

THE UNIVERSITY OF MICHIGAN
COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Communication Sciences Program

Technical Report

OUTLINE FOR AN ALGEBRAIC STUDY OF EVENT AUTOMATA

Yehoshafat Give'on

ORA Project 03105, 05662, 06689

under contract with:

U. S. ARMY RESEARCH OFFICE (DURHAM)
GRANT NO. DA-ARO(D)-31-124-G433, PROJECT NO. 4049-M

and

GRANT NO. DA-ARO(D)-31-124-G558
DURHAM, NORTH CAROLINA

and

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
CONTRACT NO. Nonr-1224(21)
WASHINGTON, D. C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

June 1964

C. ngm

U. 72

1559

ACKNOWLEDGMENT

I wish to thank S. T. Hedetniemi and J. H. Holland of the Logic of Computers Group, J. W. Thatcher of the IBM Research Center in Yorktown Heights, and M. Perles of the Hebrew University of Jerusalem, Israel, for their continuing interest in my research and for their invaluable help.

INTRODUCTION

Regular events are characterized by various means: Regular expressions, congruence relations or homomorphisms, deterministic and non-deterministic finite automata, Büchi's monadic algebras, etc. Naturally, each characterization carries with it its specific "intuition-space" of mathematical insight and concepts. In addition to these, each characterization suggests natural generalizations.

The Kleenean theory of regular events suggests the definition of a class K_W which is the smallest class of subsets of a monoid W such that it contains the finite subsets of W and it is closed under the complex-product of subsets of W and under the submonoid-generating operation in W (McKnight 1960-a, 1960-b).

The homomorphic theory of regular events suggests the definition of a class R_W of the W -regular events which are the subsets E of W such that $E = \mathcal{P}^{-1}\mathcal{P}(E)$ for some homomorphism \mathcal{P} of W with a finite range (Rabin and Scott 1959, Give'on 1963 and Medvedev 1958).

Rabin-Scott's theory of finite automata suggests the definition of finite-state automata with W as the set of input-tapes. The internal mechanism of these W -automata follows Ginsburg's studies (Ginsburg 1960) with analogous modifications which modify sequential automata into Rabin-Scott finite automata. Here, of course,

it is possible to define both deterministic and non-deterministic W-automata in a natural way.

Büchi's theory of finite automata within the framework of the theory of monadic algebras (Büchi 1960, Thatcher 1963) suggests the following definition of (pure) W-monadic algebra: a (pure) W-monadic algebra is a system

$$\mathcal{A} = \langle A, \{\alpha_w : w \in W\} \rangle$$

where

(i) A is a set, the carrier of \mathcal{A} ;

(ii) $\alpha_w : A \rightarrow A$ for any $w \in W$ with the condition that the operation

in W will be compatible with the compositions of the α_w , the operators of \mathcal{A} ;

i.e., that for any $w_1, w_2 \in W$, using the suffix-notation, we shall have:

$$\alpha_{w_1} \cdot \alpha_{w_2} = \alpha_{w_1 w_2}.$$

Now, for any $a_0 \in A$ and $F \subseteq A$ we define the event

$$\mathcal{A}[a_0, F] = \{w : a_0 \alpha_w \in F\}$$

I choose to base my theory on the homomorphic characterization of regular events. I think that it is still premature to justify this starting point though

I have the following arguments to support my choice:

(a) It seems to me that the study of automata via the study of monoids and monoid-homomorphisms brings automata closer to the main stream of algebra than any other study. Furthermore, in this way, the basic theorems of the ordinary theory of (Rabin-Scott) finite automata, become immediate results of the basic theorems of algebra.

(b) Given the vocabulary and the basic concepts of contemporary mathematics, the definition of R_W is still the most compact explicit definition of regular events (in the case where W is a free monoid). Furthermore, assuming that one of the current important trends in algebra is that of category theory, the natural way to tie the theory of regular events with it is by means of homomorphisms.

(c) Certain classes of event-automata, i.e. systems which define sets of tapes, like finite-state transductions and permutation-closures of regular events, fall directly under R_W -theory. Even context-free languages can be introduced in a fruitful manner in R_W -theory using additional combinations which are still of an algebraic nature (Chomsky and Miller 1963).

These arguments are still open to debate and in the course of the presentation several points of criticism will be discussed. Only a further and successful

development of the theory, along the lines suggested here, will be a possible conclusive justification of the theory in toto. However, in this stage of the research, only a few parts of the theory have been established, while most of it is still open to study and inquiry. Those sections which suggest directions of development, discuss open problems, or state conjectures, will be enclosed by asterisks.

1. FINITE AUTOMATA AS REGULAR SYSTEMS

In their well-known paper, Rabin and Scott already notice the relationship between finite-range homomorphisms of the free monoid and the regular events which are defined by finite automata.

Following this, we define (Giv' on 1963) a W-regular system, where now W is a free monoid generated by a finite alphabet say V , to be a system $\mathcal{A} = \langle \rho, F \rangle$ where:

(i) ρ is a homomorphism of W with a finite range, ρ is said to be the structure of \mathcal{A} ;

(ii) F is any subset of the range of ρ , the designated set of \mathcal{A} .

$T(\mathcal{A})$, the event defined by \mathcal{A} , is $\rho^{-1}(F)$.

The definition of direct-product of structures of W-regular systems follows naturally. Clearly the direct product of any finite number of structures is also a structure, i.e. it is also a homomorphism of W with a finite range. A theorem of Birkhoff (Birkhoff 1935) yields as an immediate result the well-known fact (cf. Rabin and Scott 1959) that R_W , the class of all events defined by W-regular systems, is a Boolean algebra of sets. This property of R_W is achieved by specifying an effective procedure of constructing a system which defines the given Boolean combination of the events generated by the given systems. Furthermore a good estimate for the necessary number of states of the combined system is derived. *— We wish to discuss this estimate together with the problem of minimal W-regular systems following J. Myhill and A. Nerode (as presented in Rabin and Scott 1959). — *

Following the study of the application of Boolean matrices to finite automata and of the properties of the algebra of lattice-matrices (Giv'e'on 1960, 1962), we find an isomorphism between the algebra of binary relations generated by the right translations of a finite monoid and the power-algebra (the algebra of subsets) of the monoid. Thus the power-algebra of the range of the structure of a given W-regular system serves effectively for the solution of the basic decision

problems concerning the event defined by the given system.

— There are still many directions of development which we wish to follow even in the ordinary case of the theory (i.e., for free monoids). Some of them are problems of translation from available studies of automata theory in other formulations. In particular, concerning the problem of compositions and decompositions of W -regular systems, we wish to follow Hartmanis's work on decomposition of machines (Hartmanis 1960), Krohn and Rhodes's application of Wreath-products (Krohn and Rhodes 1962) and to suggest a general framework based on the study of extensions in group theory. Holland's theory of locally effective embeddings of finite-state sequential machines in embedding spaces (Holland 1964) can serve as a machine-oriented conscience for the applicability of my algebraic approach.—

Additional properties of R_W , like the closure under the Kleenean operations, will be discussed in the general study of regular events in monoids.

2. ALGEBRAIC INTERMEZZO

The basic algebraic theory which underlies our approach is that of monoids (i.e. semigroups with identity element). Most of the algebraic material needed for our study has been developed in one way or another but not always within the context of monoid theory itself. Here I would like to present a special algebraic framework for a theory of automata which will fit my approach. This algebraic presentation will give rise to certain purely algebraic problems as well.

2.1 THE MORPHIC THEORY OF MONOIDS

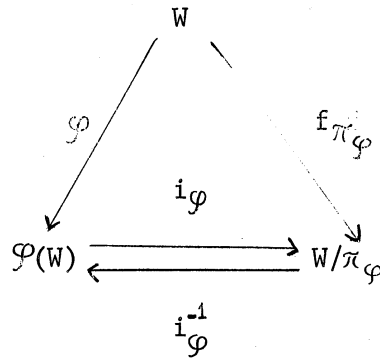
In monoid theory, due to the existence of the identity element, we can take advantage of the strong relationship which exists between congruence relations and homomorphisms.

Let φ be a homomorphism of a monoid W onto $\mathcal{P}(W)$. Denote by π_φ the equivalence relation determined by φ in W . Since π_φ is a congruence relation, an operation can be defined in W/π_φ in such a way that the natural mapping f_{π_φ} of W onto W/π_φ becomes a homomorphism. Furthermore, the equation

$$i_\varphi(\mathcal{P}(W)) = f_{\pi_\varphi}(W)$$

defines an isomorphism i_φ from $\mathcal{P}(W)$ onto W/π_φ and thus we get a commutative

diagram of homomorphisms:



Because of my specific interest in homomorphisms as mappings which define certain subsets of their domain, I can identify homomorphisms which induce the same congruence relation. Furthermore, in most cases I find that I do not have to keep the distinction between \mathcal{P} and π_φ . I shall use the term ambimorphism whenever this intentional ambiguity is used.

We can, of course, introduce this notion in a rigorous way. For example, if we denote by $[\varphi]$ the class of all homomorphisms of W which have the same induced congruence relation (as a subset of $W \times W$) like π_φ , then $[\varphi] \cup \{\pi_\varphi\}$ is said to be the ambimorphism \mathcal{P} of W .

Since \mathcal{P} and π_φ are identified, both mapping and relation notations can be used according to the specific context. The right interpretation of my notation will be the only one which will make sense at all.

Let us denote by $\mathcal{P}S$ the set of all subsets of S , for any set S . Then, by our convention, if \mathcal{P} is an ambimorphism of W then, in particular, it can be

represented by a mapping φ of W into $\mathcal{P}W$ with W/φ as its range. Thus we have

$W/\varphi \in \mathcal{P}W$ and we use the diagram

$$W \xrightarrow{\varphi} W/\varphi$$

Furthermore, we define the mapping

$$c_\varphi : \mathcal{P}W \rightarrow \mathcal{P}W$$

by

$$c_\varphi(E) =_{df} \bigcup \{ \varphi(w) : w \in E \}$$

(\bigcup is regarded as a mapping from $\mathcal{P}\mathcal{P}W$ onto $\mathcal{P}W$ which associates with any collection of subsets of W its union.)

For any mapping $f : S \rightarrow T$ we define $\mathcal{P}f : \mathcal{P}S \rightarrow \mathcal{P}T$ by $\mathcal{P}f(E) =_{df} \{ f(s) : s \in E \}$.

Thus, if φ is an ambimorphism of W we have

$$\varphi : W \rightarrow W/\varphi \in \mathcal{P}W, \quad ,$$

and

$$\mathcal{P}\varphi : \mathcal{P}W \rightarrow \mathcal{P}(W/\varphi) \in \mathcal{P}\mathcal{P}W$$

The following properties of c_φ and $\mathcal{P}\varphi$ are derived in a straight forward manner:

$$(1) \quad c_\varphi = \bigcup \mathcal{P}\varphi$$

$$(2) \quad c_\varphi \text{ is a closure operation in } \mathcal{P}W.$$

$$(3) \quad \mathcal{P}\varphi = \mathcal{P}\varphi c_\varphi$$

$$(4) \quad \mathcal{P}\varphi \text{ preserves the complex-product (which is in fact the " } \mathcal{P} \text{ of" the$$

multiplication of W); i.e. $\mathcal{P}\varphi(E_1 \cdot E_2) = \mathcal{P}\varphi(E_1) \cdot \mathcal{P}\varphi(E_2)$

Let $C_\varphi[W]$ denote the set of the fixed points of c_φ .

(5) $C_\varphi[W]$ is a Boolean algebra of sets which is generated by means of unrestricted set-unions from W/φ .

(6) $\mathcal{P}\varphi|_{C_\varphi[W]} : C_\varphi[W] \rightarrow \mathcal{P}(W/\varphi)$ is an isomorphism of Boolean algebras of sets. If we define an operation $\langle \varphi \rangle$ in $C_\varphi[W]$ by

$$E_1 \langle \varphi \rangle E_2 =_{\text{df}} c_\varphi(E_1 \cdot E_2),$$

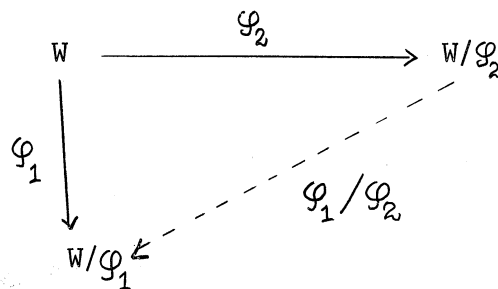
then we have also, for any $E_1, E_2 \in C_\varphi[W]$

$$\mathcal{P}\varphi(E_1 \langle \varphi \rangle E_2) = \mathcal{P}\varphi(E_1) \cdot \mathcal{P}\varphi(E_2)$$

Thus $\mathcal{P}\varphi$ restricted to $C_\varphi[W]$ is an isomorphism of the Boolean structure and $\langle \varphi \rangle$ of $C_\varphi[W]$ onto the Boolean structure and the complex product of $\mathcal{P}(W/\varphi)$.

Consider a (coinitial) pair of ambimorphisms φ_1 and φ_2 of W as presented

in the diagram:



φ_2 is said to be a morphism of φ_1 iff $\varphi_2 \subseteq \varphi_1$ (as subsets of $W \times W$).

Clearly φ_2 is a morphism of φ_1 iff the diagram can be completed into a commutative diagram. Since this completion is unique and it satisfies the equation $\xi \circ \varphi_2 = \varphi_1$ for ξ , it is natural to introduce the notation

$$\xi =_{\text{df}} \varphi_1 / \varphi_2 .$$

This notation is compatible with our ambimorphic convention which implies that

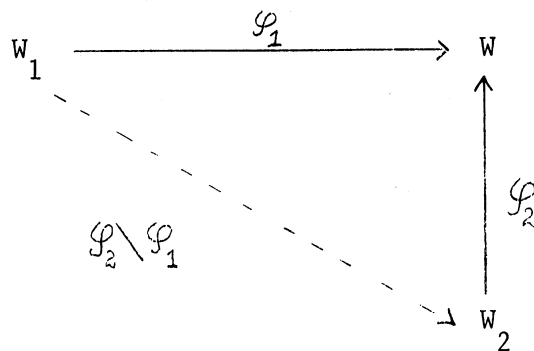
$$\varphi_1 / \varphi_2 : W / \varphi_2 \longrightarrow W / \varphi_1$$

means

$$W / \varphi_1 = (W / \varphi_2) / (\varphi_1 / \varphi_2) .$$

In addition to this we keep the form that any β -morphism of a structure α determines a quotient-structure α / β of the same type as α .

On the other hand, consider a (coterminal) pair of ambimorphisms φ_1 and φ_2 of W_1 and W_2 respectively, with $W = W_1 / \varphi_1 = W_2 / \varphi_2$ as presented in the diagram:



φ_2 is said to be a comorphism of φ_1 iff $\varphi_1^{-1} \subseteq \varphi_2^{-1}$ (as set theoretic binary relations).

Again we have that φ_2 is a comorphism of φ_1 iff the diagram can be completed

into a commutative diagram. In this case we denote the completing ambimorphism by $\varphi_2 \setminus \varphi_1$ and we have $\varphi_2 \cdot (\varphi_2 \setminus \varphi_1) = \varphi_1$. Furthermore, if we extend our morphic notation to binary relations we get the equation

$$\varphi_2 \setminus \varphi_1 = \varphi_1^{-1} / \varphi_2^{-1}$$

which makes sense. Our morphic convention is still satisfied. We have for any suitable ambimorphisms α, β and γ

$$(W / (\beta \setminus \alpha)) / \gamma = W / (\gamma \cdot (\beta \setminus \alpha)) \quad ,$$

and in our case we have

$$\varphi_2 \cdot (\varphi_2 \setminus \varphi_1) = \varphi_1 \quad ,$$

and

$$W_1 / \varphi_1 = W = W_2 / \varphi_2 \quad ,$$

$$W_2 = W_1 / (\varphi_2 \setminus \varphi_1) \quad ,$$

and thus

$$W_1 / \varphi_1 = (W_1 / (\varphi_2 \setminus \varphi_1)) / \varphi_2 \quad .$$

Note that the relations morphism-of and comorphism-of, being defined by set-inclusion, are transitive and we get, for suitable ambimorphisms α, β and γ , the following equations:

- (1) if α, β and γ are cointial and $\alpha \subseteq \beta \subseteq \gamma$ then $\alpha \subseteq \gamma$ and

$$(\alpha / \beta) (\beta / \gamma) = \alpha / \gamma \quad ,$$

- (2) if α, β and γ are coterminal and $\alpha^{-1} \subseteq \beta^{-1} \subseteq \gamma^{-1}$ then $\alpha^{-1} \subseteq \gamma^{-1}$ and

$$(\gamma\beta) \cdot (\beta\alpha) = \gamma\alpha .$$

Note that $\mathcal{P}_1/\mathcal{P}_2$ and $\mathcal{P}_2 \setminus \mathcal{P}_1$ can be both meaningful iff W/\mathcal{P}_1 and W/\mathcal{P}_2 are isomorphic to submonoids of W . That is, in the traditional terminology, iff we deal with endomorphisms.

For the introduction of products and sums of ambimorphisms, as well as for the discussion of general algebraic automata we introduce the concept of morphic diagram as follows:

A morphic diagram (of monoids) is a system

$$\mathcal{D} = \langle \mathcal{W}, \mu \rangle$$

where:

(i) \mathcal{W} is an ordered collection of monoids W_i with $i \in I$ and an index set I ;

(ii) μ is a mapping with domain $D(\mu) \subseteq I \times I$ which assigns to any $(i, j) \in D(\mu)$ an ambimorphism $\mu_{ij} : W_i \rightarrow W_i/\mu_{ij} \cong W_j$.

$g(\mathcal{D}) =_{\text{df}} \langle I, D(\mu) \rangle$ is said to be the graph of \mathcal{D} . Clearly, μ can be extended to be a mapping μ^* of the set of all directed paths $D^*(\mu)$ of $g(\mathcal{D})$ which assigns to any path $x \in D^*(\mu)$ which starts at i and ends at j , a unique ambimorphism $\mu_x^* : W_i \rightarrow W_i/\mu_x^* \cong W_j$. The definition of μ^* is given by the

following recursion:

$$\begin{aligned}\mu_{(i,j)}^* &= \mu_{ij} \quad , \\ \mu_{x(i,j)}^* &= \mu_{ij} \cdot \mu_x^* \quad .\end{aligned}$$

A morphic diagram $\mathcal{D} = \langle \mathcal{W}, \mu \rangle$ is said to be commutative iff for any

$x, y \in D^*(\mu)$ if x and y are cinitial and coterminal then $\mu_x^* = \mu_y^*$. \mathcal{D} is

said to be strongly commutative iff \mathcal{D} is commutative and for any cycle $z \in D^*(\mu)$,

μ_z^* is the identity of $W_{(z_1)_1}$.

Let \mathcal{D}_1 and \mathcal{D}_2 be two morphic diagrams, the relation $\mathcal{D}_1 \subseteq \mathcal{D}_2$ of

inclusion is naturally defined. \mathcal{D}_1 is said to be subcommutative iff there is

a commutative morphic diagram \mathcal{D}_2 which includes \mathcal{D}_1 .

Let $\mathcal{D} = \langle \mathcal{W}, \mu \rangle$ be a morphic diagram with index set I and let $W_i \in \mathcal{W}^0$.

A morphic diagram $\mathcal{D}' = \langle \mathcal{W}', \mu' \rangle$ is said to be a turn-extension of \mathcal{D} from

W_i iff:

(i) \mathcal{D}' has $J = I \cup \{j\}$, where $j \notin I$, as an index set, and $\mathcal{W}' = \mathcal{W} \cup \{W_j\}$

where W_j is any monoid.

(ii) $D(\mu') = D(\mu) \cup S_j^i D(\mu)$, where $S_j^i D(\mu)$ is derived from $D(\mu)$ by

substituting j for i ;

(iii) $\mu' \upharpoonright D(\mu) = \mu$.

In particular we say that \mathcal{D}' is a turn extension of \mathcal{D} from W_i to W_j .

Now a diagram $\mathcal{D} = \langle \mathcal{W}, \mu \rangle$ is said to be universal around W_i (for $W_i \in \mathcal{W}^0$)

iff for any monoid W_j and any turn-extension \mathcal{D}' of \mathcal{D} from W_i to W_j there is an ambimorphism

$$\varphi_{ij} : W_i \rightarrow W_i/\varphi_{ij} \subseteq W_j$$

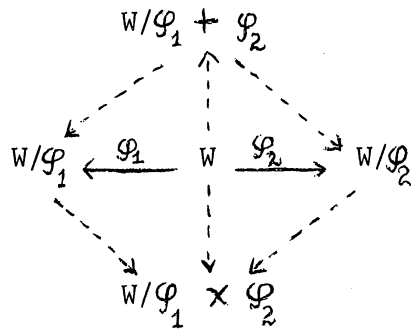
which completes \mathcal{D}' to be a commutative diagram.

\mathcal{D} is said to be couniversal around W_i iff for any monoid W_j and any turn-extension \mathcal{D}' of \mathcal{D} from W_i to W_j there is an ambimorphism

$$\varphi_{ji} : W_j \rightarrow W_j/\varphi_{ji} \subseteq W_i$$

which completes \mathcal{D}' to be a commutative diagram.

Now we define compositions of ambimorphisms. Consider a cointial pair of ambimorphisms φ_1 and φ_2 of W as presented in the diagram:



We define:

- (1) the direct product $\varphi_1 \times \varphi_2$ of φ_1 and φ_2 by

$$\varphi_1 \times \varphi_2 =_{\text{df}} \text{Max} \{ \pi : \exists \varphi_1/\pi \ \& \ \exists \varphi_2/\pi \} ;$$

- (2) the direct sum $\varphi_1 + \varphi_2$ of φ_1 and φ_2 by

$$\varphi_1 + \varphi_2 =_{\text{df}} \text{Min} \{ \sigma : \exists \sigma / \varphi_1 \ \& \ \exists \sigma / \varphi_2 \} \quad ;$$

where π and σ vary over the ambimorphisms of W and thus "Max" and "Min" regard them as set-theoretic binary relations; and " $\exists \sigma / \beta$ " means β is a morphism of α .

As an immediate result we get that the diagram above is universal around $W / \varphi_1 + \varphi_2$ and around $W / \varphi_1 \times \varphi_2$.

As for the effect of these relations and compositions of ambimorphisms we get the following results:

- (1) If φ_2 is a morphism of φ_1 then

$$c_{\varphi_1 / \varphi_2} [W / \varphi_2] = \mathcal{P} \varphi_2 c_{\varphi_1} [W]$$

- (2) (Birkhoff 1935) For any two ambimorphisms φ_1 and φ_2 of W :

(i) $\varphi_1 \times \varphi_2 = \varphi_1 \cap \varphi_2$, and therefore $W / (\varphi_1 \times \varphi_2) = \mathcal{P} \cup \{ W / \varphi_1, W / \varphi_2 \}$

- (ii) $\varphi_1 + \varphi_2$ is the transitive-closure of $\varphi_1 \cup \varphi_2$ and therefore

$$W / (\varphi_1 + \varphi_2) = (W / \varphi_1) \cap (W / \varphi_2)$$

- (iii) For any two ambimorphisms φ_1 and φ_2 and any subset of E of W

$$\text{we have } \varphi_2(E) \in c_{(\varphi_1 + \varphi_2) / \varphi_2} [W] \text{ iff } c_{\varphi_2}(E) \in c_{\varphi_1 + \varphi_2} [W]$$

* — The characterization of direct products and direct sums of cointial ambimorphisms by means of universal diagrams suggests a definition of codirect product and sums of coterminal ambimorphisms by means of couniversal diagrams. —*

* — I wish to study the formal properties of the algebra of ambimorphisms determined by the operations and relations which were suggested here (i.e., direct sums and products, morphism, codirect sums and products and comorphism). My attention is drawn to a possible relation between this algebra and other systems which were introduced in different contexts, for example the categorical grammars suggested by K. Ajdukiewicz (Ajdukiewicz 1935) and studies by Y. Bar-Hillel, Ch. Gaifman and E. Shamir (Bar-Hillel et al. 1960) have similar structure. The same holds for J. Lambek's work on syntactic types (Lambek 1958, 1961). — *

2.2 POWER ALGEBRAS

My study of regular systems as applied to finite automata (Giv'e'on 1963) led me to consider the power algebra determined by the subset-construction as applied to finite monoids. At the same time, the morphic theory of monoids, as presented in the previous section, led us to introduce operations and relations in order to study their effect on subsets of monoids.

Thus we consider the various power algebras (this term was suggested by J. W. Thatcher) which can be associated with a given monoid.

First of all we consider \mathcal{P}_W , the complex power algebra which is defined by

$$\mathcal{P}_W = \langle \mathcal{P}_W, \circ, *, \mathcal{B} \rangle$$

where:

- (i) \circ is the complex-product in $\mathcal{P}W$, i.e., \circ is the \mathcal{P} of the multiplication in W ;
- (ii) $*$ is the submonoid-generating operation in $\mathcal{P}W$, i.e., E^* is the sub-monoid of W generated by E ;
- (iii) \mathcal{B} is a class of Boolean operations in $\mathcal{P}W$.

Note that in particular $\langle \mathcal{P}W, \circ \rangle$ is a monoid with $\lambda = \{\lambda\}$ as an identity element. The distributivity of \circ over the set union makes $\mathcal{P}W$ an interesting structure which deserves special study.

Now with any ambimorphism φ of W we can associate the system

$$\mathcal{P}\langle W, \varphi \rangle =_{\text{df}} \langle \mathcal{P}W, c_{\varphi}, \mathcal{B}_{\varphi} \rangle .$$

As we shall see later, such systems will play an important role in the discussion of regular events in monoids.

— Since I am interested first in event-automata, a study of such power-systems is necessary for the algebraic framework of my theory. In particular, I should consider diagrams in which power systems appear as components and also study the effect of the various power-associations on diagrams. —

2.3 EXTENSION THEORY AND COMPOSITIONS

* — Extension theory is developed especially in group theory and recently it was translated to the general framework of category theory (cf. MacLane 1963). It is still an open problem, how much of it can be maintained in monoid theory. The relevance of extensions to automata theory is implied by the fact that they can be suitably interpreted as compositions. Thus the study of Krohn and Rhodes applies Wreath-products to derive compositions of sequential machines (Krohn and Rhodes 1962). This study suggests the hope that transformation-automata can also be studied under the same algebraic approach as that of event automata.

Appropriate notions of homomorphisms of morphic diagrams will lead to a theory of compositions of morphic diagrams as well. The general concept of composition of diagrams seems now to me to be defined as a quotient of a direct product of diagrams determined by an equivalence relation which is defined across the components of the given diagrams. However, in order to get a fruitful and interesting theory of compositions of morphic diagrams, certain restrictions should be imposed on the structure of the diagrams. Here, both category theory and the present studies of compositions of sequential machines seem to be insightful sources for such a development. — *

3. REGULAR SYSTEMS OF MONOIDS

Let W be any arbitrary monoid. A W-regular system is a system $\mathcal{A} = \langle \rho, F \rangle$

where:

- (i) ρ is an ambimorphism of W with a finite range (ρ is said to be the structure of \mathcal{A});
- (ii) F is a subset of the range of ρ (it is said to be the designated set of \mathcal{A}).

R_W , the class of regular events in W is defined by

$$R_W = \text{df } \bigcup \{ C_\rho[W] : W/\rho \text{ is finite} \} .$$

A subset E of W is said to be generated by the W -regular system $\langle \rho, F \rangle$ iff

$$E = \rho^{-1}(F), \text{ and in this case we write } E = T(\mathcal{A}).$$

The direct product construction for arbitrary monoids yields the expected result that R_W is a Boolean algebra of sets.

The problem of determining minimal W -regular systems (with respect to the size of the range of their structure) is relevant here. * — I plan to discuss it after the study of the relationships between the following systems:

- (i) W -regular systems and R_W ,
- (ii) right invariant relations in W ,
- (iii) finite-state automata with W as input,
- (iv) non-deterministic finite-state automata with W as input,
- (v) monadic algebras and relational systems with operators related to W . — *

The well-known methods employed in the theory of finite automata can be applied to these general structures with the expected results (Mezei 1962; Give'on 1964). Namely, these systems are weakly equivalent; some of them are equivalent even in certain strong senses which can be specified as certain isomorphisms of structures.

I feel that some words should be said concerning the notion of "structure" used in the literature of automata theory. Some formulations of automata yield a transparent distinction between "structure" and "behavior" which agrees with the "machine" oriented intuition. It is obvious that W -regular systems do not have behavior in any dynamic sense. On the other hand, though ρ is said to be the structure of $\langle \rho, F \rangle$ by definition, it is not a structure in the "machine" sense but in the mathematical sense. To some extent, the notion of structure and behavior are diffused in the W -regular terminology.

In spite of the equivalence between these generalizations of (free) finite automata, we find that certain properties of regular events get lost in the generalization. For example, by means of certain constructions of non-deterministic finite automata it is proven that the complex-product of regular events and the submonoid generated by a regular event are regular events. This does not hold in R_W in general, not even with the help of the equivalence of W -regular systems with non-deterministic W -automata.

Except for the free monoids only the finite monoids and the groups are known to be classes of monoids for which the classes of the regular events in them are closed under the operations mentioned above. Let us say that a monoid W is a Kleenean monoid iff R_W is closed under complex-products and the submonoid-generating operation. (In the terminology introduced in section 2.2 we get that W is Kleenean iff $\langle R_W, \circ, *, \mathcal{B} \rangle$ is a subalgebra of \mathcal{P}_W).

A close study of the proof that free monoids are Kleenean suggests the following approach:

(i) In order to define non-deterministic W -automata we have to modify Rabin-Scott's definition (of non-deterministic automata). If we adopt their definition as it stands we get:

Let B be a minimal set of generators for W . A system $\mathcal{A} = \langle S, \pi \rangle$ is a basic structure of non-deterministic W -automata iff

S is any set,

$\pi : S \times B \rightarrow \mathcal{P}S$ is any mapping.

We wish now to extend π to be a mapping

$$\pi^* : (\mathcal{P}S) \times W \rightarrow \mathcal{P}S$$

by the following recursion:

$$\pi^*(T, \lambda) = T \text{ for any } T \subseteq S$$

(*)

$$\pi^*(T, w_1 w_2) = \bigcup \{ \pi^*(\{s\}, w_2) : s \in \pi^*(T, w_1) \}$$

Contrary to the case of free monoids, the existence of π^* is not always guaranteed. * — Apparently, a certain property of the factorization of the elements of W is necessary for the existence of π^* . — *

If π^* exists then of course $\langle S, \pi^* \rangle$ is a structure of non-deterministic W -automata as defined and discussed in (Give' on 1964).

(ii) Following the construction used for defining a (non-deterministic) finite automaton which accepts the complex-product of events accepted by given automata we construct the following system:

Let ρ_1 and ρ_2 be two ambimorphisms of W and let F be a subset of W/ρ_1 .

We define the mapping

$$\langle \rho_1 F \rho_2 \rangle : W \rightarrow (W/\rho_1) \times \mathcal{P}(W/\rho_2)$$

by

$$\langle \rho_1 F \rho_2 \rangle(w) =_{\text{df}} \langle \rho_1(w), \rho_2((\rho_1^{-1}F) \setminus w) \rangle$$

where

$$S \setminus w =_{\text{df}} \{ w_2 : w_1 w_2 = w \text{ for some } w_1 \in S \} .$$

Clearly, if ρ_1 and ρ_2 are structures of W -regular systems then $\langle \rho_1 F \rho_2 \rangle$

has a finite range. Furthermore, if it is also a homomorphism of W then $\langle \rho_1 F \rho_2 \rangle$

is a structure of non-deterministic W -automata. In this case, for any $F_2 = W/\rho_2$

we get a W -automaton which defines

$$\{ w : (\langle \rho_1 F \rho_2 \rangle(w))_2 \cap F_2 \neq \emptyset \} = (\rho_1^{-1}F) \circ (\rho_2^{-1}F) .$$

Only in special cases is $\langle \rho_1 F \rho_2 \rangle$ a homomorphism of W . * — Again a

certain structure, similar to the one suggested by (i) with regard to the

factorization of elements in W , seems to be sufficient for $\langle \rho_1 F \rho_2 \rangle$ to be a

homomorphism of W . — *

(iii) Following these discussions, we are led to introduce the following property of monoids.

Let W be a monoid and $w \in W$, then we define:

$$\delta_W(w) =_{\text{df}} \{ (w_1, w_2) : w_1, w_2 \in W \text{ \& } w_1 w_2 = w \} ,$$

that is, $\delta_W(w)$ is the set of all pair-factorizations of w in W .

If we regard $\delta_W(w)$ as a subset of $W \times W$, then for any $u, v \in W$ the

expressions $(u,v) \times \delta_W(w)$ and $\delta_W(w) \times (u,v)$ are meaningful.

A monoid W is said to be sequential iff for any $w_1, w_2 \in W$ we have

$$\delta_W(w_1 w_2) = \delta_W(w_1) \times (\lambda, w_2) \cup (w_1, \lambda) \times \delta_W(w_2).$$

Thus, sequential monoids are monoids in which the pair-factorization of any element is retrievable from any of its factorizations. Clearly the free monoids are sequential and as one can prove, if W is a group then for any $w_1, w_2 \in W$:

$$\delta_W(w_1 w_2) = (w_1, \lambda) \times \delta_W(w_2) = \delta_W(w_1) \times (\lambda, w_2)$$

which shows that groups are sequential monoids.

One cannot expect that only sequential monoids will be Kleenean. The finite monoids are Kleenean and yet some of them are not sequential. * — As a conjecture, I would like to suggest that sequential monoids are Kleenean and for sequential monoids λ^* and $\langle \rho_1 F \rho_2 \rangle$, as defined in (i) and (ii), yield homomorphisms of W . — *

The definition of $\langle \rho_1 F \rho_2 \rangle$ deserves special attention since it determines a certain product of W/ρ_1 and W/ρ_2 . * — Our study of the problem of Kleenean monoids is connected therefore with the study of cross-products of monoids. — *

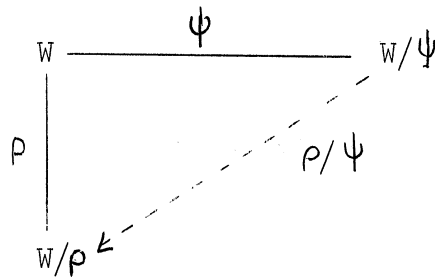
4. HOMOMORPHISMS OF REGULAR SYSTEMS

The morphism relation between ambimorphisms was introduced because of the following observations.

Let $\mathcal{A} = \langle \rho, F \rangle$ be a W -regular system and let ψ be any ambimorphism of W .

We want an appropriate definition of \mathcal{A}/ψ as an image of \mathcal{A} under ψ .

It is clear that if the diagram of ρ and ψ can be completed into a commutative diagram, then the structure of \mathcal{A}/ψ can be defined naturally. Thus



if ψ is a morphism of ρ we define

$$\mathcal{A}/\psi =_{\text{df}} \langle \rho/\psi, F \rangle .$$

As we noticed already in section 2, we get the following equality of relations

$$\psi \rho^{-1} = (\rho/\psi)^{-1}$$

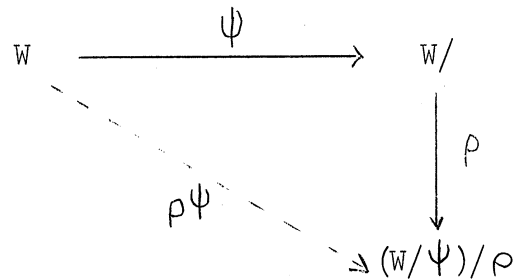
and so, for any $F \in W/\rho$ we have

$$T(\mathcal{A}/\psi) = \mathcal{P}\psi(T(\mathcal{A}))$$

where $T(\mathcal{A})$ denotes the event generated by \mathcal{A} .

If, however, ψ is not a morphism of ρ then $\mathcal{P}\psi(T(\mathcal{A}))$ is not necessarily a regular event in W/ψ . Obviously, if ψ is any ambimorphism of W and E is any

regular event in W/ψ then $\psi^{-1}(E)$ is regular in W . In particular, as the following diagram shows, if E is generated by $\mathcal{A} = \langle \rho, F \rangle$



then $\psi^{-1}(E)$ is generated by $\psi^{-1}(\mathcal{A}) =_{\text{df}} \langle \rho\psi, F \rangle$.

On the other hand, let $\mathcal{A} = \langle \rho, F \rangle$ be a W -regular system and let ψ be any ambimorphism of W then for any $E \subseteq W$ such that $c_{\psi}(E)$ is regular in W we get that $\rho\psi(E)$ is regular in W/ψ .

In order to show this, we notice that $c_{\psi}(E)$ is regular in W iff

$c_{\psi}(E) \in C_{\rho}[W]$ for some ambimorphism ρ of W with a finite range. Clearly

$c_{\psi}(E) \in C_{\psi}[W]$ and therefore

$$c_{\psi}(E) \in C_{\rho}[W] \cap C_{\psi}[W] = C_{\rho+\psi}[W]$$

which implies (by our results of section 2) that

$$\rho\psi(E) \in C_{(\rho+\psi)/\psi}[W/\psi]$$

Since $(\rho + \psi)/\psi$ has a finite range, namely $W/(\rho + \psi)$, $\rho\psi(E)$ is regular in

W/ψ . In particular if $c_{\psi}(E)$ is generated by $\mathcal{A} = \langle \rho, F \rangle$ then $\rho\psi(E)$ is

generated by

$$\psi(\mathcal{A}) =_{\text{df}} \langle (\rho + \psi)/\psi, \rho((\rho + \psi)/\rho)(F) \rangle$$

Combining these results we get the following theorem:

Let Ψ be an ambimorphism of W :

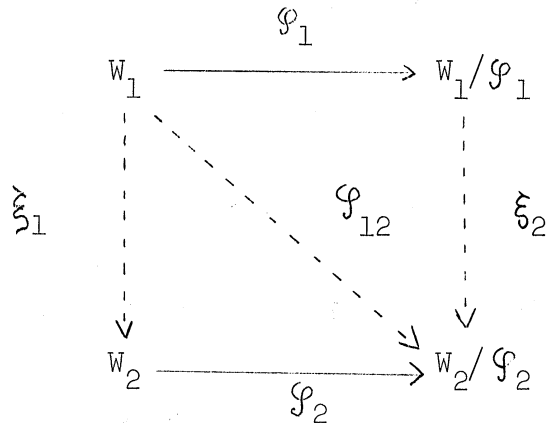
- (i) $R_{W/\Psi} \in \mathcal{P}\mathcal{P}\Psi R_W$;
- (ii) for any $E \subseteq W$: $\mathcal{P}\Psi(E)$ is regular in W/Ψ iff $c_\Psi(E)$ is regular in W ;
- (iii) $R_{W/\Psi} = \mathcal{P}\mathcal{P}\Psi R_W$ iff $\mathcal{P}c_\Psi(R_W) \subseteq R_W$,

i.e., Ψ preserves regularity (from W to W/Ψ) iff c_Ψ preserves regularity in W .

* — Analogous results should be pursued in the context of comorphisms. — *

Our method can be generalized by defining the following relation between ambimorphisms.

Let \mathcal{P}_1 and \mathcal{P}_2 be ambimorphisms of W_1 and W_2 (resp.). We say that \mathcal{P}_2 reflects \mathcal{P}_1 iff there exist ambimorphisms ξ_1 and ξ_2 which complete the following diagram into a commutative



diagram; that is, such that $\mathcal{P}_2 \xi_1 = \xi_2 \mathcal{P}_1$. Symbolically we shall denote this by

$$\langle \xi_1, \xi_2 \rangle : \mathcal{P}_1 \rightarrow \mathcal{P}_2 \quad ,$$

and the expression $\mathcal{P}_1 \rightarrow \mathcal{P}_2$ will denote that \mathcal{P}_2 reflects \mathcal{P}_1 .

Note that $\langle \xi_1, \xi_2 \rangle : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ holds iff the ambimorphism

$\mathcal{P}_{12} =_{\text{df}} \xi_2 \mathcal{P}_1 = \mathcal{P}_2 \xi_1$ has the following properties:

- (i) \mathcal{P}_{12} is a morphism of \mathcal{P}_1 ,
- (ii) \mathcal{P}_2 is a comorphism of \mathcal{P}_{12} .

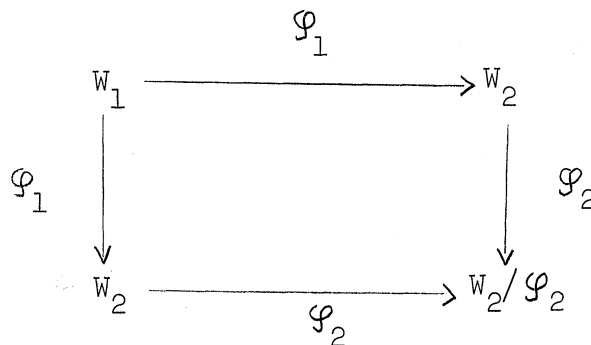
Thus $\mathcal{P}_1 \rightarrow \mathcal{P}_2$ iff there exists an ambimorphism \mathcal{P}_{12} of W_1 such that

\mathcal{P}_{12} is a morphism of \mathcal{P}_1 and \mathcal{P}_2 is a comorphism of \mathcal{P}_{12} (hence $W_1/\mathcal{P}_{12} = W_2/\mathcal{P}_2$).

Moreover, note that morphisms and comorphisms are special instances of reflection; namely, for which either ξ_1 or ξ_2 is the identity mapping. Another special instance is the case where \mathcal{P}_1 and \mathcal{P}_2 are successive ambimorphisms (i.e., such that the range of \mathcal{P}_1 is the domain of \mathcal{P}_2 : $W_2 = W_1/\mathcal{P}_1$). In this case we have

$$\langle \mathcal{P}_1, \mathcal{P}_2 \rangle : \mathcal{P}_1 \rightarrow \mathcal{P}_2 \quad ;$$

i.e., the following diagram

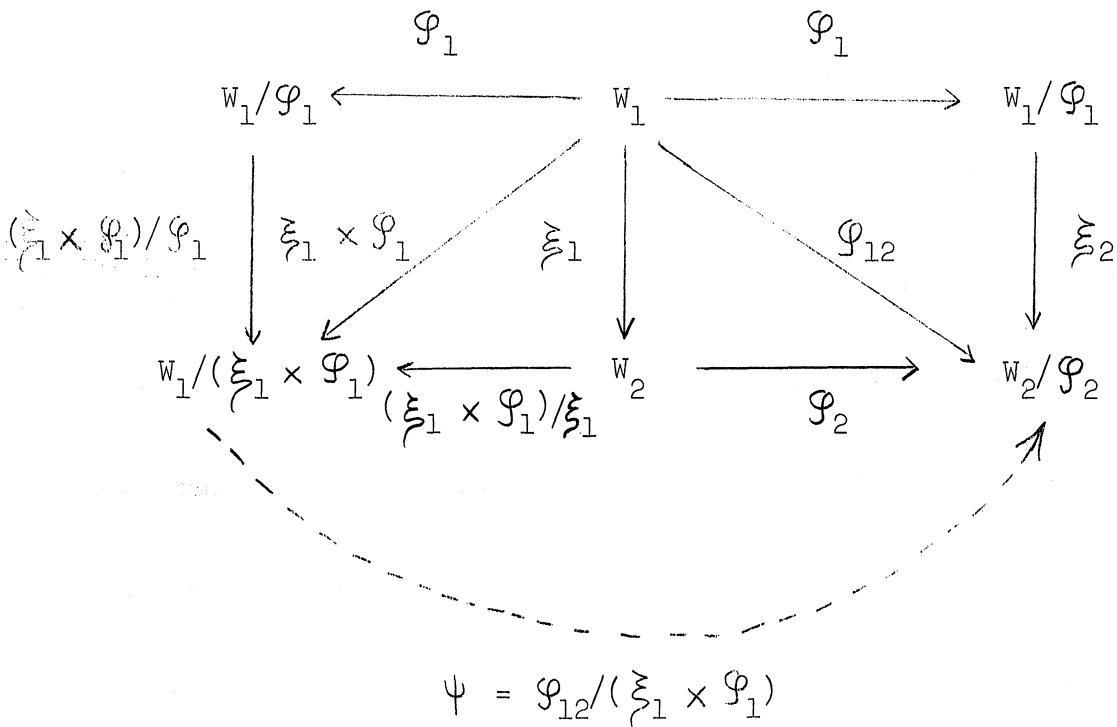


is of course commutative.

Comparing the diagram $\langle \xi_1, \xi_2 \rangle : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ with the diagram of the definition of $\xi_1 \times \mathcal{P}_1$ we find that there exists an ambimorphism

$$\psi : W_1 / (\xi_1 \times \mathcal{P}_1) \rightarrow W_2 / \mathcal{P}_2$$

which completes the following diagram into a



commutative diagram.

In particular, we always have:

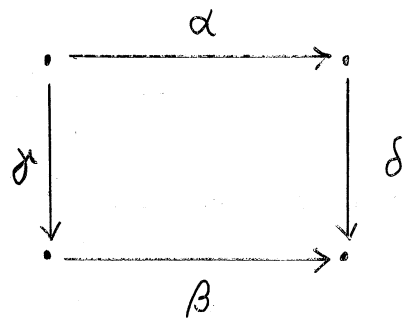
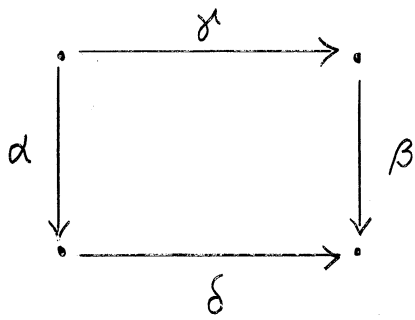
$$\langle \xi, (\xi \times \mathcal{P}) / \mathcal{P} \rangle : \mathcal{P} \rightarrow (\xi \times \mathcal{P}) / \xi ,$$

for any cointial ambimorphisms ξ and \mathcal{P} .

Note that for any appropriate ambimorphisms α, β, γ and δ :

$$\langle \alpha, \beta \rangle : \gamma \rightarrow \delta \text{ iff } \langle \gamma, \delta \rangle : \alpha \rightarrow \beta ,$$

since the following two schematic diagrams are in fact identical.



* — The implications of these discussions which are relevant to the various power algebras induced by the ambimorphisms under consideration, are quite straightforward. In particular, if ρ_1 and ρ_2 are structures of W_1 - and W_2 -regular systems (resp.) then $\langle \xi_1, \xi_2 \rangle : \rho_1 \rightarrow \rho_2$ may be considered as a general form of homomorphisms of structures of regular systems. — *

5. REPRESENTATIONS OF EVENTS BY MEANS OF REGULAR EXPRESSIONS

Our results of the previous sections led us to consider various classes of subsets of monoids. Some of these classes have been defined before and the rest are defined in the following complete list:

$$(i) \quad C_{\varphi}[W] = \{c_{\varphi}(E) : E \subseteq W\} = \mathcal{P}c_{\varphi}(\mathcal{P}W) \quad .$$

$$(ii) \quad R_W \text{ is the class of regular events in } W \quad .$$

(iii) $\mathcal{P}^2_{\varphi}(R_W) = \{\mathcal{P}\varphi(E) : E \in R_W\}$, which is usually denoted by $\varphi(R_W)$. Since $\mathcal{P}\varphi$ (restricted to R_W) is a homomorphism of the multiplicative (monoid) structure of R_W , we can use the notation of ambimorphisms, that is

$$R_W/\mathcal{P}\varphi = \mathcal{P}^2_{\varphi}(R_W) \quad .$$

(iv) $\mathcal{P}c_{\varphi}(R_W) = \{c_{\varphi}(E) : E \in R_W\}$, this is the class of the subsets in $C_{\varphi}[W]$ which are derived from the regular events in W by means of the closure operation c_{φ} .

(v) $R_W^{\varphi} =_{df} \{E \in R_W : \mathcal{P}\varphi(E) \in R_W/\varphi\}$, this is the class of those regular events in W which remain regular under $\mathcal{P}\varphi$.

Our results can be represented in the following diagram. Note that we labeled certain arrows by $\mathcal{P}\varphi$ and c_{φ} without designating that we refer to the appropriate restrictions of $\mathcal{P}\varphi$ and c_{φ} . The dotted lines appear in those cases

where the lower class of events is included in the upper class. The double arrows show our main results, i.e., they represent the cases in which the appropriate restrictions of $\mathcal{P}\varphi$ yield one-to-one correspondences.

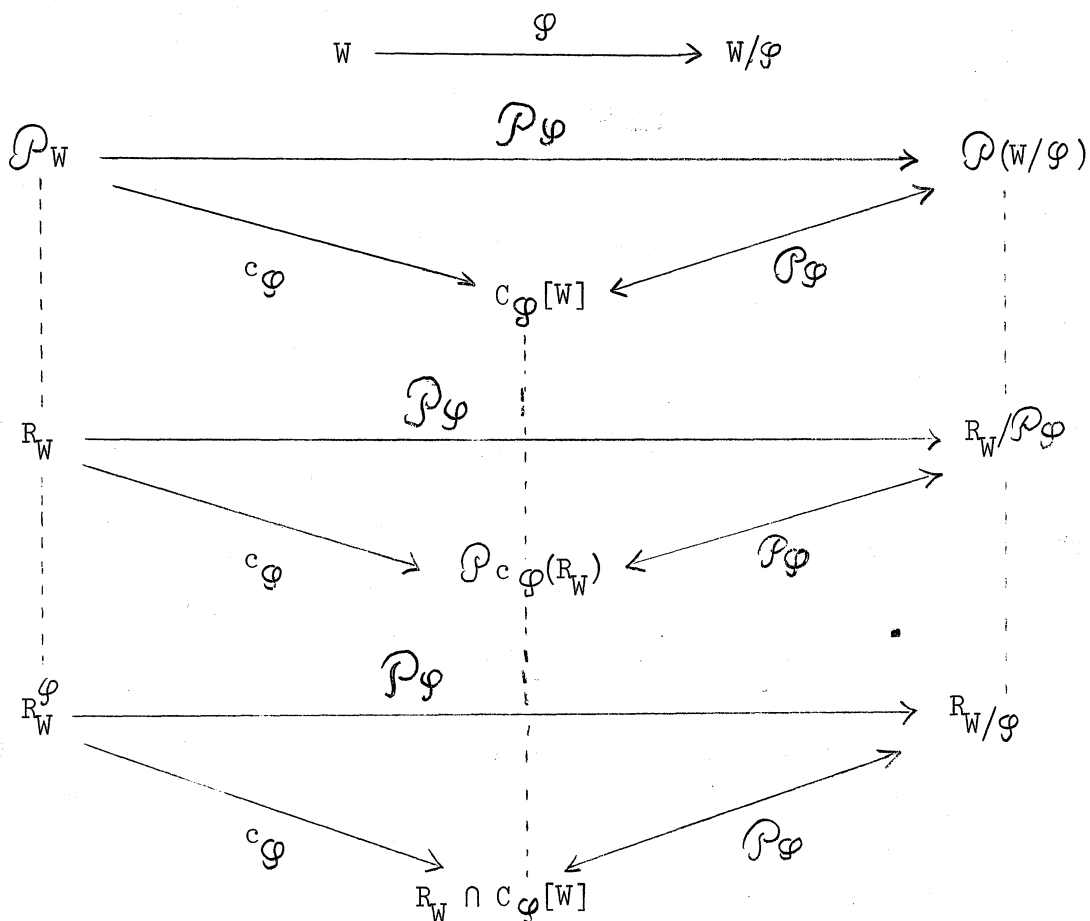
Note also that our algebraic results imply that

$$\mathcal{P}\varphi : C_{\varphi}[W] \longrightarrow \mathcal{P}(W/\varphi)$$

and

$$\mathcal{P}\varphi : R_W \cap C_{\varphi}[W] \longrightarrow R_{W/\varphi}$$

yield isomorphisms of the Boolean structures of $C_{\varphi}[W]$ and of $R_W \cap C_{\varphi}[W]$.



We proceed now to discuss some interpretations of the language of regular expressions (Kleene 1956) by which certain classes of events can be described.

Let V be a finite alphabet and let \mathcal{O} be a set of any four symbols which do not belong to V , say $0, 1, U,$ and $*$. We denote by \mathcal{E}_V the class of finite expressions over $V \cup \mathcal{O} \cup \{ (,) \}$. defined inductively by:

$$0, 1 \in \mathcal{E}_V ,$$

$$v \in \mathcal{E}_V ,$$

$$\text{if } \alpha, \beta \in \mathcal{E}_V \text{ then } (\alpha \cup \beta), \alpha\beta, (\alpha)^* \in \mathcal{E}_V .$$

Any $\alpha \in \mathcal{E}_V$ is said to be a regular expression over the alphabet V .

Regular expressions are employed in automata theory to denote certain subsets of V^* (the free monoid generated by V). This is defined as an interpretation function $e : \mathcal{E}_V \rightarrow \mathcal{P} V^*$, which will be referred to as the free interpretation and is defined inductively by the following list of equations:

(i) $e(0) = \emptyset$ (the empty event) ;

(ii) $e(1) = I = \{ \lambda \}$ (the silent event) where λ is the identity element

of V^* ;

(iii) for any $\sigma \in V$: $e(\sigma) = \{ \sigma \}$;

for any $\alpha, \beta \in \mathcal{E}_V$:

(iv) $e((\alpha \cup \beta)) = e(\alpha) \cup e(\beta)$

$$(v) \quad e(\alpha\beta) = e(\alpha) \bullet e(\beta)$$

$$(vi) \quad e((\alpha)^*) = (e(\alpha))^* \quad (\text{the submonoid of } V^* \text{ generated by } e(\alpha)).$$

Kleene's theorem of finite automata states that e is a homomorphism of the syntactical structure of \mathcal{E}_V onto the (power) algebraic structure of R_{V^*} .

In general we can define general free interpretations of \mathcal{E}_V as homomorphisms of the syntactical structure of \mathcal{E}_V onto an algebra of subsets of V^* .

An important class of general free interpretations of \mathcal{E}_V can be associated with the class of all ambimorphisms of V^* in the following manner.

Let \mathcal{P} be any ambimorphism of V^* , we define

$$e_{\mathcal{P}} : \mathcal{E}_V \longrightarrow \mathcal{P}_{c_{\mathcal{P}}}(R_{V^*})$$

by

$$e_{\mathcal{P}}(\alpha) =_{df} c_{\mathcal{P}}(e(\alpha)) \quad .$$

In words we shall say that $e_{\mathcal{P}}(\alpha)$ is the \mathcal{P} -closed event derived from α .

Kleene's theorem implies that $e_{\mathcal{P}}$ maps \mathcal{E}_V onto $\mathcal{P}_{c_{\mathcal{P}}}(R_{V^*})$ while our discussions in section 2 show that $e_{\mathcal{P}}$ maps the syntactical structure of \mathcal{E}_V onto an algebraic structure of $\mathcal{P}_{c_{\mathcal{P}}}(R_{V^*})$. Thus $e_{\mathcal{P}}$ is a general free interpretation of \mathcal{E}_V . In the next section we shall discuss the permutative events which are described by means of such an interpretation of regular expressions (Laing, Wright 1962).

So far we have been interested in interpretations of \mathcal{E}_V into $\mathcal{P}V^*$.

Given any ambimorphism \mathcal{P} of V^* with $W = V^*/\mathcal{P}$, we define

$$e^{\mathcal{P}} : \mathcal{E}_V \longrightarrow \mathcal{P}W$$

inductively by:

- (i) $e^{\mathcal{P}}(0) = \emptyset$;
 - (ii) $e^{\mathcal{P}}(1) = I$ (the identity element of $\mathcal{P}W$) ;
 - (iii) for any $\sigma \in V$: $e^{\mathcal{P}}(\sigma) = \{\mathcal{P}(\sigma)\}$;
- for any $\alpha, \beta \in \mathcal{E}_V$:
- (iv) $e^{\mathcal{P}}((\alpha \cup \beta)) = e^{\mathcal{P}}(\alpha) \cup e^{\mathcal{P}}(\beta)$
 - (v) $e^{\mathcal{P}}(\alpha\beta) = e^{\mathcal{P}}(\alpha) \circ e^{\mathcal{P}}(\beta)$
 - (vi) $e^{\mathcal{P}}((\alpha)^*) = (e^{\mathcal{P}}(\alpha))^*$ (the submonoid of W generated by $e^{\mathcal{P}}(\alpha)$).

This definition and Kleene's theorem imply that

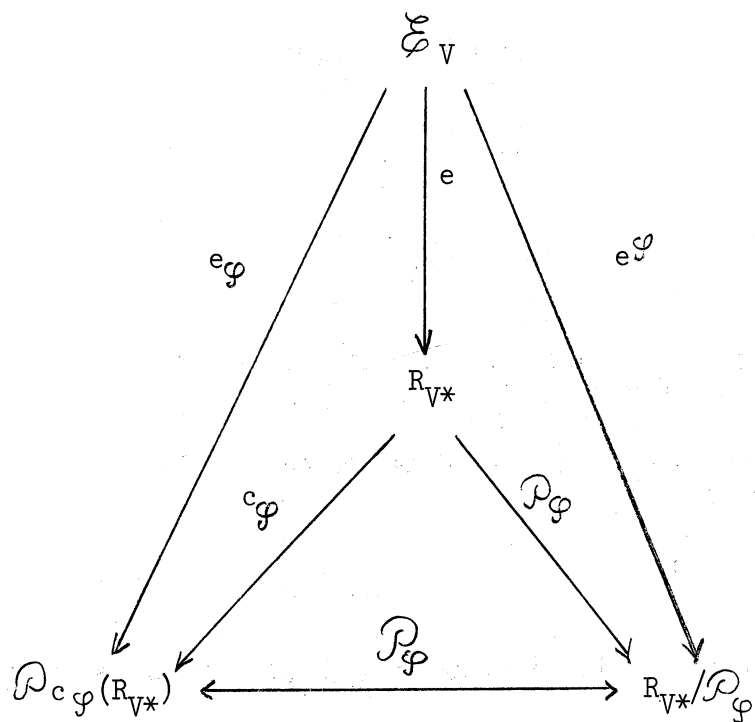
$$\mathcal{P}e^{\mathcal{P}}(\mathcal{E}_V) = R_{V^*}/\mathcal{P}\mathcal{P}$$

and since $(\mathcal{P}\mathcal{P})e = e^{\mathcal{P}}$, $e^{\mathcal{P}}$ is a homomorphism of the syntactical structure of \mathcal{E}_V onto the algebraic structure of $R_{V^*}/\mathcal{P}\mathcal{P}$.

Hence we can use the language of regular expressions to denote events in homomorphic images of V^* ; that is, in finitely generated monoids. In the next sections we shall also discuss examples of these types of interpretations.

Note that we can associate with any ambimorphism \mathcal{P} of V^* the following

commutative diagram.



6. EXAMPLES AND APPLICATIONS

6.1 COMMUTATIVE MACHINES (Laing & Wright 1962)

Let V be a finite alphabet, V^* the free monoid, and A_V the abelian free monoid generated by V . Denote by

$$\pi : V^* \longrightarrow A_V$$

the natural homomorphism of V^* onto A_V . Clearly c_π is the closure operation which extends any subset E of V^* by including for any $x \in E^*$ all the permutations of letter-tokens in x .

Our main interest here is to study $\mathcal{P}_{c_\pi}(R_{V^*})$, the class of permutative events which are derived from regular events.

First we note that π and c_π do not preserve regularity (provided that the alphabet V contains at least two letters). For, let $a, b \in V$, then the c_π -closure of $(ab)^*$ intersecting with a^*b^* yields the event $\{a^n b^n : n \geq 0\}$ which is not regular (Rabin & Scott 1959).

Applying our results of section 5, we can use the interpretation functions e_π and e^π to denote the events of $\mathcal{P}_{c_\pi}(R_{V^*})$ and of R_{V^*}/\mathcal{P}_π (respectively). Thus, while regular expressions usually refer also to their free interpretations (e.g., $(ab)^*$ and a^*b^* in the previous example) $e_\pi(\alpha)$ will refer to the c_π -closure

of $e(\alpha)$ (e.g., $e_{\pi}((ab)^*)$ is the set of all words over $\{a, b\}$ with an equal number of occurrences of a and b) and $e^{\pi}(\alpha)$ will denote the image of $e(\alpha)$ under π (e.g., $e^{\pi}((ab)^*)$ is $\{a^n b^n : n \geq 0\}$ as a subset of A_V).

Our proof that a particular event in $\mathcal{P}_{C_{\pi}}(R_{V^*})$ is not regular can be generalized as follows:

Let $V = \{a_1, \dots, a_n\}$; then for any $E \in C_{\pi}[V^*]$: E is regular in V^* iff $E \cap (a_1^* \cdot \dots \cdot a_n^*)$ is regular in V^* (Give'on 1963).

Thus we have a canonical form for the permutative events. Namely, if we choose for any $E \in C_{\pi}[V^*]$ the set $E_{\pi} =_{df} E \cap (a_1^* \cdot \dots \cdot a_n^*)$, we have the following:

- (i) $E'_{\pi} = E''_{\pi}$ iff $E' = E''$;
- (ii) E is regular iff E_{π} is regular ;
- (iii) E_{π} , viewed as a subset of A_V , is in fact $\mathcal{P}_{\pi}(E)$.

Furthermore, as presented in the work of Laing and Wright (Laing & Wright 1962), a partial characterization of $\{E_{\pi} : E \in \mathcal{P}_{\pi}(R_{V^*})\}$ is achieved by associating sentences in Presburger's language (Presburger 1930) with a class of subsets of $a_1^* \cdot \dots \cdot a_n^*$.

In particular, for any regular expression α there is a Presburger formula $Q(x_1, \dots, x_n)$ such that

$$a_1^{m_1} \cdot \dots \cdot a_n^{m_n} \in (e_{\pi}(\alpha))_{\pi} \text{ iff } Q(m_1, \dots, m_n)$$

Laing & Wright 1962).

* — It is still unknown whether this characterization is complete. In particular, the problem whether for any $\alpha_1, \alpha_2 \in \mathcal{F}_V$ there exists a $\beta \in \mathcal{F}_V$ such that $e_{\pi}(\alpha_1) \cap e_{\pi}(\alpha_2) = e_{\pi}(\beta)$, is still open. — *

Another problem which is associated with this context is that of the word problem for abelian monoids. M.O. Rabin, by means of combining the methods used to prove the solvability of the word problem for abelian groups with methods taken from automata theory, proved the solvability of the word problem for abelian monoids (Rabin 1964). * — Since the methods employed in the solution of the general word problem for abelian groups, namely that of the linear algebra over the ring of integers, were found to be useful for the study of A_V and of permutative events derived from regular expressions, they open new directions for the study of permutative events. — *

6.2 DIRECT POWERS OF V^*

The direct powers of V^* (and their submonoids) are in frequent use in automata theory. This is because direct powers algebraically represent the notion of independence of channels.

Note that we have implicitly discussed a certain submonoid of the n-th power of V^* , though not as a representation of a channel. Namely, A_V , the abelian free monoid generated by V , is obviously isomorphic to $\prod_{i=1}^n a_i^*$ and, of course, to the n-th direct power of a^* .

We denote by $D^k(V^*)$ the k-th direct power of V^* . I.e., $D^k(V^*)$ is the monoid whose carrier is the set of all k-tuples over V^* and whose operation is the component-wise concatenation.

Clearly $D^k(V^*)$ is finitely generated and has a basis consisting of the "unit-vectors" (i.e., those vectors over V^* whose entries are all λ except for one which contains a letter). Let us denote by $V_{(k)}$ the basis of $D^k(V^*)$ and so we have a natural homomorphism

$$d : V_{(k)} \longrightarrow D^k(V^*)$$

of the free monoid $V_{(k)}^*$ generated by $V_{(k)}$ onto $D^k(V^*)$.

A submonoid of $D^k(V^*)$ is said to be rectangular iff all its elements have components with an equal length. A special case of free rectangular submonoids of $D^k(V^*)$ is that of the submonoids generated by subsets of $V^{(k)}$, the k -th cartesian power of V . * — The complete characterization of the family of submonoids of $D^k(V^*)$ is still open and it deserves a special study. — *

As we shall see in the following examples the study of regular systems and their images under homomorphisms from and to submonoids of $D^k(V^*)$, seems to be a fruitful unifying framework for the study of automata which are closely related to finite automata.

6.3 REMARK ON EVENT AND TRANSMISSION AUTOMATA

Finite automata, and automata in general, are viewed in the theoretical framework in a twofold manner. As event automata they are regarded as systems which define certain subsets of the input domain (e.g., finite automata as defining regular events). In a different way, automata sometimes are regarded as transmission automata, that is, as systems which define certain functions from an input domain to an output domain (e.g., Turing machines as systems which define computable functions).

Finite automata enjoy a particular property, that in a special case these

two points of view are equivalent, and the general situation does not seem to be essentially different from this special case which will be discussed presently.

With any finite automaton \mathcal{A} over the alphabet V , we can associate the following three functions:

(i) Event function

$\chi_{\mathcal{A}} : V^* \rightarrow \{0, 1\}$, which is the characteristic function of $T(\mathcal{A})$.

(ii) Finitary-transmission function

$\chi_{\mathcal{A}}^* : V^* \rightarrow \{0, 1\}^*$ which is the "right extension" of $\chi_{\mathcal{A}}$,

determined inductively by:

$$\chi_{\mathcal{A}}^*(\lambda) = \lambda,$$

$$\chi_{\mathcal{A}}^*(x\sigma) = \chi_{\mathcal{A}}^*(x) \cdot \chi_{\mathcal{A}}(\sigma) \text{ for all } \sigma \in V \text{ and } x \in V^*.$$

(iii) ω -transmission function

$\chi_{\mathcal{A}}^{\omega} : V^{\omega} \rightarrow \{0, 1\}^{\omega}$, (where S^{ω} denotes the set of all infinite sequences over S) which is defined by:

$$(\chi_{\mathcal{A}}^{\omega}(\alpha))(i) = \chi_{\mathcal{A}}(\alpha(i))$$

(where $\alpha(i)$ is the i -th element in α).

The relationship between these three functions associated with \mathcal{A} is

obvious. Any one of them determines the other two effectively. Such a neat situation does not prevail throughout other domains of automata theory. In fact some students of finite automata theory maintain that the study of event finite automata is not sufficient for the study of transmission finite automata if their output alphabet is larger than a binary alphabet.

Since we are interested in systems operating on $D^k(V^*)$ we note that such systems can be viewed in the twofold manner (of event-and-transmission automata) in a more natural way. Trivially, any binary relation from $D^{k_1}(V^*)$ to $D^{k_2}(V^*)$ can be defined as a subset of

$$D^{k_1}(V^*) \times D^{k_2}(V^*) = D^{k_1+k_2}(V^*)$$

On the other hand, any subset $E \subseteq D^k(V^*)$ can be interpreted as a binary relation from $D^{k_1}(V^*)$ to $D^{k_2}(V^*)$ for any k_1 and k_2 such that $k_1 + k_2 = k$. This double interpretation will be employed in our study of regular systems related to $D^k(V^*)$.

6.4 SEQUENCE GENERATORS (Burks & Wright 1962)

The basic concept of sequence generator is introduced as a suggested generalization of the concepts of digital computers, finite automata and other information-processing systems (Burks & Wright 1962). Here we shall show how certain features of sequence generators, though being transmission automata, can

be studied by means of the theory of regular systems.

An algebraic sequence generator is defined to be any regular system in a rectangular submonoid of $D^k(V^*)$ generated by a subset of $\{(v_1, \dots, v_k) : v_i \in V\}$.

Obviously the theory of regular systems in free monoids is applicable here since such submonoids of $D^k(V^*)$ are finitely generated free monoids.

Several constructs, as various types of "behavior," are associated with sequence generators. However, since they are all determined by the "finitary behavior," we shall discuss mainly this one.

The finitary behavior, $\mathcal{B}^f(\Gamma)$, of an algebraic sequence generator Γ is a subset of $T(\Gamma)$ derived by means of an operation introduced and studied by I. T. Medvedev (the so called "fundamental operation $(t)_{t \in P}$ " in Medvedev, 1958):

Let W be any monoid and let $E \subseteq W$. We define:

$$E^P =_{df} \{w \in E : w = w_1 w_2 \implies w_1 \in E\} .$$

That is E^P is the set of all elements of E for which E contains also all their prefixes.

We define now:

$$\mathcal{B}^f(\Gamma) =_{df} (T(\Gamma))^P .$$

Since P preserves regularity in free monoids (Medvedev 1958), the finitary behavior of any sequence generator Γ is a regular event in the rectangular monoid

of Π . The converse does not hold, since not every regular event is in the range of ρ . There is, however, an effective procedure by which it can be determined whether or not any given regular system generates an event which is the finitary behavior of an algebraic sequence generator.

Most of the properties of sequence generators now follow from this reduction to regular systems in free monoids. Still, the particular structure of the generators of the monoids, in which sequence generators are defined as regular systems, suggests several problems which are related to other classes of events in $D^k(V^*)$ to be introduced in the next subsection.

6.5 FINITE-STATE TRANSDUCTIONS (Elgot and Mezei 1963)

A k-transduction system is a regular system in $V^*_{(k)}$ and the k-transduction defined by the k-transduction system \mathcal{A} is

$$\tau(\mathcal{A}) =_{df} (\rho_d)(T(\mathcal{A})) .$$

Any $E \in D^k(V^*)$ is a (k-)transduction iff $E = \tau(\mathcal{A})$ for some (k-)transduction system.

Our results of section 5 imply directly the following statements:

(i) The class of k-transductions is completely (and effectively) represented by $\mathcal{E}_{V(k)}$ through e^d .

(ii) The class of k-transductions is therefore the minimal class of subsets of $D^k(V^*)$ which contains the finite k-ary relations in V^* and is closed under set-

union, complex-product, and the submonoid generating operation in $D^k(V^*)$.

Another class of events in $D^k(V^*)$ is introduced by the following intuitive description:

Imagine a Rabin-Scott automaton over the alphabet $V_\beta = V \cup \{\beta\}$ where $\beta \notin V$. This automaton is reading a k -multiple input channel, which is loaded with words over the alphabet V , by means of a reading head with the following behavior. This head feeds the main automaton with the letters of the input. First the first letter of each one of the k words is fed into the main automaton and then the second and so on, according to a fixed order of the channels. Whenever the head reads a blank space (after passing through one of the words) it transmits a β to the main automaton. The head stops the operation of the system when it passes the end of the largest word among the k input words without changing the internal state of the main automaton. The k -tuple of the input words is accepted iff the main automaton, starting its operation with the initial state, ends it in a designated final state.

Formally, we have the following situation:

Let us denote by $V_\beta^{(k)}$ the cartesian k -th power of V_β ; i.e.,

$$V_\beta^{(k)} = \{(v_1, \dots, v_k) : v_i \in V_\beta\} .$$

Clearly there exists a natural monomorphism

$$\delta : (V_{\beta}^{(k)})^* \longrightarrow D^k(V_{\beta}^*)$$

which is similar to d.

Both $D^k(V^*)$ and $D^k(\beta^*)$ are submonoids of $D^k(V_{\beta}^*)$ but $D^k(V^*) \cdot D^k(\beta^*)$, their complex-product, is obviously not closed under the multiplication in $D^k(V^*)$.

Now, $R \subseteq D^k(V^*)$ is said to be a fad (for "finite automaton definable")

k-ary relation iff there is a regular event E in $(V_{\beta}^{(k)})^*$ such that:

$$(i) \text{ for all } \xi \in E : \delta(\xi) \in D^k(V^*) \cdot D^k(\beta^*) ,$$

$$(ii) R = [(\mathcal{P}\delta)(E) : D^k(\beta^*)] \text{ where } [S_1 : S_2] =_{df} \{x : xy \in S_1 \text{ for}$$

some $y \in S_2\}$.

This definition can be simplified as a result of the following observations:

1. There exists a monomorphism δ' such that the following diagram is

commutative.

$$\begin{array}{ccc} (V_{\beta}^{(k)})^* & \xrightarrow{\delta} & D^k(V_{\beta}^*) \\ & \searrow \delta' & \nearrow d \\ & (V_{\beta}^{(k)})^* & \end{array}$$

Hence for every regular event E in $(V_{\beta}^{(k)})^*$ there is a regular event E' in $(V_{\beta}^{(k)})^*$

such that $(\mathcal{P}\delta)(E) = (\mathcal{P}d)(E')$.

2. If E satisfies requirement (i) then we have

$$[(\mathcal{P}\delta)(E) : D^k(\beta^*)] = (\mathcal{P}\varepsilon_\beta)(\mathcal{P}\delta)(E)$$

where

$$\varepsilon_\beta : D^k(V_\beta^*) \rightarrow D^k(V^*)$$

is the homomorphism of $D^k(V_\beta^*)$ which amounts to the erasure of β .

3. There exists a homomorphism ε'_β (which is similar to ε_β) such

that the following diagram is commutative.

$$\begin{array}{ccccc}
 (V_\beta^{(k)})^* & \xrightarrow{\delta} & D^k(V_\beta^*) & \xrightarrow{\varepsilon_\beta} & D^k(V^*) \\
 \delta' \searrow & & \nearrow d & & \nearrow d \\
 & & (V_\beta^{(k)})^* & \xrightarrow{\varepsilon'_\beta} & V_{(k)}^*
 \end{array}$$

Hence, for any fad k-ary relation R there is a regular event E'' in $V_{(k)}^*$ for

which $R = (\mathcal{P}d)(E'')$. Therefore, every fad k-ary relation is a k-transduction.

Obviously, $R_{(V^{(k)})^*} / \mathcal{P}d$ is a proper subclass of the class of fad k-ary

relations; namely those which are "length-preserving," that is, rectangular.

Thus we have the following decreasing sequence of classes of events in

$D^k(V^*)$:

1. $R_{V_{(k)}^*} / \mathcal{P}d$, the class of k-transductions;
2. the class of fad k-ary relations;
3. $R_{(V^{(k)})^*} / \mathcal{P}d^*$, the class of rectangular fad k-ary relations;

4. the class of finitary behaviors of sequence generators.

* — As for $R_{D^k(V^*)}$, I do not know where it is located relative to this sequence. It is quite possible that it is incomparable with any class in this sequence (except for the class of k-transductions. — *

6.6 TRANSDUCTIONS AS GRAMMARS

Let us define

$$\mu : D^k(V^*) \longrightarrow V^*$$

by

$$\mu(x_1, \dots, x_k) = x_1 x_2 \dots x_k \quad .$$

This function can be used to interpret systems defining events in $D^k(V^*)$ as

grammars for languages over V . In particular we can apply μ to $R_{V^*(k)} / \mathcal{P}d$ and

get the class of k-multiple finite-state languages.

Note, however, that the k-multiple finite-state languages can be defined by means of a generalization of finite-state grammars (Bar-Hillel & Shamir 1960) like the following:

Let $V_\tau \subset V$; a k-production rule is an ordered pair (v, ξ) where $v \in (V - V_\tau)^{(k)}$ and $\xi \in D^k(V^*)$. It is a finite-state k-production rule iff

$$\xi \in (V - V_\tau)^{(k)} \cdot D^k(V_\tau^*) \quad .$$

A k-multiple finite-state grammar is a system $\mathcal{G} = \langle v_0, \Pi \rangle$ where $v_0 \in (V - V_{\sim})^{(k)}$ and Π is a finite set of finite-state k-production rules.

The definition of k-multiple finite-state languages now follows naturally.

REFERENCES

- Ajdukiewicz, K., (1935) "Die Syntaktische Konnexität, Studia Philosophica, 1, 1-27.
- Bar-Hillel, Y., Ch. Gaifman & E. Shamir, (1960) "On Categorical and Phrase Structure Grammars," Bull. Res. Council of Israel, 9F, 1-16.
- Bar-Hillel, Y., M Perles & E. Shamir, (1961) "On Formal Properties of Simple Phrase Structure Grammars," Zeitschrift für Phonetik, Sprachwissenschaft und Kommunikationsforschung, 14, 143-172.
- Bar-Hillel, Y., & E. Shamir, (1960) "Finite State Languages: Formal Representation and Adequacy Problems," Bull. Res. Council of Israel, 8F, 155-166.
- Birkhoff, G., (1935) "Structure of Abstract Algebras," Proc. Cambridge Philos. Soc., 29, 441-464.
- Büchi, J. R., (1960) "Mathematical Theory of Automata," Notes on material presented by J. B. Wright and J. R. Büchi, Communication Science 403, Fall 1960, The University of Michigan.
- Burks, A. W., & J. B. Wright (1962) "Sequence Generators and Digital Computers," in Recursive Function Theory, Proc. of Symposia in Pure Math., 5, A.M.S., 139-199.
- Chomsky, N., & G. A. Miller, (1963) "Introduction to Formal Analysis of Natural Languages," in Handbook of Mathematical Psychology, II, John Wiley and Sons Inc., 269-418.
- Ginsburg, S., (1960) "Some Remarks on Abstract Machines," Trans. of the A.M.S., 96, 3, 400-444.
- Give'on, Y., (1960) "Boolean Matrices and Their Application to Finite Automata," Tech. Report No. 5, Applied Logic Branch, The Hebrew University of Jerusalem.
- , (1962) "Lattice Matrices and Finite Automata," Tech. Report No. 8, Applied Logic Branch, The Hebrew University of Jerusalem. (To be published in part in Information and Control under the title "Lattice Matrices".)
- , (1963) "The Application of Algebraic Systems to Finite Automata," Tech. Report 04995-1-T, ORA, The University of Michigan.
- , (1964) "The Theory of Algebraic Automata I: Morphisms and Regular Systems," Tech. Note 05662, 03105-27-T, ORA, The University of Michigan.

- Hartmanis, J., (1960) "Symbolic Analysis of a Decomposition of Information Processing Machines," Information and Control, 3, 2, 154-178.
- Holland, J. H., (1964) "Universal Embedding Spaces for Automata," Tech. Report 06114-1-T, ORA, The University of Michigan. (To be published in a festschrift for Norbert Wiener, Netherlands Central Institute for Brain Research, Amsterdam.)
- Krohn, K. B., & J. L. Rhodes, (1963) "Algebraic Theory of Machines," in Mathematical Theory of Automata, Microwave Research Institute Symposia Series, Volume XII, Polytechnic Press, Brooklyn, N.Y., 341-384.
- Laing, R., & J. B. Wright, (1962) "Commutative Machines," Tech. Note 04422, 03105-24-T, ORA, The University of Michigan.
- Lambek, J., (1958) "The Mathematics of Sentence Structure," American Math. Monthly, 65, 154-170.
- , (1961) "On the Calculus of Syntactic Types," in Structure of Language and its Mathematical Aspects, Proc. 12th Symp. in Appl. Math., Providence, R.I.: American Mathematical Society, 166-178.
- McKnight, J. D., (1963a) "Fine Quotients of Semigroups," G.E. Computer Lab. Report 63-CDL-8.
- , (1963b) "Generalized Kleene Quotient Theorems," G.E. Computer Lab. Report 63-CDL-10.
- Medvedev, I. T., (1958) "Finite Automata and the Representation of Events," (Trans. from Russian by Jacques J. Schorr-Kon) Lincoln Laboratory Group Report 34-73, Lexington, Mass.
- Mezei, J., (1963) "Structure of Monoids with Applications to Automata," in Mathematical Theory of Automata, Microwave Research Institute Symposia Series, Volume XII, Polytechnic Press, Brooklyn, N.Y., 267-300.
- , (1964) "Congruence Relations on Abstract Algebras," IBM Research Note NC-352.
- Rabin, M. O., (1964) "On the Word Problem for Abelian Monoids," (Private Communication)
- Rabin, M. O. & D. Scott, (1959) "Finite Automata and Their Decision Problems," IBM Journal, 3, 2, 114-125.
- Thatcher, J. W., (1963) "Notes on Mathematical Automata Theory," Tech. Note 05662, 03105-26-T, ORA, The University of Michigan.

UNIVERSITY OF MICHIGAN



3 9015 02825 9995