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Technical Report

A HOMOMORPHIC THEORY OF CONTEXT-FREE LANGUAGES AND ITS GENERALIZATIONS

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ABSTRACT

Usually, and naturally, context-free languages are defined and studied by means of grammars. In the course of study of these languages several algebraic characterizations were found. In this paper we want to regard one of these characterizations (namely, the homomorphic characterization that was established by Chomsky and Schützenberger) as the definition of this family of languages and to show how one can derive some of their well known properties directly from this "redefinition."

In addition to simplification of proofs we find that the arguments involved lead us naturally to some generalizations that have some bearing on mathematical linguistics. The families of languages derived by means of these generalizations exhibit some features which are too complex for the context-free model and yet these features are of the type that one encounters in the study of natural languages. We discuss very briefly some examples of these generalizations.

1. Introduction

Usually, and naturally, context-free languages are defined and studied by means of grammars (Chomsky 1956, Bar-Hillel et al 1960, Chomsky & Miller 1963). In the course of study of these languages several algebraic characterizations were found. In this paper we want to regard one of these characterizations (namely, the homomorphic characterization that was established by Chomsky and Schützenberger (cf. Chomsky & Miller 1963)) as the definition of this family of languages and to show how one can derive <u>some</u> of their well known properties directly from this "redefinition."

In addition to simplification of proofs we find that the arguments involved lead us naturally to some generalizations that have some bearing on mathematical linguistics. The families of languages derived by means of these generalizations exhibit some features which are too complex for the context-free model and yet these features are of the type that one encounters in the study of natural languages. We discuss very briefly some examples of these generalizations.

In what follows we assume a certain acquaintance with the theory of regular events and with the elementary theory of monoids.

1.1 We define a family of languages to be called CF-languages:

Let V_T be a fixed finite alphabet. For any finite alphabet V we denote by $V_{(2)}$ the basis of the monoid $V^* x V^*$.

That is, the elements of $V_{(2)}$ are of the form (a,λ) or (λ,a), where λ is the empty-word and $a \in V$.

Let K_V be the kernel of the homomorphism

$$\gamma : V^*_{(2)} \longrightarrow F_G(V)$$

of $V^*_{(2)}$, the free monoid generated by $V_{(2)}$, onto $F_G(V)$, the free group generated by V, with

$$\gamma(a,\lambda) = a \text{ , and } \gamma(\lambda,a) = a^{-1} \text{ , for all } a \in V \text{ .}$$

That is, $K_V = \{x \in V^*_{(2)} : \gamma(x) = 1 \in F_G(V)\}$.

For our purposes we are interested in alphabets V which include V_T. For any such alphabet $V \supseteq V_T$, we have a natural homomorphism

$$\varepsilon_V : V^*_{(2)} \longrightarrow V^*_T \text{ ,}$$

which is defined by:

$$\varepsilon_V(\alpha) = \begin{cases} a & \text{if } \alpha = (a,\lambda) \text{ and } a \in V_T , \\ \lambda & \text{otherwise ;} \end{cases} \qquad \text{for all } \alpha \in V_{(2)} .$$

Obviously, ε_V amounts to the erasure of letters except for $V_T x \{\lambda\}$ which is interpreted as V_T . No confusion is caused by denoting all the homomorphisms ε_V , for all $V_T \subseteq V$, by ε .

We denote by $R_{V^*_{(2)}}$ the class of the regular events over $V_{(2)}$ as an alphabet.

1.1.1 A subset L of V_T^* is said to be a <u>CF-language defined by V</u> iff

$$L = \varepsilon(E \cap K_V) \text{ for some } E \in R_{V^*_{(2)}} .$$

We denote by $\mathcal{L}_V$ the class of all CF-languages defined by V .

We may express now the characterization of the <u>context-free</u> languages by:

1.1.2 <u>THEOREM (Chomsky & Schützenberger)</u>: A subset L of V_T^* is a context-free language iff there exists a finite alphabet $V \subseteq V_T$, such that $L \in \mathcal{L}_V$.

1.2 Our main goal is to prove the following well known properties of context-free languages for the CF-languages as defined in 1.1.1:

1.2.1 Every regular event over V_T is a CF-language.

1.2.2 The intersection of a CF-language with a regular event over V_T, is a CF-language.

1.2.3 For any homomorphism $\psi : V_T^* \longrightarrow V_T^*$ and any CF-language L, $\psi(L)$ is a CF-language.

1.2.4 The class of all CF-languages (over V_T) is closed under the Kleenean operations of events.

1.2.5 If V_T is a single-letter alphabet then every CF-language is regular; if not, then there exists a CF-language which is not regular.

2. The properties of $\mathcal{L}_V$

2.1 <u>LEMMA:</u> If $V_T \subseteq V_1 \subseteq V_2$ then $\mathcal{L}_{V_1} \subseteq \mathcal{L}_{V_2}$.

<u>Proof:</u> $K_{V_1} \subseteq K_{V_2}$ and $R_{(V_1)^*_{(2)}} \subseteq R_{(V_2)^*_{(2)}}$.

2.1.1 <u>COROLLARY:</u> For any $V_T \subseteq V_1, V_2$:

$$\mathcal{L}_{V_1} \cup \mathcal{L}_{V_2} \subseteq \mathcal{L}_{V_1 \cup V_2} .$$

2.2 <u>PROPOSITION:</u> $R_{V_T^*} \subseteq \mathcal{L}_{V_T}$, hence for any $V_T \subseteq V$, $R_{V_T^*} \subseteq \mathcal{L}_V$.

<u>Proof:</u> K_{V_T} is obviously a submonoid of $V^*_{(2)}$. Define the homomorphism $g : V_T^* \longrightarrow K_{V_T}$ by

$$g(a) = (a,\lambda)(\lambda,a) \quad \text{for all } a \in V_T .$$

Clearly, for any $x \in V_T^*$: $\varepsilon(g(x)) = x$. Hence for any $E \subseteq V_T^*$:

$$E = \varepsilon(g(E)) = \varepsilon(g(E) \cap K_{V_T}) .$$

If $E \in R_{V_T^*}$, then $g(E) \in R_{(V_T)^*_{(2)}}$, and therefore $E \in \mathcal{L}_{V_T}$.

2.3 <u>PROPOSITION:</u> For any $V_T \subseteq V$, if $E \in R_{V_T^*}$ and $L \in \mathcal{L}_V$, then $E \cap L \in \mathcal{L}_V$.

<u>Proof:</u> Let $L = \varepsilon(E_L \cap K_V)$ for $E_L \in R_{V^*_{(2)}}$:

$$E \cap L = E \cap \varepsilon(E_L \cap K_V)$$

$$= \varepsilon(\varepsilon^{-1}(E)) \cap \varepsilon(E_L \cap K_V)$$

$$= \varepsilon((\varepsilon^{-1}(E) \cap E_L) \cap K_V) \ .$$

But $E \in R_{V_T^*}$ and $E_L \in R_{V^*_{(2)}}$ imply that $\varepsilon^{-1}(E) \cap E_L$ is regular in $V^*_{(2)}$ and therefore $E \cap L \in \mathcal{L}_V$.

2.4 <u>PROPOSITION:</u> Let $\psi : V_T^* \longrightarrow V_T^*$ be any homomorphism, $V_T \subsetneq V$, $b \notin V$, and $V_b = V \cup \{b\}$. Then for any $L \in \mathcal{L}_V$: $\psi(L) \in \mathcal{L}_{V_b}$.

<u>Proof:</u> Let i_1 and i_2 be the two natural monomorphisms of V^* into $V^*_{(2)}$; namely, those which are determined by

$$i_1(a) = (a,\lambda) \ , \text{ and } \ i_2(a) = (\lambda,a) \ , \text{ for all } a \in V \ .$$

By combining ψ with i_1 , and i_2 we get two homomorphisms of V_T^* into $V^*_{(2)}$:

$$\psi_1 = i_1 \circ \psi \ , \text{ and } \ \psi_2 = i_2 \circ \psi \ .$$

We "extend" $\psi : V_T^* \longrightarrow V_T^*$ into a homomorphism $\psi_b : V^*_{(2)} \longrightarrow (V_b)^*_{(2)}$ by:

$$\left.\begin{aligned} \psi_b(a,\lambda) &= (b,\lambda)\psi_1(a) \\ \psi_b(\lambda,a) &= \psi_2(a)(\lambda,b) \end{aligned}\right\} \text{ for all } a \in V_T \ ,$$

$$\left.\begin{aligned}\psi_b(a,\lambda) &= (a,\lambda)\\ \psi_b(\lambda,a) &= (\lambda,a)\end{aligned}\right\}\qquad \text{for all } a\in V-V_T .$$

We have now:

(i) for any $E\in R_{V^*_{(2)}}$: $\psi_b(E)\in R_{(V_b)^*_{(2)}}$;

(ii) $x\in K_V$ iff $\psi_b(x)\in K_{V_b}$, hence for any $S \subseteq V^*_{(2)}$:

$$\psi_b(S)\cap K_{V_b} = \psi_b(S\cap K_V) ;$$

(iii) for any $x\in V^*_{(2)}$: $\varepsilon(\psi_b(x)) = \psi(\varepsilon(x))$.

Thus, if $L = \varepsilon(E_L\cap K_V)$ for $E_L\in R_{V^*_{(2)}}$, then:

$$\begin{aligned}\psi(L) &= \psi(\varepsilon(E_L\cap K_V))\\ &= \varepsilon(\psi_b(E_L\cap K_V))\\ &= \varepsilon(\psi_b(E_L)\cap K_V) ,\end{aligned}$$

which shows that $\psi(L)\in\mathcal{L}_{V_b}$.

2.5 **PROPOSITION:** For any $V_T \subseteq V$, if $L_1,L_2\in\mathcal{L}_V$ then $L_1\cup L_2\in\mathcal{L}_V$.

Proof: $$\begin{aligned}L_1\cup L_2 &= \varepsilon(E_{L_1}\cap K_V)\cup\varepsilon(E_{L_2}\cap K_V)\\ &= \varepsilon((E_{L_1}\cap K_V)\cup(E_{L_2}\cap K_V))\\ &= \varepsilon((E_{L_1}\cup E_{L_2})\cap K_V) .\end{aligned}$$

2.5.1 <u>COROLLARY:</u> For any $V_T \subseteq V_1, V_2$, if $L_1 \varepsilon \mathcal{L}_{V_1}$ and $L_2 \varepsilon \mathcal{L}_{V_2}$, then $L_1 \cup L_2 \varepsilon \mathcal{L}_{V_1 \cup V_2}$.

2.6 <u>PROPOSITION:</u> Let $V_T \subseteq V$, $b \notin V$, and $V_b = V \cup \{b\}$. If $L_1, L_2 \varepsilon \mathcal{L}_V$ then $L_1 \cdot L_2 \varepsilon \mathcal{L}_{V_b}$.

<u>Proof:</u> We have $L_i = \varepsilon(E_{L_i} \cap K_V)$ for $E_{L_1}, E_{L_2} \varepsilon R^*_{V(2)}$.

Define

$$E = E_{L_1} \cdot (b,\lambda) \cdot E_{L_2} \cdot (\lambda,b) \; ;$$

then $E \varepsilon R^*_{(V_b)(2)}$ and

$$E \cap K_{V_b} = (E_{L_1} \cap K_V) \cdot (b,\lambda) \cdot (E_{L_2} \cap K_V) \cdot (\lambda,b) \; .$$

Hence,

$$\begin{aligned} \varepsilon(E \cap K_{V_b}) &= \varepsilon((E_{L_1} \cap K_V) \cdot (b,\lambda) \cdot (E_{L_2} \cap K_V) \cdot (\lambda,b)) \\ &= \varepsilon((E_{L_1} \cap K_V) \cdot (E_{L_2} \cap K_V)) \\ &= L_1 \cdot L_2 \, , \end{aligned}$$

and therefore $L_1 \cdot L_2 \varepsilon \mathcal{L}_{V_b}$.

2.7 <u>PROPOSITION:</u> Let $V_T \subseteq V$, $b \notin V$, and $V_b = V \cup \{b\}$. If $L \varepsilon \mathcal{L}_V$ then $L^* \varepsilon \mathcal{L}_{V_b}$.

Proof: Let $L = \varepsilon(E_L \cap K_V)$ for $E_L \in R_{V^*_{(2)}}$. We define

$$E = (b,\lambda)^* \cdot (E_L \cdot (\lambda,b))^* ,$$

and clearly we have $E \cap K_{V_b} = \bigcup_{k=0}^{\infty} (b,\lambda)^k \cdot ((E_L \cap K_V) \cdot (\lambda,b))^k$.

Hence,

$$\begin{aligned} \varepsilon(E \cap K_{V_b}) &= \varepsilon(\bigcup_{k=0}^{\infty} (b,\lambda)^k \cdot ((E_L \cap K_V) \cdot (\lambda,b))^k) \\ &= \bigcup_{k=0}^{\infty} (\varepsilon(E_L \cap K_V))^k \\ &= L^* . \end{aligned}$$

2.8 If $V_T = \{a\}$ then K_{V_T} enjoys a particular structure.

For any set of words E, denote by $\pi(E)$ the permutation-closure of E (i.e., the set of all permutations of the words that are in E).

Obviously,

$$K_{V_T} = \pi(((a,\lambda) \cdot (\lambda,a))^*) ,$$

and for any $S \subseteq V^*_{T(2)}$:

$$S \cap K_{V_T} = \pi(S) \cap K_{V_T} .$$

This leads us to employ Presburger's formulas and their relationships with the commutative-events that are derived from regular events by means of π (cf. Laing & Wright 1962, Presburger 1930).

2.8.1 LEMMA: If $V_T = \{a\}$ then $\mathcal{L}_{V_T} = R_{V_T}{}^* = R_a{}^*$.

Proof: For any word x , over an alphabet that contains the letter b , we denote by $n_b(x)$ the number of occurrences of b in x .

For any $E \in R_{(V_T)}{}^*_{(2)}$ there exists a Presburger's formula $Q(m,n)$ such that

$$x \in \pi(E) \; K_{V_T} \quad \text{iff} \quad Q(n_{(a,\lambda)}(x), n_{(\lambda,a)}(x)) \; ;$$

hence for any $L \in \mathcal{L}_{V_T}$ there exists a Presburger's formula $Q(n)$ such that

$$L = \{a^n : Q(n)\} \; .$$

Since $\{n : Q(n)\}$ is ultimately periodic, L must be regular.

2.8.2 Unfortunately I cannot prove the stronger true statement:

if $V_T = \{a\}$, then for any $V_T \subseteq V$: $\mathcal{L}_V = R_a{}^*$;

without referring to Theorem 1.1.2.

2.8.3 LEMMA: If V_T contains at least two letters then $\mathcal{L}_{V_T}$ contains languages which are not regular.

Proof: Let $a, b \in V_T$; we define

$$E = ((a,\lambda)\cdot(\lambda,b))^{*}\cdot((\lambda,a)\cdot(b,\lambda))^{*} .$$

Obviously, $E \cap K_{V_T} = \{((a,\lambda)\cdot(\lambda,b))^{k}\cdot((\lambda,a)\cdot(b,\lambda))^{k} : k \geq 0\}$,

and therefore

$$\varepsilon(E \cap K_{V_T}) = \{a^k b^k : k \geq 0\} \in \mathcal{L}_{V_T} .$$

3. Generalizations: Abstract CF-systems

3.1 An abstract CF-system is a system

$$\mathcal{C} = \langle W_T, \mathcal{W}, P, \varepsilon \rangle$$

where:

(i) W_T is a monoid (the terminal monoid of $\mathcal{C}$) ;

(ii) $\mathcal{W}$ is a set of monoids $\{W_\alpha : \alpha \in A\}$ (the auxiliary monoids of $\mathcal{C}$) where A is an indexing set;

(iii) P is a mapping which associates with any auxiliary monoid W_α , a submonoid P_α of W_α ;

(iv) ε is a functional which associates with any auxiliary monoid W_α a homomorphism, also denoted by $\varepsilon : W_\alpha \longrightarrow W_T$; subjected to conditions which will be specified presently.

First, for any monoid W , we denote by R_W , the class of the regular-events in W . (A subset E of W is said to be regular in W iff there exists a homomorphism $\psi : W \longrightarrow M$ with a finite range, such that $E = \psi^{-1}(F)$ for some $F \subseteq M$.)

The conditions imposed on $\mathcal{C}$ are as follows:

3.1.1 For any $\alpha, \beta \epsilon A$ there exists $\gamma \epsilon A$ such that

$$W_\alpha \cup W_\beta \subseteq W_\gamma .$$

3.1.2 If $W_\alpha \subseteq W_\beta$ then $R_{W_\alpha} \subseteq R_{W_\beta}$.

3.1.3 $W_\alpha \subseteq W_\beta$ implies $W_\alpha \cap P_\beta = P_\alpha$.

3.1.4 $\varepsilon(P_\alpha) = W_T$ for all $\alpha \epsilon A$.

3.2 For the basic properties of regular events in arbitrary monoids we refer to the literature (Give'on, 1964a,b). In particular, we shall make use of the following basic properties:

3.2.1 For any monoid W , R_W is a Boolean algebra of sets.

3.2.2 For any epimorphism $\psi : W_1 \longrightarrow W_2$, if $E \epsilon R_{W_2}$ then $\psi^{-1}(E) \epsilon R_{W_1}$.

3.3 Let $\mathcal{C}$ be a fixed abstract CF-system.

For any $\alpha \epsilon A$ we define $\mathcal{L}_\alpha$ to be the set of all subsets L of W_T of the form

$$L = \varepsilon(E \cap P_\alpha) \quad \text{for some} \quad E \in R_{W_\alpha} .$$

3.4 PROPOSITION: If $W_\alpha \subseteq W_\beta$ then $\mathcal{L}_\alpha \subseteq \mathcal{L}_\beta$.

Proof: Let $L = \varepsilon(E \cap P_\alpha)$ for $E \in R_{W_\alpha}$. Since $W_\alpha \subseteq W_\beta$, it follows by 3.1.2 and 3.1.3 that $E \in R_{W_\beta}$ and $W_\alpha \cap P_\beta = P_\alpha$. Hence

$$E \cap P_\alpha = E \cap (P_\beta \cap W_\alpha) = (E \cap W_\alpha) \cap P_\beta = E \cap P_\beta ,$$

and therefore $L \in \mathcal{L}_\beta$.

3.5 PROPOSITION: For any $\alpha \in A$: $R_{W_T} \subseteq \mathcal{L}_\alpha$.

Proof: From $\varepsilon(P_\alpha) = W_T$ (3.1.4) follows that for any subset $S \subseteq W_T$ we have

$$\varepsilon(\varepsilon^{-1}(S) \cap P_\alpha) = S .$$

Hence, for any $E \in R_{W_T}$ we have $E = \varepsilon(\varepsilon^{-1}(E) \cap P_\alpha)$, and since $\varepsilon^{-1}(E) \in R_{W_\alpha}$ (3.2.2), it follows that $E \in \mathcal{L}_\alpha$.

3.6 PROPOSITION: For any $\alpha \in A$, $E \in R_{W_T}$ and $L \in \mathcal{L}_\alpha$, we have $E \cap L \in \mathcal{L}_\alpha$.

Proof: Let $L = \varepsilon(E_L \cap P_\alpha)$ for $E_L \in R_{W_\alpha}$, then

$$E \cap L = E \cap \varepsilon(E_L \cap P_\alpha)$$

$$= \varepsilon(\varepsilon^{-1}(E)) \cap \varepsilon(E_L \cap P_\alpha)$$

$$= \varepsilon((\varepsilon^{-1}(E) \cap E_L) \cap P_\alpha) \ .$$

By 3.2.1 and 3.2.2 follows that $\varepsilon^{-1}(E) \cap E_L \in R_{W_\alpha}$ and therefore $E \cap L \in \mathcal{L}_\alpha$.

3.7 <u>PROPOSITION:</u> For any $\alpha \in A$ and $L_1, L_2 \in \mathcal{L}_\alpha : L_1 \cup L_2 \in \mathcal{L}_\alpha$.

<u>Proof:</u> Let $L_i = \varepsilon(E_i \cap P_\alpha)$ for $L_1, L_2 \in R_{W_\alpha}$, then

$$L_1 \cup L_2 = \varepsilon(E_1 \cap P_\alpha) \cup \varepsilon(E_2 \cap P_\alpha)$$

$$= \varepsilon((E_1 \cup E_2) \cap P_\alpha) \ .$$

By 3.2.1 follows that $L_1 \cup L_2 \in \mathcal{L}_\alpha$.

3.7.1 <u>COROLLARY:</u> For any $\alpha, \beta \in A$ there exists a $\gamma \in A$ such that for all $L_1 \in \mathcal{L}_\alpha$, and $L_2 \in \mathcal{L}_\beta : L_1 \cup L_2 \in \mathcal{L}_\gamma$.

3.8 <u>PROPOSITION:</u> Let $\tau_T : W_T \longrightarrow W_T$ be a mapping such that for some $\alpha, \beta \in A$ there exists a mapping $\tau_{\alpha\beta} : W_\alpha \longrightarrow W_\beta$ with the following properties:

(i) $\tau_T \varepsilon = \varepsilon \tau_{\alpha\beta}$,

(ii) for any $E \in R_{W_\alpha} : \tau_{\alpha\beta}(E) \in R_{W_\beta}$, (iii), for all $\omega \epsilon W_\alpha$: $\omega \epsilon P_\alpha$ iff $\tau_{\alpha\beta}(\omega) \epsilon P_\beta$.

Then for any $L \epsilon \mathcal{L}_\alpha$: $\tau_T(L) \epsilon \mathcal{L}_\beta$.

Proof: Let $L = \varepsilon(E \cap P_\alpha)$ for $E \epsilon R_{W_\alpha}$. By (i) we have

$$\tau_T(L) = \tau_T \varepsilon(E \cap P_\alpha) = \varepsilon \tau_{\alpha\beta}(E \cap P_\alpha) .$$

By (iii) we have $\tau_T(L) = \varepsilon(\tau_{\alpha\beta}(E) \cap P_\beta)$, which by (ii) implies that $\tau_T(L) \varepsilon \mathcal{L}_\beta$.

4. Examples: Generalized CF-systems

In order to make our generalization more concrete and more applicable to the study of formal languages, we impose on the abstract CF-systems additional conditions.

4.1 A <u>generalized CF-system</u> (over V_T with auxiliary alphabets $\mathcal{V}$) is an abstract CF-system

$$\mathcal{G} = \langle V_T^*, F(\mathcal{V}_{(2)}), P, \varepsilon \rangle$$

where

(i) $F(\mathcal{V}_{(2)})$ is the set of the free monoids $V_{(2)}^*$, for any $V \in \mathcal{V}$;

(ii) for any $V \in \mathcal{V}$: $\varepsilon : V_{(2)}^* \longrightarrow V_T^*$ is the erasing homomorphism (as defined in 1.1).

(iii) $\mathcal{V}$ is rich enough in the following sense: for any finite cardinality n which is larger than the cardinality of V_T , there exists a $V \in \mathcal{V}$ with cardinality n .

4.1.1 Note that from (iii) follows that $\mathcal{V}$ can be taken as the <u>class</u>

of all finite alphabets, and by 3.1.3 and 3.1.4, we can even assume that $\mathcal{V}$ is the class of all finite alphabets that include V_T, without changing the languages defined by means of $\mathcal{G}$.

Note however, that in the definition of abstract CF-systems we required (cf. 3.1 (ii)) that the class of the auxiliary monoids of any abstract CF-system must be a set. But this is, in our case, a baroque difference.

4.1.2 Obviously, the CF-languages as defined in 1.1.1 are defined by means of a generalized CF-system where for any $V \in \mathcal{V}$: $p(V) = K_V$. On the other hand, by means of other choices for p we may derive new types of languages over V_T, and in this section we suggest and discuss briefly some examples of such types.

4.2 Our notation, namely $V_{(2)}$, hints that in fact we have in mind some other generalizations of the concept of CF-languages, which fall under the machinery of abstract systems but <u>not</u> under the generalized CF-systems.

For example, let $V_{(3)}$ denote the basis of $V^* x V^* x V^*$. Denote by $B_V^{(3)}$ the set of all words in $V_{(3)}^*$ which reduce to the empty-word by means of successive (and successful) applications of the rule

$$(a,\lambda,\lambda)(\lambda,a,\lambda)(\lambda,\lambda,a) \longrightarrow \lambda \qquad \text{for all } a \in V .$$

Define $L \subseteq V_T^*$ to be a <u>$CF^{(3)}$-language</u> iff there exists $E \in R_{V_{(3)}^*}$ such that

$$L = \varepsilon(E \cap B_V^{(3)}) .$$

(The definition of ε is evident.)

The generalized notion of $CF^{(k)}$-languages is obvious. Note that the $CF^{(1)}$-languages are the regular languages over V_T, and the $CF^{(2)}$-languages are precisely the CF-languages. It is easily verified that for any $k \geq 1$, the $CF^{(k)}$-languages are defined by means of appropriate abstract CF-systems with V_T^* as their terminal monoid, and $(V^*)^k$ as their auxiliary monoids.

4.2.1 <u>PROPOSITION:</u> Let $k \geq 1$, $V_T \subseteq \{a_1,\ldots,a_k\}$, then $\{a_1^n a_2^n \ldots a_k^n : n \geq 0\}$ is a $CF^{(k)}$-language.

<u>Proof</u>: For any $1 \leq i \leq k$ define $E_i \subseteq V_{(k)}^*$ by:

$$E_i = ((a_i,\lambda,\ldots,\lambda)\cdot(\lambda,a_{i+1},\lambda,\ldots,\lambda)\cdot\ldots\cdot(\lambda,\lambda,\ldots,\lambda,a_{i+k-1}))^*$$

where $i + j = (i + j)$ modulo k . Define now

$$E = E_1 \cdot E_2 \cdot \ldots \cdot E_k$$

and clearly

$$\varepsilon(E \cap B_{V_T}^{(k)}) = \{a_1^n a_2^n \ldots a_k^n : n \geq 0\} .$$

4.3 For any alphabet V we have the natural homomorphism

$$d : V_{(2)}^* \longrightarrow V^* x V^*$$

which is determined by the identity on $V_{(2)}$.

By means of d we define the following two submonoids of $V_{(2)}^*$:

4.3.1 $E_V = \{x \epsilon V_{(2)}^* : d(x) = (y,y)\}$.

4.3.2 $P_V = \{x \epsilon V_{(2)}^* : d(x) = (y,\sigma(y))$ for some permutation σ of $y\}$.

(4.3.3 In general, any submonoid W of $V^* x V^*$ determines in this fashion a submonoid $d^{-1}(W)$ of $V_{(2)}^*$.)

4.4 It is quite easy to verify:

4.4.1 If $V_1 \subseteq V_2$ then $(V_1)_{(2)}^* \cap E_{V_2} = E_{V_1}$, and $(V_1)_{(2)}^* \cap P_{V_2} = P_{V_1}$.

4.4.2 For all $V \subseteq V_T : \varepsilon(E_V) = \varepsilon(P_V) = V_T^*$.

Hence we have two types of generalized CF-systems. We denote by $\mathcal{L}_V^E$ (resp., by $\mathcal{L}_V^P$) the set of languages of the form

$$L = \varepsilon(E \cap E_V) \quad \text{for} \quad E \in R_{V^*_{(2)}}$$

$$(\text{resp.,} \quad L = \varepsilon(E \cap P_V) \quad \text{for} \quad E \in R_{V^*_{(2)}}) .$$

4.5 Note that if $\alpha : V^*_{(2)} \longrightarrow F_{AG}(V)$ is the natural homomorphism of $V^*_{(2)}$ onto the free abelian group generated by V , then $P_V = \mathrm{Ker}\ \alpha$.

4.5.1 The relationship between E_V and K_V is not clear, though obviously, $E_V \subseteq P_V$ and $K_V \subseteq P_V$.

For example, let

$$x_1 = (a,\lambda)(b,\lambda)(\lambda,a)(\lambda,b) \text{ , and}$$

$$x_2 = (\lambda,b)(a,\lambda)(\lambda,a)(b,\lambda) \text{ ;}$$

then $x_1 \notin K_V$ but $x_1 \in E_V$, and $x_2 \in K_V$ but $x_2 \notin E_V$.

4.6 In addition to the properties of $\mathcal{L}_V^E$ and $\mathcal{L}_V^P$ which were derived in the previous section for abstract CF-systems, we can prove some of the properties of the CF-languages to hold for $\mathcal{L}_V^E$ as well.

For example, 2.6 holds for $\mathcal{L}_V^E$; substitute E_V instead of K_V in the proof. The same holds for 2.4 and 2.8.1.

4.7 PROPOSITION: Let $\{a_1,\dots,a_k\} \subseteq V_T$ then $\{a_1^n a_2^n \dots a_k^n : n \geq 0\}$ is both in $\mathcal{L}_{V_T}^E$ and in $\mathcal{L}_{V_T}^P$.

Proof: We define

$$E = (a_1,\lambda)^*\cdot((\lambda,a_1)\cdot(a_2,\lambda))^*\cdot((\lambda,a_2)\cdot(a_3,\lambda))^*\cdot\ldots\cdot((\lambda,a_{k-1})(a_k,\lambda))^*\cdot(\lambda,a_k)^* ,$$

then $E \cap P_V = E \cap E_V =$

$$= \{(a_1,\lambda)^n((\lambda,a_1)\cdot(a_2,\lambda))^n\cdot\ldots\cdot((\lambda,a_{k-1})\cdot(a_k,\lambda))^n\cdot(\lambda,a_k)^n : n \geq 0\} ,$$

and therefore

$$\varepsilon(E \cap P_V) = \varepsilon(E \cap E_V) = \{a_1^n a_2^n\cdot\ldots\cdot a_k^n : n \geq 0\} \in \mathcal{L}_{V_T}^E \cap \mathcal{L}_{V_T}^P .$$

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