omega_c t - 1}{\pi t} \tag{57}$$

The coefficients b_n (the $a_n = 0$) are given from (36) and (57) by

$$b_n = 2\int_0^1 \frac{\cos \omega_c t - 1}{\pi t} \sin \pi n t \, dt$$

$$= \frac{1}{\pi} [S_i(\pi n + \omega_c) - 2S_i(\pi n) + S_i(\pi n - \omega_c)] \tag{58}$$

and the relative mean square error by

$$E_{rel} = 1 - \frac{\pi}{\omega_c} \sum_{n=1}^{N} b_n^2 \tag{59}$$

In this case, E_{rel} is near its smallest value at $\omega_c = 3\pi$ when $N = 4$ (four non-zero terms in h*) where $E_{rel} = .119$. The relatively large E_{rel} is probably due to the discontinuity in H at $\omega = 0$ in addition to those at $\omega = \pm\omega_c$. Figure 7 gives the b_n and shows $h^*(t)$ and $H^*_I(j\omega)$ for $\omega_c = 3\pi$ and $N = 4$.

Both of the realizations in Figure 3 and Figure 4 have parallel signal paths, one of which is direct and a second which is through a delay line. Gain matching in the two paths must be exact or cancellation of $h_1(t)$ and $h_2(t - 2T)$ will not occur for $t > 2T$. In the case of the Legendre realization any difference between $h_1(t)$ and $h_2(t - 2T)$ grows as t^N which causes a large error in the realized frequency response at zero frequency; in the case of sines or cosines realization the difference between $h_1(t)$ and $h_2(t - 2T)$ oscillates with small amplitude at frequencies $\omega = \pi n$ which causes undesired resonances in the realized frequency response. A practical expedient for reducing the criticalness in gain matching is to shift the singularities of H^* slightly to the left in complex p plane (by replacing $j\omega$ in $H^*(j\omega)$ by $j\omega + \sigma_o$). This causes $h^*(t)$ to become multiplied by $e^{-\sigma_o t}$, a damping factor which is effective in reducing the error for large t. The shift σ_o should be made as small as possible consistent with the obtainable accuracy in gain matching as the shift causes H^* to depart from the desired real or imaginary characteristics.

CONCLUSIONS

In this paper the synthesis of real and imaginary transfer functions has been described. Realization is only possible within a time lag T, and requires a delay element filter with a finite memory of 2T. The approximation problem has been solved in both the frequency domain and time domain by the introduction of two sets of orthogonal approximating functions, one based on Legendre polynomials in the time domain and the other on sines and cosines in the time domain. It appears that the latter approximating functions have the following realization advantages: 1) realization is simple in form, consisting of $(1 - e^{-j\omega 2T})$ and H_3 in cascade and 2) at most, only one pole is required at $p = 0$.

APPENDIX

Development of Recursion Formula for Legendre Function $\Theta_n(j\omega)$

From (19)

$$\Theta_n(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, \Theta_n\, dt$$

$$= \frac{1}{2^n\, n!} \int_{-1}^{1} \frac{d^n (t^2 - 1)^n}{dt^n}\, e^{-j\omega t}\, dt \qquad (A - 1)$$

Integration by parts of (A - 1) gives (making the derivative of $e^{-j\omega t}$ appear in the new integrand)

$$\Theta_n(j\omega) = \frac{1}{2^n\, n!} \left\{ \frac{d^{n-1} (t^2 - 1)^n}{dt^{n-1}}\, e^{-j\omega t} \right\} \Big|_{-1}^{1}$$

$$+\, j\omega \int_{-1}^{1} \frac{d^{n-1} (t^2 - 1)^n}{dt^{n-1}}\, e^{-j\omega t}\, dt, \qquad (A - 2)$$

but $\dfrac{d^{n-1} (t^2 - 1)^n}{dt^{n-1}} = 0$ for $t = \pm 1$. Hence

$$\Theta_n = j\omega \int_{-1}^{1} \frac{d^{n-1} (t^2 - 1)^n}{dt^{n-1}}\, e^{-j\omega t}\, dt \qquad (A - 3)$$

Repeating the integration by parts a total of n times yields

$$\Theta_n = \frac{(j\omega)^n}{2^n\, n!} \int_{-1}^{1} (t^2 - 1)^n\, e^{-j\omega t}\, dt \qquad (A - 4)$$

Integration by parts of (A - 4) gives (making the integral of $e^{-j\omega t}$ appear in the new integrand)

$$\Theta_n = \frac{(j\omega)^n}{2^n\, n!} \left\{ (t^2 - 1)^n \frac{e^{-j\omega t}}{(-j\omega)} \right\} \Big|_{-1}^{1} + \frac{(j\omega)^{n-1}}{2^{n-1}(n-1)!} \int_{-1}^{1} (t^2 - 1)^{n-1}\, t\, e^{-j\omega t}\, dt$$

$$= \frac{(j\omega)^{n-1}}{2^{n-1}(n-1)!} \int_{-1}^{1} (t^2 - 1)^{n-1}\, t\, e^{-j\omega t}\, dt, \quad n > o \qquad (A - 5)$$

Finally, a similar integration by parts of (A - 5) gives

$$\Theta_n = \frac{(j\omega)^{n-1}}{2^{n-1}\,(n-1)!} \quad \left\{ (t^2 - 1)^{n-1}\, t\, \frac{e^{-j\omega t}}{(-j\omega)} \right\} \Big|_{-1}^{1} + \frac{2(n-1)}{j\omega} \int_{-1}^{1} (t^2 - 1)^{n-2}\, t^2\, e^{-j\omega t}\, dt$$

$$+ \frac{1}{j\omega} \int_{-1}^{1} (t^2 - 1)^{n-1}\, e^{-j\omega t}\, dt \quad = \frac{(j\omega)^{n-2}}{2^{n-2}\,(n-2)!} \int_{-1}^{1} (t^2 - 1)^{n-2}\, t^2\, e^{-j\omega t}\, dt$$

$$+ \frac{(j\omega)^{n-2}}{2^{n-1}\,(n-1)!} \int_{-1}^{1} (t^2 - 1)^{n-1}\, e^{-j\omega t}\, dt, \; n > 1 \qquad (A - 6)$$

Adding and subtracting $\frac{(j\omega)^{n-2}}{2^{n-2}\,(n-2)!} \int_{-1}^{1} (t^2 - 1)^{n-2}\,(t^2 - 1)\, e^{-j\omega t}\, dt$,

$$\Theta_n = \frac{(j\omega)^{n-2}}{2^{n-2}\,(n-2)!} \int_{-1}^{1} (t^2 - 1)^{n-2}\,(t^2 - 1)\, e^{-j\omega t}\, dt$$

$$+ \frac{j\omega^{n-2}}{2^{n-1}\,(n-1)!} \int_{-1}^{1} (t^2 - 1)^{n-1}\, e^{-j\omega t}\, dt$$

$$+ \frac{(j\omega)^{n-2}}{2^{n-2}\,(n-2)!} \int_{-1}^{1} (t^2 - 1)^{n-2}\, e^{-j\omega t}\, dt \qquad (A - 7)$$

which can be written as (because of (A - 4)

$$\Theta_n = \frac{2(n-1)}{j\omega}\,\Theta_{n-1} + \frac{1}{j\omega}\,\Theta_{n-1} + \Theta_{n-2} = \frac{2n-1}{j\omega}\,\Theta_{n-1} + \Theta_{n-2} \qquad (A - 8)$$

This is the desired result.

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TABLE I

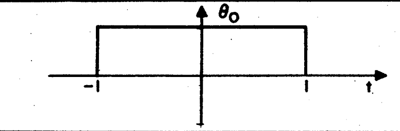
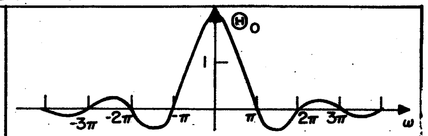
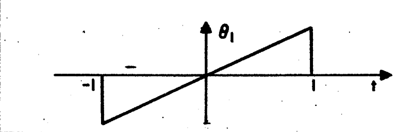
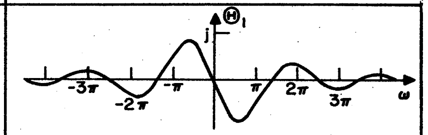
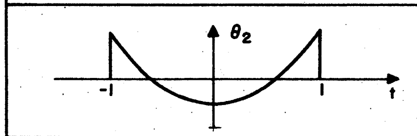
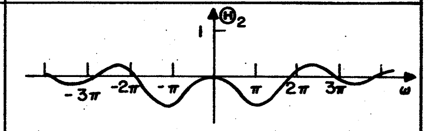
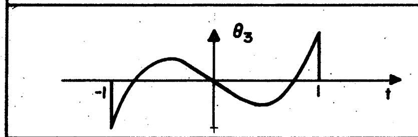
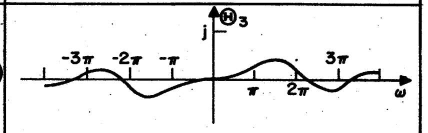
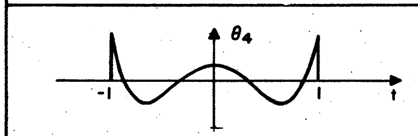
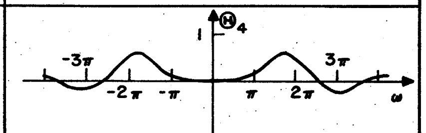
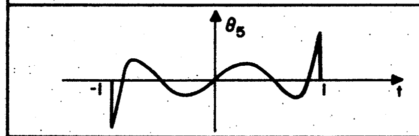
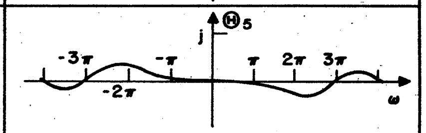
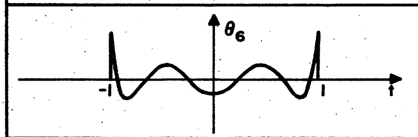
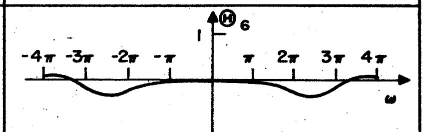
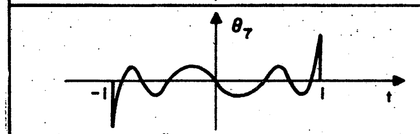
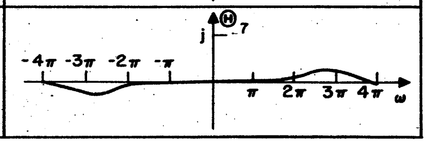
TIME DOMAIN		FREQUENCY DOMAIN	
	$\theta_0 = 1$	$\Theta_0 = \frac{2 \sin \omega}{\omega}$	
	$\theta_1 = t$	$\Theta_1 = j\left(-\frac{2 \sin \omega}{\omega^2} + \frac{2 \cos \omega}{\omega}\right)$	
	$\theta_2 = \frac{1}{2}(3t^2 - 1)$	$\Theta_2 = -\frac{6 \sin \omega}{\omega^3} + \frac{6 \cos \omega}{\omega^2} + \frac{2 \sin \omega}{\omega}$	
	$\theta_3 = \frac{1}{2}(5t^3 - 3t)$	$\Theta_3 = j\left(\frac{30 \sin \omega}{\omega^4} - \frac{30 \cos \omega}{\omega^3} - \frac{12 \sin \omega}{\omega^2} + \frac{2 \cos \omega}{\omega}\right)$	
	$\theta_4 = \frac{1}{8}(35t^4 - 30t^2 + 3)$	$\Theta_4 = \frac{210 \sin \omega}{\omega^5} - \frac{210 \cos \omega}{\omega^4} - \frac{90 \sin \omega}{\omega^3} + \frac{20 \cos \omega}{\omega^2} + \frac{2 \sin \omega}{\omega}$	
	$\theta_5 = \frac{1}{8}(63t^5 - 70t^3 + 15t)$	$\Theta_5 = j\left(\frac{-1890 \sin \omega}{\omega^6} + \frac{1890 \cos \omega}{\omega^5} + \frac{840 \sin \omega}{\omega^4} - \frac{210 \cos \omega}{\omega^3} - \frac{30 \sin \omega}{\omega^2} + \frac{2 \cos \omega}{\omega}\right)$	
	$\theta_6 = \frac{1}{16}(231t^6 - 315t^4 + 105t^2 - 5)$	$\Theta_6 = \frac{-20790 \sin \omega}{\omega^7} + \frac{20790 \cos \omega}{\omega^6} + \frac{9450 \sin \omega}{\omega^5} - \frac{2520 \cos \omega}{\omega^4} - \frac{420 \sin \omega}{\omega^3} + \frac{42 \cos \omega}{\omega^2} + \frac{2 \sin \omega}{\omega}$	
	$\theta_7 = \frac{1}{16}(429t^7 - 693t^5 + 315t^3 - 35t)$	$\Theta_7 = j\left(\frac{270270 \sin \omega}{\omega^8} - \frac{270270 \cos \omega}{\omega^7} - \frac{124740 \sin \omega}{\omega^6} + \frac{34650 \cos \omega}{\omega^5} + \frac{6300 \sin \omega}{\omega^4} - \frac{756 \cos \omega}{\omega^3} - \frac{56 \sin \omega}{\omega^2} + \frac{2 \cos \omega}{\omega}\right)$	

Table II

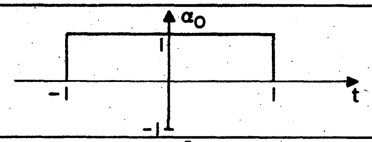
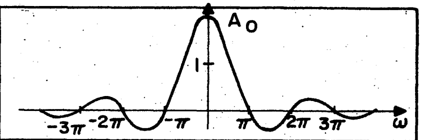
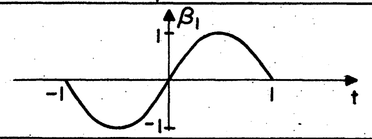
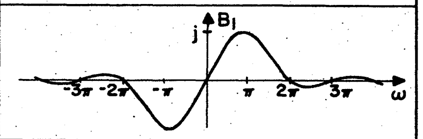
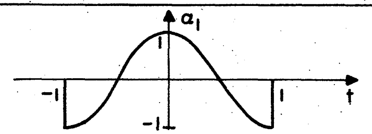
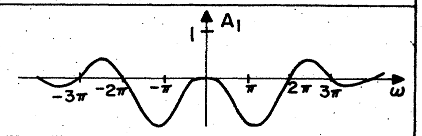
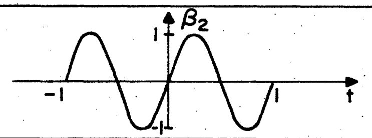
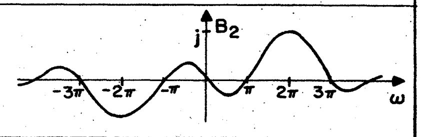
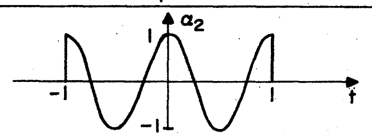
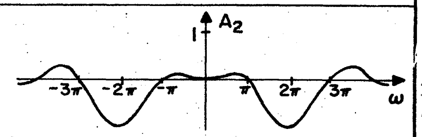
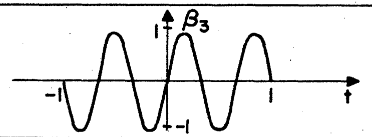
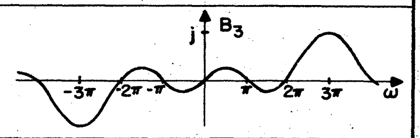
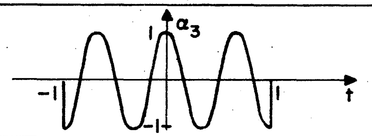
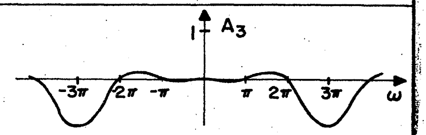
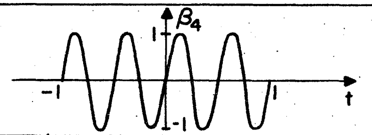
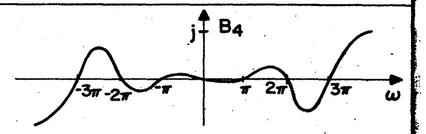
TIME DOMAIN		FREQUENCY DOMAIN	
	$\alpha_0(t) = 1$	$A_0(\omega) = \frac{2 \sin \omega}{\omega}$	
	$\beta_1(t) = \sin \pi t$	$B_1(\omega) = +j \frac{2\pi \sin \omega}{\pi^2 - \omega^2}$	
	$\alpha_1(t) = \cos \pi t$	$A_1(\omega) = - \frac{2\omega \sin \omega}{\pi^2 - \omega^2}$	
	$\beta_2(t) = \sin 2\pi t$	$B_2(\omega) = -j \frac{4\pi \sin \omega}{4\pi^2 - \omega^2}$	
	$\alpha_2(t) = \cos 2\pi t$	$A_2(\omega) = \frac{2\omega \sin \omega}{4\pi^2 - \omega^2}$	
	$\beta_3(t) = \sin 3\pi t$	$B_3(\omega) = +j \frac{6\pi \sin \omega}{9\pi^2 - \omega^2}$	
	$\alpha_3(t) = \cos 3\pi t$	$A_3(\omega) = - \frac{2\omega \sin \omega}{9\pi^2 - \omega^2}$	
	$\beta_4(t) = \sin 4\pi t$	$B_4(\omega) = -j \frac{8\pi \sin \omega}{16\pi^2 - \omega^2}$	

TABLE III

E_{rel} for Cosine Function Approximation of Real Low Pass Filter

ω_c

N	π	2π	3π	3.5π	4π	4.5π	5π	6π	7π
3	.0978	.0511	.0363	.0307	.0869	.2176	.3125	-	-
4	.0976	.0506	.0345	.0287	.0275	.0230	.0702	.2600	-
5	.0975	.0504	.0340	.0286	.0260	.0221	.0222	.0589	.2226

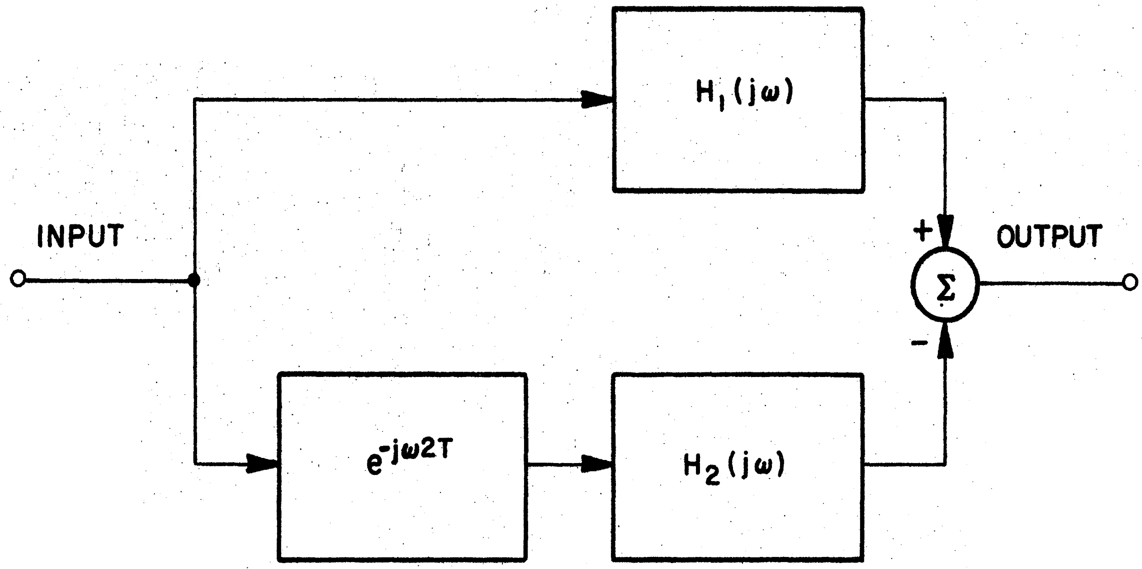


Figure 1. Single Delay Element Realization of H*(jω).

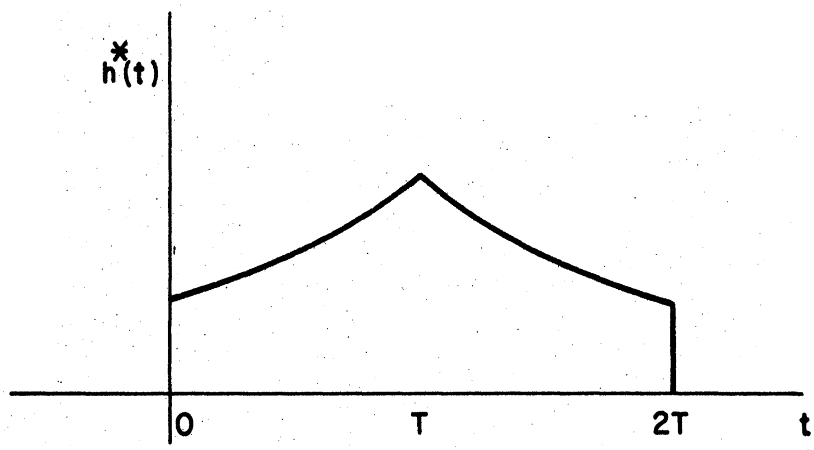


Figure 2. An h*(t) Utilizing More Than One Delay Element.

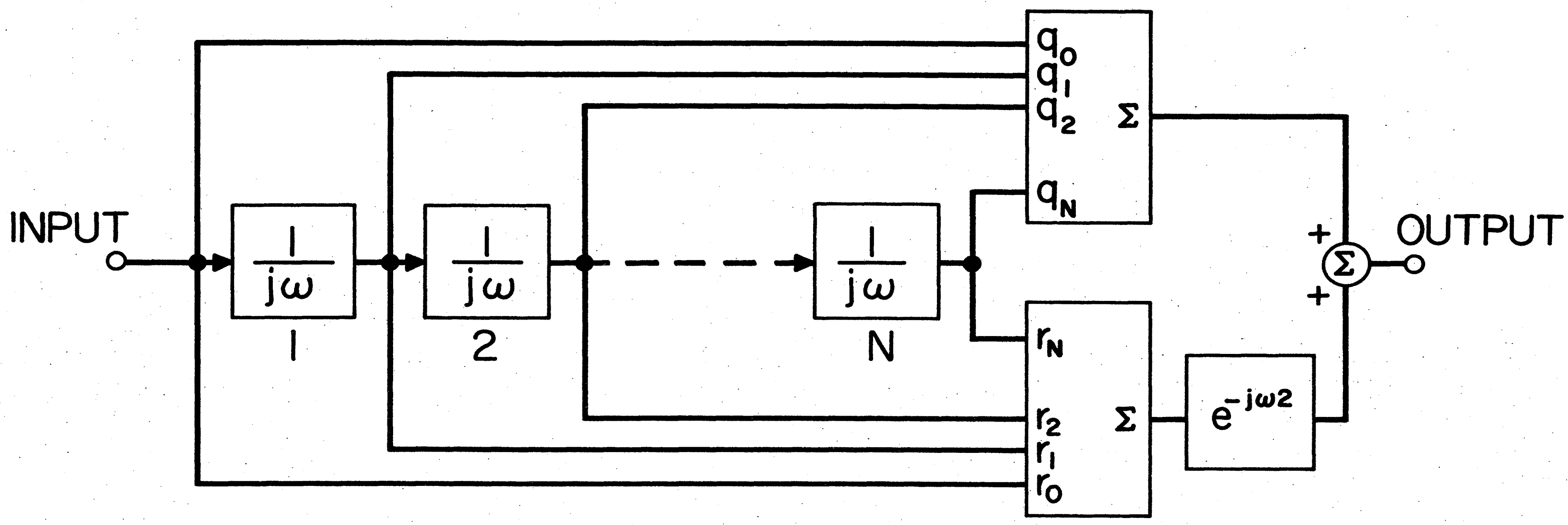


Figure 3. Realization of Real or Imaginary Filter Based on Legendre Function Approximation.

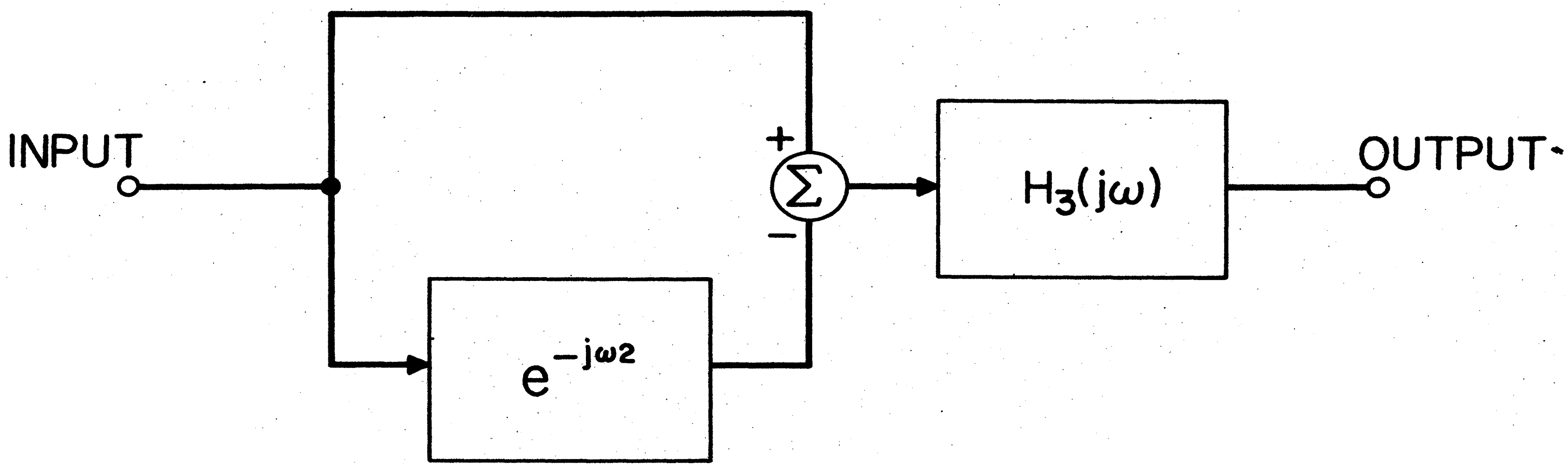


Figure 4. Realization of Real or Imaginary Filter Based on Fourier Series Approximation.

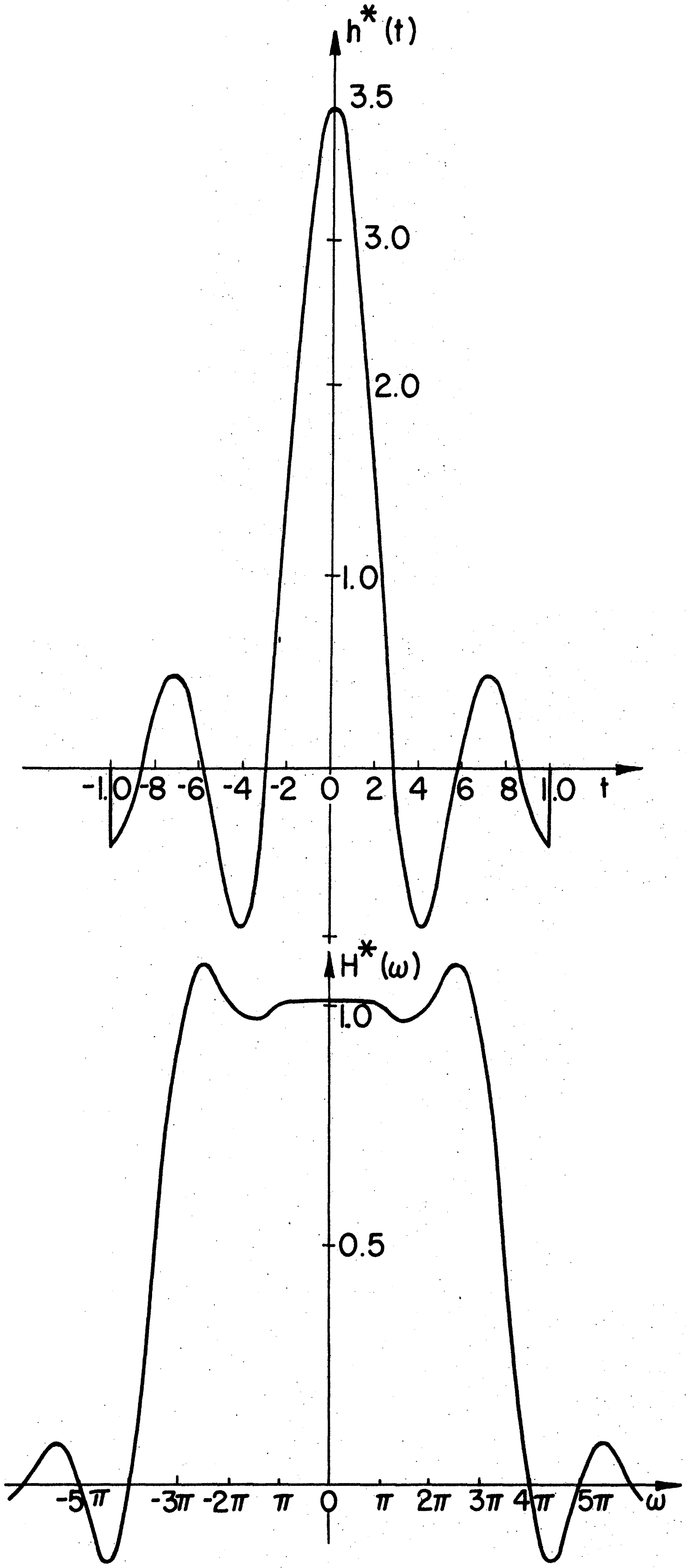


Figure 5. H* and h* for Cosine Function Approximation of Real Low Pass Filter.

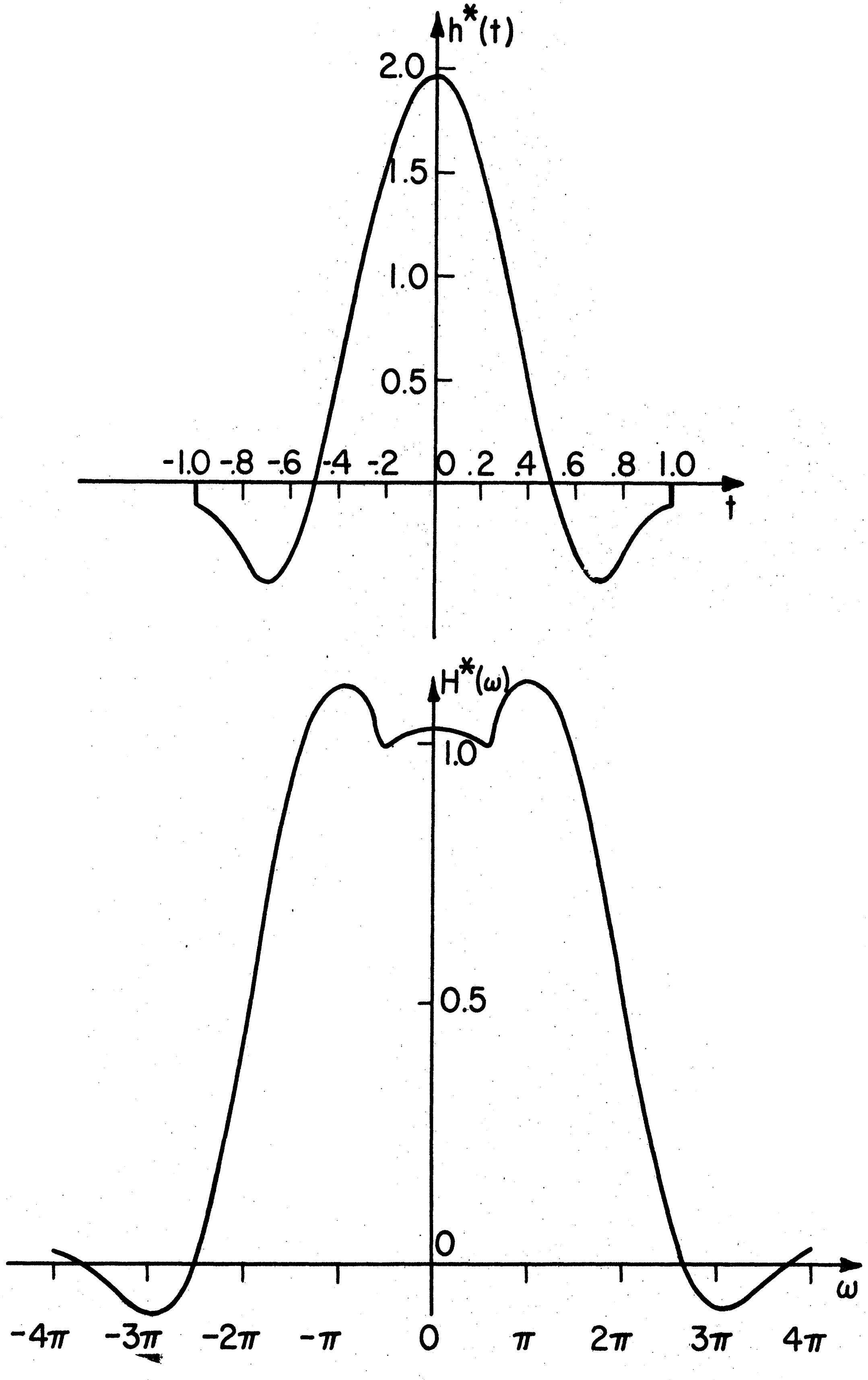


Figure 6. H* and h* for Legendre Function Approximation of Real Low Pass Filter.

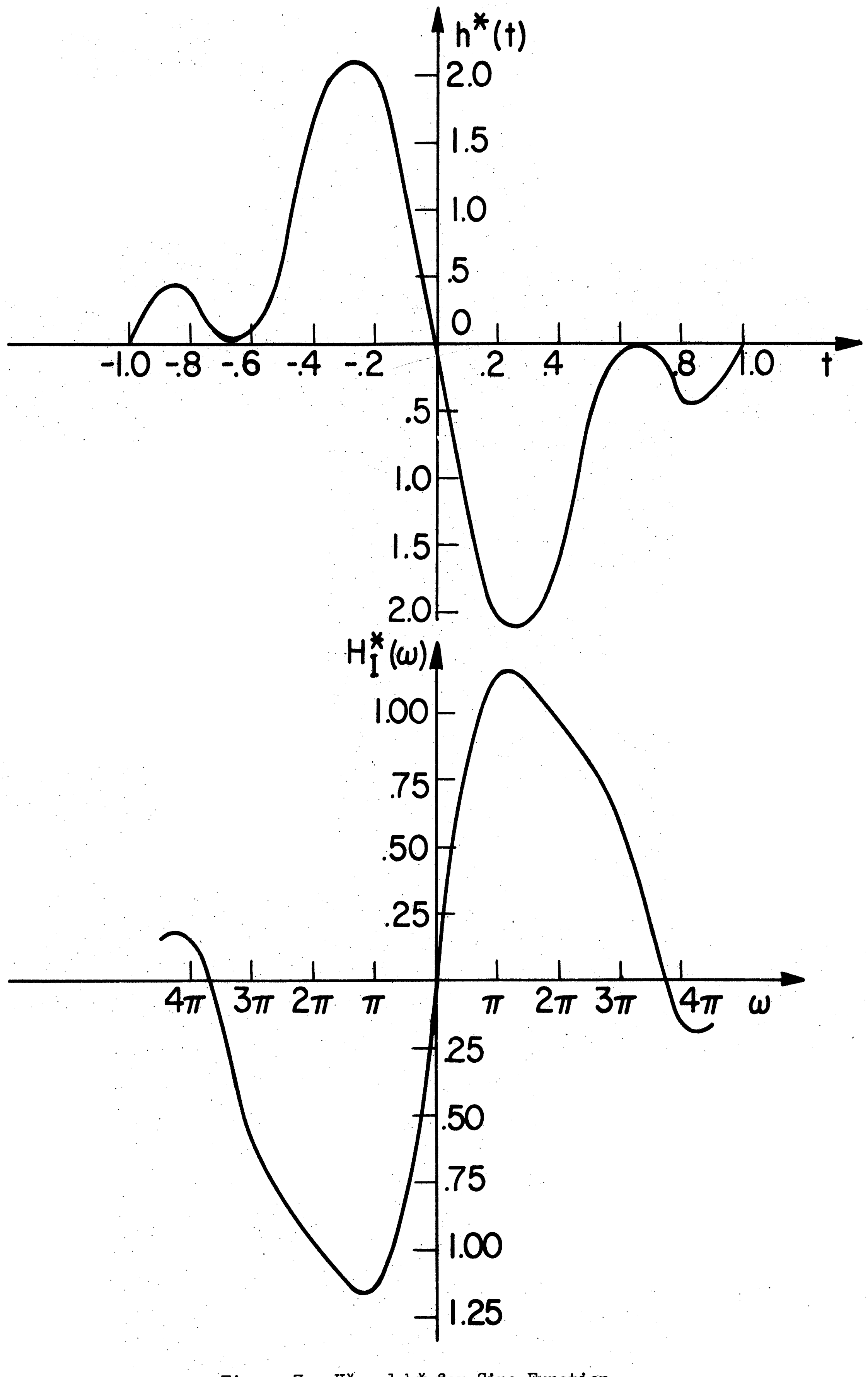


Figure 7. H* and h* for Sine Function Approximation of Imaginary Low Pass Filter.