THE UNIVERSITY OF MICHIGAN

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Communication Sciences Program

Technical Note

# NORMAL MONOIDS AND FACTOR MONOIDS OF COMMUTATIVE MONOIDS

Yehoshafat Give'on

ORA Projects 03105 and 05662

under contract with:

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
CONTRACT NO. Nonr-1224(21)
WASHINGTON, D.C.

and

U. S. ARMY RESEARCH OFFICE (DURHAM)
BOX CM, DUKE STATION
CONTRACT NO. DA-31-124-ARO(D)-G433
DURHAM, NORTH CAROLINA

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

May 1963

## §1 Introduction

The relevance of the theory of monoids to automata theory has recently become more and more apparent. (See, for example, Mezei [2] with respect to finite automata, and Laing and Wright [1] with respect to the theory of commutative machines.) In this paper certain properties of commutative monoids are discussed; some of them are directly relevant to the theory of commutative automata. In particular, we are interested in finitely generated monoids and in finite factor monoids.

We begin with a study of three closure operations on submonoids of a commutative monoid which provides us with tools for the study of factor monoids. Next we discuss some properties of factor monoids and give certain conditions for finite factor monoids. We conclude with the proof that those closure operations lead to finitely generated monoids when they are applied on any submonoid of any finitely generated free commutative monoid. In particular this implies that any normal submonoid of any finitely generated free commutative monoid is finitely generated.

The notation used in this paper partially follows the notation used in [1] for employing regular expressions to denote commutative events. The customary notation of abelian algebras is also used. Thus $B^*$, $x^*$, and $\lambda^*$ denote the commutative monoids generated by B (any non-empty set of elements of a given commutative monoid,) x (any element of a given commutative monoid) and $\lambda$ (the identity element of the monoid under discussion). But "+" denotes the operation of the monoid and therefore $x+B^*$ denotes the coset determined by the monoid generated by B with x as a leader.

The problems discussed in this paper were suggested by J. B. Wright; I wish to thank him for prompting this research and for his continuous interest.

## §2 The closure operators $N_F$, $S_F$ and $H_F$.

Let F be a fixed commutative monoid.

__Definition 1:__ Let M be a submonoid of F. We denote by __$N_F(M)$__ the set of all elements y of F for which $x+y \in M$ for some $x \in M$. $N_F(M)$ is said to be the __normal extension of M in F__ and M is said to be a __normal__ submonoid of F (in short __normal__) iff $N_F(M) = M$.

__Lemma 1:__ Let M be a submonoid of F, then:

(i) $N_F(M)$ is the minimal normal submonoid of F which includes M;

(ii) M is normal iff $x, x+y \in M$ implies $y \in M$ for any $x,y \in F$.

__Remark:__ For the proof of (i) we shall prove that $N_F$ is a closure operation on the submonoids of F.

__Proof:__ (1) $M \subseteq N_F(M)$.

Let $y \in M$ then $y+\lambda \in M$. Since $\lambda \in M$ we get $y \in N_F(M)$.

(2) $N_F(M)$ is a submonoid of F.

Let $x_1, x_2 \in N_F(M)$, then $x_1+y_1, x_2+y_2 \in M$ for some $y_1, y_2 \in M$ and therefore $(x_1+x_2) + (y_1+y_2) \in M$ where $y_1+y_2 \in M$, which shows that $x_1+x_2 \in N_F(M)$. Since $\lambda \in N_F(M)$ we have that $N_F(M)$ is a submonoid of F.

(3) $N_F(N_F(M)) = N_F(M)$, hence $N_F(M)$ is normal.

By (1) we have that $N_F(M) \subseteq N_F(N_F(M))$. Let $y \in N_F(N_F(M))$, then $y+x_1 \in N_F(M)$ for some $x_1 \in N_F(M)$. But $y+x_1 \in N_F(M)$ implies $y+x_1+x_2 \in M$ for some $x_2 \in M$ and $x_1 \in N_F(M)$ implies $x_1+x_3 \in M$ for some $x_3 \in M$. Hence, $y+(x_1+x_2+x_3) \in M$ where $x_1+x_2+x_3 \in M$, which shows that $y \in N_F(M)$. Thus we have $N_F(M) = N_F(N_F(M))$.

(4) Let $M_1, M_2$ be submonoids of F; if $M_1 \subseteq M_2$ then $N_F(M_1) \subseteq N_F(M_2)$.

Let $x \in N_F(M\ )$ then $x+y \in M_1 \subseteq M_2$ for some $y \in M_1 \subseteq M_2$ which shows that $x \in N_F(M_2)$.

To complete the proof of (i), let M' be any normal submonoid of F which includes M, then $N_F(M) \subseteq N_F(M') = M'$. This shows that $N_F(M)$ is the minimal normal submonoid of F which includes M.

The proof of (ii) follows immediately from Df. 1.

Corollary: Let $M_1$ be any intermediate submonoid of F between M and $N_F(M)$, (i.e., $M \subseteq M_1 \subseteq N_F(M)$), then $N_F(M_1) = N_F(M)$.

Proof: Immediate .

Lemma 2: Let $\phi: F \to F_1$ be a homomorphism of F onto a monoid $F_1$. If $M_1$ is a normal submonoid of $F_1$ then $\phi^{-1}(M_1)$ is a normal submonoid of F. In particular, the kernel of $\phi$ is a normal submonoid of F.

Remark: Note that from the commutativity of F it follows that $F_1$ is commutative.

Proof; Since $\phi$ is a homomorphism it follows immediately that $\phi^{-1}(M_1)$ is a submonoid of F. Let $x, x+y \in \phi^{-1}(M_1)$ where $x, y \in F$, then we have $\phi(x), \phi(x+y) = \phi(x)+\phi(y) \in M_1$. Since $M_1$ is normal we have that $\phi(y) \in M_1$ and therefore $y \in \phi^{-1}(M_1)$. Hence $\phi^{-1}(M_1)$ is a normal submonoid of F.

Since $\lambda^* = \{\lambda\}$ is a normal submonoid of $M_1$ we get that $\ker \phi = \phi^{-1}(\lambda) = \phi^{-1}(\lambda^*)$ is a normal submonoid of F.

<u>Lemma 3</u>: Let M be a submonoid of F; then $(x+M) \cap (y+N) \neq \emptyset$ iff $(x+N_F(M)) \cap (y+N_F(M)) \neq \emptyset$

<u>Proof</u>: From $M \subsetneq N_F(M)$ it follows that $(x+M) \cap (y+M) \neq \emptyset$ implies $(x+N_F(M)) \cap (y+N_F(M)) \neq \emptyset$. On the other hand, let $(x+N_F(M)) \cap (y+N_F(M)) \neq \emptyset$ then we have $x+x_1 = y+y_1$ for some $x_1, y_1 \in N_F(M)$; hence $x_1+u_1 = u_2$ and $y_1+v_1 = v_2$ for some $u_1, v_1, u_2, v_2 \in M$. Therefore, $x+x_1+u_1+v_1 = x+u_2+v_1$ and $x+x_1+u_1+v_1 = y+y_1+v_1+u_1 = y+v_2+u_1$, which imply $x+(u_2+v_1) = y+(v_2+u_1)$ and this shows that $(x+M) \cap (y+M) \neq \emptyset$ since $u_2+v_1, v_2+u_1 \in M$.

<u>Lemma 4</u>: Let M and N be two submonoids of F. If $M \subsetneq N$ and N is normal then

$$x \in N \;\&\; (x+M) \cap (y+M) \neq \emptyset \text{ implies } y \in N.$$

<u>Proof</u>: We have $y+m_2 = x+m_1$ where $m_1, m_2 \in M \subsetneq N$. Hence $m_2, y+m_2 \in N$ and therefore $y \in N$.

<u>Definition 2</u>: Let $M_1$ and $M$ be two submonoids of $F$. We say that $M_1$ is <u>subtractive onto M</u> iff for any $x \in M_1$ there is $y \in M_1$ such that $x+y \in M$. We denote by <u>$S_F(M)$</u> the set of all elements $x \in F$ such that $x+y \in M$ for some $y \in F$.

<u>Lemma 5</u>: Let $M$ be a submonoid of $F$, then $S_F(M)$ is the maximal submonoid of $F$ which is subtractive onto $M$.

<u>Proof</u>: Clearly $\lambda \in S_F(M)$. Let $x,y \in S_F(M)$, then we have $x+x_1, y+y_1 \in M$ for some $x_1, y_1 \in F$. Hence $(x+y)+(x_1+y_1) = (x+x_1)+(y+y_1) \in M$, which shows that $x+y \in S_F(M)$. Therefore $S_F(M)$ is a submonoid of $F$. By the definition of $S_F(M)$ it follows that any submonoid of $F$ which is subtractive onto $M$ is included in $S_F(M)$.

<u>Corollary</u>: $M \subseteq S_F(M)$.

<u>Proof</u>: $M$ itself is subtractive onto $M$.

<u>Lemma 6</u>: Let $M$ be a submonoid of $F$ then $S_F(M)$ satisfies the following condition:

for any $x,y \in F$ : $x+y \in S_F(M)$ implies $x \in S_F(M)$. (In particular, $x+y \in M$ implies $x \in S_F(M)$.)

<u>Proof:</u> Let $x+y \in S_F(M)$ then $x+(y+z) \in M$ for some $z \in F$, hence $x \in S_F(M)$.

<u>Corollary:</u> $S_F(M)$ is a normal submonoid of F. Hence, $x \in S_F(M)$ & $(x+M) \cap (y+M) \neq \emptyset$ implies $y \in S_F(M)$.

<u>Proof:</u> Immediate; the result follows from Lemma 4.

<u>Lemma 7:</u> $S_F$ is a closure operation on the submonoids of F; i.e.,

(i) $M \subseteq S_F(M)$ and $S_F(M)$ is a submonoid of F,

(ii) $M_1 \subseteq M_2$ implies $S_F(M_1) \subseteq S_F(M_2)$,

(iii) $S_F(S_F(M)) = S_F(M)$.

<u>Proof;</u> We had (i) as a corollary to Lemma 5. The proof of (ii) is immediate by Df. 2. From (i) follows that $S_F(M) \subseteq S_F(S_F(M))$; so let $x \in S_F(S_F(M))$ then we have $x+y \in S_F(M)$ for some $y \in F$ and so we have $x+(y+z) \in M$ for some $y,z \in F$, thus $x \in S_F(M)$.

<u>Corollary</u>: Let $M_1$ be an intermediate submonoid of F between M and $S_F(M)$ then $S_F(M_1) = S_F(M)$.

<u>Proof</u>: Immediate.

<u>Lemma 8</u>: Let $M_1$ be an intermediate submonoid of F between M and $S_F(M)$. If $M_1$ is normal then it is subtractive onto M.

<u>Proof</u>: Let $x \in M_1 \subseteq S_F(M)$, then $x+y \in M$ for some $y \in S_F(M)$. But $M \subseteq M_1$ and so $x, x+y \in M_1$ which implies that $y \in M_1$. Hence for any $x \in M_1$ there is $y \in M_1$ such that $x+y \in M$.

Lemma 6 implies the following relation between the sets of generators for F and for $S_F(M)$

<u>Lemma 9</u>: Let W be a set of generators for F then

$W' =_{df} \{w \in W : (w+F) \cap M \neq \emptyset\}$ is a set of generators for $S_F(M)$. In particular, if W is a basis of F then W' is a basis of $S_F(M)$.

<u>Proof</u>: From the definition of $W'$ it is clear that $W' \subseteq S_F(M)$. Let $x \in S_F(M) \subseteq F$ then $x$ is a finite sum of elements of $W$. Let $w$ be any element of $W$ which is a summand of $x$, then we have $x = w+y$ which implies by Lemma 6 that $w \in S_F(M)$ and therefore $w \in W'$. Hence $x$ is a finite sum of elements of $W'$ which shows that $S_F(M)$ is generated by $W'$.

<u>Corollary</u>: (i) If $F$ is free then $S_F(M)$ is free.

(ii) If $F$ is finitely generated then $S_F(M)$ is finitely generated.

<u>Proof</u>: Immediate.

The proof of the following lemma is now obvious and so is not given here.

<u>Lemma 10</u>: Let $M$, $M_1$ and $M_2$ be submonoids of $F$.

(i) $M_1 \subseteq M_2$ implies that if $M$ is subtractive onto $M_1$ then it is subtractive onto $M_2$.

(ii) $M \subseteq M_1$ implies that $M_1$ is subtractive onto $M$ iff it is subtractive onto $N_F(M)$.

<u>Definition 3</u>: Let $M$ be a submonoid of $F$. The submonoid <u>spanned</u> by $M$ in $F$ is denoted by <u>$H_F(M)$</u> and defined by

$$H_F(M) =_{df} \{x \in F : x = \lambda \vee x^* \cap N_F(M) \neq \lambda^*\} .$$

<u>Lemma 11</u>: Let $M$ be a submonoid of $F$. $H_F(M)$ is a normal submonoid of $F$ which includes $M$ and is subtractive onto $M$ and so $M \subseteq N_F(M) \subseteq H_F(M) \subseteq S_F(M) \subseteq F$.

<u>Proof</u>: Let $x,y \in H_F(M)$ then we have $k_1 x, k_2 y \in N_F(M)$ for for some positive integers $k_1$ and $k_2$. Therefore, $k_1 k_2(x+y) = k_2(k_1 x)+k_1(k_2 y) \in N_F(M)$, hence $x+y \in H_F(M)$. Since $\lambda \in H_F(M)$ by definition, $H_F(M)$ is a submonoid of $F$, and from the definition of $H_F(M)$ it follows directly that $M \subseteq H_F(M)$.

Let $x,x+y \in H_F(M)$, that is $k_1 x, k_2(x+y) \in N_F(M)$ for some positive integers $k_1$ and $k_2$. Hence $k_2(k_1 x), k_1 k_2(x+y) \in N_F(M)$, that is, $k_1 k_2 x$, $k_1 k_2 x+k_1 k_2 y \in N_F(M)$ which implies $k_1 k_2 y \in N_F(M)$ which shows that

$y \in H_F(M)$. Thus $H_F(M)$ is normal.

Let $x \in H_F(M)$ then $kx \in N_F(M)$ for some positive integer k. Hence $x+(k-1)x \in N_F(M) \subseteq S_F(M)$ which implies by Lemma 6, that $x \in S_F(M)$. Thus $H_F(M) \subseteq S_F(M)$ and so by Lemma 8, $H_F(M)$ is subtractive onto M.

Corollary: $x \in H_F(M)$ & $(x+M) \cap (y+M) \neq \emptyset$ implies $y \in H_F(M)$.

Proof: Immediate by Lemma 4.

Remark: The relation $x^* \cap N_F(M) \neq \lambda^*$ can be interpreted as the "linear" dependence of x on M. This interpretation is in particular obvious in the case where F is a finitely generated free commutative monoid. In this case, F can be embedded in a linear space over the rationals, $R^n$ (where n is the number of the free generators of F) and $H_F(M)$ is the intersection of F with the subspace of $R^n$ spanned by M.

Lemma 12: $H_F$ is a closure operation on the submonoids of F; i.e.,

(i) $M \subseteq H_F(M)$ and $H_F(M)$ is a submonoid of F,

(ii) $M_1 \subseteq M_2$ implies $H_F(M_1) \subseteq H_F(M_2)$,

(iii) $H_F(H_F(M)) = H_F(M)$ .

<u>Proof:</u> From Lemma 11 we have (i). (ii) follows directly from the definition of $H_F(M)$. From (i) we infer that $H_F(M) \subseteq H_F(H_F(M))$, so let $x \in H_F(H_F(M))$. If $x = \lambda$ then $x \in H_F(M)$. If $x \neq \lambda$ then $x^* \cap N_F(H_F(M)) \neq \lambda^*$. By Lemma 11 we have that $H_F(M)$ is normal and therefore we have for $x \neq \lambda$ and $x \in H_F(H_F(M))$ that $x^* \cap H_F(M) \neq \lambda^*$. This implies $k_1 x \in H_F(M)$ for some positive integer $k_1$; hence $k_2 k_1 x \in N_F(M)$ for some positive integers $k_1$ and $k_2$, which shows that $x \in H_F(M)$.

The algebraic relations among $N_F$, $H_F$ and $S_F$ are summarized in the following Lemma, some of them are implied directly by our previous discussions.

<u>Lemma 13:</u> The three operations $N_F$, $H_F$ and $S_F$ are commutative, idempotent and satisfy the following relations:

(i) $H_F \circ N_F = H_F$

(ii) $S_F \circ N_F = S_F \circ H_F = S_F$ .

In other words, the semigroup of operations $\mathcal{C}$ generated by $N_F$, $H_F$, $S_F$ is a commutative idempotent monoid with a zero, in which $N_F$ is the identity element and $S_F$ is its zero element.

<u>Proof:</u> All we need to show is that the following table is the multiplication table for $\mathcal{C}$:

| $\circ$ | $N_F$ | $H_F$ | $S_F$ |
|---|---|---|---|
| $N_F$ | $N_F$ | $H_F$ | $S_F$ |
| $H_F$ | $H_F$ | $H_F$ | $S_F$ |
| $S_F$ | $S_F$ | $S_F$ | $S_F$ |

By Lemma 1, corollary of Lemma 6 and Lemma 11, we have the following relations:

$N_F \circ N_F = N_F$, $N_F \circ S_F = S_F$ and $N_F \circ H_F = H_F$ .

From the corollary of Lemma 7 and Lemma 11 we get the relations:

$S_F \circ N_F = S_F \circ H_F = S_F$ .

From Df. 3 and the relation $N_F \circ N_F = N_F$ we get the relation:

$H_F \circ N_F = H_F$ .

By Lemma 7 and Lemma 12 we have the relations:

$$S_F \circ S_F = S_F \quad \text{and} \quad H_F \circ H_F = H_F .$$

Hence, we need to prove only that $H_F \circ S_F = S_F$ holds. By Lemma 12 we know that $S_F(M) \subseteq H_F(S_F(M))$; so let $x \in H_F(S_F(M))$, then we have $kx \in N_F(S_F(M)) = S_F(M)$ for some positive integer k. But $kx = x+(k-1)x$ and so by Lemma 6 we have that $x \in S_F(M)$. Hence we have $H_F \circ S_F = S_F$ .

## §3 Factor monoids.

The method by which factor groups are defined in group theory cannot be applied directly to our context in order to get a definition of factor monoids. This is due to the fact that in our case we do not have the property that two cosets are either disjoint or identical. However, by defining a suitable equivalence relation in F we can define F/M to be the abstraction of F by that equivalence relation and the term "factor monoid" will be appropriate for F/M in the sense that factor groups become special cases of factor monoids and the theory of factor monoids will be similar to theory of factor groups. With this aim in mind we follow the suggestion of Mezei [2] for the equivalence relation $\rho_M$ and introduce the following definition.

<u>Definition 4</u>: Let M be a submonoid of F, we define a binary relation $\rho_M$ in F by

$$x\rho_M y \text{ iff } (x+M)\cap(y+M) \neq \emptyset .$$

Furthermore, we shall use the following notations:

(i) for any equivalence relation $\rho$ which is defined in F: $\rho(x) =_{df} \{y \in F : x\rho y\}$.

(ii) for any submonoid M' of F we define

$$M'/M =_{df} M'/\rho_M = \{\rho_M(x) : x \in M'\} .$$

<u>Theorem 1</u>: (Mezei) For any submonoid M of F, $\rho_M$ is the minimal congruence relation $\rho$ which is defined in F such that $\rho(\lambda) = N_F(M)$.

<u>Proof</u>: From the definition of $\rho_M$ it follows immediately that $\rho_M$ is symmetric and reflexive. Let $x \rho_M y$ and $y \rho_M z$, then we have $x + m_1 = y + m_2$ and $y + m_3 = z + m_4$ for some $m_1, m_2, m_3, m_4 \in M$. Hence $x + m_1 + m_3 = z + m_2 + m_4$ and $m_1 + m_3, m_2 + m_4 \in M$ which imply $x \rho_M z$. Thus $\rho_M$ is an equivalence relation.

Let $x \rho_M y$ and z be any element of F, then we have $x + m_1 = y + m_2$ for some $m_1, m_2 \in M$. Hence $(x+z) + m_1 = (y+z) + m_2$ which shows that $(x+z)\rho_M(y+z)$. Therefore $\rho_M$ is a congruence relation defined in F.

By Lemma 3 we have that $x \in \rho_M(\lambda)$ iff $(x + N_F(M)) \cap N_F(M) \neq \emptyset$. But since $N_F(M)$ is normal we have $(x + N_F(M)) \cap N_F(M) \neq \emptyset$ iff $x \in N_F(M)$. Hence $\rho_M(\lambda) = N_F(M)$.

Now let $\rho$ be any congruence relation which is defined in F such that $\rho(\lambda) = N_F(M)$. By Lemma 3 we have that $x \rho_M y$ implies

$x + m_1 = y + m_2$ for some $m_1, m_2 \in N_F(M) = \rho(\lambda)$. Since $\rho$ is a congruence relation we have $x + m_1 \, \rho \, x$ and $y + m_2 \, \rho \, y$, which together with the equality $x + m_1 = y + m_2$ imply $x \rho y$. Hence $x \rho_M y$ implies $x \rho y$.

Another connection among congruence relations which are defined in F, is given in the following lemma.

Lemma 14: Let $M_1$ and $M_2$ be two submonoids of F such that $M_1 \subseteq M_2$ and let $x, y \in F$, then $x \rho_{M_1} y$ implies $x \rho_{M_2} y$.

Proof: Immediate by Df. 4 .

Definition 5: Let M be a submonoid of F, we define a binary operation $\oplus$ in F/M by:

$$\rho_M(x) \oplus \rho_M(y) =_{df} \rho_M(x+y)$$

Theorem 2: Let M be a submonoid of F, then:

(i) $\langle F/M, \oplus \rangle$ is a commutative monoid with $\rho_M(\lambda)$ as its identity element;

(ii) $\langle F/M, \oplus \rangle$ is the image of $\langle F, + \rangle$ under the homomorphism $r_M(x) =_{df} \rho_M(x)$ whose kernel is $N_F(M)$;

(iii) the maximal submonoid of $\langle F/M, \oplus \rangle$ which is a group is $\langle S_F(M)/M, \oplus \rangle$ ;

(iv) if F is a cancellative monoid (i.e., if $x + z = y + z$ implies $x = y$ for all $x, y, z \in F$,) then $\langle S_F(M)/M, \oplus \rangle$ is the maximal subsemigroup of $\langle F/M, \oplus \rangle$ which is a group.

Remark: As it is usually done, we shall use the symbols "$F/M$", "$S_F(M)/M$" and "$H_F(M)/M$" to denote the sets of the equivalence classes and the algebraic systems consisting of these sets and the operation $\oplus$.

Proof: Since $\rho_M$ is a congruence relation we get that $x_1 \; \rho_M \; x$ and $y_1 \; \rho_M \; y$ imply $(x_1 + y_1) \; \rho_M \; (x+y)$ and therefore $\oplus$ is well defined.

(i) Clearly F/M is a commutative semigroup, and $\rho_M(x) \oplus \rho_M(\lambda) = {}_M(x)$ holds for any $x \in F$ since $\rho_M$ is a congruence relation. Hence F/M is a commutative monoid with $\rho_M(\lambda)$ as its identity element.

(ii) Immediate.

(iii) $\rho_M(x)$ is a unit in F/M iff there is $y \varepsilon F$ such that $\rho_M(x+y) = \rho_M(\lambda)$. Hence, by Lemma 6 and Lemma 11, $\rho_M(x)$ is a unit in F/M iff $x \varepsilon S_F(M)$. Since $S_F(M)/M$ forms a group we get that it is the maximal submonoid of F/M which is a group.

(iv) We have to show that in the case where F is cancellative, $\rho_M(x) + \rho_M(u) = \rho_M(x)$ implies $\rho_M(u) = \rho_M(\lambda)$. From $(x+u)\ \rho_M\ x$ follows $x + u + m_1 = x + m_2$ (for some $m_1, m_2 \varepsilon M$), hence, by the law of concellation we get $u + m_1 = m_2$ which shows that $\rho_M(u) = \rho_M(\lambda)$.

We can strengthen the result stated in Theorem 2 (iii) as follows:

<u>Lemma 15</u>: Let M and $M_1$ be submonoids of F, then $M_1/M$ is a submonoid of F/M which is a group iff M is subtractive onto $N_F(M)$.

<u>Proof</u>: If $M_1/M$ is a group then for any $x \varepsilon M_1$ there is $y \varepsilon M_1$ such that $\rho_M(x+y) = \rho_M(\lambda)$. Thus, for any $x \varepsilon M_1$ there is $y \varepsilon M_1$ such that $x + y \varepsilon \rho_M(\lambda) = N_F(M)$. Hence M is subtractive onto $N_F(M)$.

If $M_1$ is subtractive onto $N_F(M)$ then for any $x \varepsilon M_1$ there is $y \varepsilon M_1$ such that $x + y \varepsilon N_F(M)$. Hence for any $a \varepsilon M_1/M$ there is

$b \in M_1/M$ such that $a \oplus b = \rho_M(\lambda)$ and since $M_1/M$ is a commutative monoid, this implies that $M_1/M$ is a group.

<u>Corollary</u>: If $M \subseteq M_1$ then $M_1/M$ is a group iff $M_1$ is substractive onto M.

<u>Proof</u>: Immediate by Lemma 10 (ii) .

Most of the expected connections between homomorphisms of monoids, factor monoids, congruence relations and normal submonoids can be established similarly to the corresponding results of group theory.

<u>Theorem 3</u>: Let F and $F_1$ be commutative monoids and let $f : F \to F_1$ be a homomorphism of F onto $F_1$ .

(i) $K_f$, the kernel of f is a normal submonoid of F.

(ii) The relation $\rho_f$ determined **in** F by f:

$$x\rho_f y \underset{df}{\Longleftrightarrow} f(x) = f(y) \quad ,$$

is a congruence relation and $\rho_f(\lambda) = K_f$ .

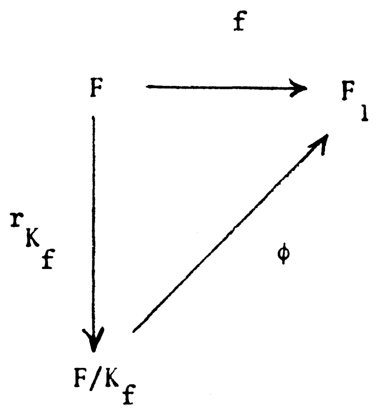
(iii) F is unit-free (i.e., $u + v = \lambda$ implies $u = \lambda$ for all $u, v \in F$ ,) iff $S_F(K_f) = K_f$.

<u>Proof</u>: (i) See Lemma 2.

(ii) Immediate.

(iii) Let $x + y \in K_f$ then in $F_1$ we have $f(x) + f(y) = \lambda$. Hence, if $F_1$ is unit-free then $x + y \in K_f$ implies $x \in K_f$ which shows that $S_F(K_f) = K_f$ . On the other hand if we have in $F_1$ $u + v = \lambda$ then $u = f(x)$ and $v = f(y)$ for some $x, y \in F$ and $f(x+y) = f(x) + f(y) = u + v = \lambda$; i.e., $x + y \in K_f$. Hence if $S_F(K_f) = K_f$ then we have $x \in K_f$ and therefore $u = f(x) = \lambda$ and thus $F_1$ is unit-free.

<u>Theorem 4</u>: Let $F$ and $F_1$ be commutative monoids and let $f: F \to F_1$ be a homomorphism of $F$ onto $F_1$ . There exists a unique homomorphism $\phi: F/K_f \to F_1$ such that the following diagram is commutative,




that is, $\phi \circ r_{K_f} = f$ (where $r_{K_f}(x) =_{df} \rho_{K_f}(x)$). Moreover, $\phi$ is onto and it has a trivial kernel, namely $K_\phi = \{K_f\}$. However, $\phi$ is an isomorphism of $F/K_f$ onto $F_1$ iff $\rho_f = \rho_{K_f}$.

<u>Proof:</u> Clearly, we define $\phi$ by $\phi(\rho_{K_f}(x)) =_{df} f(x)$.

By Theorem 1 and according to Theorem 3(i), we have that $\rho_{K_f}$ is the minimal congruence relation $\rho$ which is defined in F such that $\rho(\lambda) = K_f$. By Theorem 3(ii) we have that $\rho_f$ is a congruence relation which is defined in F such that $\rho_f(\lambda) = K_f$. Hence $\rho_{K_f}$ implies $\rho_f$, which shows that $\phi$ is well defined. It is obvious that $\phi$ is the only mapping from $F/K_f$ to $F_1$ which satisfies the relation $\phi \circ r_{K_f} = f$.

From the fact that $r_{K_f}$ and f are both homomorphisms of F and f is onto, and from the relation $\phi \circ r_{K_f} = f$ it follows that $\phi$ is a homomorphism of $F/K_f$ onto $F_1$.

Note that

$$\rho_{K_f}(x) \in K_\phi \quad \text{iff} \quad f(x) = \lambda \quad \text{in } F_1 ,$$

$$\text{i.e., iff} \quad x \in K_f ,$$

$$\text{i.e., iff} \quad \rho_{K_f}(x) = K_f .$$

Hence, $K_\phi = \{K_f\}$.

Now, if $\rho_f = \rho_{K_f}$ and $\phi(\rho_{K_f}(x)) = \phi(\rho_{K_f}(y))$, then $f(x) = f(y)$ and therefore $\rho_f(x) = \rho_{K_f}(x) = \rho_{K_f}(y) = \rho_f(y)$. Hence $\rho_f = \rho_{K_f}$ implies that $\phi$ is an isomorphism of $F/K_f$ onto $F_1$. On the other hand, if $\phi$ is an isomorphism then

$$f(x) = \phi(\rho_{K_f}(x)) = \phi(\rho_{K_f}(y)) = f(y) \text{ implies } \rho_{K_f}(x) = \rho_{K_f}(y)$$

that is, that $\rho_f$ implies $\rho_{K_f}$. Since always $\rho_{K_f}$ implies $\rho_f$, we get that if $\phi$ is an isomorphism then $\rho_f = \rho_{K_f}$.

We end this section with a discussion on certain conditions for F/M to be finite.

<u>Lemma 16</u>: Let M be a submonoid of F. If F is cancellative and $S_F(M) \neq F$ then F/M is infinite (and so F is infinite too).

<u>Proof</u>: Let $x \in F$ and let $k_1 > k_2$ be two distinct non-negative integers. Assume that $\rho_M(k_1x) = \rho_M(k_2x)$, then $k_1x + m_1 = k_2x + m_2$ for some $m_1, m_2 \in M$. Hence $k_2x + (k_1-k_2)x + m_1 = k_2x + m_2$, which by cancellation implies $(k_1 - k_2)x + m_1 \in M$. Since $k_1 > k_2$ we get $k_1-k_2 \geq 1$ and so we have $x + ((k_1-k_2-1)x + m_1) \in M$ which shows that $x \in S_F(M)$. Hence if $x \notin S_F(M)$ then $\rho_M(k_1x) \neq \rho_M(k_2x)$ for any distinct non-negative integers $k_1$, $k_2$, and thus F/M is infinite.

<u>Lemma 17</u>: Let F be cancellative and M be a submonoid of F. If $x \in F$ but $x \notin H_F(M)$ then $\rho_M(k_1x) = \rho_M(k_2x)$ implies $|k_1-k_2|x = \lambda$. In particular, if $k_1 \neq k_2$ and yet $\rho_M(k_1x) = \rho_M(k_2x)$, (where $x \in F$ but $x \notin H_F(M)$, ) then x is a unit of F (i.e., there is $y \in F$ such that $x + y = \lambda$ ).

<u>Proof</u>: As in the proof of Lemma 16 we get that $|k_1-k_2|x \in N_F(M)$. Thus $x \notin H_F(M)$ implies $|k_1-k_2|x = \lambda$.

<u>Corollary</u>: Let F be cancellative and unit-free and let M be a submonoid of F. If $H_F(M) \neq F$ then F/M is infinite.

<u>Remark</u>: Note that since $H_F(M) \subseteq S_F(M) \subseteq F$ holds, $H_F(M) = F$ implies $S_F(M) = F$.

<u>Lemma 18</u>: Let F be cancellative and unit-free, M be a submonoid of F and $w \in F$; then $w^* \cap N_F(M)$ is a normal submonoid of F generated by a unique element.

<u>Proof</u>: Let $x, x + y \in w^*$ for $x, y \in F$, then $x = k_1w$ and $x + y = k_2w$ for some non-negative integers $k_1$, $k_2$.

If $k_2 \geq k_1$ then let $k_2 = k_1 + k$ and we have $k_1w + kw = k_1w + y$ and so by cancellation we get $y = kw \in w^*$. If $k_2 < k_1$ then let $k_1 = k_2 + k$ and we have $x + y + kw = x$ and so by cancellation we get $y + kw = \lambda$. But F is unit-free and therefore we have $y = \lambda \in w^*$. Thus $w^*$ is a normal submonoid of F. Hence $w^* \cap N_F(M)$ as an intersection of normal submonoids of F is a normal submonoid of F too.

If $w^* \cap N_F(M) = \lambda^*$ then $\lambda$ is the generator of $w^* \cap N_F(M)$. If $w^* \cap N_F(M) \neq \lambda^*$ then let $k_0$ be the minimal positive integer k such that $kw \in (w^* \cap N_F(M))$. Clearly $(k_0w)^* \subseteq (w^* \cap N_F(M))$ and we shall show that $(k_0w)^* = (w^* \cap N_F(M))$.

Let $kw \in (w^* \cap N_F(M))$ and let $k = pk_0 + r$ where p,r are non-negative integers such that $0 \leq r < k_0$. Then we have $kw = p(k_0w) + rw$ and $kw$, $p(k_0w) \in (w^* \cap N_F(M))$. But $w^* \cap N_F(M)$ is normal and therefore $rw \in (w^* \cap N_F(M))$; hence, by the choice of $k_0$ it follows that $r = 0$ and $kw = p(k_0w) \in (k_0w)^*$. Thus $w^* \cap N_F(M) = (k_0w)^*$.

As for the uniqueness of the generator of $w^* \cap N_F(M)$ we shall show that $x_1^* = x_2^*$ implies $x_1 = x_2$. From $x_1^* = x_2^*$ follows that $x_1 = k_2x_2$ and $x_2 = k_1x_1$ for some positive integers $k_1$, $k_2$. Hence $x_1 = k_1k_2x_1$ and $x_1 = x_1 + (k_1k_2-1)x_1$

which by cancellation implies $(k_1k_2-1)x_1 = \lambda$ and this by the fact that F is unit-free, implies $k_1k_2 = 1$ and thus $x_1 = x_2$ .

<u>Definition 6</u>: Let F be cancellative and unit-free, and M be a submonoid of F. For any set $B = \{..w_i..\}$ of generators for $H_F(M)$ we associate the set $B_M = \{...\varepsilon_i ...\}$ where $\varepsilon_i = k_iw_i$, for any i, is the generator of $w_i^* \cap N_F(M)$.

We denote by $[B_M]$ the set of all elements of $H_F(M)$ of the form $x = \sum_i a_iw_i$ where for any i : $0 \leqq a_i < k_i$ (and $a_i \neq 0$ only for a finite set of values for i).

<u>Lemma 19</u>: Let M be a submonoid of F where F is cancellative and unit-free, and let B be a set of generators for $H_F(M)$. Then:

(i) $B_M$ is a set of elements of $N_F(M)$,

(ii) for any $x \in H_F(M)$ there is $y \in [B_M]$ such that $x + B_M^* \subseteq y + B_M^*$,

(iii) $(x_1 + B^*_M) \cap (x_2 + B^*_M) \neq \emptyset$ and $x_1 \in N_F(M)$ imply $x_2 \in N_F(M)$.

Proof: From the definition of $B_M$ it follows directly that $B_M$ is a set of elements of $N_F(M)$. Hence $B^*_M \subseteq N_F(M)$ and this implies (iii).

Let $x \in H_F(M)$ then $x = \sum_i a_i w_i$. For any i there are non-negative integers $b_i$ and $c_i$ such that $a_i = b_i k_i + c_i$ and $0 \leq c_i < k_i$; so $x = \sum_i b_i \varepsilon_i + \sum_i c_i w_i$. Let $m = \sum_i b_i \varepsilon_i$ and $y = \sum_i c_i w_i$ then $x \in B^*_M$, $y \in [B_M]$ and $x = y + m$. Hence $x + B^*_M \subseteq y + B^*_M$.

Corollary: (i) $[B_M] \cup B_M$ is a set of generators for $H_F(M)$.

(ii) $H_F(M) = \bigcup \{y + B^*_M : y \in [B_M]\}$.

(iii) $([B_M] \cap N_F(M)) \cup B_M$ is a set of generators for $N_F(M)$.

(iv) $N_F(M) = \bigcup \{y + B^*_M : y \in ([B_M] \cap N_F(M))\}$.

In order to apply these results to factor monoids we need the following lemma.

Lemma 20: Let $M_1$ and $M_2$ be submonoids of F such that $M_1 \subseteq M_2$. If $F/M_1$ is finite then $F/M_2$ and $M_2/M_1$ are finite too (hence, $N_F(M_2)/M_1$ is also finite.)

Proof: Let $F/M_1 = \{\rho_{M_1}(x_1), \ldots, \rho_{M_1}(x_k)\}$. By Lemma 14 we have that for any $x \in F$ : $\rho_{M_1}(x) \subseteq \rho_{M_2}(x)$.

Hence $F = \bigcup_{i=1}^{k} \rho_{M_2}(x_i)$ and therefore $F/M_2$ contains at most k elements.

Let $\rho_1(x)$ be an element of $M_2/M_1$ then $\rho_1(x) \subseteq \rho_{M_1}(x) \in F/M_1$.

Let $\rho_1(x)$ and $\rho_1(y)$ be any elements of $M_2/M_1$ which are included in $\rho_{M_1}(z)$ for some $z \in F$, then $x\rho_{M_1}y$ and therefore $\rho_1(x) = \rho_1(y)$. Hence $M_2/M_1$ contains at most the same number of elements as $F/M_2$.

By taking $M_2 = N_F(M_2)$ we get that $N_F(M_2)/M_1$ is finite.

Theorem 5: Let F be cancellative and unit-free and M be a submonoid of F. If $H_F(M)$ is finitely generated then $N_F(M)$ is finitely generated and $H_F(M)/M$ is finite.

Proof: Let B be a finite set of generators for $H_F(M)$, then clearly $B_M$ and $[B_M]$ are finite.

From the corollary of Lemma 19 it follows that $N_F(M)$ is generated by a subset of $[B_M] \cup B_M$ and so it is finitely generated. From the same corollary follows that $H_F(M)/B^*_M$ is finite, but $B^*_M \subseteq N_F(M)$ and so by Lemma 20 we have that $H_F(M)/N_F(M)$ is finite but $H_F(M)/N_F(M) = H_F(M)/M$ and so $H_F(M)/M$ is finite.

Combining Theorem 5 with the corollary of Lemma 17 we get a necessary and sufficient condition for F/M to be finite in the case where F is finitely generated, cancellative and unit-free, e.g., in the case where F is a finitely generated free commutative monoid.

Theorem 6: Let F be a finitely generated cancellative and unit-free commutative monoid and let M be a submonoid of F then F/M is finite iff $H_F(M) = F$.

## §4 Normal submonoids of $F_n$

Let $F_n$ be the free commutative monoid generated by $E = \{e_1, \ldots, e_n\}$ . The special case of the finitely generated submonoids of $F_n$ is of importance to the study of commutative events since these are the submonoids of $F_n$ which are denoted by regular expressions over $E$ as an alphabet. For a detailed discussion on this connection the reader is referred to [1].

Lemma 21: Let $M$ be a submonoid of $F_n$. There is a finitely generated free submonoid $N$ of $N_F(M)$ which is normal in $F_n$ and $H_{F_n}(M) = H_{F_n}(N)$.

Proof: Let $R^n$ be the n-dimensional vector-space over the rationals; then, as one can easily verify, $H_{F_n}(M) = F_n \cap V(M)$ where $V(M)$ is the sub-vector-space spanned by $M$ and clearly $F_n$ is the first orthant of $R^n$.

From linear algebra we know that for any sub-vector-space $V$ of $R^n$ which has a basis in the first orthant of $R^n$ , one can find such a basis $\{v_1, \ldots, v_k\}$ with the additional property that $v$ is a vector of $V$ with non-negative components only iff $v = \sum_{i=1}^{k} r_i v_i$ where for all $1 \leq i \leq k : r_i \geq 0$.

Since $M \subseteq F_n$, $V(M)$ has such a basis in the first orthant of $R^n$ say $\{v_1, \ldots, v_k\}$. For any $v_i$ in this basis there is positive integer $p_i$ such that $p_i v_i \in F_n$ and therefore $p_i v_i \in H_{F_n}(M)$. But this implies that $p_i q_i v_i \in N_{F_n}(M)$ for some positive integer $q_i$. So let $w_i = k_i v_i$ be the first non-zero point on the line determined by $v_i$ which is an element of $N_{F_n}(M)$. Since $\{v_i, \ldots, v_k\}$ is an independent set of vectors in $R_n$ we get that $W = \{w_1, \ldots, w_k\}$ generates a free submonoid N of $N_F(M)$.

Let $x, x+y \in N$ for some $y \in F_n$ then clearly $y \in V(M)$ and therefore $y \in H_{F_n}(M)$. So let $x = \sum_{i=1}^{k} a_i w_i$ $y = \sum_{i=1}^{k} r_i v_i$ and $x + y = \sum_{i=1}^{k} b_i w_i$ then we have

$$r_i = (b_i - a_i)k_i \geq 0 \quad \text{for all } 1 \leq i \leq k$$

which shows that $y \in N$ and therefore N is normal.

From $V(M) = V(N)$ follows $V(M) \cap F_n = V(N) \cap F_n$, that is, $H_{F_n}(M) = H_{F_n}(N)$.

Following some of the ideas which were discussed in the last part of the previous section we define:

<u>Definition 7</u>: Let N be a normal free submonoid of $F_n$ generated by the basis $W = \{w_1, \ldots, w_k\}$.

<u>a</u>: We denote by $[W]_H$ the set of all elements x of $F_n$ for which in $R^n$ we have $x = \sum_{i=1}^{k} r_i w_i$ where for all $1 \leq i \leq k$ : $r_i$ is rational and $0 \leq r_i < 1$.

<u>b</u>: We define a binary operation <+> in the first orthant of V(N) with regard to W:

$$x = \sum_{i=1}^{k} r_i w_i \text{ and } y = \sum_{i=1}^{k} s_i w_i \text{ imply } x <+> y =_{df} \sum_{i=1}^{k} (\max(r_i, s_i)) w_i .$$

Certain properties of <+> are summarized in the following lemma; the proof is straightforward and will not be given.

<u>Lemma 22</u>: (i) <+> is associative, commutative and idempotent.

(ii) $x, y \in [W]_H$ iff $x <+> y \in [W]_H$ .

Similarly to Lemma 19 we have the following theorem which establishes the relation between N and $H_{F_n}(N)$ in more detail.

<u>Theorem 7</u>: Let $N$ be a normal free submonoid of $F_n$ generated by the basis $W$ and let $H = H_{F_n}(N)$.

Then:

(i) $[W]_H$ is a finite set of elements of $H$,

(ii) for any $x \in H$ there is $y \in [W]_H$ such that $x + N \subseteq y + N$,

(iii) $(x+N) \cap (y+N) \neq \emptyset$ (i.e., $x \rho_N y$,) for $x,y \in H$ implies:

(1) $(x+N) \cap (y+N) = (x <+> y) + N$,

(2) $x, y, x <+> y \in \rho_N(x)$ ,

(3) if $x, y \in [W]_H$ then $x = y$.

<u>Proof</u>: (i) From the definition of $[W]_H$ it follows that if $x \in [W]_H$ then $px \in N$ for a suitable positive integer $p$ and since $x \in F_n$ it follows that $x \in H$. Clearly $[W]_H$ is finite.

Let $W = \{w_1, \ldots, w_i\}$ . From the definition of $H_F$ (see section 1, Df. 3) it follows that, in our case, for $x \in F_n$ , $x \in M$ iff $x = \sum_{i=1}^{k} r_i w_i$ where for all $1 \leq i \leq k$ : $r_i$ is a non-negative <u>rational</u>.

(ii) Let $x \in H$ then we have $x = \sum_{i=1}^{k} r_i w_i$ for non-negative rationals $r_i$. For any $1 \leq i < k$ let $a_i$ be a non-negative integer and $s_i$ be a non-negative rational such that $r_i = a_i + s_i$ and $0 \leq s_i < 1$; and so we have

$$x = \sum_{i=1}^{k} a_i w_i + \sum_{i=1}^{k} s_i w_i \quad .$$

Let $w = \sum_{i=1}^{k} a_i w_i$ and $y = \sum_{i=1}^{k} s_i w_i$, then we have $w \in N$ and $x = w + y$. From $x, w \in F_n$ follows that $y \in F_n$ (in other words, $F_n$ is a normal submonoid of the first orthant of $R^n$ which is a commutative monoid) and so $y \in [W]_H$ and $x + N \subseteq y + N$.

(iii) Let $x = \sum_{i=1}^{k} r_i w_i$, $y = \sum_{i=1}^{k} s_i w_i$ and $z = \sum_{i=1}^{k} t_i w_i \in (x+N) \cap (y+N)$ for non-negative rationals $r_i$, $s_i$ and $t_i$. Hence we have

$$z = \sum_{i=1}^{k} t_i w_i = \sum_{i=1}^{k} r_i w_i + \sum_{i=1}^{k} a_i w_i = \sum_{i=1}^{k} s_i w_i + \sum_{i=1}^{k} b_i w_i \quad ,$$

or

$$z = \sum_{i=1}^{k} (r_i + a_i) w_i = \sum_{i=1}^{k} (s_i + b_i) w_i$$

for non-negative integers $a_i$ and $b_i$. Since $W$ is a linear basis of $V(N)$ we get

$$t_i = r_i + a_i = s_i + b_i \qquad \text{for all } 1 \leq i \leq k \quad ,$$

which implies $\quad t_i = \max(r_i, s_i) + \min(a_i, b_i)$

and $\quad a_i - b_i = s_i - r_i \qquad \text{for all } 1 \leq i \leq k \quad .$

Let $w = \sum_{i=1}^{k} (\min(a_i,b_i))w_i$ then clearly $w \in N$ and we have $z = (x <+> y) + w$ which implies $(x + N) \cap (y + N) \subseteq (x <+> y) + N$ .

On the other hand we have:

$$x <+> y = \sum_{i=1}^{k} r_i w_i + \sum_{s_i \geq r_i} (s_i - r_i)w_i = x + \sum_{a_i \geq b_i} (a_i - b_i)w_i$$

and

$$x <+> y = \sum_{i=1}^{k} s_i w_i + \sum_{r_i \geq s_i} (r_i - s_i)w_i = y + \sum_{b_i \geq a_i} (b_i - a_i)w_i \quad .$$

Hence $x <+> y \in (x + N) \cap (y + N)$ and this concludes the proof of (iii),(1). Furthermore, the last equalities show that $(x <+> y)\rho_N x$ and $(x <+> y)\rho_N y$ and since we assume $x\rho_N y$ we have proved (iii), (2).

The same equalities yield

$$x <+> y = x + \sum_{s_i \geq r_i} (s_i - r_i)w_i = y + \sum_{r_i \geq s_i} (r_i - s_i)w_i \quad ,$$

and so $x,y \in [W]_H$ implies

$$0 = |r_i - s_i| < 1 \text{ for all } 1 \leq i \leq k$$

and yet $|r_i - s_i|w_i \in N$ which is possible only if $|r_i - s_i| = 0$ and therefore $x = y$ .

<u>Corollary</u>: (i) $[W]_H \cup W$ is a set of generators for $H$ and therefore $H$ is finitely generated.

(ii) $H/N = \{y + N : y \in [W]_H\}$ and so $H/N$ is finite.

Now, by Lemma 21 and Theorem 5, Theorem 7 yields the following result:

Theorem 8: For any submonoid $M$ of $F_n$, if $M$ is normal then it is finitely generated. In particular $M$ is a finite union of disjoint cosets of a normal free submonoid of $M$.

Proof: By Lemma 21 there is a normal free submonoid $N$ of $M$ such that $H_{F_n}(M) = H_{F_n}(N)$. By Theorem 7 we get that $H_{F_n}(M)$ is finitely generated and so by Theorem 5 we get that $N_{F_n}(M) = M$ is finitely generated.

From the relations $N \subseteq M \subseteq H_{F_n}(M) = H_{F_n}(N)$ it follows that $x \in M$ and $x\rho_N y$ imply $y \in N_{F_n}(M) = M$. Hence we get $M = \bigcup\{y + N : y \in [W]_H \cap M\}$ and so by Theorem 7 we get that $M$ is a finite union of disjoint cosets of $N$.

Thus Theorem 8, the corollary of Theorem 7 and Lemma 9 show us that the operators $N_{F_n}$, $H_{F_n}$ and $S_{F_n}$ have in their range only finitely generated submonoids of $F_n$. This is implied directly from Theorem 8 "alone" since $H_{F_n}$ and $S_{F_n}$ have in their range only normal submonoids of $F_n$.

* * *

## References

[1] R. Laing and J. B. Wright, "Commutative Machines," Technical Note, The University of Michigan (ORA), December 1962.

[2] J. Mezei, "Structure of Monoids with Applications to Automata," Technical Report, IBM Corporation

DISTRIBUTION LIST

(One copy unless otherwise noted)

Assistant Secretary of Defense
for Research and Engineering — 2
Information Office Library Branch
Pentagon Building
Washington 25, D.C.

Armed Services Technical
Information Agency — 10
Arlington Hall Station
Arlington 12, Virginia

Chief of Naval Research — 2
Department of the Navy
Washington 25, D.C.
Attn: Code 437, Infor. Syst. Br.

National Security Agency
Fort George G. Meade, Maryland
Attn: R-4, Howard Campaigne

Director, Naval Research Labs. — 6
Tech. Infor. Officer, Code 2000
Washington 25, D.C.

Commanding Officer — 10
Office of Naval Research
Navy No. 100, Fleet Post Office
New York, New York

Commanding Officer, ONR Br. Office
346 Broadway
New York 13, New York

Commanding Officer, ONR Br. Office
495 Summer Street
Boston 10, Massachusetts

Chief of Naval Operations
OP-07T-12
Navy Department
Washington 25, D.C.

Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Code 607A, NTDS

Bureau of Naval Weapons
Department of the Navy
Washington 25, D.C.
Attn: RAAV, Avionics Division

Bureau of Ships
Department of the Navy
Washington 25, D.C.
Attn: Communications Br., Code 686

Naval Ordnance Laboratory
White Oaks
Silver Spring 19, Maryland
Attn: Technical Library

Massachusetts Institute of Technology
Research Laboratory of Electronics
Cambridge, Massachusetts
Attn: Professor W. McCulloch

David Taylor Model Basin
Washington 7, D.C.
Attn: Technical Library

Naval Electronics Laboratory
San Diego 52, California
Attn: Technical Library

University of Illinois
Control Systems Laboratory
Urbana, Illinois
Attn: D. Alpert

University of Illinois
Digital Computer Laboratory
Urbana, Illinois
Attn: Dr. J. E. Robertson

Technical Information Officer
U. S. Army Signal R&D Laboratory
Fort Monmouth, New Jersey
Attn: Data Equipment Branch

U. S. Naval Weapons Laboratory
Dahlgren, Virginia
Attn: Head, Computation Division
G. H. Gleissner

George Washington University
Washington, D.C.
Attn: Prof. N. Grisamore

Aberdeen Proving Ground, BRL
Aberdeen Proving Ground, Maryland
Attn: J. H. Giese, Chief Comput. Lab.

Office of Naval Research
Resident Representative
The University of Michigan

Ann Arbor, Michigan

Commanding Officer
ONR, Branch Office
John Crerar Library Bldg.
86 East Randolph Street
Chicago 1, Illinois

Commanding Officer
ONR, Branch Office
1030 E. Green Street
Pasadena, California

Commanding Officer
ONR, Branch Office
1000 Geary Street
San Francisco 9, California

National Bureau of Standards
Washington 25, D.C.
Attn: Mr. R. D. Elbourn

Naval Ordnance Laboratory
Corona, California
Attn: H. H. Weider

Massachusetts Inst. of Technology
Cambridge, Massachusetts
Attn: D. W. Baumann, Dynamic
Analysis and Control Lab.

Burroughs Corporation
Research Center
Paoli, Pennsylvania
Attn: R. A. Tracey

National Bureau of Standards
Data Processing Systems Division
Room 239, Bldg. 10, Attn: A. K. Smilow
Washington 25, D. C.

Lockheed Missiles and Space Company
3251 Hanover Street
Palo Alto, California
Attn: W. F. Main

The University of Michigan
Ann Arbor, Michigan
Attn: Professor A. W. Burks,
Dept. of Philosophy

Census Bureau
Washington 25, D.C.
Attn: Office of Assistant Director
for Statistical Services,
Mr. J. L. McPherson

National Science Foundation
Program Dir. for Documentation Res.
Washington 25, D.C.
Attn: Helen L. Brownson

University of California
Los Angeles 24, California
Attn: Department of Engineering,
Professor Gerald Estrin

Columbia University
New York 27, New York
Attn: Department of Physics,
Professor L. Brillouin

University of Illinois
Champaign Urbana, Illinois
Attn: John R. Pasta

Naval Research Laboratory
Washington 25, D.C.
Attn: Security Systems
Code 5266, Mr. G. Abraham

Diamond Ordnance Fuze Laboratory
Connecticut Ave. & Van Ness St.
Washington 25, D.C.
Attn: ORDTL-012, E. W. Channel

Harvard University
Cambridge, Massachusetts
Attn: School of Applied Science
Dean Harvey Brook

The University of Chicago
Institute for Computer Research
Chicago 37, Illinois
Attn: Mr. Nicholas C. Metropolis

Wright Air Development Division
Electronic Technology Laboratory
Wright-Patterson AFB, Ohio
Attn: Lt. Col. L.M. Butsch, Jr.
ASRUEB

Laboratory for Electronics, Inc.
1079 Commonwealth Ave.
Boston 15, Massachusetts
Attn: Dr. H. Fuller

Stanford Research Institute
Computer Laboratory
Menlo Park, California
Attn: H. D. Crane

General Electric Co.
Schenectady 5, New York
Attn: Library, L.M.E. Dept.

Hunter College
New York 21, New York
Attn: Dean Mina Rees

The RAND Corp.
1700 Main St.
Santa Monica, California
Attn: Numerical Analysis Dept.
Willis H. Ware

General Electric Research Laboratory
P. O. Box 1088
Schenectady, New York
Attn: Information Studies Section
R. L. Shuey, Manager

Mr. Sidney Kaplan
1814 Glen Park Ave.
Silver Spring, Maryland

University of Pennsylvania
Institute of Co-operative Research
Philadelphia, Pennsylvania
Attn: Dr. John O'Conner

Stanford Research Institute
Menlo Park, California
Attn: Dr. Charles Rosen
Applied Physics Laboratory

Northeastern University
360 Huntington Ave.
Boston, Massachusetts
Attn: Prof. L. O. Dolansky

New York University
New York, New York
Attn: Dr. J. H. Mulligan, Jr.
Chairman of E. E. Dept.

Marquardt Aircraft Co.
16555 Saticoy St.
P. O. Box 2013, South Annex
Van Nuys, California
Attn: Dr. Basun Chang
Research Scientist

Texas Technological College
Lubbock, Texas
Attn: Paul G. Griffith
Dept. of Elec. Eng.

Dr. Stanley Winkler
IBM Corporation
Federal Systems Division
326 E. Montgomery Ave.
Rockville, Maryland

Post Office Department
Office of Research and Engineering
12th and Pennsylvania Ave.
Washington 25, D.C.
Attn: Mr. R. Kopp
Res. & Dev. Division

L. G. Hanscom Field
AF-CRL-CRRB
Bedford, Massachusetts
Attn: Dr. H. H. Zschirnt

Department of the Army
Office of the Chief of Res. & Dev.
Pentagon, Room 3D442
Washington 25, D.C.
Attn: Mr. L. H. Geiger

Bell Telephone Laboratories
Murray Hill Laboratory
Murray Hill, New Jersey
Attn: Dr. Edward F. Moore

National Biomedical Res. Found., Inc.
8600 16th St., Suite 310
Silver Spring, Maryland
Attn: Dr. R. S. Ledley

University of Pennsylvania
Moore School of Elec. Eng.
200 South 33rd St.
Philadelphia 4, Pennsylvania
Attn: Miss Anna Louise Campion

Division of Automatial Data
Processing /AOP/
Department of State
Washington 25, D.C.
Attn: F. P. Diblasi, 19A16

Auerbach Electronics Corp.
1634 Arch St.
Philadelphia 3, Pennsylvania

University of Pennsylvania
Mechanical Languages Projects
Moore School of Elec. Eng.
Philadelphia 4, Pennsylvania
Attn: Dr. Saul Gorn, Director

Applied Physics Laboratory
Johns Hopkins University
8621 Georgia Ave.
Silver Spring, Maryland
Attn: Document Library

Bureau of Supplies and Accounts, Chief
Navy Department
Washington, D.C.
Attn: Code W3

National Aeronautics & Space Admin.
Goddard Space Flight Center
Greenbelt, Maryland
Attn: Chief, Data Systems Div.

Federal Aviation Agency
Bureau of Research & Development
Washington 25, D.C.
Attn: RD-375, Mr. Harry Hayman

Mr. Donald F. Wilson
Naval Research Laboratory
Washington 25, D.C.
Attn: Code 5144

David Taylor Model Basin
Washington 7, D.C.
Attn: Aerodynamics Laboratory, Code 628
Miss Cravens

Lincoln Laboratory
Mass. Institute of Technology
Lexington 73, Massachusetts
Attn: Library

Dr. C. R. Porter
Psychology Department
Howard University
Washington 1, D.C.

Electronics Research Laboratory
University of California
Berkeley 4, California
Attn: Director

Institute for Defense Analysis
Communications Research Division
Von Neumann Hall
Princeton, New Jersey

Hebrew University
Jerusalem, Israel
Attn: Prof. Y. Bar-Hillel

National Physical Laboratory
Teddington, Middlesex
England
Attn: Dr. A. M. Uttley, Supt.
Autonomics Division