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Technical Report

TOWARD A HOMOLOGICAL ALGEBRA OF AUTOMATA IV
5. The Characterization of Projective Automata

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ABSTRACT

The problem of characterizing the projective objects of a studied category is a natural one. In our case of the category theoretic study of automata, the importance of this characterization is derived from the importance of the characterization of M_W, the input monoid regarded as an automaton.

The methods of homological algebra appear to be sufficient for a complete characterization of the projective automata. This characterization implies a possible application of automata theory to the theory of monoids (in analogy to the fruitful interaction between ring theory and module theory). It also provides a characterization of M_W for finite input monoids, and thus the categorical-completeness theorem for $\mathcal{A}_W$ for any finite input monoid W.

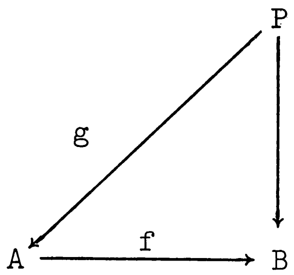
0. INTRODUCTION

0.1 Following the main problem of our first paper in this series, we want to characterize the projective objects of $\mathcal{A}_W$, the category of automata (as state diagrams) with W as input.

0.2 Naturally, we follow our notation and terminology set up in our previous papers (Give'on 1964).

1. BASIC NOTIONS

1.1 We recall that an automaton P is <u>projective</u> iff for any morphism $P \overset{h}{\to} B$ and any epic $A \overset{f}{\to} B$ there exists a morphism $P \overset{g}{\to} A$ for which the following diagram is commutative.



1.2 For any automaton A and any <u>set</u> T we define $T \cdot A$ to be the automaton whose set of states is

$$T \times S_A = \{(t, s) : t \in T \ \& \ s \in S_A\}$$

and whose transition function is defined by

$$(t, s) \cdot w =_{df} (t, s \cdot w) \quad .$$

Obviously $T \cdot A = \bigoplus_{t \in T} (\{t\} \cdot A)$ and $\{t\} \cdot A \approx A$.

1.3 It is easy to verify that for any set T, $T \cdot M_W$ is free and projective.

1.4 We are trying to follow the study of projective modules as closely as possible.

1.4.1 An <u>epic</u> $A \overset{f}{\to} B$ is said to <u>split</u> iff there is a morphism $B \overset{g}{\to} A$ for which

$$B \overset{g}{\to} A \overset{f}{\to} B = i_B \quad .$$

1.4.2 A <u>monic</u> $B \overset{g}{\to} A$ is said to <u>split</u> iff there is a morphism $A \overset{f}{\to} B$ for which

$$B \overset{g}{\to} A \overset{f}{\to} B = i_B \quad .$$

1.4.3 In general we say a morphism f that it <u>splits</u> iff it is a unit in the composition algebra of the morphisms of $\mathcal{A}_W$. Obviously a morphism splits iff it is either a monic or an epic which splits.

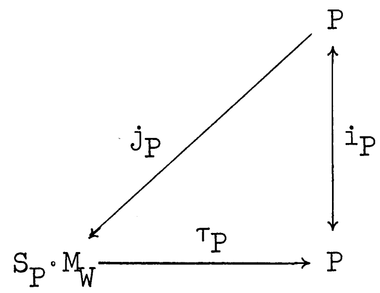
1.4.4 A sequence $A \overset{g}{\to} B \overset{f}{\to} A$ is said to be a <u>splitting sequence of A through B</u> iff $fg = i_A$.

Clearly in this case f is an epic and g is a monic which split.

1.5 <u>LEMMA</u>. An automaton P is projective iff it has a splitting sequence through some free automaton $T \cdot M_W$.

<u>Proof</u>: We always have an epic $S_P \cdot M_W \overset{\tau_P}{\longrightarrow} P$ defined by $\tau_P(s, w) = s \cdot w$. Assume that P is projective then for $P \overset{i_P}{\longrightarrow} P$ there exists a morphism

$P \xrightarrow{j_P} S_P \cdot M_W$ for which the following diagram is commutative,



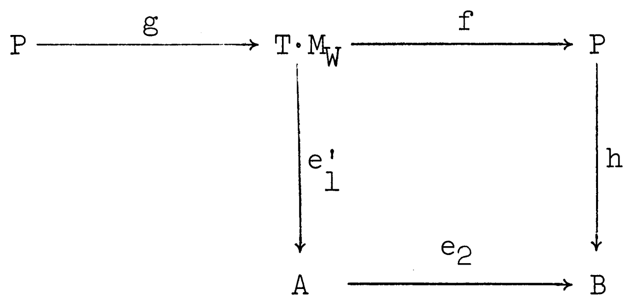
this yields a splitting sequence

$$P \xrightarrow{j_P} S_P \cdot M_W \xrightarrow{\tau_P} P$$

of P through $S_P \cdot M_W$. We shall refer to this sequence as <u>the canonical splitting sequence of P.</u> On the other hand, assume that

$$P \xrightarrow{g} T \cdot M_W \xrightarrow{f} P$$

is a splitting sequence. Let $P \xrightarrow{h} B$ be any morphism and $A \xrightarrow{e_2} B$ be an epic. Since $T \cdot M_W$ is projective, there exists a morphism $T \cdot M_W \xrightarrow{e_1'} A$ for which the following diagram is commutative.



Hence, for $e_1 = e_1'g$ we have

$$e_2 e_1 = e_2 e_1' g = hfg = h$$

which proves that P is projective.

1.5.1 REMARK: Our arguments show in fact that P_1 is projective iff it has a splitting sequence through some projective automaton P_2.

1.5.2 COROLLARY: Any projective automaton is a direct sum (i.e., disjoint union) of monogenic (i.e., epic images of M_W) projective automata.

Proof: We refer again to the canonical splitting sequence of P

$$P \xrightarrow{j_P} S_P \cdot M_W \xrightarrow{\tau_P} P$$

defined in the proof of Lemma 1.5.

Since $S_P \cdot M_W = \underset{s}{\oplus}\{s\} \cdot M_W$ and $P \xrightarrow{j_P} S_P \cdot M_W$ is monic, we have a decomposition of P as a direct sum $P = \underset{s}{\oplus} P_s$ where

$$S_{P_s} = \{s' \epsilon S_P : j_P(s') \epsilon s \cdot W\} \quad ,$$

and a family of monic morphisms

$$P_s \xrightarrow{j_P^s} \{s\} \cdot M_W$$

defined by

$$j_P^s \; : \; S_{P_s} \rightarrow \{s\} \times W \; : \; s' \rightarrow j_P(s') \quad .$$

On the other hand we define a family of morphisms

$$\{s\} \cdot M_W \xrightarrow{\tau_P^s} P$$

by

$$\tau_P^s \; : \; \{s\} \times W \rightarrow S_P \; : \; (s, w) \rightarrow s \cdot w \quad .$$

Obviously, for any $s \in S_P$ we have:

$$i_{P_s} = i_P|P_s = \tau_P j_P|P_s = \tau_P j_P^s = \tau_P^s j_P^s \quad ,$$

hence $\tau_P^s(\{s\}\cdot M_W) = P_s$ and therefore P_s is monogenic.

1.5.3 COROLLARY: (The converse of Cor. 1.5.2)

A direct sum of projective automata is projective.

Proof: By Lemma 1.5 we have for $P = \bigoplus_s P_s$ a family of splitting sequences

$$P_s \xrightarrow{g_s} T_s \cdot M_W \xrightarrow{f_s} P_s$$

and we can assume that all the T_s are disjoint. Hence we have the splitting sequence

$$\bigoplus_s P_s \xrightarrow{\bigoplus_s g_s} (\bigcup_s T_s)\cdot M_W \xrightarrow{\bigoplus_s f_s} \bigoplus_s P_s$$

where $\bigoplus_s g_s$ and $\bigoplus_s f_s$ are morphisms defined as the unions of $\{g_s\}$ and of $\{f_s\}$ (resp.). Thus P is projective.

1.6 Thus we have that a direct sum of automata is projective iff each summond is projective. Furthermore, an automaton is projective iff it is a direct sum of monogenic projective automata.

Hence, in order to characterize the projective objects of $\mathcal{A}_W$ we have only to characterize the monogenic projective automata.

2. MONOGENIC PROJECTIVE AUTOMATA

2.1 An automaton P is monogenic iff there exists an epic $M_W \overset{e}{\to} P$.

If P is projective we have a splitting sequence

$$P \overset{j}{\to} M_W \overset{e}{\to} P$$

of P through M_W. Hence, if P is a monogenic projective automaton it is isomorphic to a monogenic subautomaton of M_W.

2.2 An element $u \in W$ is said to be <u>projective</u> iff the subautomaton $M_W(u)$ of M_W generated by u is projective.

2.2.1 Note that the set of states of $M_W(u)$ is the principal right ideal of W generated by u:

$$I(u) = \{uw \ : \ w \in W\} \ .$$

2.3 A rephrasing of Lemma 1.5, as applied to monogenic projective automata, expressed in terms of the elements of W, yields the following proposition.

2.3.1 <u>PROPOSITION:</u> An element $u \in W$ is projective iff there is $j(u) \in W$ for which

(i) $uw_1 = uw_2$ iff $j(u)w_1 = j(u) \cdot w_2$;

and

(ii) $u = uj(u)$.

Proof: By Lemma 1.5 and by 2.1, $M_W(u)$ is projective iff there exists a splitting sequence of $M_W(u)$ through M_W:

$$M_W(u) \xrightarrow{j} M_W \xrightarrow{e_u} M_W(u)$$

where $M_W \xrightarrow{e_u} M_W(u)$ is the canonical morphism determined by

$$e_u \ : \ W \to I(u) \ : \ w \to uw \ .$$

Hence, if $M_W(u)$ is projective, the morphism $M_W(u) \xrightarrow{j} M_W$ is monic. That is,

$$j(uw_1) = j(uw_2) \ \text{ iff } \ uw_1 = uw_2 \ .$$

But $j(uw_i) = j(u)w_i$, hence (i). In addition to this, $e_u j = i_{M_W(u)}$, and so

$$u = (e_u \circ j)(u) = e_u(j(u)) = uj(u) \ .$$

On the other hand, assume (i) and (ii). Define

$$M_W(u) \xrightarrow{j} M_W$$

by

$$j \ : \ I(u) \to W \ : \ uw \to j(u)w \ ,$$

by (i), j is monic. For the canonical epic $M_W \xrightarrow{e_u} M_W(u)$ we get by (ii)

$$\begin{aligned}(e_u \circ j)(uw) &= e_u(j(uw)) = e_u(j(u)w) \\ &= (uj(u))w = uw \ ,\end{aligned}$$

hence

$$M_W(u) \overset{j}{\rightarrow} M_W \overset{e_u}{\longrightarrow} M_W(u)$$

is a splitting sequence and therefore $M_W(u)$ is projective.

2.4 REMARKS: It follows immediately from Prop. 2.3.1 that if u is an idempotent of W then it is projective; set $j(u) = u$.

However, this is not a necessary condition for u being projective. For example, if u is a left-cancellative element of W (i.e., if $uw_1 = uw_2$ always implies $w_1 = w_2$), then u is projective. In fact if u is such an element of W, $M_W \overset{e_u}{\rightarrow} M_W(u)$ is an isomorphism.

Hence in particular, if W is a group or a free monoid, or any left cancellative monoid, then the only monogenic projective automaton is M_W (up to an isomorphism, of course).

To be more general, if u is an element of W for which there is x in W such that $u = uxu$, then u is projective; set $j(u) = xu$. For example, if $u = u^n$ for some $n > 1$.

2.5 From the characterization of the projective elements in W given by Prop. 2.3.1, it follows that if u is projective then j(u) is an idempotent of W:

$$uj(u) = u1_W, \quad \text{hence} \quad j(u)j(u) = j(u)\cdot 1_W \quad .$$

This observation yields the following complete characterization of the monogenic projective automata.

2.5.1 THEOREM: A monogenic automaton is projective iff it is isomorphic to a monogenic subautomaton of M_W generated by an idempotent of W.

Proof: The splitting sequence

$$M_W(u) \overset{j}{\to} M_W \xrightarrow{e_u} M_W(u)$$

implies that $M_W(u) \overset{j}{\to} j(M_W(u))$ is an isomorphism. But obviously

$$j(M_W(u)) = M_W(j(u)) \quad .$$

2.5.2 REMARK: Thus, even though the class of the projective elements in W may be larger than the class of the idempotents of W, the latter class is sufficient for the characterization of the projective elements of W and of the projective monogenic automata.

2.6 The characterization of monogenic projective automata by means of splitting sequences through M_W yields Theorem 2.5.1 directly by considering the idempotents of $End(M_W)$ as follows.

An automaton P is monogenic and projective iff it has a splitting sequence $P \overset{j}{\to} M_W \overset{e}{\to} P$ through M_W. Hence, if P is monogenic and projective, there exists an endomorphism $M_W \xrightarrow{je} M_W$ which is an idempotent of $End(M_W)$,

$$jeje = j(ej)e = je \quad ,$$

and P is isomorphic to $M_W(je(1_W))$. From the idempotence of je follows

$$je(1_W) = je(je(1_W)) = je(1_W)\ je(1_W) \quad ,$$

i.e., $je(1_W)$ is idempotent in W.

On the other hand, if $M_W \overset{\sigma}{\to} M_W$ is an idempotent endomorphism of M_W and $P = M_W(\sigma(1_W))$, then

$$P \overset{j}{\to} M_W \xrightarrow{\sigma'} P$$

for the inclusion morphism j and for

$$\sigma' : W \to \sigma(1_W)\ W : w \to \sigma(1_W) \cdot w = \sigma(w) \quad ,$$

is a splitting sequence of P through M_W.

The relationship between the idempotents of W and of $End(M_W)$ was established by the representation theorem for $\mathcal{A}_W$ (Give'on 1964) since W is isomorphic to $End(M_W)$.

3. THE CHARACTERIZATIONS OF M_W FOR A FINITE W

3.1 In our first paper in this series (Give'on 1964) we showed that if M_W can be identified (up to an isomorphism) by means of categorical predicates then $\mathcal{A}_W$ is categorically complete. There we succeeded to characterize M_W for the case where W is a unit-commutative monoid (i.e., where $w_1w_2 = 1_W$ always implies $w_1w_2 = w_2w_1$).

3.2 Our results in this paper enable us to characterize M_W for the case where W is any finite monoid. Thus one can hope to apply the categorical study of $\mathcal{A}_W$ to the theory of finite monoids.

3.2.1 <u>THEOREM</u>: If W is a finite monoid then an object X of $\mathcal{A}_W$ is isomorphic to M_W iff X satisfies the following two conditions:

(i) X is monogenic;

(ii) for any monogenic projective automaton P there exists a monic $P \overset{j}{\rightarrow} X$.

<u>Proof</u>: By our previous results we know that M_W satisfies these conditions. Assume now that X is an automaton which satisfies them as well.

Since X is monogenic there exists an epic $M_W \overset{e}{\to} X$ and therefore the cardinality of S_X is not larger than that of W. Since M_W is projective we have a monic $M_W \overset{j}{\to} X$ and therefore j is surjective and it determines an isomorphism of M_W and X.

3.2.2 <u>COROLLARY</u>: If W is finite then $\mathcal{A}_W$ is categorically complete.

BIBLIOGRAPHY

Give'on, Y., "Toward a Homological Algebra of Automata, I, II & III," Tech. Reports in process, ORA, The University of Michigan (1964-1965).