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Technical Note

THE THEORY OF ALGEBRAIC AUTOMATA I:
MORPHISMS AND REGULAR SYSTEMS

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INTRODUCTION

This is a preliminary report on a study of an algebraic gener-
alization of the concept of regular events. Among all the possible ways
of generalization we chose the one suggested by the characterization of
regular events by means of homomorphisms with finite ranges (cf. [RS]
and [YG] 1963).

Following this suggestion, our attention is directed to the study
of homomorphisms of monoids and their effect on subsets of monoids.
Furthermore, we are able now to suggest a general algebraic framework in
which several and various domains in the area of automata theory (like
finite-state transductions, commutative machines, and context-free languages)
can be studied and generalized uniformly.

In this report, we present the study of the basic and immediate
properties of regular systems in monoids and the effect of homomorphisms
on such systems.

1.1 ALGEBRAIC PRELIMINARIES AND NOTATIONS

Let W be any monoid (i.e., a semigroup with identity). A binary relation π defined in W is said to be <u>algebraic in W</u> iff it is compatible with the operation in W; i.e., iff

$$(w_1, w_2) \in \pi \text{ implies } (ww_1, ww_2), (w_1 w, w_2 w) \in \pi$$
$$\text{for any } w, w_1, w_2 \in W.$$

π is said to be a <u>congruence (relation) in W</u> iff π is an algebraic equivalence relation defined in W.

If π is a congruence in W, then an operation can be well defined in W/π by

$$\pi(w_1) \cdot \pi(w_2) =_{df} \pi(w_1 \cdot w_2) \quad ,$$

(where $\pi(w)$ denotes the equivalence class in W/π which contains w) and a mapping

$$\hat{\pi} : W \to W/\pi$$

by

$$\hat{\pi}(w) =_{df} \pi(w)$$

As one can easily prove, $\hat{\pi}$ is a homomorphism and thus it is said to be the <u>homomorphism (of W) induced by π</u>.

On the other hand, for any homomorphism ϕ of W we denote by $\hat{\phi}$ the binary relation defined in W by

$$(w_1, w_2) \in \hat{\phi} \text{ iff}_{df} \phi(w_1) = \phi(w_2)$$

As one can easily prove, $\hat{\phi}$ is a congruence in W (it is said to be the <u>congruence induced by ϕ in W</u>) and from $\phi = \hat{\pi}$ follows $\hat{\phi} = \pi$.

Studies of the relationship between the homomorphisms of W and the congruences in W can be found in the literature on monoids and semigroups ([CP], [JM]).

Since the relations $\pi \leftrightarrow \hat{\pi}$ and $\phi \leftrightarrow \hat{\phi}$ induce a certain duality within the context of this paper, and since the traditional distinction between the homomorphisms of W and the congruences induced by them in W can be easily proved to be unnecessary, we shall often identify the homomorphism ϕ of W with the congruence $\hat{\phi}$ induced by it in W and use the term morphism ϕ to denote simultaneously ϕ and $\hat{\phi}$. Moreover, we shall choose to use either the mapping notation or the relation notation for the morphisms according to the convenience of each notation in the specific context.

Let ϕ be a morphism of W and let E be a subset of W. The subset of W, $c_\phi(E)$, defined by

$$c_\phi(E) =_{df} \bigcup\{\phi(w) : w \in E\}$$

is said to be the ϕ-closure of E and E is said to be ϕ-closed iff $c_\phi(E) = E$.

Clearly, the operation c_ϕ on the subsets of W is a closure operation, i.e.:

(i) $E \subseteq c_\phi(E)$,

(ii) $E_1 \subseteq E_2$ implies $c_\phi(E_1) \subseteq c_\phi(E_2)$,

(iii) $c_\phi(c_\phi(E)) = c_\phi(E)$.

We denote by $C_\phi[W]$ the class of all subsets of W which are ϕ-closed. For example, if I is the identity on W then $C_I[W]$ is $\mathcal{P}(W)$, the class of all subsets of W.

As an immediate property of $C_\phi[W]$ we have the following lemma:

LEMMA 1: $C_\phi[W]$ is a Boolean algebra of sets which is isomorphic to $\mathcal{P}(W/\phi)$ under ϕ^ε (which is ϕ as a mapping operating on the subsets of W).

<u>REMARK</u>: We shall denote both ϕ and ϕ^{ε} and even $\phi^{\varepsilon\varepsilon}$ by "ϕ".

We shall be interested in the following three operations on morphisms:

The <u>direct product</u> of the morphisms ϕ_1 and ϕ_2 of W is denoted by $\phi_1 \boxtimes \phi_2$ and defined by

$$\phi_1 \boxtimes \phi_2 =_{df} \phi_1 \cap \phi_2 \quad .$$

The <u>direct sum</u> of ϕ_1 and ϕ_2 is denoted by $\phi_1 \oplus \phi_2$ and is defined by

$$\phi_1 \oplus \phi_2 =_{df} (\phi_1 \cup \phi_2)^*$$

where π^* denotes the transitive-closure of π.

Obviously, these operations can be extended by induction to n-ary operations for any n. The following theorem summarizes the main properties of these operations.

<u>THEOREM 2:</u> Let $\phi_1, \ldots, \phi_n$; be n morphisms of W, then:

(i) $\underset{i=1}{\overset{n}{\boxtimes}} \phi_i$ and $\underset{i=1}{\overset{n}{\oplus}} \phi_i$ are also morphisms of W;

(ii) $\underset{i=1}{\overset{n}{\boxtimes}} \phi_i$ is the maximal morphism of W which is included in all the ϕ_i, since $\underset{i=1}{\overset{n}{\boxtimes}} \phi_i = \underset{i=1}{\overset{n}{\cap}} \phi_i$;

(iii) $\underset{i=1}{\overset{n}{\oplus}} \phi_i$ is the minimal morphism of W which includes all the ϕ_i, and in fact, $\underset{i=1}{\overset{n}{\oplus}} \phi_i = (\underset{i=1}{\overset{n}{\cup}} \phi_i)^*$;

(iv) $C_{\underset{i=1}{\overset{n}{\boxtimes}} \phi_i}[W] = \underset{i=1}{\overset{n}{\cap}} {}^{\varepsilon}C_{\phi_i}[W]$ where $\underset{i}{\cap}{}^{\varepsilon}C_i$ denotes the class of

the intersections $\underset{i}{\cap} E_i$ of all $E_i \in C_i$;

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(v) $\displaystyle C_{\underset{i=1}{\overset{n}{\oplus}} \phi_i}[W] = \bigcap_{i=1}^{n} C_{\phi_i}[W]$;

(vi) $\otimes$ and $\oplus$ are commutative and associative operations;

(vii) if W/ϕ_i is finite for all $1 \leq i \leq n$ then so are

$$W/\overset{n}{\underset{i=1}{\otimes}} \phi_i \text{ and } W/\overset{n}{\underset{i=1}{\oplus}} \phi_i .$$

The <u>cartesian product</u> of the morphisms ϕ_1 and ϕ_2 of W_1 and W_2 respectively, is denoted by $\phi_1 \times \phi_2$ and defined to be a morphism of $W_1 \otimes W_2$ by

$$(\phi_1 \times \phi_2)(w_1, w_2) =_{df} \phi_1(w_1) \times \phi_2(w_2)$$

(where "x" denotes the operation of the cartesian product of sets.)

Again this operation can be extended by induction.

THEOREM 3: Let $\phi_1, \ldots, \phi_n$; be morphisms of $W_1, \ldots, W_n$ respectively, then:

(i) $\displaystyle \overset{n}{\underset{i=1}{\times}} \phi_i$ is a morphism of $\displaystyle \overset{n}{\underset{i=1}{\otimes}} W_i$;

(ii) $\displaystyle C_{\overset{n}{\underset{i=1}{\times}} \phi_i}[\overset{n}{\underset{i=1}{\otimes}} W_i] = \overset{n}{\underset{i=1}{\times}} {}^{\varepsilon}C_{\phi_i}[W_i]$ where $x^{\varepsilon}C_i$ denotes the class

of the cartesian products $\underset{i}{\times} E_i$ of all $E_i \in C_i$;

(iii) the cartesian product of morphisms is an associative and essentially commutative operation;

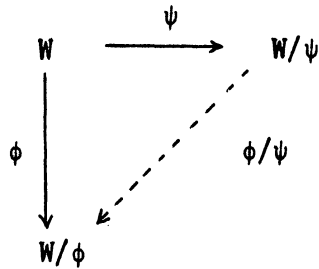(iv) if W_i/ϕ_i is finite for any $1 \leq i \leq n$, then so is

$$\overset{n}{\underset{i=1}{\otimes}} W_i / \overset{n}{\underset{i=1}{\times}} \phi_i .$$

Finally, let ϕ be a morphism of W. A morphism ψ of W is said to be a morphism of ϕ iff ψ is included in ϕ.

The significance of this relation is given in the following theorem:

THEOREM 4: If ψ is a morphism of the morphism ϕ of W then the following diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\ \psi\ } & W/\psi \\
\phi \downarrow & \swarrow & \phi/\psi \\
W/\phi & &
\end{array}
$$

can be completed by $\phi/\psi : W/\psi \to W/\phi$ to be a commutative diagram, i.e.,

$$(\phi/\psi) \circ \psi = \phi,$$

in a unique way.

Furthermore, we have

$$C_{\phi/\psi}[W/\psi] = \psi C_\phi[W]$$

and in particular, for any $F \subseteq W/\phi$

$$(\phi/\psi)^{-1}(F) = \psi(\phi^{-1}(F)) .$$

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1.2   REGULAR EVENTS IN MONOIDS

We wish to characterize a family of subsets of a monoid W in a fashion similar to the way the regular events are characterized as certain subsets of a finitely generated free monoid ([RS], [YG] 1963).

DEFINITION 1:   A **W-regular system** is a system $\mathcal{A}$ = <ρ, F> where:

(i) ρ is a morphism of W with a finite range,

(ii)   F is a subset of the range of ρ.

The morphism ρ is said to be the **structure of** $\mathcal{A}$ and $T(\mathcal{A})$ $=_{df}$ $\rho^{-1}(F)$ = $\bigcup F$ is **the event generated by** $\mathcal{A}$. A subset E of W is said to be **regular in** W iff E is the event generated by some W-regular system.

We denote by R_W the class of all subsets of W which are regular in W, that is,

$$R_W =_{df} \bigcup \{C_\rho[W]:W/\rho \text{ is finite}\} \quad .$$

As an immediate result of Df. 1 we get:

LEMMA 5: R_W is closed under set complementation.

From the properties of the direct product of morphisms we derive the following construction of Boolean combinations of W-regular systems:

For any Boolean set function β_S we denote by β_ρ the corresponding Boolean proposition function which is isomorphic to β_S and thus the following relationship holds:

let β_S be a Boolean set function of n variables and let $S_1, \ldots, S_n$ be any n subsets of S, then

$$\beta_S(S_1, \ldots, S_n) = \{s \in S : \beta_\rho(s \in S_1, \ldots, s \in S_n)\} \quad .$$

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Given n W-regular systems $\mathcal{A}_i = <\rho_i, F_i>$ and a Boolean set function β_S of n variables, we define:

$$\beta(\mathcal{A}_1, \ldots, \mathcal{A}_n) =_{df} <\overset{n}{\underset{i=1}{\text{\&}}} \rho_i, \; \beta^*(F_1, \ldots, F_n)>$$

where

$$\beta^*(F_1, \ldots, F_n) =_{df} \{(\overset{n}{\underset{i=1}{\text{\&}}} \rho_i)(w) : \beta_P(\rho_1(w) \in F_1, \ldots, \rho_n(w) \in F_n)\} \; .$$

Obviously,

$$T(\beta(\mathcal{A}_1, \ldots, \mathcal{A}_n)) =$$

$$= \beta^*(F_1, \ldots, F_n) \quad ,$$

$$= \{w : \beta_P(w \in \bigcup F_1, \ldots, w \in \bigcup F_n)\} \quad ,$$

$$= \beta_S (\bigcup F_1, \ldots, \bigcup F_n) \quad ,$$

$$= \beta_S(T(\mathcal{A}_1), \ldots, T(\mathcal{A}_n)) \quad .$$

Thus we have:

THEOREM 6: R_W is a Boolean algebra of sets.

In the rest of this section we shall explore, with the expected results, the relationships between the following concepts:

    (i)   W-regular systems and R_W;

    (ii) right invariant relations in W;

    (iii) finite-state automata with W as their input;

    (iv) non-deterministic finite-state automata with W as their

            input.

The methods used to establish the expected relationships are taken from the study of ordinary finite automata (cf. [RS], [RB], and [YG] 1960).

Let $\hat{f}$ be an equivalence defined in W which is <u>right-invariant</u> (i.e., $(w_1, w_2) \in \hat{f}$ implies $(w_1 w, w_2 w) \in \hat{f}$ for any $w, w_1, w_2 \in W$). We associate with $\hat{f}$ the following system $\mathcal{Q}_f = \langle S, s_0, \tau_f \rangle$ where:

(i) $S = W/\hat{f}$,

(ii) $s_0 = \hat{f}(\lambda)$, where λ is the identity element of W,

(iii) $\tau_f : S \times W \to S$ is defined by $\tau_f(\hat{f}(w_1), w) =_{df} \hat{f}(w_1 w)$.

[Note that τ_f is well defined just because $\hat{f}$ is right-invariant.]

Now, for any $S_F \subseteq S$ we have:

$$T(\mathcal{Q}_f, S_F) =_{df} \{w \in W : \tau_f(s_0, w) \in S_F\} \quad ,$$

$$= \{w \in W : \hat{f}(w) \in S_F\} \quad ,$$

$$= \bigcup S_F \quad .$$

Hence, if $E \subseteq W$ is a union of certain equivalence classes of a right-invariant equivalence $\hat{f}$ defined in W, then E is defined by an automaton (which is finite state iff $W/\hat{f}$ is finite), with W as its input (cf. [SG]) in a manner similar to that by which ordinary regular events are defined by Rabin-Scott finite automata.

On the other hand, let $\mathcal{Q} = \langle S, s_0, \tau \rangle$ be a system with:

(i) S is a set and $s_0 \in S$,

(ii) $\tau : S \times W \to S$ is a mapping satisfying:

$\tau(s, \lambda) = s$ for all $s \in S$,

$\tau(s, w_1 w_2) = \tau(\tau(s, w_1), w_2)$ for all $s \in S, w_1, w_2 \in W$;

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[(iii)-optional - for each $s \in S$ there is $w \in W$ such that

$$\tau(s_0, w) = s].$$

We define the binary relation $\hat{f}_\tau$ in W by:

$$(w_1, w_2) \in \hat{f}_\tau \quad \text{iff}_{df} \quad \tau(s_0, w_1) = \tau(s_0, w_2) \quad .$$

Clearly, $\hat{f}_\tau$ is an equivalence and the cardinality of $W/\hat{f}_\tau$ is less than or equal to that of S. If S is finite, then equality holds iff requirement (iii) is satisfied. Furthermore, $\hat{f}_\tau$ is right-invariant.

Now let $S_F \subseteq S$. The system $< \mathcal{Q}, S_F >$ is said to be a <u>W-automaton</u> (which is a <u>finite-state</u> W-automaton iff S is finite) with <u>the</u> <u>structure</u> $\mathcal{Q}$. The set:

$$T(\mathcal{Q}, S_F) =_{df} \{w \in W : \tau(s_0, w) \in S_F\}$$

is said to be the <u>event</u> <u>defined</u> by $< \mathcal{Q}, S_F >$.

Clearly, we have:

$$T(\mathcal{Q}, S_F) = \bigcup \{\{\tau(s_0, w_1) : \tau(s_0, w_1) = \tau(s_0, w)\} : \tau(s_0, w) \in S_F\}$$

$$= \bigcup \{\hat{f}_\tau(w) : \tau(s_0, w) \in S_F\}$$

$$= \bigcup \hat{S}_F \quad \text{where} \quad \hat{S}_F =_{df} \{\hat{f}(w) : \tau(s_0, w) \in S_F\} \quad .$$

In conclusion we have:

<u>LEMMA 7</u>: A subset E of W is a union of certain equivalence classes of a right-invariant equivalence (with a finite index) in W, iff E is defined by some (finite-state) W-automaton.

The connection with W-regular systems is established by the following lemma:

LEMMA 8:   Let E be a subset of W, then E is regular in W iff E is defined by a finite-state W-automaton.

PROOF: Let $<\mathcal{U}, S_F>$ be a finite-state W-automaton with $\mathcal{U} = <S, s_0, \tau>$. Define in W the binary relation ρ_τ by :

$$(w_1, w_2) \varepsilon \rho_\tau \quad \text{iff}_{df} \quad \text{for all } s \varepsilon S : \tau(s, w_1) = \tau(s, w_2) \quad .$$

Clearly, ρ_τ is a morphism of W with a finite range (which is less than or equal to the cardinality of $S \times S$,) and ρ_τ is included in $\hat{f}_\tau$, i.e.,

$$\rho_\tau(w_1) = \rho_\tau(w_2) \text{ implies } \tau(s_0, w_1) = \tau(s_0, w_2) \text{ for all } w_1, w_2 \varepsilon W.$$

Hence, if we define

$$F =_{df} \{\rho_\tau(w) : \tau(s_0, w) \varepsilon S_F\}$$

we get a W-regular system $<\rho_\tau, F>$ which generates $T(\mathcal{U}, S_F)$.

The converse follows directly by Lemma 7.

Before we turn to establish the identity of the W-regular events and the events defined by non-deterministic W-automata, we wish to give an alternative proof of Lemma 8 by the means of the construction of the transition monoid $M(\mathcal{U})$ associated with the structure $\mathcal{U}$ of W-automata. (For the case of the free monoid cf. [RB], [YG] 1960 and 1962.)

Let $\mathcal{U} = <S, s_0, \tau>$ be a structure of W-automata. We define $M(\mathcal{U})$ to be the set of all functions

$$\tau_w : S \rightarrow S$$

defined for any $w \varepsilon W$ (using the suffix notation for functions) by:

$$s\tau_w =_{df} \tau(s, w) \qquad .$$

$M(\mathcal{U})$ is a monoid of functions since τ_λ (where λ is the identity element of W) is the identity on S and for any w_1, w_2 ϵ W we have:

$$s\tau_{w_1 w_2} = s\tau_{w_1} \cdot \tau_{w_2} \qquad .$$

This implies the existence of a morphism ρ_τ of W with $W/\rho_\tau = M(\mathcal{U})$ which is defined by:

$$\rho_\tau(w) =_{df} \tau_w \qquad .$$

Clearly, if S is finite then $M(\mathcal{U})$ is finite and for any $S_F \subseteq S$ if we define

$$S_F^\tau =_{df} \{\tau_w : s_0 \tau_w \epsilon S_F\}$$

we get, for a finite S, a W-regular system $<\rho_\tau, S_F^\tau>$ which generates $T(\mathcal{U}, S_F)$.

Let us define a system $\mathcal{N} = <S, \pi>$ to be a <u>structure of</u> <u>(finite-state)</u> <u>non-deterministic W-automata</u> iff:

(i)   S is a (finite) set,

(ii) $\pi : S \times W \rightarrow \mathcal{P}(S)$ is a mapping from $S \times W$ into the class

of all subsets of S, $\mathcal{P}(S)$, satisfying

$\pi(s, w_1 w_2) = \bigcup\{\pi(s', w) : s' \epsilon \pi(s, w_1)\}$ for all $s \epsilon S$

and w_1, w_2 ϵ W.

We associate with $\mathcal{N}$ the monoid $M(\mathcal{N})$ of the binary relations π_w defined in S for any w ϵ W by:

$$(s_1, s_2) \epsilon \pi_w \quad iff_{df} \quad s_2 \epsilon \pi(s_1, w) \qquad .$$

From requirement (ii) follows that for any w_1, w_2 ϵ W we have:

$$\pi_{w_1 w_2} = \pi_{w_1} \cdot \pi_{w_2} \qquad ,$$

and therefore $M(\mathcal{M})$ is closed under the complex-product of binary relations and π_λ is its identity element. Thus $M(\mathcal{M})$ is indeed a monoid of binary relations and it is the morphic image of W under ρ_π which is defined by:

$$\rho_\pi(w) =_{df} \pi_w \quad .$$

For any S_O, $S_F \subseteq S$ we say that the <u>non-deterministic</u> <u>W-automaton</u> $<S_O, \mathcal{M}, S_F>$ <u>defines</u> the event

$$T(S_O, \mathcal{M}, S_F) =_{df} \{w \varepsilon W : \pi(s, w) \varepsilon S_F \text{ for some } s \varepsilon S_O\} \quad .$$

Obviously, we get:

$$T(S_O, \mathcal{M}, S_F) = \{w \varepsilon W : \pi_w \cap (S_O \times S_F) \neq \emptyset\}$$

$$= \{w \varepsilon W : \rho_\pi(w) \varepsilon [S_O, \pi, S_F]\}$$

where $[S_O, \pi, S_F] =_{df} \{\pi(w) \varepsilon M(\mathcal{M}) : \pi(w) \cap (S_O \times S_F) \neq \emptyset\}$.

Hence, if S if finite then $T(S_O, \mathcal{M}, S_F)$ is generated by the W-regular system $<\rho_\pi, [S_O, \pi, S_F]>$. Thus, similar to the "free" automata, we can regard the non-deterministic (finite-state) W-automata (as defined here) as a special case of W-regular systems where the range of their structure is a monoid of binary relations defined in a certain (finite) set.

1.3 THE EFFECT OF HOMOMORPHISMS ON W-REGULAR SYSTEMS

Theorem 4 leads us to the following definition:

DEFINITION 2: A morphism ψ of W is said to be <u>a morphism of the W-regular</u> <u>system</u> $\mathcal{A} = <\rho, F>$ iff ψ is a morphism of ρ. In this case, the W-regular system $<\rho/\psi, F>$ is said to be the <u>morphic image of</u> $\mathcal{A}$ (under ψ), or alternatively, the <u>ψ-image of</u> $\mathcal{A}$ and will be denoted by $\mathcal{A}/\psi$.

And it implies directly:

LEMMA 9: Let ψ be a morphism of the W-regular system $\mathcal{A} = <\rho, F>$ and let $E \subseteq W$; then E is generated by $\mathcal{A}$ iff $\psi(E)$ is generated by $\mathcal{A}/\psi$.

In this section we study the effects of the morphisms of W on R_W. Later we shall study the properties of the morphisms of W-regular systems.

Our main result is summarized by the following theorem:

THEOREM 10: Let ϕ be a morphism of W and let $E \subseteq W$. Then $\phi(E)$ is regular in W/ϕ iff $c_\phi(E)$ is regular in W.

In particular we have:

(i) if E_2 is generated by the W/ϕ-regular system $\mathcal{A}' = <\rho, F>$ then $\phi^{-1}(E_2) = c_\phi(\phi^{-1}(E_2))$ is generated by the W-regular system $\mathcal{A} = <\rho \circ \phi, F>$;

(ii) if E is any subset of W such that $c_\phi(E)$ is generated by the W-regular system $\mathcal{A} = <\rho, F>$ then $\phi(E) = \phi(c_\phi(E))$ is generated by the W/ϕ-regular system $\mathcal{A}' = <\phi/(\phi \; \Theta \; \rho), (\rho/(\phi \; \Theta \; \rho))(F)>$.

PROOF: Immediate; for the proof of (ii) apply Theorem 4, Lemma 9 and Theorem 2 ((iii) & (v)).

As an immediate but important corollary we get:

COROLLARY 10.1: If W is a finitely generated monoid, say by V, and E is a

subset of W then: E is regular in W iff $\phi^{-1}(E)$ is a ("free") regular event

over V as an alphabet, where ϕ is the natural homomorphism of the free monoid

generated by V onto W.

These results motivate us to consider the two classes of events introduced

in the following two definitions.

DEFINITION 3: Let ϕ be a morphism of W; we denote by $\underline{\phi R_W}$ the class of the

ϕ-images of the regular events in W. That is:

$$\phi R_W =_{df} \{\phi(E) : E \; \epsilon \; R_W\} \qquad .$$

Clearly, Theorem 10 implies:

COROLLARY 10.2: If ϕ is a morphism of W then

$$R_{W/\phi} \subseteq \phi R_W \qquad .$$

In Lemma 1 we noticed that ϕ^ϵ, which is ϕ operating on the subsets of

W, is a one-to-one correspondence between $C_\phi[W]$ and $\wp(W/\phi)$. Now, from

Theorem 10 we infer:

COROLLARY 10.3: If ϕ is a morphism of W then ϕ^ϵ (restricted appropriately)

is a one-to-one correspondence between $R_W \cap C_\phi[W]$ and $R_{W/\phi}$.

DEFINITION 4: Let ϕ be a morphism of W; we denote by $C_\phi(R_W)$ the class of

all the ϕ-closures of the regular events in W. That is:

$$C_\phi(R_W) =_{df} \{c_\phi(E) : E \; \epsilon \; R_W\} \qquad .$$

Clearly, we have:

LEMMA 11: If ϕ is a morphism of W then ϕ^ϵ (restricted appropriately) is a one-to-one correspondence between $C_\phi(R_W)$ and ϕR_W.
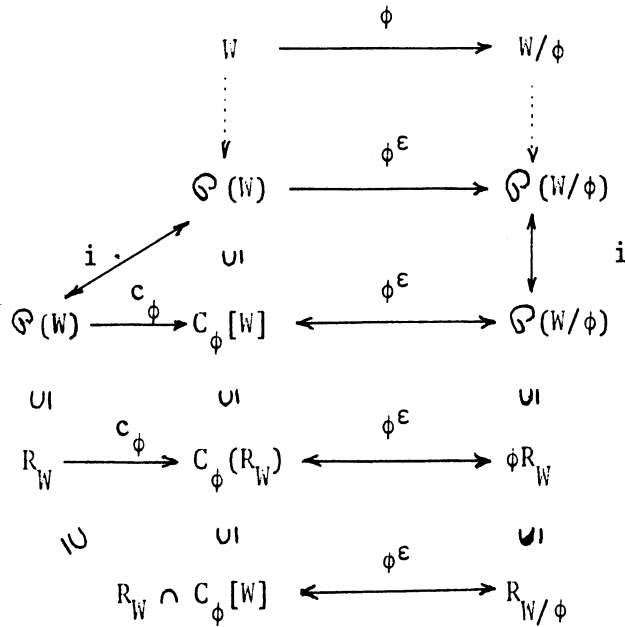
Thus, we get the following conclusion of Theorem 10:

THEOREM 12: Let ϕ be a morphism of W then

$$\phi R_W = R_{W/\phi} \qquad \text{iff} \qquad C_\phi(R_W) = R_W \qquad ;$$

that is, iff c_ϕ is regularity-preserving in W.

We can summarize these results by the following diagram:

where:

(i) ϕ^ϵ is ϕ operating on the subsets of W, i.e.,

$$\phi^\epsilon(S) =_{df} \{\phi(s) : s \in S\} \qquad ;$$

(ii) i is the identity on its contexts;

(iii) c_ϕ is the operation of ϕ-closure;

(iv) $\longleftrightarrow$ shows one-to-one correspondence.

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